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1990

A SIMPLE DUALITY MODEL OF PRODUCTION INCORPORATING RISK AVERSION AND PRICE UNCERTAINTY

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Risk

Invited Paper Session

CAEFMS/AAEA Annual Meeting

Vancouver, Canada

August 6, 1990

AAEA 1990

p.46

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ABSTRACT

Proceeding within the framework of a linear mean-variance utility function, this paper develops a duality model of production that incorporates risk aversion and price uncertainty. In contrast to risk models based on an expected utility function, this model provides a practical alternative to standard duality models for econometric research.

#8464

A SIMPLE DUALITY MODEL OF PRODUCTION

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Empirical applications of the dual approach to modelling producer behavior in agriculture have invariably assumed that producers are risk neutral. This reflects the widely held opinion, based on the results of studies conducted within the framework of expected utility maximization (e.g. Pope, Epstein), that duality models cannot readily incorporate risk aversion and uncertainty. Indeed this presumed inability to model risk has been a major criticism of the dual approach.

In contrast the present paper demonstrates that duality theory can easily accomodate price uncertainty within the context of a linear mean-variance model of choice under risk. The resulting dual model retains the essential advantages of price certainty models in terms of econometric estimation, hypothesis testing and policy inference (Fuss and McFadden), and the standard price certainty model is nested within this model. Thus the model developed here provides a practical alternative to standard price certainty models whenever risk aversion and price uncertainty are important determinants of behavior.

Although the assumption of linearity is highly restrictive, linear mean-variance models have had extensive application in agricultural economics. Moreover the analysis can be extended to more general nonlinear mean-variance models of choice in a manner that remains tractable for empirical research, and it has recently been demonstrated that nonlinear mean-variance models often provide a close approximation to empirical expected utility models (Sinn, Meyer).

The Model

The mean-variance approach has usually been applied in terms of a utility function that is linear in expected profits $E\pi$ and variance of profits $V\pi$:

$$(1) \quad U = E\pi - (\alpha/2)\, V\pi$$

where $\alpha>0$ assuming risk aversion (e.g. Chavas and Pope). Profits are defined as

$$(2) \quad \pi = py - wx$$

where y,x denote the vectors of output and input levels and p,w denote the corresponding price vectors. Assuming that outputs y, inputs x and input prices w are nonstochastic, the expected value and variance of profits conditional on (y,x) are

$$(3) \quad E\pi(y,x) = \bar{p}y - wx$$

$$V\pi(y,x) = y^T\, Vp\, y$$

where $\bar{p}$ denotes the vector of expected output prices p and Vp denotes the covariance matrix of prices p. For example in the case of a single output $V\pi = y^2 Vp$, and in the case of two outputs $V\pi = (y^1)^2 \operatorname{var}(p^1) + (y^2)^2 \operatorname{var}(p^2) + 2y^1y^2 \operatorname{cov}(p^1,p^2)$.

Substituting (3) into (1), the producer's choice problem is expressed as

$$(4) \quad U^*(\bar{p},w,Vp) = \max_{x,y \in T} \; U(x,y) \equiv p\bar{y} - wx - (\alpha/2)\, y^T\, Vp\, y \; \ldots$$

where $U^*(\bar{p},w,Vp)$ denotes the producer's dual indirect utility function, i.e. the relation between maximum feasible utility U^* and exogenous variables $\bar{p}$,w,Vp. Only the M(M+1)/2 distinct elements of the symmetric matrix Vp are included as arguments of the dual. The properties of this utility function are summarized as follows.

Proposition 1. Assume existence of the utility function (4).

