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DYNAMIC RESOURCE AND AGRICULTURAL ECONOMICS USING

STOCHASTIC OPTIMAL CONTROL: AN INTRODUCTION

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by

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Abstract

This paper analyzes the optimum control of stochastic processes. An investment model is used to introduce stochastic differential equations, interpret Ito's lemma, and derive Bellman's equation. An economic model of soil conservation where erosion is a stochastic process is then used to derive and interpret the stochastic maximum principle.

DYNAMIC RESOURCE AND AGRICULTURAL ECONOMICS USING STOCHASTIC OPTIMAL CONTROL: AN INTRODUCTION

I. Introduction

Most problems in agricultural and resource economics involve the management of "assets" that evolve stochastically over time. Plants, pest populations, and wildlife grow according to random biological growth functions. Farmer debt changes with future and uncertain interest rates, expenditures, and revenues. Soil erosion depends on weather, and groundwater contamination occurs through uncertain fate and transport dynamics. Research and development firms must allocate expenditures to acquire knowledge that may or may not lead to future marketable products. The need for analyzing stochastic and dynamic problems is clear. For example, Trapp (p.2) writes that "dynamics and stochastics go hand-in-hand, just as statics and perfect knowledge do. I do not believe we will go very far forward in dynamic theory without further considerations of questions of imperfect knowledge and stochasticness."

In the agricultural and resource economics literature, discrete-time stochastic dynamic programming seems to be the technique of choice to analyze stochastic and dynamic problems, even though continuous-time techniques are often used in the deterministic case. To complement the discrete-time approach, this paper introduces the basic techniques for the optimum control of continuous stochastic processes. Although more complete developments can be found in, for example, Brock, Chow, or Kushner, two specific economic examples are utilized here to develop the necessary background and intuition to include stochastic optimum control as another working tool for the agricultural economist. While it is assumed that the reader is familiar with control techniques for the deterministic case, no background in stochastic calculus is necessary.

The paper proceeds as follows. In Section II, a dynamic investment model of the firm when future prices and capital stocks are uncertain is used to introduce stochastic differential equations, interpret Ito's lemma for the differential of a function of stochastic processes, and identify Bellman's equation. Bellman's equation and the envelope theorem are then used to derive the stochastic and dynamic analogue of Hotellings lemma. Thus, it is shown that stochastic dynamic duality can be used to model firm behavior in the presence of Markovian price expectations. This result, which contradicts the conclusions in Taylor, is just one example where stochastic calculus may yield insights that are less apparent with discrete-time stochastic techniques. In Section III, an economic model of soil conservation with stochastic changes in the soil stock is used to derive and interpret the stochastic maximum principle. A method is also outlined for deriving the expected dynamics of the optimal choice functions, which can then be used to investigate the sensitivity of the optimal choices to parameters changes.

Brock (p.1) suggests that, "Economists use ordinary calculus every day without understanding the intricate details of Lebesque or Riemann integrals..... There is no reason why the same cannot be done for the stochastic calculus as well." This introduction to the optimum control of stochastic processes provides the necessary background to begin further exploration into the dynamic and stochastic nature of agricultural and resource problems.

II. Firm Investment under Uncertainty

The study of capital investment decisions has a long history in economics, and the links between investment and resource models are well recognized. In general investment models, the change in the capital stock $K(t)$ over time is often modelled as a deterministic differential equation $dK(t)/dt=I(t)-\delta K(t)$, where $I(t)$ is gross investment and $\delta K(t)$ is depreciation.

For the purposes of this paper, the change in the capital stock over time is modelled as a <u>stochastic</u> differential equation:

$$(1) \qquad dK(t) = [I(t) - \delta K(t)]dt + \sigma(K(t))dW(t)$$

where $I(t)$ is gross investment, $\delta K(t)$ is depreciation, $\sigma(K(t))$ is a function, and $W(t)$ is a random variable. More specifically, it is assumed that $W(t)$ is a <u>Wiener process</u> (or <u>Brownian motion process</u>).

The Wiener process $W(t)$ is characterized by the following assumptions:

(1) $W(t)$ is distributed normal with zero mean;

(2) $E[dW(t)]=0$;

(3) $E[dW(t)dW(t)]= dt$; and

(4) $E[dW(t)dW(s)]=0$ for s not equal to t.

