Asymmetric Parametric Division Rules

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Abstract

We describe and characterize the family of asymmetric parametric division rules for the adjudication of conflicting claims on a divisible homogeneous good. As part of the characterization, we present two novel axioms which restrict how a division rule indirectly allocates between different versions of the same claimant. We also show that such division rules can alternately be represented as the maximization of an additively separable social welfare function.

1 Introduction

When a firm goes bankrupt, how should its liquidated value be distributed among creditors? How should an estate be divided among heirs when more is promised in a will than is available? How should the cost of a project be shared among the group of beneficiaries? What is a fair way to tax citizens?

The first two questions are known as *conflicting claims problems*, or simply *claims problems*. The question is how to distribute fairly some good when there is an insufficient amount of the good to satisfy all claims on it. A solution to the problem is a *division rule*, which assigns to every claims problem an allocation, or *award*, to the claimants. We classify division rules by the axioms they satisfy. The problem is as old as human history, and specific examples with proposed awards are even offered in the Talmud. O'Neill (1982) was the first to formalize the problem, and since then numerous rules and axioms have been proposed.[1] Formally, the claims problem is identical to the problems of cost-sharing and fair taxation, though we will primarily use the claims interpretation.

We characterize a family of division rules which we call (*asymmetric*) *parametric rules*, a generalization of Young's (1987) class of symmetric parametric rules, and

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[1]See Thomson (2003) for a survey.

which was first introduced by Thomson (2006, p.99). Parametric rules divide as follows: There is a collection of continuous monotone functions $\{f_i(c_i, \cdot)\}$ indexed by all possible claimants i and all possible claims c_i. Each $f_i(c_i, \cdot)$ represents a schedule of possible awards that specifies how much claimant i with claim c_i is awarded over all possible values of a parameter. Thus for parameter value λ, the amount awarded to i with claim c_i is $f_i(c_i, \lambda)$. For a given claims problem, a common parameter is chosen for all claimants so that all of the good is distributed. Intuitively, one can think of the parameter as some sort of measure of fairness, and the function $f_i(c_i, \cdot)$ is then simply the translation of this measure to an award. The choice of a common parameter implies that the claimants are being treated equitably with respect to this standard of fairness.

As part of our characterization of parametric rules, we use two axioms that are widely used in the literature: Consistency and Resource Monotonicity. Consistency states that if a division rule chooses an allocation for a group of claimants, then it should not choose to reallocate the awards of any subgroup when considered as a separate problem. Resource Monotonicity states that if the amount to be divided increases, then no claimant's award should decrease.

Another common axiom is Symmetry, which states that claimants with equal claims should receive equal awards. Obviously, parametric rules do *not* generally satisfy Symmetry. We study asymmetric division rules for normative and positive reasons. For reasons of fairness, we may not want a rule to be symmetric. As an informal example, parents may treat their children differently because they recognize each child is different and has different needs, even though the children might protest that they are being treated asymmetrically. More generally, there may be considerations outside of the model (e.g. rights, needs, history) that require a rule to treat claimants asymmetrically. Thus, depending on the context of the problem, fairness may require a rule to be asymmetric. From a descriptive stand point, there are many rules, especially real-life rules, that are not symmetric. For example, U.S. bankruptcy law stipulates that when a firm goes bankrupt, taxes owed to the federal government must be paid before other claims.[2] The forming of a queue (e.g., to purchase tickets to a popular concert or sporting event, to withdraw money during a bank run) is another example. Thus by not imposing Symmetry we are able to better understand these common division rules.

We introduce two novel axioms as part of our characterization. To understand the role these axioms play, observe that an asymmetric parametric rule could potentially allocate intrapersonally. That is, one could observe how a rule might allocate between different versions of the same claimant by comparing, say, $f_i(c_i, \cdot)$ to $f_i(c'_i, \cdot)$, where c_i and c'_i are different claims i could have. However in the traditional formulation of a claims problem, a division rule does not allocate intrapersonally, as a problem cannot have the same claimant with two different claims. But a division rule does *indirectly* allocate intrapersonally. That is, one could observe how a rule allocates between i when his claim is c_i and a second "go-between" claimant, j, with claim c_j, and then compare that to how the rule allocates between i with claim c'_i and j with claim c_j.