Then

a) U^* is increasing in $\bar{p}$, decreasing in w, decreasing in Vp;

b) U^* is linear homogeneous in $(\bar{p},w,Vp)$,

i.e. $U^*(\lambda\bar{p},\lambda w,\lambda Vp) = \lambda U^*(\bar{p},w,Vp)$ for $\lambda>0$;

c) U^* is convex in $(\bar{p},w,Vp)$, i.e. $U^*(\lambda\bar{p}^a+(1-\lambda)\bar{p}^b,\lambda w^a+(1-\lambda)w^b,\lambda Vp^a +(1-\lambda)Vp^b) \le \lambda U^*(\bar{p}^a,w^a,Vp^a) + (1-\lambda)U^*(\bar{p}^b,w^b,Vp^b)$ for $0\le\lambda\le1$; and

d) (assuming $U^*(.)$ is differentiable)

(5) $\quad \partial U^*(\bar{p},w,Vp)/\partial\bar{p}^j = y^{j*} \qquad j=1,.,M$

(6) $\quad \partial U^*(\bar{p},w,Vp)/\partial w^i = -x^{i*} \qquad i=1,.,N$

(7) $\quad \partial U^*(\bar{p},w,Vp)/\partial Vp_{jj} = -(\alpha/2)(y^{j*})^2 \qquad j=1,.,M$

$\quad \partial U^*(\bar{p},w,Vp)/\partial Vp_{ij} = -\alpha\, y^{i*}y^{j*} \qquad i\ne j;j=1,.,M$

where x^*,y^* denotes the solution to (4) conditional on $(\bar{p},w,Vp)$, and Vp_{ij} denotes $\mathrm{cov}(p^i,p^j)$.

Proof. 1-a is obvious. Since U(x,y) is linear in $(\bar{p},w,Vp)$ by (4), x^* and y^* are homogeneous of degree zero in $(\bar{p},w,Vp)$, and in turn 1-b is established. Suppose (x^a,y^a) solves (4) given $(\bar{p}^a,w^a,Vp^a)$, (x^b,y^b) solves (4) given $(\bar{p}^b,w^b,Vp^b)$, and (x^c,y^c) solves (4) given $(\bar{p}^c,w^c,Vp^c) \equiv \lambda(\bar{p}^a,w^a,Vp^a)+(1-\lambda)(\bar{p}^b,w^b,Vp^b)$. Then

$$U^*(\bar{p}^c,w^c,Vp^c) = \bar{p}^c y^c - w^c x^c - (\alpha/2)\, y^{cT}\, Vp^c\, y^c$$
$$= \lambda\, [\bar{p}^a y^c - w^a x^c - (\alpha/2)\, y^{cT}\, Vp^a\, y^c]$$
$$+ (1-\lambda)\, [\bar{p}^b y^c - w^b x^c - (\alpha/2)\, y^{cT}\, Vp^b\, y^c]$$
$$\le \lambda\, U^*(\bar{p}^a,w^a,Vp^a) + (1-\lambda)\, U^*(\bar{p}^b,w^b,Vp^b)$$

since (x^c,y^c) does not generally solve (4) given $(\bar{p}^a,w^a,Vp^a)$ or $(\bar{p}^b,w^b,Vp^b)$, e.g. $U^*(\bar{p}^a,w^a,Vp^a) \ge \bar{p}^a y^c - w^a x^c - (\alpha/2)\; y^{cT}\; Vp^a\; y^c$. This establishes 1-c. 1-d is obtained by applying the envelope theorem to problem (4) (e.g. Takayama, pp. 137-9). (7) follows from the symmetry restriction on Vp, i.e. $Vp_{ij}=Vp_{ji}$ $(i,j=1,.,M)$.

Q.E.D.

Proposition 1-c,d implies the following comparative static properties for the output supply and factor demand relations $y^* = y(\bar{p},w,Vp)$, $x^* = x(\bar{p},w,Vp)$: the matrix of second derivatives

$$(8) \qquad \nabla^2 U^*(\bar{p},w,Vp) \equiv \begin{bmatrix} \partial y/\partial\bar{p} & \partial y/\partial w & \partial y/\partial Vp \\ -\partial x/\partial\bar{p} & -\partial x/\partial w & -\partial x/\partial Vp \\ -\partial Q/\partial\bar{p} & -\partial Q/\partial w & -\partial Q/\partial Vp \end{bmatrix}$$

$$Q \equiv (\alpha/2)yy^T$$

is symmetric positive semidefinite. These restrictions on (8) imply, as in the standard theory of the competitive risk-neutral firm,

$$(9) \qquad \partial y^j(\bar{p},w,Vp)/\partial\bar{p}^j \geq 0 \qquad j=1,.,M$$