Using these four assumptions, the two terms in equation (1) have straightforward interpretations: $[I(t) - \delta K(t)]dt$ is the expected change in the capital stock; and $\sigma(K(t))dW(t)$ is the unexpected change. And, since the variance in the change in the capital stock $Var(dK(t)) = E\{[dK(t)-E(dK(t))]^2\}$ $= E[\sigma(K(t))^2 dW(t)^2] = \sigma(K(t))^2 E[dW(t)^2] = \sigma(K(t))^2 dt$, the term $\sigma(K(t))^2$ is the variance of $dK(t)$ over the period dt.

For example, $\sigma(K(t))dW(t)$ could represent unexpected depreciation in the capital stock that is either less than or greater than expected depreciation. Thus, while $\delta K(t)$ would be expected depreciation of the firm, the term $\delta K(t) + \sigma(K(t)dW(t)$ would be total depreciation over the period dt. The term $\sigma(K(t))dW(t)$ could also be interpreted as general technological improvements that increase the effective stock of capital. Alternatively, if the variance of $dK(t)$ was modelled as $\sigma(K(t),I(t))^2 dt$, then $I(t)$ could represent nominal investment, while $\sigma(K,I)dW$ could represent unexpected technical change embodied in new investment.

Armed with the basic definition of a Wiener process, consider the following problem for the risk-neutral firm that must choose capital investments when future prices and capital stocks are unknown:

$$(2) \quad J(t,k,p) \equiv \max_{I(\tau)} E_t \int_t^\infty e^{-r\tau}[f(K(\tau),I(\tau)) - P(\tau)'K(\tau)]d\tau$$

$$\text{s.t.} \qquad dK(\tau) = [I(\tau) - \delta K(\tau)]d\tau + \sigma(K(\tau))dZ(\tau) \;;\; K(t) = k$$

$$dP(\tau) = g(P(\tau))d\tau + \alpha(P(\tau))dW(\tau) \;;\; P(t) = p$$

where $K(\tau)\geq 0$ is an N-vector of capital stocks; $I(\tau)\geq 0$ is an N-vector of gross investments; δ is an NxN diagonal matrix of depreciation rates; $P(\tau)$ is an N-vector of capital rental rates normalized by the output price; f is the production technology; r is the discount rate; $g(P(\tau))$ is a N-vector of functions $g^i(P(\tau))$ that represent expected price changes at τ; $\sigma(K(\tau))$ is an NxN matrix of functions σ^{ij}; $\alpha(P(\tau))$ is an NxN matrix of functions α^{ij}; E_t is the expectations operator conditional on information at initial time t; and the variables $Z(\tau)$and $W(\tau)$ are each N-vectors of Wiener processes, with $E[dZ(\tau)] = E[dW(\tau)] = 0$, $E[dZ(\tau)dZ(\tau)'] = Sdt$, $E[dW(\tau)dW(\tau)'] = Bdt$, and $E[dZ(\tau)dW(\tau)'] = 0$; and the random variables dZ and dW are independent across different time periods. The NxN matrix S is the correlation matrix for the vector $dZ(\tau)$, and the NxN matrix B is the correlation matrix for $dW(\tau)$. The model could be easily extended to the case where $dZ(\tau)$ and $dW(\tau)$ are correlated, which would imply that changes in the capital stocks and the capital rental prices are correlated.

Problem (2) states that the firm chooses investment to maximize the expected discounted value of firm profits over the period τ= t to ∞, given unexpected changes in prices and the capital stock. The value function for the firm's problem, J(t,k,p), is the expected maximized discounted value of firm profits. The value function J(t,k,p) is analogous to the expected profit function for the static case with uncertainty and risk neutrality.

Bellman's principle of optimality, which is valid for deterministic or stochastic control problems, implies that (see,e.g., Intrilligator):

$$(3.1)\quad J(t,K(t),P(t)) = \mathrm{Max}_I\{e^{-rt}[f(k,I) - pk]dt + E_t[J(t+dt,K(t+dt),P(t+dt))]\}$$

which can be rearranged to yield:

$$(3.2)\quad 0 = \mathrm{Max}_I\{e^{-rt}[f(k,I) - pk] + (1/dt)E_t[dJ]\}$$

where dJ is the differential of the value function J(t,k,p).