[2]See Kaminski (2000, 2006).

This would reveal how the rule allocates intrapersonally.

Our two axioms put restrictions on how the rule allocates intrapersonally.[3] The first axiom, Intrapersonal Consistency, states that how the rule indirectly allocates between different versions of claimant i will not change when the go-between's claim c_j changes. However, it may be that there are some intrapersonal allocations that cannot be compared, meaning there is no go-between claimant that would reveal how the rule intrapersonally allocates. The second axiom, Non-comparability Continuity in Claims at Priority Points (or N-Continuity for short), states that the non-comparability of two allocations is a continuous relation with respect to small changes in the claim.

The axioms that characterize parametric rules are Continuity, N-Continuity, a weaker version of Consistency known as Bilateral Consistency, Intrapersonal Consistency, and Resource Monotonicity. We also show that if Resource Monotonicity is strengthened to Strict Resource Monotonicity, the resulting characterization can be derived without Intrapersonal Consistency and N-Continuity.

In his paper, Young also showed that there is a connection between a symmetric parametric rule and a rule that can be written as the result of maximizing an additively separable and symmetric social welfare function. We generalize this result as well for asymmetric rules. That is, we show that any parametric rule maximizes a strictly convex and additively separable social welfare function.

There is a growing literature on asymmetric division rules. Moulin (2000) derives a rich family of asymmetric rules that satisfy Consistency, as well as axioms not considered here, namely Upper Composition, Lower Composition, and Homogeneity. Chambers (2006) studies a similar family, though without imposing Homogeneity. Naumova (2002) characterizes an asymmetric version of Young's (1988) family of equal sacrifice rules. The key axioms there are Consistency, Upper Composition, and Strict Resource Monotonicity. Hokari and Thomson (2003) characterize a family of asymmetric rules which generalize the Talmud rule, and derive the consistent extensions of these rules. Ju, Miyagawa, and Sakai (2007) accommodate the US bankruptcy rule by expanding a claims problem to allow for multi-dimensional claims. However their focus is on rules that give no benefit to claimants who transfer claims between themselves (an axiom called Reallocation-proofness).

Kaminski (2006) also accommodates division rules like the US bankruptcy rule by expanding the definition of a claims problem, though he does this even more generally than Ju et al. (2007). Instead of a claim, each individual has a "type" (of which a claim may be a part). Though Symmetry is assumed, this is with respect to types, meaning claimants with the same type receive the same award. Thus, individuals with identical claims may receive different awards if their types differ in other respects.

Interestingly, Kaminski's results can be used to provide an alternative characterization of the family of asymmetric parametric rules that we consider here. We discuss this in more detail in the conclusion. But to summarize briefly, this requires defining a claims problem to include ones where one claimant appears multiple times with different claims. The advantage of this approach is that only "standard" axioms are

[3]Obviously when Symmetry is assumed, how a rule allocates intrapersonally is not an issue as this can be inferred from how the rule allocates interpersonally.

needed. The disadvantage is that it uses a definition of a claims problem which is unrealistic. As a result, it hides the issue of intrapersonal allocation which we discussed earlier.

The paper proceeds as follows. Section 2 formalizes the claims problem, and describes several examples of division rules. Section 3 has the main results. Here, the family of parametric rules and the axioms are discussed. Examples are given to demonstrate the necessity of the axioms. The characterization result is stated. Also, the equivalence of the family of parametric rules and those that maximize an appropriate social welfare function is shown. Section 4 concludes with a discussion of how Kaminski (2006) relates to our results. Proofs are collected in the appendix.

2 The Model

2.1 Definitions

The conflicting claims problem is simple: there is a group of people, each of whom has a claim on some divisible homogeneous good, but there is an insufficient amount of the good to satisfy all of the claims. Formally, a (*claims*) *problem* is a tuple (N, c, E) where $N \subset \mathbb{N}$ is a finite group of claimants, $c = (c_i)_{i \in N} \in \mathbb{R}^N_{++}$ is the vector of claims, and $E \in \mathbb{R}_+$ is the endowment to be divided, all satisfying $E \leq \sum_N c_i$. We will typically write a problem as (c, E), as the group of claimants N is implicit in the claims vector c. An *awards vector* for a problem (c, E) is an allocation $x \in \mathbb{R}^N_+$ satisfying $\sum_N x_i = E$ and $0 \leq x_i \leq c_i$ for every $i \in N$. Let $X(c, E)$ denote the set of awards vectors for the problem (c, E). A *division rule* is a function S that maps every problem to an awards vector.