$$\partial x^i(\bar{p},w,Vp)/\partial w^i \leq 0 \qquad i=1,.,N$$

$$(10) \qquad \partial y^j(\bar{p},w,Vp)/\partial\bar{p}^i = \partial y^i(\bar{p},w,Vp)/\partial\bar{p}^j \qquad i,j=1,.,M$$

$$\partial x^i(\bar{p},w,Vp)/\partial w^j = \partial x^j(\bar{p},w,Vp)/\partial w^i \qquad i,j=1,.,N$$

$$\partial y^j(\bar{p},w,Vp)/\partial w^i = -\partial x^i(\bar{p},w,Vp)/\partial p^j \qquad i=1,.,N;j=1,.,M.$$

Thus output supplies are increasing in own expected prices, factor demands are decreasing in own prices, and standard reciprocity relations are satisfied.

In addition the above restrictions on (8) imply that $(\alpha/2)\partial(yy^T)/\partial Vp$ is symmetric negative semidefinite and in turn $\alpha y^j \partial y^j(\bar{p},w,Vp)/\partial Vp_{jj} \leq 0$ $(j=1,.,M)$. Thus in the case of risk aversion $(\alpha>0)$,

$$(11) \qquad \partial y^j(\bar{p},w,Vp)/\partial Vp_{jj} \leq 0 \qquad j=1,.,M.$$

In other words, an increase in the variance of output price p^j leads (ceterus paribus) to a reduction in supply of output j by the risk averse firm. In the case of a single output, risk aversion and uncertainty about the output price leads to a reduction in output supply relative to the case of risk neutrality or price certainty.

We can also deduce from Proposition 1 that the impacts of output price uncertainty on factor demands are ambiguous.[1] Since all terms in the submatrix $\partial x(\bar{p},w,Vp)/\partial Vp$ are off the diagonal of matrix (8), all of these terms are unsigned by convexity of the dual.

The impacts of output price uncertainty on output supplies and factor demands are related to the impacts of expected prices as follows. Symmetry of (8) implies the following reciprocity relations in addition to (10):

$$(12)\ \partial y^i(\bar{p},w,Vp)/\partial Vp_{jk} = -(\alpha/2)[y^j \partial y^k(\bar{p},w,Vp)/\partial \bar{p}^i + y^k \partial y^j(\bar{p},w,Vp)/\partial \bar{p}^i] \quad i,j,k=1,.,M$$

$$\partial x^i(\bar{p},w,Vp)/\partial Vp_{jk} = (\alpha/2)[y^j \partial y^k(\bar{p},w,Vp)/\partial w^i + y^k \partial y^j(\bar{p},w,Vp)/\partial w^i] \quad i=1,.,N; j,k=1,.,M.$$

In the case of a single output, (12) reduces to

$$(13)\ \partial y(\bar{p},w,Vp)/\partial Vp = -\alpha\, y\, \partial y(\bar{p},w,Vp)/\partial \bar{p}$$

$$\partial x^i(\bar{p},w,Vp)/\partial Vp = \alpha\, y\, \partial y(\bar{p},w,Vp)/\partial w^i \qquad i=1,.,N.$$

Thus if input i is normal in the sense that $\partial y(\bar{p},w,Vp)/\partial w^i<0$ and the producer is risk averse, then $\partial x^i(\bar{p},w,Vp)/\partial Vp<0$. In this case the producer's demand for input i decreases as uncertainty about output price increases.

The above discussion demonstrates that all comparative static properties of major interest for model (4) correspond to first and second derivatives of the dual utility function $U^*(\bar{p},w,Vp)$. This result is consistent with the standard theory of the competitive risk neutral firm. Epstein (1977) also reaches a similar conclusion in the context of expected utility maximization and zero flexibility in production (as in this model, all input decisions are made prior to exact knowledge of the relevant output prices).