While (3.2) just restates the principle of optimality, the derivation of Bellman's equation involves substituting for the differential of the value function. However, because J is a function of the stochastic processes, i.e. capital stocks K and prices P, the change in the value function depends on uncertain changes in prices and capital stocks. As a result, the ordinary rules of calculus cannot be used to determine the differential dJ. While an infinite set of stochastic calculi can be defined (Brock), Ito's lemma provides the rule for stochastic differentials that will be used here and seems to be most commonly used in economics.

To derive the stochastic differential dJ and indirectly provide a justification for Ito's lemma, the first step is to expand J(t,k,p) in a Taylor series of order dt:

$$(4)\ dJ = J(t+dt,K(t+dt),P(t+dt)) - J(t,K(t),P(t)) =$$

$$J_t dt + J_k' dk + .5dk' J_{kk} dk + J_p' dp + .5dp' J_{pp} dp + dk' J_{kp} dp + o(dt)$$

where subscripts denote partial derivatives, and o(dt) is the remainder for terms of order higher than dt. The quadratic terms in k and p are included in the Taylor expansion because they are of order dt or less after taking expected values. It is also assumed that in the limit $E_t[o(dt)]/dt = 0$ as dt goes to 0.

The next step is to find $E_t[dJ]$, which involves evaluating the expected values of the terms in equation (4). Using the definitions of the Wiener processes W and Z, these expected values are:

(5.1) $E_t[J_t dt] = J_t dt$

(5.2) $E_t[J_k' dk] = J_k'(I - \delta k)dt$

(5.3) $E_t[J_p' dp] = J_p' g(p)dt$

(5.4) $E_t[dk' J_{kk} dk] = E_t[((I-\delta k)dt + \sigma(k)dZ)' J_{kk}((I-\delta k)dt + \sigma(k)dZ)]$

$= E_t[\sigma(k)' J_{kk} \sigma(k) dZdZ']$

$= \text{trace}[\sigma(k)' J_{kk} \sigma(k) S]dt,$

where the derivation in (5.4) uses the assumption that higher order terms than dt enter the remainder function o(dt).

Following the steps in (5.4),

(5.5) $E_t[dp' J_{pp} dp] = \text{trace}[\alpha(p)' J_{pp} \alpha(p) B]dt$

and

(5.6) $E_t[dk' J_{kp} dp] = 0$ since W and Z are independent.

Therefore, equations (4) and (5.1) - (5.6) yield after a little rearranging;

(6.1) $E_t[dJ] = \{J_t + J_k'(I-\delta k) + J_p' g(p) +$

$.5\text{trace}[\sigma' J_{kk} \sigma S] + .5\text{trace}[\alpha' J_{pp} \alpha B]\}dt + E_t[o(dt)]$

and letting dt approach 0 yields;

(6.2) $(1/dt)E_t[dJ] = J_t + J_k'(I-\delta k) + J_p' g(p) +$

$.5\text{trace}[\sigma' J_{kk} \sigma S] + .5\text{trace}[\alpha' J_{pp} \alpha B]$

since by assumption $E_t[o(dt)]/dt = 0$ as dt approaches 0.

Equation (6.2) is defined as the <u>differential generator</u> of the function $J(t,k,p)$ and gives the expected change through time of a function of stochastic processes [Chow]. This differential generator for the stochastic case is analogous to the total time derivative dJ/dt for the deterministic case.

Substituting the differential generator (6.2) into equation (3) yields Bellman's equation for problem (2):

$$(7) \quad -J_t = \mathrm{Max}_I\{e^{-rt}(f(k,I) - pk) + J_k'(I-\delta k) + J_p'g(p) + .5\mathrm{trace}[\sigma' J_{kk}\sigma S] + .5\mathrm{trace}[\alpha' J_{pp}\alpha B]\}$$

Bellman's equation evaluated at the optimal choice of investment, $I^* = I^*(t,k,p)$, is also known as the Hamilton-Jacobi equation (Kamien and Schwartz, Intrilligator).

Equation (6) can now be used to interpret <u>Ito's lemma</u>, which states that the differential of the function $J(t,k,p)$ is:

$$(8) \quad dJ = E_t[dJ] + J_k'\sigma dz + J_p'\alpha dw + o(dt)$$

Thus, Ito's lemma (8) for the differential of a function of stochastic processes just implies that the change in a function J includes the expected change $E_t[dJ]$, plus the unexpected change about the mean $J_k'\sigma dz + J_p'\alpha dw$. This unexpected change is directly related to the variance of dJ, since $\mathrm{Var}(dJ) = E[(dJ - E(dJ))^2] = E[(J_k'\sigma dz + J_p'\alpha dw)^2] = \mathrm{trace}(\sigma J_k J_k'\sigma S)dt + \mathrm{trace}(\alpha J_p J_p'\alpha B)dt$.