A convenient way of graphically representing a division rule is the path of awards it generates. For a fixed set of claimants N and for a fixed claims vector c for those claimants, the path of awards of a rule S is the graph of all possible allocations awarded by S as E varies from 0 to the sum of claims $\sum_N c_i$. See Figure 1.

Alternatively, a problem (c, E) can be interpreted as a cost-sharing problem or a taxation problem. Under the cost-sharing interpretation, c is the vector of monetary benefits each individual will receive from the shared project and E is the cost of the project. The restriction $E \leq \sum_N c_i$ means that the project is socially beneficial. Under the taxation interpretation, c is the vector of incomes and E is the total tax to be raised.

2.2 Examples of Division Rules

The following are simple examples of division rules. Figure 2 illustrates the paths of awards for each of these division rules.

- *The Proportional Rule.* For every (c, E) and for every $i \in N$,

$$P_i(c, E) \equiv \lambda c_i,$$

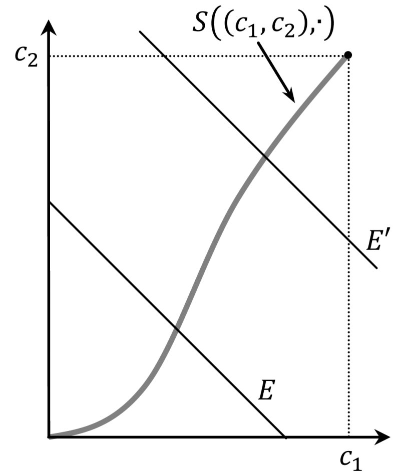

Figure 1: ***Path of awards.*** *The horizontal axis measures claimant 1's claim and award. The vertical axis measures claimant 2's. The path of awards is the set of all awards as E varies.*

where λ is chosen so that $\sum_N P_j(c, E) = E$. (Note then that $\lambda = E \big/ \sum_N c_j$.) P gives to each claimant the same proportion of his respective claim. For example, if there is only enough of the endowment to cover half of the total claims, then P will give each claimant half of his claim.

- *The Constrained Equal Awards Rule.* For every (c, E) and for every $i \in N$,

$$CEA_i(c, E) \equiv \min\{c_i, \lambda\}$$

where λ is chosen so that $\sum_N CEA_j(c, E) = E$. CEA gives to each claimant the same award, with the exception of those claimants who would otherwise receive more than their respective claims.

- *The Constrained Equal Losses Rule.* For every (c, E) and for every $i \in N$,

$$CEL_i(c, E) \equiv \max\{0, c_i - \lambda\},$$

where λ is chosen so that $\sum_N CEL_j(c, E) = E$. CEL equalizes losses (i.e. the difference between an agent's claim and his award) across claimants, with the exception of those claimants who would otherwise receive a negative award.

- *The Dictatorial Rule with priority* $\succ$ (where $\succ$ is a strict linear order over $\mathbb{N}$). For every (c, E) and for every $i \in N$,

$$Dic_i^{\succ}(c, E) \equiv \begin{cases} c_i, & \text{if } i \succ \lambda \\ E - \sum_{j \in N: j \succ i} c_j, & \text{if } i = \lambda \\ 0, & \text{if } \lambda \succ i \end{cases},$$

where $\lambda \in \mathbb{N}$ is chosen so that $\sum_N Dic_j^{\succ}(c, E) = E$. $Dic^{\succ}$ distributes the endowment by lining up the claimants according to $\succ$, and then going down the line and giving each claimant his full claim until the endowment is exhausted.

