Application of the Nash Bargaining Model to a Problem of Efficient Resources Use and Cost-Benefit Allocation

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Abstract

This paper presents an application of the Nash model to a combined problem of optimal use of a common facility and the related cost-benefit allocation considerations. Two alternative modifications and a diagrammatically procedure are developed. The conditions for an optimal Nash solution are derived. The Procedure is applied to a case where cooperation between farms can be considered.

Introduction

In the evaluation of a common facility it is necessary to determine both the facility's optimal capacity and the allocation of common cost and profit among users, such that the common cost will be fully covered and the supply of the facility's services will meet the demand.

The usual procedure in such cases is to determine first the capacity of the facility, and then to charge the consumers of the plant's services in a given way, assuming that the plant will be occupied to its full capacity and that the consumers will cover the total cost. This approach sometimes leads to situations where the supply does not meet the demand for the services, or - in a case of economies of scale - where the service fees do not cover the cost (Coase; Samuelson; Laughlin). The plant's budgetary deficit must then be covered by the government or any authority which is interested in its continued operation. In other words, individuals not using the plant have to subsidize its services.

The Nash solution (Nash,1953) is an axiomatic unique solution to a cooperative bargaining problem between two players. It has been modified (Harsanyi, 1963,1977) to allow for N players. The advantage of using the Nash solution for problems such as described above is that the solution is both unique and stable (Nash, 1953). Some doubts have been expressed (Nash, 1950; Luce and Raiffa; Harsanyi, 1966; Bachrach) regarding its equity in cases of asymmetry in the utility levels at the non-cooperative point.

In this paper the Nash model is modified to an allocation problem which is also concerned with optimization of resource use. In the following section two alternative models are developed. Then a diagrammatical exposition of the suggested solution is developed and the uniqueness of the solution is also proved. The models are then applied to a case where two agricultural farms consider cooperation with regard to water resources use.

Presentation of the problem - two alternative models

Consider two rational producers each having known production function of one input and one product. For each producer the quantity of available input is limited. In order to obtain more units of the limited input, the producers must cooperate by establishing a common plant.

In the negotiating process the producers decide on the capacity of the plant as well as on an allocation scheme for the plants's output and its common cost (the later to be covered by themselves only). It is assumed that the utility of producer i is a linear function of his income. The Nash model for this case, in terms of net income, is

$$\max f = [Y_1(w_1) - \underline{Y}_1 - D_1][Y_2(w_2) - \underline{Y}_2 - D_2]$$

s.t. (1) $D_1 + D_2 - C(w_1 + w_2) = 0$

(2) $Y_1(w_1) - D_1 \geq \underline{Y}_1$

(3) $Y_2(w_2) - D_2 \geq \underline{Y}_2$

where $Y_i(w_i)$ is the production function in terms of gross income for producer i, and $Y_i(w_i) = Y_i(w_i + \underline{w}_i)$.

w_i is the quantity of input that producer i receives from the common plant.

$\underline{w}_i$ is the quantity of input that producer i uses from his own resources only.

$\underline{Y}_i$ is the net income of producer i without cooperation (the conflict point).

$C(w_1 + w_2)$ is the cost function for the common plant.

D_i is the share of producer i in the common cost, i.e. the fee to be paid by producer i for using the plant.

Note that constraint (1) satisfies the condition that the common cost be covered by the users, while constraints (2) and (3) ensure individual rationality. For the purpose of the analysis it is assumed that, in the relevant range of production, $Y_i(w_i)$ is a monotonically increasing function which is continuous and differentiable; $C(w_1 + w_2)$ is continuous and is differentiable with $dc/dw > 0$.

It can be shown that if $w_1 + w_2 > 0$, then an optimal solution for f will occur when there are strong inequalities in constraints (2) and (3). In this case, the shadow price for these constraints is zero; since the Nash solution satisfies individual rationality (Nash, 1953), it is obvious that constraints (2) and (3) are redundant and can therefore be ignored (see Appendix).

The reduced model will then be (Model A):

$\max f = [Y_1 - \underline{Y}_1 - D_1][Y_2 - \underline{Y}_2 - D_2]$

s.t. $D_1 + D_2 = C$

which the Lagrange expression for it is

$L = f - b(D_1 + D_2 - C).$

b, which is the Lagrange multiplier can be interpreted as the change in the value of the objective function if the total amount of fees is greater then the common cost by one unit.