The main objectives of this section are complete: stochastic differential equations and Wiener processes have been introduced; Bellman's equation and the Hamilton-Jacobi equation have been derived; and Ito's lemma has been defined and interpreted. See, for example, Pindyck or Stefanou for economic applications to agricultural and natural resource problems where state equation dynamics are modelled as stochastic differential equation.

Another potential use of the above results is considered here. Since the Hamilton-Jacobi equation plays a central in dynamic duality theory for the deterministic case (e.g., see, Epstein), it is not surprising that Bellman's equation (7) can be used to derive the dynamic and stochastic analogue of Hotelling's lemma. First, note that since problem (2) is autonomous, the value function $J(t,k,p) = e^{-rt}V(k,p)$, where V is the firm's intertemporal value of profit discounted to the initial time t [Kamien and Schwartz]. As a result, $-J_t = re^{-rt}V$, and Bellman's equation (7) can be written as:

$$(9) \quad rV(k,p) = \text{Max}_I \{f - pk + V_k'(I - \delta k) + V_p'g(p) + .5\text{trace}[\sigma' V_{kk} \sigma S] + .5\text{trace}[\alpha' V_{pp} \alpha B]\}$$

Applying the envelope theorem to the Hamilton-Jacobi equation (9), after rearranging, yields the stochastic and dynamic analogue of Hotelling's lemma:

$$(10.1) \quad I^*(k,p) = V_{kp}^{-1}\{rV_p + k - V_{pp}g(p) - V_p' g_p(p) - .5\text{trace}[\sigma(k)' V_{kkp} \sigma(k) S] - .5\text{trace}[\alpha(p)' V_{ppp} \alpha(p) B] - \text{trace}[\alpha(p)' V_{pp} \alpha_p(p) B]\} + \delta k$$

While equation (10.1) is rather complicated, it is a straightforward generalization of the dynamic analogue of Hotellings lemma for the dynamic and deterministic case. For example, if it is assumed that there is no uncertainty, but prices are expected to follow the deterministic differential equations $dP/dt = g(p)$, then equation (10.1) becomes:

$$(10.2) \quad I^{*}(k,p) = V_{kp}^{-1}\{rV_{p} + k - V_{pp}g(p) - V_{p}'g_{p}(p)\} + \delta k$$

and if there is no uncertainty and prices are assumed to be constant over time, as in Epstein, then equation (10.1) becomes:

$$(10.3) \quad I^{*}(k,p) = V_{kp}^{-1}\{rV_{p} + k\} + \delta k$$

Thus, in contrast to the assertions of Taylor, equation (10.1) shows that stochastic dynamic duality can be used to model investment behavior when capital stock dynamics are uncertain and prices evolve according to a Markovian process. As with the static and deterministic case, the matrix of second derivatives of the value function $V(k,p)$ is symmetric, and the value function is homogenous of degree zero in normalized prices. Unlike the static and deterministic case, and as Stefanou notes, fourth-order derivatives are in general necessary to characterize the concavity/convexity of the value function and to investigate the response of the optimal choices functions $I^{*}(k,p)$ to price changes. For empirical analysis, where degrees of freedom and multicollinearity problems will probably restrict analyses to quadratic-like value functions, equation (10.1) along with simple forms for $g(p)$, $\sigma(k)$, and $\alpha(p)$ may be tractable.

III. The Stochastic Maximum Principle

A generalization of the model of soil conservation developed in McConnell is used in this section to derive the stochastic maximum principle. Specifically, it is assumed that the depth of the top soil stock evolves according to a stochastic differential equation. Since the change in the soil stock over time is influenced by farmer choices, such as inputs, crops grown, tillage practices, as well as weather, the evolution of the soil stock over time is a stochastic process.