Proposition 1 implies more complex relations between derivatives of the dual than in standard price certainty models. The envelope relations (5) and (7) imply

$$(14)\quad \partial U^*(\bar{p},w,Vp)/\partial Vp_{jj} = -(\alpha/2)\ (\partial U^*(\bar{p},w,Vp)/\partial \bar{p}^j)^2$$

$$\partial U^*(\bar{p},w,Vp)/\partial Vp_{ij} = -\alpha\ \partial U^*(\bar{p},w,Vp)/\partial \bar{p}^i\ \partial U^*(\bar{p},w,Vp)/\partial \bar{p}^j \qquad i\neq j;j=1,..,M.$$

Thus derivatives of the dual with respect to price covariances Vp are simple nonlinear functions of derivatives of the dual with respect to expected prices $\bar{p}$.

These restrictions (14) imply that in practice it is easier to specify functional forms for the output supply and factor demand equations (5)-(6) and then work backwards to the dual utility function rather than vice-versa. This contrasts with standard price certainty models, where it is simpler to specify a functional form for the dual and then derive output supply and factor demand equations using the envelope theorem. First functional forms are specified for the derivatives of the dual with respect to prices $\bar{p}$,w:

$$(15)\quad \partial U^*(\bar{p},w,Vp)/\partial \bar{p}^j = a^j(\bar{p},w,Vp) \qquad j=1,..,M$$

$$\partial U^*(\bar{p},w,Vp)/\partial w^i = b^i(\bar{p},w,Vp) \qquad i=1,..,N$$

where these functions are homogeneous of degree zero (Proposition 1-b) and satisfy the restriction that the M+N dimensional matrix $\partial^2 U^*(\bar{p},w,Vp)/\partial\bar{p}\partial w$ is symmetric. Restrictions (14) imply

$$(16)\quad \partial U^*(\bar{p},w,Vp)/\partial Vp_{jj} = -(\alpha/2)\ a^j(\bar{p},w,Vp)^2$$

$$\partial U^*(\bar{p},w,Vp)/\partial Vp_{ij} = -\alpha\ a^i(\bar{p},w,Vp)\ a^j(\bar{p},w,Vp) \qquad i\neq j;j=1,..,M.$$

Proposition 1-b also implies, by Euler's theorem,

$$(17)\quad U^*(\bar{p},w,Vp) = \Sigma_j\ \partial U^*(\bar{p},w,Vp)/\partial\bar{p}^j\ \bar{p}^j + \Sigma_i\ \partial U^*(\bar{p},w,Vp)/\partial w^i\ w^i + \Sigma_j\Sigma_{k\geq j}\ \partial U^*(\bar{p},w,Vp)/\partial Vp_{jk}\ Vp_{jk}$$
$$= \Sigma_j\ a^j(\bar{p},w,Vp)\ \bar{p}^j + \Sigma_i\ b^i(\bar{p},w,Vp)\ w^i - (\alpha/2)\ \Sigma_j\ a^j(\bar{p},w,Vp)^2\ Vp_{jj} - \alpha\ \Sigma_j\Sigma_{k>j}\ a^i(\bar{p},w,Vp)\ a^j(\bar{p},w,Vp)\ Vp_{jk}$$

by (15)-(16). Equation (17) defines a dual utility function that satisfies linear homogeneity (Proposition 1-b), symmetry and restrictions (14).[2]

For example, consider the case where one output supply and N factor demand equations are specified as follows:

$$(18)\quad y^* = \partial U^*(\bar{p},w,Vp)/\partial\bar{p} = a_{00} + \Sigma_k\ a_{0k}\ (w^k/\bar{p})^{1/2} + a_{0,N+1}\ (Vp/\bar{p})^{1/2}$$
$$-x^{i*} = \partial U^*(\bar{p},w,Vp)/\partial w^i = a_{i0}\ (\bar{p}/w^i)^{1/2} + \Sigma_k\ a_{ik}\ (w^k/w^i)^{1/2} + a_{i,N+1}\ (Vp/w^i)^{1/2} \qquad i=1,..,N$$

where the matrix of coefficients $[a_{ij}]$ is symmetric. Then (14) and (17) imply the following functional form for the dual utility function:

$$(19)\quad U^*(\bar{p},w,Vp) = a_{00}\ \bar{p} + \Sigma_k\ a_{0k}\ \bar{p}^{1/2}(w^k)^{1/2} + a_{0,N+1}\ \bar{p}^{1/2}Vp^{1/2} + \Sigma_i\ a_{i0}\ \bar{p}^{1/2}(w^i)^{1/2} + \Sigma_i\Sigma_k\ a_{ik}\ (w^i)^{1/2}(w^k)^{1/2} + \Sigma_i\ a_{i,N+1}\ (w^i)^{1/2}Vp^{1/2} - (\alpha/2)\ [a_{00} + \Sigma_k\ a_{0k}\ (w^k/\bar{p})^{1/2} + a_{0,N+1}\ (Vp/\bar{p})^{1/2}]^2\ Vp.$$