First order conditions for maximization yield

(4) $dY_1/dw_1 = dC/dw = dY_2/dw_2$

which means that in the optimal Nash solution, for both producers the marginal income produced by additional input from the common plant is equal to the marginal cost associated with producing this additional input in the plant.

If $Y^o{}_i(w^o{}_i)$ is an optimal solution for producer i, then from first order conditions

(5) $Y^o{}_2 - \underline{Y}_2 - D_2 = Y^o{}_1 - \underline{Y}_1 - D_1$

which means equal additional net income for each producer with respect to the conflict point.

The allocation of the common cost between the producers is derived as follows:

$$D_1 = 0.5[\ C + (Y^o{}_1 - \underline{Y}_1) - (Y^o{}_2 - \underline{Y}_2)]$$

(6)

$$D_2 = 0.5[\ C + (Y^o{}_2 - \underline{Y}_2) - (Y^o{}_1 - \underline{Y}_1)]$$

In the optimal cooperative solution, the capacity of the common plant (G) is defined by the aggregate demand for w

(7) $G = w^o{}_1 + w^o{}_2$

One can thus see two stages in the solution of this problem:

(i) Efficiency allocation of the common input according to its marginal value, without taking into consideration the common cost (the gross income stage);

(ii) Equity allocation of the common cost using Nash's axioms (the net income stage).

The above procedure does not provide an efficient solution when more than two participants are involved. A possible solution is the modification suggested by Harsanyi (1963). An efficient solution can also be achieved empirically by applying the above two stages in the following way (Model B).

$\max g = Y_1(w_1) - C(w_1 + w_2)$

s.t. $Y_2(w_2) = K$

This model maximizes the gross income of producer 1 minus the common cost, where the net income of producer 2 is fixed at level K (an exogenous parameter indicating the desired net income for producer 2). Technically, K can be changed in an iterative process in order to maximize g. Although the model is presented for the case of 2 players, it is easily adjusted to a case with N players; in this case the number of constrains will be N-1.

the Lagrange expression for Model B is

$L = Y_1(w_1) - C(w_1 + w_2) - e[Y_2(w_2) - K]$

which yields , using to first-order conditions (after rearrangement):

(8) $dC/dw = dY_1/dw_1 = -edY_2/dw_2$.

Thus, (8) is the optimal condition for Model B, while (4) is the optimal condition for Model A. Identical solutions for the two models are obtained when e=-1, where e is the ratio between the marginal cost of production of an additional unit of w in the common plant and the marginal income from the use of that unit by producer 2. The parameter e regulates any distortion resulting from the constraint imposed on the net income level of producer 2.

In case where this constraint leads producer 2 to use w in excess (more than its marginal productivity), then $|e| > 1$, and vice versa. In practice, the problem of allocating w produced in the common plant to the participants is solved by changing the level of K such that e=-1, and $K=K^*$. At this stage one can allocate the common cost between the producers using several allocating schemes. The Nash solution provides results similar to Model A by using K^* instead of $Y^o_2 - \underline{Y}_2$ in (6) for the solution of Model B.

Diagrammatical exposition of Nash solution including determination of the plant capacity

Figure 1 presents the income possibility curve (A_1A_2) for the two producers in terms of additional gross income (without subtraction of the common cost). Different points on A_1A_2 are associated with the same capacity of a given common facility but with different allocations of the facility's services between the producers. Each point on A_1A_2 therefore represents a combination of gross income and cost to be allocated. In the negotiation process the producers have to decide on (1) their location on the income possibility curve, and (2) how to allocate the common cost.

The Nash solution will be on a ray (OS) of slope 45^o from the origin (Figure 1). The point m on A_1A_2 is related to the common cost of m′m; The producers′ gross incomes are $y^m{}_1$ and $y^m{}_2$; the point p^m on the ray OS is the solution which maximizes the product of the net incomes of the producers (proved later). The point p^m can be obtained by the following diagrammatical procedure: from m on A_1A_2 a horizontal line gm and a vertical line hm, both equal in length to m′m, are drawn towards the Y_1 and Y_2 axis, respectively. From g or h, a line is drawn perpendicular to OS; the point p^m on OS is then the intersection point of the line gh with OS. At p^m the net incomes of producer 1 and 2 are $y^{om}{}_1$ and $y^{om}{}_2$, respectively, where $y^{om}{}_1 = y^{om}{}_2$. The shares of producer 1 and 2 in the common cost are $y^m{}_1 - y^{om}{}_1$ and $y^m{}_2 - y^{om}{}_2$, respectively. Producer 2 pays $y^m{}_2 - y^{om}{}_2 = ml$ (since $lg + ml = mg$ and $lg = lp^m$) and producer 1 pays $y^m{}_1 - y^{om}{}_1 = lp^m$. Construction of the point p^m in this manner provides a guaranty that $lp^m + ml = m'm$, or, in other words, that the cost is covered by the users. The point m on A_1A_2 thus, provide a solution without side payments.