Consider the following generalization of the McConnell model:

$$(11)\quad J(x,t) = \max_{s,z} E_t\{\int_t^T e^{-r\tau}[pf(s(\tau),x(\tau),z(\tau))g(\tau) - cz(\tau)]dt + R[x(T)]e^{-rT}]\}$$

$$\text{s.t. } dx(\tau) = (k-s(\tau))d\tau + \sigma(s(\tau),x(\tau),z(\tau))dw(\tau);\quad x(t) = x$$

where all the variables are scalars and p is the output price; $s(\tau)$ is soil erosion; $x(\tau)$ is soil depth with initial value x; $z(\tau)$ is a variable input with price c; $g(\tau)$ is an index of neutral technical change and $fg(\tau)$ is the technology; r is the discount rate; R is the resale value of the land at the terminal time T given a terminal soil stock x(T); k is the natural regeneration of the soil stock; and $w(\tau)$ is a Wiener process, where $\sigma(s(\tau),x(\tau),z(\tau))^2$ is the variance of $dx(\tau)$ over the period $d\tau$; and J(x,t) is the value function. For the purposes of this section, the value function J is not written as a function of the other parameters of the problem (p,c,r,T,k) for notational convenience.

Using the process outlined in Section II, Bellman's equation for problem (11) is:

(12) $-J_t = \text{Max}_{s,z} \{[pg(t)f(s,x,z) - cz]e^{-rt} + J_x(k-s) + .5\sigma^2 J_{xx}\}$

The technology parameter $g(\tau)$ in the production function implies that problem (11) in not autonomous. Therefore, following Larson, equation (12) is written in current value terms, using $J(x,t) = e^{-rt}V(x,t)$, as;

(13) $rV(x,t) - V_t(x,t) = \text{Max}_{s,z}\{pg(t)f(s,x,z) - cz + V_x(k-s) + .5\sigma^2 V_{xx}\}$

Defining the expected marginal value of the stock V_x as μ, the change in the expected marginal value of the stock with a change in the initial stock (V_{xx}) as μ_x, and substituting μ and μ_x into equation (13), the current-value Hamiltonian for problem (11) is:

(14) $H(s,x,z,\mu,\mu_x) = pg(t)f - cz + \mu(k-s) + .5\sigma^2\mu_x$

The Hamiltonian for the continuous stochastic case has a similar interpretation as that for the deterministic case, i.e. the expected change in the value function at time t includes current returns $pg(t)f - cz$, expected capital gains $\mu(k-s)$, plus the new term $.5\sigma^2\mu_x$ which is the cost of uncertainty to the firm at time t. Notice that the variance of the random variable enters the Hamiltonian even though the firm is assumed to be risk neutral. If $V(x,t)$ is concave in x, $V_{xx} \equiv \mu_x$ is less than or equal to zero, and uncertainty tends to reduce the expected value of changes in the value function. However, $V(x,t)$ is not necessarily concave in x, which implies that uncertainty could increase the value of the Hamiltonian. Stefanou suggests a process for determining the shape of the value function in x.

The stochastic maximum principle is directly analogous to the deterministic case. From equation (15), the stochastic maximum principle implies the following optimality conditions:

(16) $H_s = pgf_s - \mu + \sigma_s\mu_x = 0$

(17) $H_z = pgf_z - c + \sigma_z\mu_x = 0$

In contrast to the deterministic case analyzed by McConnell, the marginal cost of soil erosion includes the marginal current value of the stock μ plus the extra term $\sigma_s\mu_x$, which takes account of the marginal effect of soil erosion on the standard deviation of dx over the period dt. This additional term begins to look like an adjustment for risk preferences found in a static expected utility-maximization framework. If it is assumed that $\sigma_s > 0$ and $\mu_x < 0$, so that additional erosion increases uncertainty as measured by the variance of dx, then the total term $\sigma_s\mu_x < 0$ and farmers would choose less erosion under uncertainty than for the deterministic case. On the other hand, if $\mu_x>0$, the total term $\sigma_s\mu_x > 0$ and farmers would choose more erosion under uncertainty than for the deterministic case. Similar implications follow from equation (17). Thus, from equations (16) and (17), farmers acting in a dynamic and uncertain world may provide one explanation why static duality analyses often fail to satisfy theoretical restrictions. No "psychological" motivation to explain "irrational" behavior, as analyzed in Opaluch and Segerson, may be necessary.