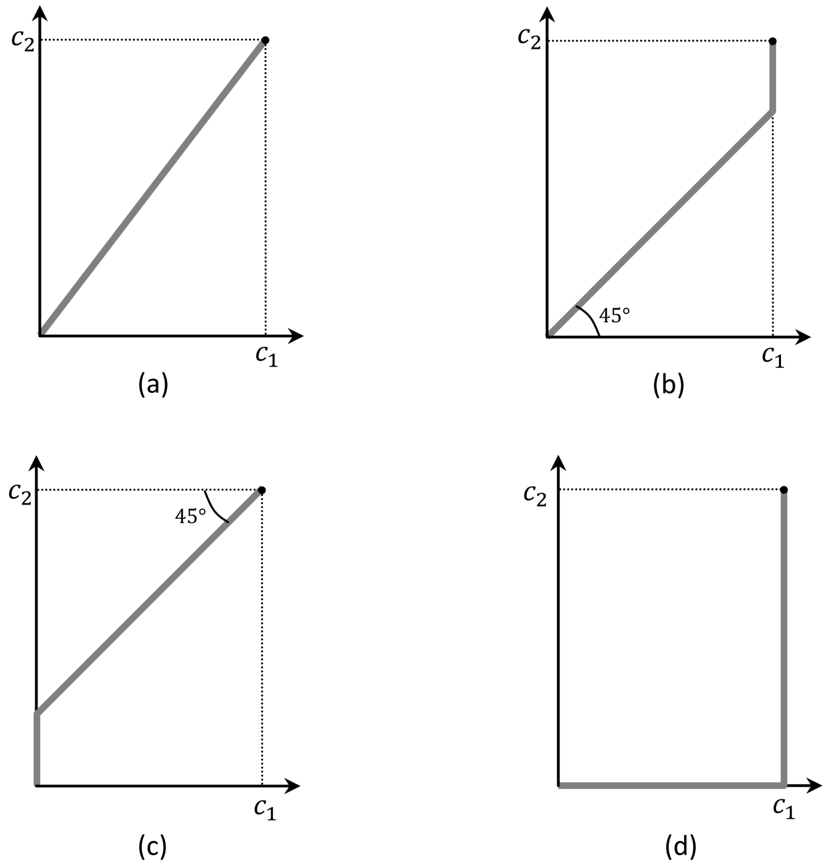

Figure 2: ***Paths of awards for examples.*** *The paths of awards for (a) P, (b) CEA, (c) CEL, and (d) $Dic^{1 \succ 2}$.*

3 Main Results

3.1 Asymmetric Parametric Division Rules

We characterize a family of division rules that we call (*asymmetric*) *parametric* rules. To understand this family, consider again the examples above. In each case, the award given to a claimant is determined by his claim c_i and a parameter λ. For a given problem, a common parameter is chosen for all claimants so that the sum of awards equals the endowment.

Formally, let $\mathcal{F}$ be the family of functions $f : \mathbb{N} \times \mathbb{R}_{++} \times [a, b] \to \mathbb{R}_+$, where $-\infty \leq a < b \leq \infty$, such that (i) f is weakly increasing in the third argument, (ii) f is continuous in the third argument, and (iii) for every $i \in \mathbb{N}$ and $c_0 \in \mathbb{R}_{++}$ we have $f(i, c_0, a) = 0$ and $f(i, c_0, b) = c_0$. From here on, we will write $f(i, c_i, \lambda)$ as $f_i(c_i, \lambda)$. One can alternatively think of $f \in \mathcal{F}$ as the collection of functions $\{f_i\}_{\mathbb{N}}$, where each $f_i : \mathbb{R}_{++} \times [a, b] \to \mathbb{R}_+$ satisfies the above three criteria.

Observe that for any $f \in \mathcal{F}$ and for any claims vector c, the function $\sum_N f_i(c_i, \cdot)$ is continuous and weakly increasing. So by the Intermediate Value Theorem, for every (c, E) there exists $\lambda \in [a, b]$ such that $\sum_N f_i(c_i, \lambda) = E$. Moreover, if λ' is such that $\sum_N f_i(c_i, \lambda') = E$ as well, then for every $i \in N$, we have $f_i(c_i, \lambda') = f_i(c_i, \lambda)$. Hence,