Consider now the case with side payments. The point b on the gross income possibility curve (Figure 1) is related to a cost of b′b of the common plant. As in the former case (point p^m) the point p^b here represents a Nash solution where the producers′ net incomes are $y^{ob}{}_1$ and $y^{ob}{}_2$ (where $y^{ob}{}_1 = y^{ob}{}_2$). The share of producer 1 in the common cost is $y^b{}_1 - y^{ob}{}_1$ which is greater than the common cost b′b, while producer 2 does not pay for using the common plant. Instead, producer 2 receives side payments $y^{ob}{}_2 - y^b{}_2$ from producer 1 such that $y^b{}_1 - y^{ob}{}_1 - b'b = y^{ob}{}_2 - y^b{}_2$.

A solution in the case of Model B is depicted in Figure 2. Two income possibility curves are shown: A_1A_2 is the gross income possibility curve between producers 1 and 2, and B_1B_2 is the net income possibility curve, assuming that producer 1 pays for all the common costs.

Each point on B_1B_2 corresponds to a point which is vertically along to it on A_1A_2 minus the common cost at that point. For example the point m^o on B_1B_2 corresponds to m on A_1A_2 minus mm^o. The Nash solution on B_1B_2 is at the intersection between OS and the tangent MM to B_1B_2 at m^o, which is perpendicular to OS. It can be shown that e from equation (8) is interpreted as the slope of the tangent MM. The intersection between MM and OS occurs at T^m; if m'm (Figure 1) equals mm^o (Figure 2), then a slope of -1 for MM means that the solution at T^m equals the solution at p^m. The interpretation of the results is the same as in the previous section.

One important result to emerge from this discussion is that the outcome of the negotiation process (the point T^m) is better in terms of both equity and efficiency than point m^o which was obtained directly by subtraction of the common cost from producer 1's income. The point T^m is outside the net income possibility curve B_1B_2 which means that the solution in T^m assigns additional income to the producers as compared to the point m^o which is on that curve. The uniqueness of that solution is proved in the following corollary.

Corollary: If m presents an optimal solution on a gross income possibility curve, which is obtained by the diagrammatical procedure, then m is a unique allocation scheme satisfying Nash's axioms.

Proof: The proof is based on geometric considerations (Figure 3). Suppose that a solution m on the gross income possibility curve A_1A_2 is associated with a common cost mm_2. Let m_2 be any solution obtained by subtraction of the common cost from the income of producer 1. Thus, m_2 satisfies the conditions for efficiency and for covering the common cost. Let R^m be the solution (with side payments) obtained by the diagrammatic procedure. The product of the net incomes is then $Og_1 \times Og_2$ at R^m and $Od_1 \times Od_2$ at m_2. Both R^m and m_2 are the result of the location of m on A_1A_2. The product $Og_1 \times Og_2$ is represented by the area $Og_2R^mg_1$ (termed rectangle B) and the product $Od_1 \times Od_2$ by the area $Od_2m_2d_1$ (termed rectangle H). R^m represents a unique Nash solution if and only if area B $\geq$ area H. Since the area Od_2ag_1 is common to both B and H, one needs to refer only to the differences between B and Od_2ag_1 (termed B′) and between H and Od_2ag_1 (termed H′). Therefore it has to be shown only that area B′ $\geq$ area H′.

Triangle m_2R^mb is both right-angled and isoceles (by definition the slope of OS is 45^o so OS $\perp$ m_1R^m and $md_2 \perp OA_2$). aR^m is a bisector of the right angle m_2R^mb, so $m_2a = ab = aR^m$.

For each point a on the line g_1R^m we have

$ag_1 \leq g_1R^m$.

From geometric considerations

$ad_2 = R^mg_2$ and $aR^m = d_2g_2$,

$am_2 = d_1g_1$ and $d_1m_2 = ag_1$,

and also $ad_2 = g_1R^m$.

Since $am_2 = aR^m$ and $ag_1 \leq ad_2$, the product $ag_1 \times am_2$ is smaller than $ad_2 \times aR^m$ and therefore area B′ $\geq$ area H′.