If Vp=0, then (19) reduces to a standard Generalized Leontief profit function under price certainty. Note that all coefficients of the dual utility function (19) except for α are included directly in the output supply and factor demand equations (18). Moreover, as will be discussed below, α can be calculated indirectly from estimates of these equations.

Furthermore a second order differential approximation to a true dual utility function $U^*(\bar{p},w,Vp)$, and knowledge of α in the linear mean-variance utility function (1), provides a second order differential approximation to a corresponding true production or transformation function. This statement is established by the following Proposition, which also demonstrates that second derivatives of the production or transformation function at an equilibrium combination of inputs and outputs can easily be calculated from the dual utility function and from knowledge of α, expected prices $\bar{p}$ and the covariance matrix of prices Vp. The following notation is adopted in this Proposition. Assume m outputs, n+1 inputs and define the vector of net outputs $z=(y^1,..,y^m,-x^1,..,-x^n)$. Define the producer's transformation function in the form $z^o \equiv -x^{n+1}=f(z)$. Define the vector of prices $v=(p^1,..,p^m,w^1,..,w^n)$ and $v^o=w^{n+1}$. Assume that at least factor price v^o is nonstochastic (i.e. known with certainty when all input decisions are made), let $\bar{v}$ denote expected values for prices v, and let Vv denote the covariance matrix of prices v. Then the producer's maximization problem can be expressed in terms of the dual utility function

$$(20) \quad U^*(v^o,\bar{v},Vv) = \max_z \; \bar{v}z + v^o f(z) - (\alpha/2)\, z^T\, Vv\, z \, .$$

Proposition 2. Assume an interior solution $z^* >> 0$ to problem (20), $U^*(v^o,\bar{v},Vv)$ twice differentiable, and $f(z)$ twice differentiable at z^*. Then

$$[v^o\, f_{zz}(z^*) - \alpha\, Vv] = -[U^*_{\bar{v}\bar{v}}(v^o,\bar{v},Vv)]^{-1}$$

where f_{zz} and $U^*_{\bar{v}\bar{v}}$ denote matrices of second derivatives.[9]

Proof. The first order conditions for an interior solution $z^* >> 0$

to (20) are

(21) $\bar{v}^i + v^o\ \partial f(z^*)/\partial z^i - \alpha\ Vv_i\ z\ = 0$ $\qquad$ i=1,.,m+n

where Vv_i denotes the i'th row of Vv. Total differentiating (21) with respect to $\bar{v}$,

(22) $I + v^o\ f_{zz}(z^*)\ \partial z^*/\partial\bar{v} - \alpha\ Vv\ \partial z^*/\partial\bar{v}\ = 0$

where I denotes an identity matrix. Proposition 1-d implies $U^*_{\bar{v}\bar{v}}(v^o,\bar{v},Vv)\ =\ \partial z^*/\partial\bar{v}$ which has full rank without loss of generality (i.e. without violating the homogeneity conditions in Proposition 1-b). Substituting $U^*_{\bar{v}\bar{v}}(v^o,\bar{v},Vv)$ for $\partial z^*/\partial\bar{v}$ in (22) and rearranging establishes Proposition 2. Q.E.D.