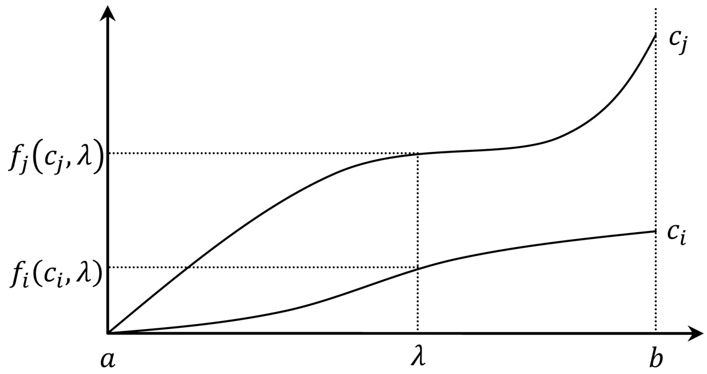

Figure 3: ***Parametric functions.*** *An asymmetric parametric division rule would divide E between i and j (when their respective claims are c_i and c_j) by finding λ such that $f_i(c_i, \lambda) + f_j(c_j, \lambda) = E$.*

for any $f \in \mathcal{F}$, we can define a division rule S^f as follows. For every (c, E) and for every $i \in N$,

$$S_i^f(c, E) \equiv f_i(c_i, \lambda), \tag{1}$$

where λ is chosen so that $\sum_N f_i(c_i, \lambda) = E$. We say a rule S has a *parametric representation* if there exists $f \in \mathcal{F}$ such that $S = S^f$. If f is continuous then we say S has a *continuous* parametric representation. See Figure 3.

A special case is a rule that has a parametric representation $f = \{f_0\}$, i.e. where $f_i = f_0$ for every $i \in \mathbb{N}$. We say S has a *symmetric parametric representation* f_0 if $\{f_0\} \in \mathcal{F}$ is a parametric representation of S. Such rules were characterized by Young (1987), and we discuss his axioms shortly.

Of our examples above, P, CEA, and CEL all have symmetric parametric representations, while $Dic^{\succ}$ has an asymmetric parametric representation. For P, the function

$$f_0(c_i, \lambda) = \lambda c_i$$

is a parametric representation where $a = 0$ and $b = 1$. For CEA, the function

$$f_0(c_i, \lambda) = \min\{c_i, \lambda\}$$

is a parametric representation where $a = 0$ and $b = \infty$. For CEL, the function

$$f_0(c_i, \lambda) = \max\{0, c_i + \lambda\}$$

is a parametric representation where $a = -\infty$ and $b = 0$. For $Dic^{<}$ (note that $\succ$ is the "less than" relation), the collection of functions

$$f_i(c_i, \lambda) = \begin{cases} 0, & \lambda < i - 1 \\ (1 + \lambda - i)\, c_i, & i - 1 \leq \lambda < i \\ c_i, & \lambda \geq i \end{cases}$$

is a parametric representation where $a = 0$ and $b = \infty$. Figure 4 illustrates.