The same proof holds for the case without side payments.

The procedure for finding a global optimum solution is demonstrated in Figure 4. Suppose that the two producers are able to cooperate and move from the conflict point O to a point either on the gross income possibility curve A_1A_2 or on D_1D_2. The curve A_1A_2 is characterized by a common facility with a cost of m'm ($=m'_0m_0$) and D_1D_2 is characterized by another facility with a cost of d'd (d'd > m'm).

The producers find a point on A_1A_2 or on D_1D_2 which will maximize the Nash product of net income; this point lays on the ray OS. The point m_0 on A_1A_2 is related to A^{m_0} on OS, and is preferable to m on the same curve which is related to A^m on OS. For the same reason m_0 on A_1A_2 is preferable to d (which is the optimal solution on D_1D_2). The global solution in this example is provided by a facility associated with a common cost of m'_0m_0, a given resources allocation represented by m_0 on A_1A_2 and net income distribution presented by A^{m_0} on OS.

Application

The two models (A and B) were adjusted and applied to an agricultural cooperation problem. Two agricultural producers with limited water resources for irrigation and a given technology and cropping patterns consider the possibility to cooperate in order to develop a new water source (e.g. well). Assume that they have to negotiate only over the well's capacity, allocation of the additional water between them and allocation of the common cost of the well. The well's cost function is characterized with economies of scale , which make cooperation between the farmers more attractive. Each farmer has a typical production function, limited land , constraints on annual water consumption and on consumption in the peak month (Table 1 and equation 12).

At the first stage $\underline{Y}_i$ is maximized, assuming no cooperation between the producers, such that each one optimizes the use of his own limited resources.

$$\max \underline{Y}_i = c'_i X_i$$

$$\text{s.t.} \quad R_i X_i \le u_i$$

$$X_i \ge 0 \qquad i=1,2$$

where c'_i is a vector of net profit coefficients; X_i is a vector of crop activity levels; u_i is a vector of constraint levels (land and water). The relevant data and the non-cooperative solution values are presented in Table 1.

At the second stage Models A and B are formulated and solved using nonlinear mathematical programming techniques, since the common cost function and Model A's objective function are nonlinear.

Model A was adjusted as follows:

$$\max F^A = (Y_1 - \underline{Y}_1 - D_1)(Y_2 - \underline{Y}_2 - D_2)$$

s.t. (9) $Q^A{}_i X_i - W^A{}_i - W^J{}_i \le b^A{}_i$

(10) $Q^J{}_i X_i - W^J{}_i \le b^J{}_i$

(11) $t' X_i \le L_i$

(12) $C = V^{0.4565}$

(13) $Y_i = c'_i X_i$

(14) $V = W^A{}_1 + W^A{}_2$

(15) $(1/12)V = W^J{}_1 + W^J{}_2$

(16) $Y_i - D_i \ge \underline{Y}_i$

(17) $C = D_1 + D_2$

$D_1, D_2 \gtreqless 0$

$X_i, Y_i, V, C, W^A{}_i, W^J{}_i \ge 0$

where V is the annual capacity of the well; $W^A{}_i$ and $W^J{}_i$ are respectively the water quantities purchased annually and in the peak month by producer i, $b^A{}_i$ is a vector of annual water constraints, $b^J{}_i$ is a vector of water constraints in the peak month; $Q^A{}_i$ and $Q^J{}_i$ are matrices of water coefficients, t' is a vector of 1's and L_i is the land constraint vector. The model permits side payments by allowing D_i to be negative.

Model B was adjusted as follows:

$\max F^B = Y_1 - C$

s.t. (9) to (15)

(16) $Y_2 = K$

$X_i, Y_i, V, C, W^A{}_i, W^J{}_i \geq 0$

K was changed parametrically within the range $\underline{Y}_2$ to $2\underline{Y}_2$. F^B was maximized when a value of \$14311.3 for Y_2 was used. Results for the two models are identical and are presented in Table 2.

Concluding comments

In this paper Nash's solution for a cooperative bargaining problem was applied to an optimization model with allocation of inputs and common cost. An alternative model for the simple two-person problem was modified, a procedure for allocating common costs was developed and incorporated into the model. It was proved that the solution which includes the allocation of the common cost is unique and fulfills Nash conditions. The existence of the proposed solution is guaranteed over the entire range of the income possibility curve. The model was applied to a typical agricultural example.