From the Hamiltonian, $H_\mu = (k-s)$, and the state equation is found by:

(18) $dx = H_\mu dt + \sigma(k)dw, \quad x(t) = x.$

And, since the costate variable $\mu \equiv V_x$ is a function of the stochastic process x, Ito's differentiation rule (8) is used to find the <u>costate equation:</u>

$$(19)\quad d\mu = (r\mu - H_x)dt + \sigma\mu_x dw + o(dt) = [r\mu - pgf_x - \sigma\sigma_x\mu_x]dt + \sigma\mu_x dw + o(dt)$$

with terminal condition $\mu(T) = R_x[x(T)]$.

Thus, the maximum principle for continuous stochastic processes implies that the optimal paths of z,s,x, and μ satisfy equations (16)-(19). The analogous conditions in the deterministic case are equations (6) - (11) in McConnell.

Finally, when terminal time T is a choice variable, maximizing (11) with respect to T yields:

$$(20)\quad \{pg(T)f(T) - cz(T) + \mu(T)(k-s(T)) + .5\sigma(T)^2\mu_x(T)\} = rR[x(T)]$$

Thus, from (20), the optimal time to sell occurs when the returns from remaining on the land equal the opportunity cost of remaining on the land. While asset replacement has been analyzed in a deterministic context for many years (see, e.g., Goundrey, Perrin, or Samuelson), equation (20) is also an asset replacement criterion for general stochastic control models. If the value function is concave in x, so that $\mu_x<0$, then the rotation length--time to sell--is shorter for the stochastic case. Equation (20) generalizes the result found in Reed, where the risk of fire reduces the optimal forest rotation.

In a deterministic model, where there is no difference between planned and realized rates of soil loss, McConnell shows under what conditions the rate of soil loss increases or decreases overtime.[7] For the dynamic and uncertain model analyzed here, Ito's lemma implies that the expected change in soil erosion $(1/dt)E_t[ds]$ will differ from the observed change due to the realization of the random variable dw and any adaptation on the part of the farmer. Thus, for example, observing increasing soil loss in any time period does not necessarily imply that a farmer planned such a change.

Two mathematical steps must be followed to derive the expected change in the optimal choice of erosion $s(x,t)$ over the period dt, $(1/dt)E_t[ds]$. First, Ito's is applied to the optimality conditions (16) and (17). And second, the two-equation system for $(1/dt)E_t[ds]$ and $(1/dt)E_t[dz]$. (This process is outlined in Appendix A for a simplified example.) The optimality conditions (16) and (17) imply that:

$$(21)\quad (1/dt)E_t[d(pgf_s)] - (1/dt)E_t[d\mu] + (1/dt)E_t[d(\sigma_s\mu_x)] = 0$$

$$(22)\quad (1/dt)E_t[d(pgf_z)] - c_t + (1/dt)E_t[d(\sigma_z\mu_x)] = 0$$

The individual terms in equations (21) and (22) are evaluated in Appendix B, using Ito's lemma and the process outlined in Appendix A. Substituting equations (B1) - (B5) from Appendix B into equation (21) and (22), after some mundane (if tedious), algebra yields:

$$(28)\quad \begin{bmatrix} (1/dt)E_t[ds] \\ (1/dt)E_t[dz] \end{bmatrix} = A\ [\ B + C - D\ ]$$

where,

$$A = \begin{bmatrix} pgf_{ss} + \mu_x\sigma_{ss}, & pgf_{sz} + \mu_x\sigma_{sz} \\ pgf_{zs} + \mu_x\sigma_{zs}, & pgf_{zz} + \mu_x\sigma_{zz} \end{bmatrix}^{-1}$$

$$B = \begin{bmatrix} pgf_s[r - (f_x/f_s) - (g_t/g) - (p_t/p) - (k-s)(f_{sx}/f_s)] \\ pgf_z[(c_t/(c-\sigma_z\mu_x)) - (g_t/g) - (p_t/p) - (k-s)(f_{zx}/f_z)] \end{bmatrix}$$

$$C = \begin{bmatrix} r\mu_x\sigma_s - \sigma\sigma_x\mu_x - \sigma_s[\mu_{xt} + \mu_{xx}(k-s) + .5\mu_{xxx}\sigma^2] \\ r\mu_x\sigma_z - \sigma\sigma_x\mu_x - \sigma_z[\mu_{xt} + \mu_{xx}(k-s) + .5\mu_{xxx}\sigma^2] \end{bmatrix}$$