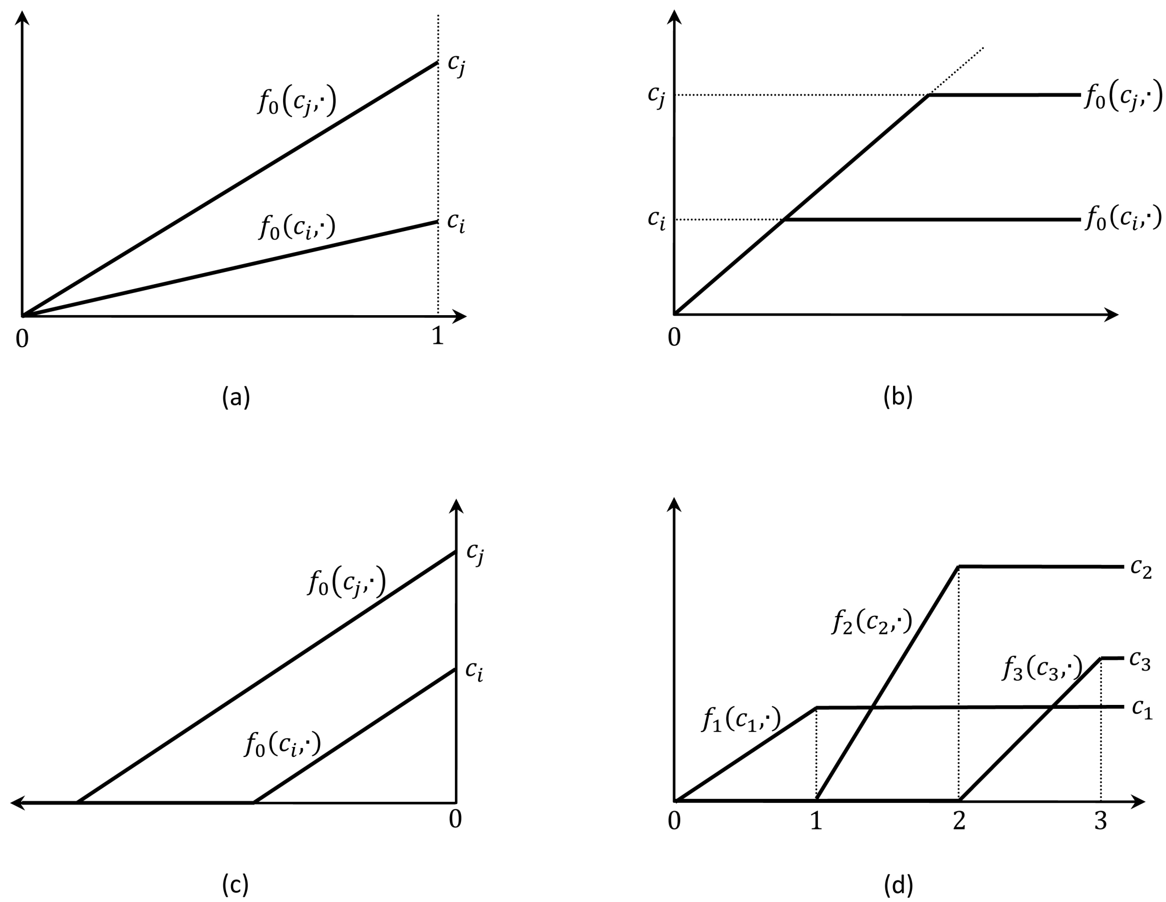

Figure 4: ***Parametric representations for examples.*** *Parametric representations of (a) P, (b) CEA, (c) CEL, and (d) $Dic^{<}$.*

3.2 Young (1987)

As parametric rules are a generalization of Young's (1987) symmetric parametric rules, we first introduce the axioms used by Young. One of the most prominent axioms studied in the literature is Consistency, which deals with how the rule allocates when the group of claimants shrinks.

Consistency. *For every (c, E), if $N' \subset N$, then*

$$S_{N'}(c, E) = S\left(c_{N'}, \textstyle\sum_{N'} S_i(c, E)\right).$$

A slightly weaker version is Bilateral Consistency, which is Consistency applied only to two-person subgroups N'.

To understand Consistency, consider the following. Suppose a rule chooses an allocation for a problem. Some claimants are given their respective awards and leave. Suppose now the rule is asked to reconsider its original allocation for those claimants who remain. That is, the rule is given the opportunity to reallocate what remains of the endowment between the claimants who have not yet received their awards.[4] The

[4] This is also a claims problem since $\sum_{N'} S_i(c, E) \leq \sum_{N'} c_i$ for every $N' \subset N$.

rule can choose to distribute according to the original allocation, or it can choose a new allocation. What Consistency says is that the rule will always choose the original allocation. All of the examples given above are consistent, though it would not be difficult to construct a rule that is not. For example, a rule that distributed according to CEA when there are three claimants and CEL when there are two claimants would not satisfy Consistency.

Symmetry. *For every* (c, E) *and for every* $\{i, j\} \subset N$*, if* $c_i = c_j$*, then* $S_i(c, E) = S_j(c, E)$*.*

This states that if two agents have the same claim, then they will get the same award. Of the examples above, $Dic^{\succ}$ is not symmetric.

