Safety First Programming -- A Zero-One Approach

by

Myles J. Watts, Associate Professor
Agricultural Economics and Economics
Montana State University

Joseph A. Atwood, Assistant Professor
Agricultural Economics and Economics
Montana State University

Glenn A. Helmers, Professor
Department of Agricultural Economics
University of Nebraska

Larry J. Held, Associate Professor
Department of Agricultural Economics
University of Wyoming

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Introduction

Safety-first methods have been proposed as an approach to making decisions in a risky environment (see Roy, Kataoka, Telser, Roummasset, and others). In general the methods to implement safety-first programming have required assumptions of tractable multivariate distributions (Pyle and Turnovsky) or the use of conservative stochastic inequalities (Telser, Sengupta, or Atwood). The methods presented in this paper allow safety-first modeling with finitely discrete multivariate populations or samples. The method uses exact probabilities or estimates of probabilities rather than the usually conservative probability bounds of the stochastic inequality methods presented by Atwood, or Atwood et al. The method is not without cost, however, in that a zero-one optimization algorithm is required.

This paper is organized as follows. A brief discussion of three safety first criteria is followed by a presentation of the mathematical models for each of the criteria. An empirical example concludes the paper.

Safety-First Criteria

Safety first models attempt to minimize (or are constrained by) the probability of failing to achieve certain goals of a decision maker. The probability and goal of concern can be denoted as

$$Pr(z < g) \tag{1}$$

where $Pr(\cdot)$ denotes the probability of the event $(\cdot)$,

z denotes a random variable (usually income), and

g is a goal level for the income random variable.

Several forms of safety-first behavior have been defined and discussed. Roy proposed that decision makers might choose from among their feasible alternative that alternative which minimizes (1) or the $\Pr(z < g)$. If mixtures of alternatives are available (a common assumption in portfolio analysis), let x_i denote the level of the i^{th} alternative in an activity mix $i = 1,2,...k$. These x_i values can be listed in a $k \times 1$ vector $\underline{x}$. Roy's criterion becomes to select an activity mix $\underline{x}$ from the set of feasible activity mixes X which minimizes the probability of aggregate income falling below some goal g. If $\underline{c}$ is a $k \times 1$ vector of per unit income levels, aggregate income z can be written as $z = \underline{c}^t\underline{x}$. Further assume for this paper that the feasible set X can be described by a set of conventional linear inequalities or $A\underline{x} \leq \underline{b}$ and $\underline{x} \geq \underline{0}$. Roy's criterion can be written as

Roy's Criterion

$$\underset{\underline{x}}{\text{Minimize}} \Pr(z = \underline{c}^t\underline{x} < g) \tag{2}$$

$$\text{Subject to } A\,\underline{x} \leq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

with $\underline{c}^t$ a transposed $k \times 1$ random vector of per unit income levels,

$\underline{x}$ a $k \times 1$ vector of activity levels,

g a fixed goal,

A an $m \times k$ matrix of technical coefficients,

$\underline{b}$ an $m \times 1$ vector of available resource levels, and

$\underline{0}$ an appropriately dimensioned vector of zeros.

Kataoka suggested an alternative version of safety-first behavior. Kataoka suggested that a decision maker might select an activity mix which generates the largest income goal for which there is only a given probability of falling below that goal. By modifying system (2) slightly, Kataoka's criterion can be written as

Kataoka's Criterion

$$\underset{\underline{x}}{\text{Maximize}}\ g \qquad (3)$$

$$\text{Subject to } A\,\underline{x} \leq \underline{b}$$

$$\Pr(z = \underline{c}^t\underline{x} < g) \leq \delta$$

$$\underline{x} \geq \underline{0},$$

with δ a probability limit (fixed exogenously) and the other parameters and variables one as previously defined.