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Table 1: Main Characteristics of the Producers

	Producer 1	Producer 2
Available farm land (ha)	200	220
Annual water quota (m^3)	310000	270000
June water quota (m^3)	90000	90000
Net profit for crop I ($/ha)	1500	1500
Net Profit for crop II ($/ha)	1900	1000
$\underline{Y}_i$ ($)	171000	122727

Table 2: Results for a Cooperative Allocation Game

	Model A (Nash) Producer		Model B Producer	
	1	2	1	2
Crop I (ha)	0	95.4	0	95.4
Crop II (ha)	200	0	200	0
Y_i ($)	380000	143113	380000	143113
Annual water from well (m^3)	280000	29900	280000	29899.9
June water from well (m^3)	110000	14950	110000	14949.9
D_i ($)	127313	-61313		
$Y_i - D_i$ ($)	252687	204426		
$Y_i - D_i - \underline{Y}_i$ ($)	816.87	81699		
Well's capacity (m^3/year)	1500000		1499999.9	
Common cost ($)	66000		66000	

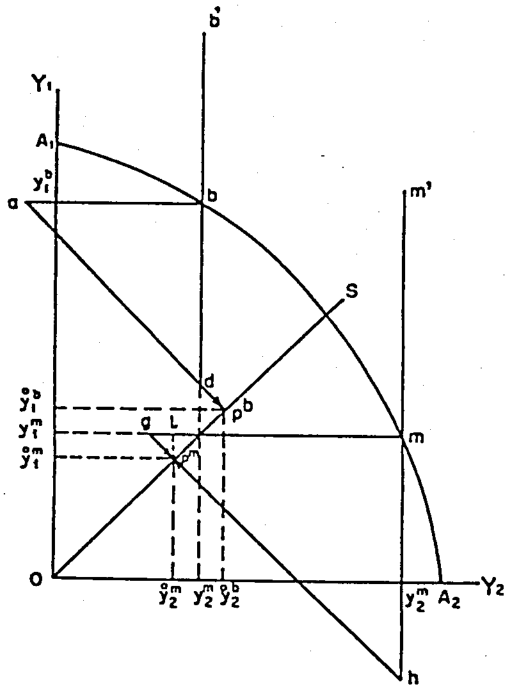

Fig. 1: The solution procedure for Model A.

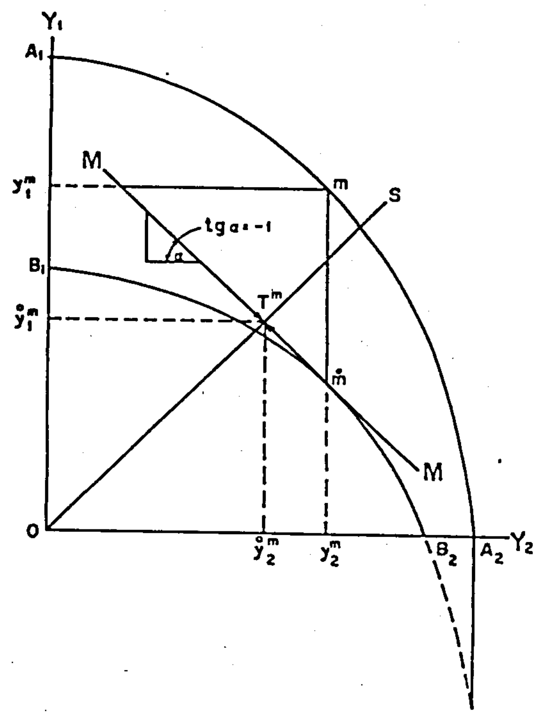

Fig. 2: The solution procedure for Model B.

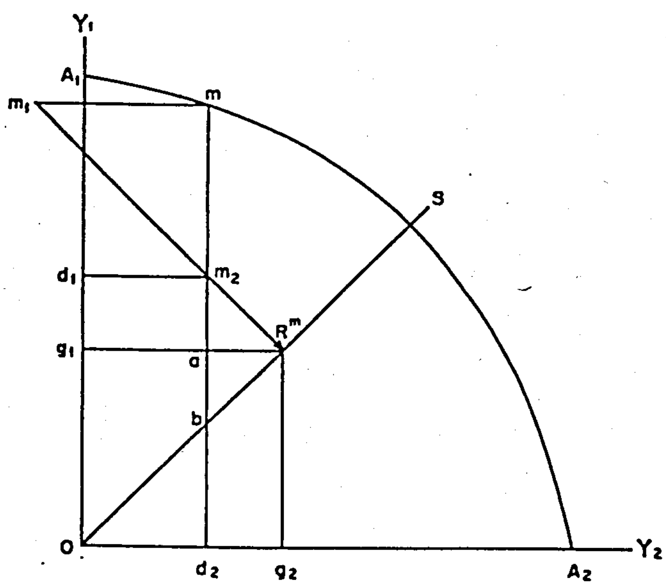

Fig. 3: Geometric exposition for the uniqueness of the optimal solution in R^m.