Continuity. *For every* (c, E)*, for every sequence of problems* $\left\{\left(c^k, E^k\right)\right\}$ *with the same group of claimants* N*, if* $\left(c^k, E^k\right) \to (c, E)$*, then* $S\left(c^k, E^k\right) \to S(c, E)$*.*

Young's Theorem states that a division rule has a continuous symmetric parametric representation if and only if it satisfies Continuity, Bilateral Consistency, and Symmetry. An important step in the proof is showing that these three axioms imply the following.

Resource Monotonicity. *For every* (c, E) *and* (c, E')*, if* $E < E'$*, then for every* $i \in N$ *we have* $S_i(c, E) \leq S_i(c, E')$*.*

Resource Monotonicity states that if the endowment increases, no claimant should get a smaller award. A stronger version is Strict Resource Monotonicity, the only difference being a strict rather than weak inequality in the statement of the axiom. All of the examples given above are resource monotonic, while P is also strictly resource monotonic.

3.3 Intrapersonal Consistency

We motivate our first novel axiom with the following example, which illustrates that simply dropping Symmetry from Young's theorem is not enough to characterize parametric rules.

Example 1 *We construct a continuous, consistent, and resource monotonic rule and show that it does not have a parametric representation.*

Consider the rule S *defined over any claims problem where* $N = \{1, 2\}$*. Fix* c_1*,* c_1'*,* c_2*, and* c_2' *where* $c_1 < c_1'$ *and* $c_2 < c_2'$*. Let* S *satisfy* $S((c_1, c_2), c_1) = (c_1, 0)$*,* $S((c_1', c_2), c_2) = (0, c_2)$*,* $S((c_1, c_2'), c_2') = (0, c_2')$*, and* $S((c_1', c_2'), c_1') = (c_1', 0)$*. Hence, if claims are either* (c_1, c_2) *or* (c_1', c_2') *then* S *gives priority to claimant* 1 *over claimant* 2*. If claims are either* (c_1', c_2) *or* (c_1, c_2') *then* S *gives priority to* 2 *over* 1*. Figure 5 illustrates the paths of awards for this rule.*

In the appendix, we show that S *can be extended to a continuous, consistent, and resource monotonic rule over all problems.*

Now we show that there is no parametric representation of S*. If there were, then panel (a) of Figure 5 would imply that there exist* $\lambda < \lambda'$ *such that* $f_1(c_1, \cdot)$ *is weakly*