$$D = \begin{bmatrix} \mu_x\sigma_{sx}(k-s) + \sigma^2\big[.5(pgf_{sxx} + \mu_x\sigma_{sxx}) + .5(pgf_{sss} + \mu_x\sigma_{sss})s_x^2 + \\ .5(pgf_{szz} + \mu_x\sigma_{szz})z_x^2 + (pgf_{ssz} + \mu_x\sigma_{ssz})s_xz_x + (pgf_{ssx} + \mu_x\sigma_{ssx})s_x + \\ (pgf_{szx} + \mu_x\sigma_{szx})z_x\big] \\ \\ \mu_x\sigma_{zx}(k-s) + \sigma^2\big[.5(pgf_{zxx} + \mu_x\sigma_{zxx}) + .5(pgf_{zss} + \mu_x\sigma_{zss})s_x^2 + \\ .5(pgf_{zzz} + \mu_x\sigma_{zzz})z_x^2 + (pgf_{zsz} + \mu_x\sigma_{zsz})s_xz_x + (pgf_{zsx} + \mu_x\sigma_{zsx})s_x + \\ (pgf_{zzx} + \mu_x\sigma_{zzx})z_x\big] \end{bmatrix}$$

While equation (23) is rather complicated in total, the individual components can be readily interpreted and compared to the analogous conditions for the deterministic case. And, as found in Stefanou and the investment model in Section II, the implications of the stochastic and dynamic model include that of the deterministic model as a special case. There are three main sets of terms in equation (23). First, the matrix A is the inverse of the Hessian matrix of the current-value Hamilton (14). The analogous term for the deterministic case is the also just the appropriate Hessian matrix. Second, the vector B is identical to the deterministic case found in McConnell, except that $c_t/(c-\sigma_z\mu_x)$ rather than c_t/c is included to take account of the full marginal cost of the variable input z. Thus, the product of the matrix A and the vector B is essentially equation (12) in McConnell.

The third main sets of terms in equation (23), the vectors C and D, include increasingly complicated effects that are not found in the deterministic model. The vector C takes into account third- and fourth-order changes in the value function due to a change in the state x. The vector D takes into account the third-order effects of firm choices on the production function and the variance function.

Equation (23) provides the expected dynamics of the optimal choice functions s(x,t) and z(x,t) for general functional forms and generalizes the results in Stefanou. However, more specific assumptions can greatly simplify this general result. For example, if it is assumed that firm choices only influence the expected change in the stock but not the unexpected change, as in Stefanou, then $\sigma_s = \sigma_z = 0$ and the vector C reduces to $C'=(-\sigma\sigma_x\mu_x, -\sigma\sigma_x\mu_x)$, the vector D reduces to the production functions terms, and the vector B and matrix A become identical to the deterministic case. If it is assumed that the production function f and the variance function σ are quadratic in s, z, and x, then the vector D collapses to $D'=(\mu_x\sigma_{sx}(k-s), \mu_x\sigma_{zx}(k-s))$.

IV. CONCLUSIONS

This paper introduces the basic techniques for the optimum control of stochastic processes. An investment model of the firm is used to introduce stochastic differential equations, interpret Ito's lemma, and derive Bellman's equation. A stochastic and dynamic analogue of Hotelling's lemma is derived. An economic model of soil conservation where erosion is a stochastic process is then used to derive and interpret the stochastic maximum principle. Since most agricultural and resource management problems involve stochastic and dynamic decision making, the potential applications of stochastic control are numerous.

Deterministic models are special cases of their stochastic counterparts. As a result, the differences between stochastic and deterministic models can always be readily compared. Thus, at a minimum, analyzing a problem at the theoretical level in a dynamic and stochastic framework highlights the assumptions underlying deterministic models (or static models with uncertainty). As a final note, possible avenues for empirical research within a continuous-time stochastic framework are suggested by developments in macroeconomics (see, e.g., Gandolfo and Padoan).