Telser offered a third safety-first condition in which expected income is maximized subject to satisfying a probabilistic constraint on an income goal. Telser's criterion can be written as

Telser's Criterion

$$\underset{\underline{x}}{\text{Maximize}}\ \underline{E}^t\underline{x} \qquad (4)$$

$$\text{Subject to } A\underline{x} \leq \underline{b}$$

$$\Pr(z = \underline{c}^t\underline{x} < g) \leq \delta$$

$$\underline{x} \geq \underline{0},\ \delta \text{ and } g \text{ fixed.}$$

with $\underline{E}$ a k x 1 vector of expected per unit income levels and the other variables as previously defined.

In each of the above models, the income vector $\underline{c}^t$ is assumed to be a random vector. Let there be n discrete possible states of nature and $\underline{c}_i$ be a k x 1 vector of income levels should state i occur for i = 1,2,...n. Given a choice vector $\underline{x}$, aggregate income in state i can be denoted as $z_i = \underline{c}_i^t\underline{x}$. Define the matrix C as $C = [\underline{c}_1,\underline{c}_2,\ldots\underline{c}_n]^t$. A vector $\underline{z}$ of possible aggregate income values can be constructed as $\underline{z} = C\underline{x}$. Let the probability of state i be r_i and let the vector $\underline{r}$ be an n x 1 vector of these probability levels. Expected aggregate income can be written as $E_z = \underline{r}^t\underline{z} = \underline{r}^tC\underline{x}$.[1/] The probability that $z = \underline{c}^t\underline{x}$ falls below g can be computed as

$$Pr(z = \underline{c}^t\underline{x} < g) = \sum_{i=1}^{n} r_i \, I_{(-\infty,g)}(z_i) = \sum_{i=1}^{n} r_i \, I_{(-\infty,g)}(\underline{c}_i^t\underline{x}) \qquad (5)$$

where $I_{(-\infty,g)}(z_i)$ is an indicator or zero-one function which multiplies by 1 if $-\infty < \underline{c}_i^t\underline{x} < g$ or 0 if $g \leq z_i = \underline{c}_i^t\underline{x}$. The following system allows the computation of (5) by effectively constructing a zero-one function. The example presents Roy's criterion

Roys Criterion

$$\underset{\underline{x}}{\text{Minimize}} \; Pr(z = \underline{c}^t\underline{x} < g) = \underline{r}^t\underline{d} \qquad (6)$$

$$\text{Subject to } A\underline{x} \leq \underline{b} \qquad (7)$$

$$C\underline{x} - \underline{1}\,g + mI\,\underline{d} \geq \underline{0} \qquad (8)$$

$$\underline{x} \geq \underline{0}, \; d_i = 0 \text{ or } 1, \; g \text{ fixed}$$

where $\underline{d}$ is an n x 1 vector whose elements are 0 or 1,

$\underline{1}$ is an n x 1 vector of ones

m is a number larger than the worst possible loss

I is an n x n identity matrix

the other variables and parameters are as previously defined.

The i^{th} inequality of (8) is

$$\underline{c}_i^t\underline{x} - g + m\,d_i \geq 0 \qquad (9)$$

with m a large number and $d_i = 0$ or 1. When $\underline{c}_i^t\underline{x} < g$, (9) can be satisfied only with $d_i = 1$. When $\underline{c}_i^t\underline{x} \geq g$, (9) can be satisfied with $d_i = 0$. If the value $\underline{r}^t\underline{d}$ from (6) is minimized or constrained, the i^{th} element of $\underline{d}$ will be 1 only when $\underline{c}_i^t\underline{x} < g$. The expression $\underline{r}^t\underline{d}$ then becomes

$$\underline{r}^t\underline{d} = \sum_{i=1}^{n} r_i \; I_{(-\infty,g)} \, (\underline{c}_i^t\underline{x}) = \Pr(\underline{c}^t\underline{x} < g) \tag{10}$$

By slightly modifying expressions (6) -(8), Kataoka's and Telser's criteria can be modeled as follows:

Kataoka's Criterion

$$\underset{\underline{x}}{\text{Maximize}} \; g$$

$$\text{Subject to } A\underline{x} \leq \underline{b}$$

$$C\underline{x} - \underline{1}g + mI\,\underline{d} \geq \underline{0}$$

$$\underline{r}^t\underline{d} \leq \delta$$

$$\underline{x} \geq \underline{0},\; d_i = 0 \text{ or } 1,\; g \text{ free.}$$

Telser's Criterion

$$\underset{\underline{x}}{\text{Maximize}} \; \underline{E}_c^t\underline{x}$$

$$\text{Subject to } A\underline{x} \leq \underline{b}$$

$$C\underline{x} - \underline{1}g + mI\,\underline{d} \geq \underline{0}$$

$$\underline{r}^t\underline{d} \leq \delta$$

$$\underline{x} \geq \underline{0},\; d_i = 0 \text{ or } 1,\; g \text{ fixed.}$$

An Example

To demonstrate the potential of the above models, zero-one safety first solutions will be contrasted to results reported by Atwood et al. Atwood et al. generated safety-first solutions using a linear stochastic inequality. Table 1 replicates their Table 3 with modification so as to obtain zero-one Telser criterion solutions. The probability of income falling below $95000 is constrained to be less than 20%. Table 2 presents solutions obtained using the zero-one algorithm for varying levels of probability. For all solutions when the probability limit exceeded .1 the objective function of exact solutions exceed those generated with the stochastic inequality as reported by Atwood et al. For example when $\delta = .1$ the stochastic inequality model selected a solution with an objective value equal to $154074 while the zero-one solution's objective value equaled $158232. In general solutions obtained with the zero-one algorithm, being exact, will usually be less conservative than those generated with the stochastic inequality model.

At this point the reader might be concerned with the computational time required to obtain solutions to zero-one safety first models. This is a legitimate concern given the state of the art in mixed integer and zero-one algorithms. A limited Monte Carlo experiment was conducted to examine tradeoffs in computational time versus conservativeness of solutions between the stochastic inequality model and the zero-one model. Five multivariate normal samples of sizes 5, 10, 15, and 20 were generated and used for the C matrix in the above system. To avoid infeasabilities Kataoka's criterion was used with a probability limit of .25. Table 3 presents the population mean vector, variance covariance matrix used in generating the multivariate sample. Table 4 presents the mean solution time (using MICP87 on a personal computer) for the continuous stochastic inequality versus the zero-one model. Table 4 also presents average proportions of the approximate solutions divided by the exact solutions objective values. The reader will note that obtaining the continuous solutions required substantially less computation time for all sample sizes. As sample size increases, the required computation time increases much faster for the zero-one method than for the continuous method. The reader will also note, however, that the Kataoka values obtained with the continuous model were conservative at about 90 percent of the actual optimal objective value.

Summary and Conclusions

This paper has presented a method whereby exact solutions to probabilistically constrained problems can be obtained given a finitely discrete multivariate distribution of states. The solutions will generally be less conservative than those obtained using linear stochastic inequalities. However the model requires a zero-one algorithm to obtain the more accurate solutions. As sample size increases, the required computational time increases much faster for the zero-one model as contrasted to the model using the stochastic inequality and continuous linear programming code. This suggests that the researcher will need to consider tradeoffs in accuracy of solutions versus computer time and expense. For very large (or nonlinear) problems the ability to obtain solutions using the zero-one algorithm may be quire limited. In such cases, the ability to impose probabilistic constraints by enforcing continuous linear constraints may be attractive. However for smaller scale problems which are linear in the objective and constraints, the zero-one safety first model is a viable and more accurate alternative.

Footnotes

1. Suppose the decision maker or researcher possess a independently and identically distributed random sample of size n. Replacing the vector $\underline{r}$ with an n x 1 vector $\frac{1}{n}\,\underline{1}$ and the matrix C with the sample values allows the use of nonparametrically unbiased and strongly convergent estimators in the following models.