Thus the properties of the dual utility function (4) are largely analogous to the properties of the standard dual profit function in the absence of risk aversion (α=0) (e.g. Diewert). This suggests the following simple procedure for estimating output supply and factor demand relations under the assumption of a linear mean-variance utility function and stochastic output prices. First specify, in the manner discussed above, a second order flexible functional form for the dual utility function $U^*(\bar{p},w,Vp)$ that is linear homogeneous in $(\bar{p},w,Vp)$ and satisfies restrictions (14), e.g. a modified Generalized Leontief or Normalized Quadratic. Corresponding output supply and factor demand equations typically are linear in coefficients of the utility function.[4] These equations are consistent with the functional form for the dual utility function provided that the reciprocity conditions (10) are satisfied.[5]

Within this context a simple nested test of the hypothesis of risk neutrality (α=0) is available: under risk neutrality all coefficients with price covariance terms in the output supply and factor demand equations are insignificant, i.e. the dual utility function (4) reduces to a standard dual profit function $\pi(\bar{p},w) = \max \bar{p}y - wx$. If the hypothesis of risk neutrality is rejected, then the coefficient α for the linear mean-variance utility function can be calculated from estimates of equations (5)-(6) as follows. Differentiating (7) with respect to $\bar{p}^j$,

$$(23)\quad \partial^2U^*(.)/\partial\bar{p}^j\partial Vp_{jj} = -\alpha\, y^j\ \partial y^j(.)/\partial\bar{p}^j$$
$$= -\alpha\ \partial U^*(.)/\partial\bar{p}^j\ \ \partial^2U^*(.)/\partial\bar{p}^{j2}$$

by (5). Rearranging (23), α can be calculated from estimates of the output supply equations as follows:

$$(24)\quad \alpha = \frac{-\ \partial^2U^*(.)/\partial\bar{p}^j\partial Vp_{jj}}{\partial U^*(.)/\partial\bar{p}^j\ \ \partial^2U^*(.)/\bar{p}^{j2}}\ .$$

In addition α can be estimated directly by respecifying an output supply equation (5) as (using 23)

$$(25)\quad y^j = -\ (1/\alpha)\ \frac{\partial^2U^*(.)/\partial\bar{p}^j\partial Vp_{jj}}{\partial^2U^*(.)/\bar{p}^{j2}}\ .$$

Alternatively, if there are sufficient degrees of freedom, the dual utility function can be estimated jointly with (5)-(6) as

$$(26)\quad \bar{p}y - wx = (\alpha/2)\ y^T\ Vp\ y + U^*(\bar{p},w,Vp)\ .$$

<u>Conclusion</u>

It is apparent here that duality models of production can accomodate risk aversion and price uncertainty in a manner that is tractable for empirical research. Indeed in the case of a linear mean-variance utility function, which is the most popular form of the mean-variance model in agricultural research, price uncertainty does not complicate or compromise the dual approach in any essential manner. The dual approach remains tractable, albeit more complex, for a nonlinear mean-variance utility function.

FOOTNOTES

1. As in standard duality theory for the competitive risk-neutral agent, Proposition 1 exhausts the implications of the maximization hypothesis (4) for comparative statics.

2. As a check on the consistency of equations (17) and (15), note that differentiating $\tilde{U} \equiv U^*_{\bar{p}}\, \bar{p} + U^*_w\, w + U^*_{Vp}\, Vp$ (17) with respect to (e.g.) $\bar{p}$ yields

$$\tilde{U}_{\bar{p}} = U^*_{\bar{p}} + U^*_{\bar{p}\bar{p}}\, \bar{p} + U^*_{\bar{p}w}\, w + U^*_{\bar{p}Vp}\, Vp$$
$$= U^*_{\bar{p}}$$

since the derivative $U^*_{\bar{p}}$ is homogeneous of degree zero in $(\bar{p},w,Vp)$ by Proposition 1-b and Euler's theorem.

3. An analogous relation between a dual profit function and a transformation function is derived by Lau for the standard case of a competitive firm and nonstochastic prices. Proposition 2 reduces to Lau's result if either $\alpha=0$ or $Vv=0$.

4. In contrast, a Translog dual utility function implies that expected shares $\bar{p}y^*/U^*, wx^*/U^*$ depend on the coefficient α of the linear mean-variance utility function (1). In this case the output supply and factor demand equations cannot be estimated by linear methods.

5. Output supply and factor demand equations (5)-(6), satisfying reciprocity conditions (10), integrate up to a class of functions that includes the hypothesized functional form for the dual utility function (Epstein 1982). Estimating these equations jointly with the dual utility function of course restricts this class to the hypothesized utility function.

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