V. APPENDIX A

The process for deriving the expected soil erosion dynamics $(1/dt)E_t[dS]$ from the optimality conditions of the stochastic maximum principle is outlined in this appendix. The basic problem involves using Ito's Lemma to determine the differential of a function $y = F(x,t,s(x,t)) = 0$, where $s(x,t)$ is the optimal choice of s at time t given x. For the purposes of this appendix, the simplifying assumptions are made that the state x evolves according to the stochastic differential equation $dx = (k-s)dt + \sigma dw$, where σ is a scaler. Using the process outlined in the text, equations (4) - (6.1), a second-order Taylor approximation of order dt implies that:

(A1) $dy = F_t dt + F_x dx + .5F_{xx}dx^2 + F_s ds + .5F_{ss}ds^2 + F_{xs}dsdx + o(dt)$

where

(A2.1) $dx = (k-s)dt + \sigma dw$

(A2.2) $dx^2 = [(k-s)dt + \sigma dw]^2 = \sigma^2 dw^2 + o(dt)$

(A2.3) $ds = s_t dt + s_x dx + .5s_{xx}dx^2 + o(dt)$

(A2.4) $ds^2 = s_x^2\sigma^2 dw^2 + o(dt)$

(A2.5) $dsdx = s_x\sigma^2 dw^2 + o(dt)$

Using equations (A2.1) - (A2.5), the differential generator of the function $y = F(x,t,s(x,t))$ is:

(A4) $(1/dt)E_t[dy] = F_t + F_x(k-s) + .5F_{xx}\sigma^2 + .5F_{ss}s_x^2\sigma^2 + F_{sx}s_x\sigma^2 +$
$F_s(1/dt)E_t[ds]$,

which when set equal to zero can be rearranged to yield:

(A5) $(1/dt)E_t[ds] = -(1/F_s)[F_t + F_x(k-s) + .5F_{xx}\sigma^2 + .5F_{ss}s_x^2\sigma^2 + F_{sx}s_x\sigma^2]$

VI. APPENDIX B

Following the process outlined in Appendix A, the terms in equations (21) and (22) can be evaluated as:

$$\text{(B1)}\quad (1/dt)E_t[d(pgf_s)] = (p_t g + pg_t)f_s + pg[f_{sx}(k-s) + \sigma^2(.5f_{sxx} + .5f_{sss}s_x{}^2$$
$$.5f_{szz}z_x{}^2 + f_{ssz}s_x z_x + f_{ssx}s_x + f_{szx}z_x)] +$$
$$pgf_{ss}(1/dt)E_t[ds] + pgf_{sz}(1/dt)E_t[dz]$$

$$\text{(B2)}\quad (1/dt)E_t[d\mu] = r\mu - pgf_x - \sigma\sigma_x\mu_x$$

$$\text{(B3)}\quad (1/dt)E_t[d(\sigma_s\mu_x)] = \mu_x(1/dt)E_t[d(\sigma_s)] + \sigma_s(1/dt)E_t[\mu_x]$$

where

$$(1/dt)E_t[d(\sigma_s)] = \sigma_{ss}(1/dt)E_t[ds] + \sigma_{sz}(1/dt)E_t[dz] + \sigma_{sx}(k-s)$$
$$+ \sigma^2[.5\sigma_{sss}s_x{}^2 + .5\sigma_{szz}z_x{}^2 + .5\sigma_{sxx} +$$
$$\sigma_{ssz}s_x z_x + \sigma_{ssx}s_x + \sigma_{szx}z_x]$$

and

$$(1/dt)E_t[\mu_x] = \mu_{xt} + \mu_{xx}(k-s) + .5\mu_{xxx}\sigma^2$$

$$\text{(B4)}\quad (1/dt)E_t[d(pgf_z)] = (p_t g + pg_t)f_z + pg[f_{zx}(k-s) + \sigma^2(.5f_{zxx} + .5f_{zss}s_x{}^2$$
$$.5f_{zzz}z_x{}^2 + f_{zsz}s_x z_x + f_{zsx}s_x + f_{zzx}z_x)] +$$
$$pgf_{zs}(1/dt)E_t[ds] + pgf_{zz}(1/dt)E_t[dz]$$

$$\text{(B5)}\quad (1/dt)E_t[d(\sigma_z\mu_x)] = \mu_x(1/dt)E_t[d(\sigma_z)] + \sigma_z(1/dt)E_t[\mu_x]$$

where

$$(1/dt)E_t[d(\sigma_z)] = \sigma_{zs}(1/dt)E_t[ds] + \sigma_{zz}(1/dt)E_t[dz] + \sigma_{zx}(k-s)$$
$$+ \sigma^2[.5\sigma_{zss}s_x{}^2 + .5\sigma_{zzz}z_x{}^2 + .5\sigma_{zxx} +$$
$$\sigma_{zsz}s_x z_x + \sigma_{zsx}s_x + \sigma_{zzx}z_x]$$

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