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Table 1

Partial Tableau for Example Problem

AJAEINT   OBJECTIVE: MAX   VARIABLES: 12   DATE 03-03-1989
INTEGERS: 0   CONSTRAINTS: 27   SLACKS: 26   TIME 08:16:26

	X1	X2	X3	X4	X5	X6	G	D1	D2	• • •	D9	D10		RHS	
LOWER															LOWER
UPPER															UPPER
RETURN	538.64	318.88	260.78	188.11	123.04	20.590								.0000000	RETURN
R1	1.0000	1.0000	1.0000	1.0000	1.0000								<=	400.0000	R1
R2	2.9700	1.7700	1.8200	1.8500	1.9000	.41000							<=	1084.000	R2
R3	1.0800	1.0900	1.2500	1.2800	.25000								<=	1127.000	R3
R4	2.8400	3.7100	3.0800	3.1400	.96000								<=	1611.000	R4
R5	2.3900	3.9100	4.4300	.64000	1.2300								<=	1232.000	R5
R6	5.6800			1.7400		.27000							<=	1084.000	R6
R7	2.7200	1.5600	1.6100	1.6300	.67000								<=	805.0000	R7
R8	1.0400	1.9900	1.2000	1.2200	.08000								<=	768.0000	R8
R9	.57000	.88000	.30000	.83000	.36000								<=	1230.000	R9
R10	.19000	3.1500	3.6400	.08000	.15000								<=	904.0000	R10
R11	5.3000			1.5800									<=	897.0000	R11
R12						1.0000							<=	300.0000	R12
R13			-1.0000			.07800							<=	.0000000	R13
R14				-1.0000		.10100							<=	.0000000	R14
R15					-.80000	.10100							<=	.0000000	R15
C1	516.52	217.99	296.50	132.14	106.22	-50.160	-1.0000	999999					>=	.0000000	C1
C2	781.51	412.95	343.04	203.08	126.16	-92.120	-1.0000		999999				>=	.0000000	C2
C3	620.07	322.18	213.42	114.53	111.55	200.49	-1.0000						>=	.0000000	C3
C4	250.77	139.00	166.14	105.55	101.09	141.89	-1.0000			•			>=	.0000000	C4
C5	332.24	407.41	198.00	108.88	65.790	-9.6300	-1.0000			•			>=	.0000000	C5
C6	273.25	117.71	339.72	174.31	173.26	62.760	-1.0000			•			>=	.0000000	C6
C7	507.20	274.63	262.26	273.91	139.97	-50.020	-1.0000						>=	.0000000	C7
C8	1137.6	669.96	287.19	348.87	194.90	-143.17	-1.0000						>=	.0000000	C8
C9	801.75	490.10	313.96	302.70	158.44	119.93	-1.0000				999999		>=	.0000000	C9
C10	335.62	136.89	187.58	117.73	53.510	26.070	-1.0000					999999	>=	.0000000	C10
DEVIATES								.10000	.10000	• • •	.10000	.10000	<=	.2000000	DEVIATES
GOAL							1.0000						=	95000.00	GOAL
	X1	X2	X3	X4	X5	X6	G	D1	D2	• • •	D9	D10		RHS	

Table 2
Selected Solutions to Zero-One Example Problem
With an Income Goal of $95000

Probability Level	Zero-One Expected Income	Activity Levels X_1	X_2	X_3	X_4	X_5	X_6	Continuous Model Expected Income
0	154074	163	91	100	20	26	203	154074
.1	158232	164	153	47	16	20	159	154074
.2	161088	165	195	10	13	16	128	157531

Table 3
Population Mean and Variance-Covariance Matrix for
Monte Carlo Generation of Multivariate Discrete States of Nature

		Variate				
		X_1	X_2	X_3	X_4	X_5
Mean Vector		100	120	120	130	115
Variance/Covariance Matrix	X_1	650	325	-445	-920	-390
	X_2	325	2678	475	-1020	620
	X_3	-445	475	2622	700	510
	X_4	-920	-1020	700	5125	-403
	X_5	-390	620	510	-403	850

Table 4
Average Computation Time Required to Obtain Solutions and
Proportion of Optimal Objective Value Obtained with
Inequality Approximation by Sample Size

	Sample Size			
	5	10	15	20
Mean Solution Time (seconds)				
Inequality Model	3.2	5.8	10.6	15.2
Zero-One Model	23.8	106.	357	696
Mean Proportion of Objective Achieved	.878	.909	.885	.896