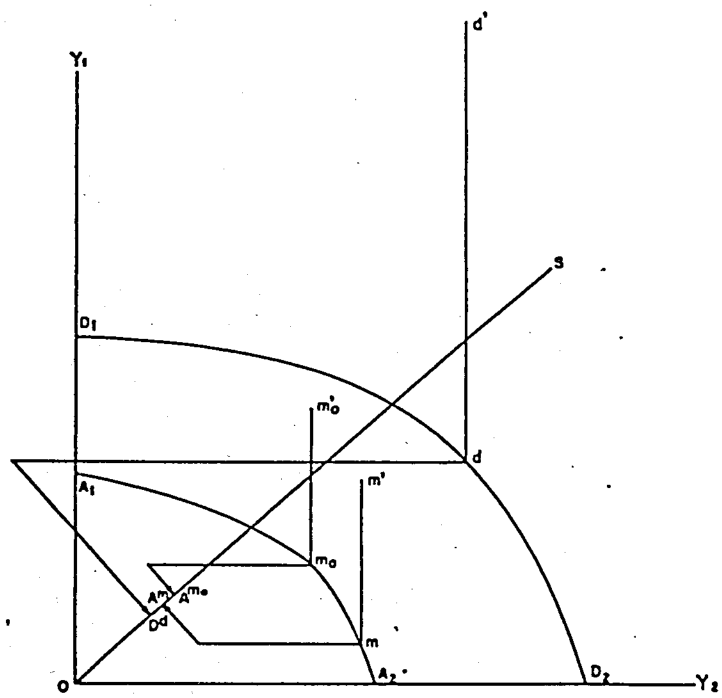

Fig. 4: The global optimal solution procedure.

Appendix: Conditions for Individual rationality constraints

It is obvious that if constraints (2) and (3) contain equalities than there is no cooperation between the producers. Therefore assume that $w_1+w_2>0$, and that only one individual constraint (e.g. for producer 1) shows equality

$$Y_1-D_1=\underline{Y}_1.$$

It will be shown that this leads to a contradiction which means that a solution to the cooperative problem exist only if there are inequalities in (2) and (3).

The Lagrange expression for this case is

$$L=f-\lambda_1(D_1+D_2-C)-\lambda_2(Y_1-\underline{Y}_1-D_1)$$

First order conditions for maximization are

(A1) $\partial L/\partial w_1=\partial Y_1/\partial w_1[Y_2-\underline{Y}_2-D_2]+\lambda_1(\partial C/\partial w_1)+\lambda_2(\partial Y_1/\partial w_1)=0$

(A2) $\partial L/\partial w_2=\partial Y_2/\partial w_2[Y_1-\underline{Y}_1-D_1]+\lambda_1(\partial C/\partial w_2)=0$

(A3) $\partial L/\partial D_1=-[Y_2-\underline{Y}_2-D_2]-\lambda_1\cdot\lambda_2=0$

(A4) $\partial L/\partial D_2=-[Y_1-\underline{Y}_1-D_1]-\lambda_1=0$

(A5) $\partial L/\partial \lambda_1=D_1+D_2-C=0$

(A6) $\partial L/\partial \lambda_2=Y_1-\underline{Y}_1-D_1=0$

From (A4) $\lambda_1=-[Y_1-\underline{Y}_1-D_1]$

and therefore $\lambda_1\leq 0$.

From (A2) $\partial Y_1/\partial w_2[Y_1-\underline{Y}_1-D_1]=-\lambda_1(\partial C/\partial w_2)$

but $\partial C/\partial w_2\geq 0$, and $\partial Y_2/\partial w_2\geq 0$, and since the producer is rational, he will produce where $\partial Y_2/\partial w_2>0$. In this case a solution can be reached only if $\lambda_1<0$, which is obtained only when $Y_1-\underline{Y}_1-D_1>0$.

But from (A6) $Y_1-\underline{Y}_1-D_1=0$, and this is a contradiction. Therefore it is necessary for both (2) and (3) to have strong inequalities in order to obtain a cooperative solution.