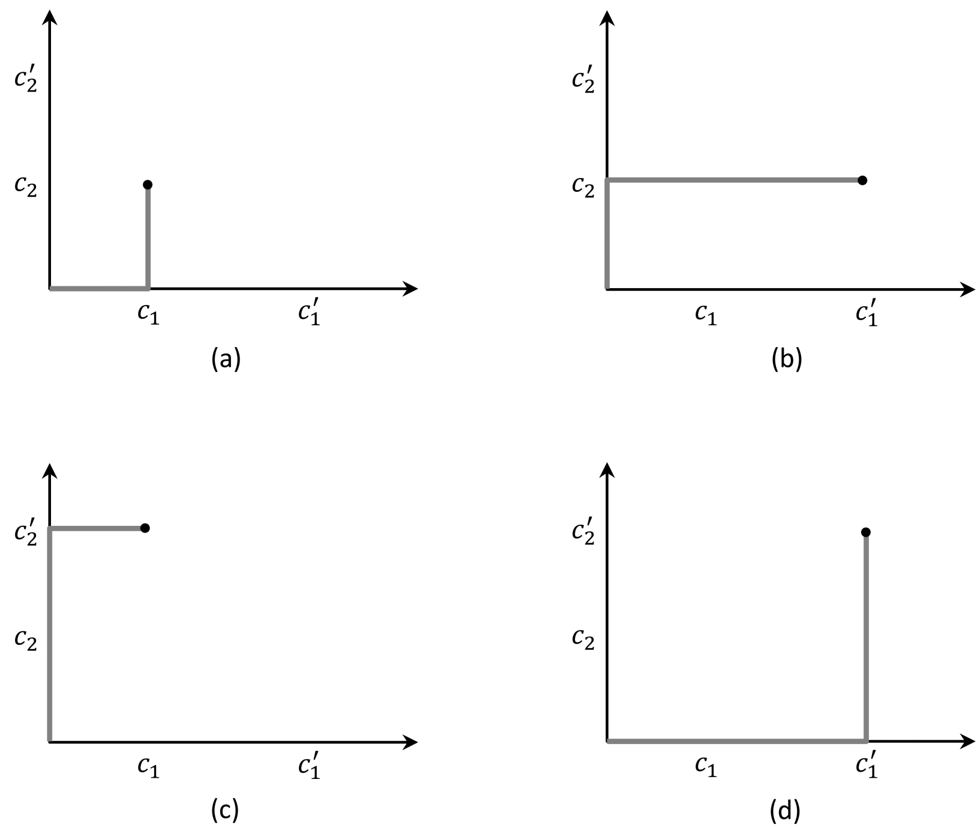

Figure 5: ***Paths of awards for Example 1.*** *Panel (a) shows that when 1 has claim c_1 and 2 has claim c_2, then 1 is given priority over 2. Panel (b) shows that when 1 has a claim of c_1' and 2 has a claim of c_2, then 2 is given priority over 1. Etc.*

increasing on $[\lambda, \lambda']$, $f_1(c_1, \lambda) = 0$, $f_1(c_1, \lambda') = c_1$, and where $f_2(c_2, \cdot)$ is zero on $[\lambda, \lambda']$. Similarly, panel (b) would imply that there exists $\lambda'' > \lambda'$ such that $f_2(c_2, \cdot)$ is weakly increasing on $[\lambda', \lambda'']$, $f_2(c_2, \lambda'') = c_2$, and where $f_1(c_1', \cdot)$ is zero on $[\lambda, \lambda'']$. Panel (d) would imply that there exists $\lambda''' > \lambda''$ such that $f_1(c_1', \cdot)$ is weakly increasing on $[\lambda'', \lambda''']$, $f_1(c_1', \lambda''') = c_1'$, and where $f_2(c_2', \cdot)$ is zero on $[\lambda, \lambda''']$. Finally, panel (c) would imply that $f_1(c_1, \cdot)$ is zero on $[\lambda, \lambda''']$, which is a contradiction since we already showed that $f_1(c_1, \lambda') = c_1$.

The key to understanding this example is recognizing that a rule can implicitly reveal how it allocates between different claims of one claimant. For example, panel (a) of Figure 5 shows that S gives priority to c_1 over c_2, while panel (b) shows that S gives priority to c_2 over c_1'. Hence, S implicitly gives priority to c_1 over c_1'. However, S fails to have an asymmetric parametric representation because panels (c) and (d) imply the opposite: that S gives priority to c_1' over c_1. The reason why Consistency does not preclude such rules is because it has no force with two-person groups. What is needed is an axiom that requires a rule to be *intrapersonally* consistent.

The following definitions will make it easier to formalize this axiom. Define:

$$Y \equiv \{(i, c_i, x_i) : i \in \mathbb{N}, c_i \in \mathbb{R}_{++}, x_i \in [0, c_i]\}.$$

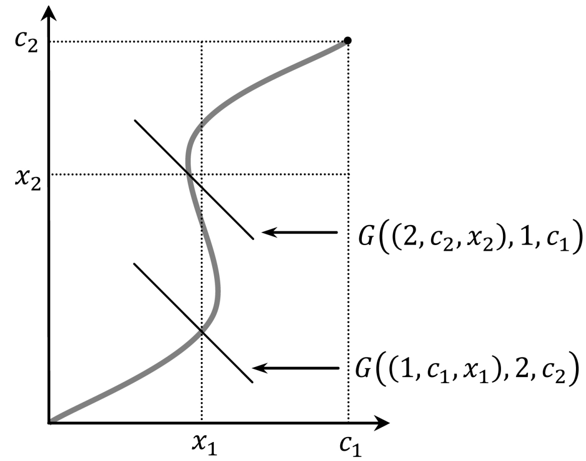

Figure 6: ***Illustration of*** G**.**

We think of (i, c_i, x_i) as describing an agent, his claim, and his award. Following Young (1994, p.76), we will call (i, c_i, x_i) a *situation*.[5] For a given rule S, for every situation $(i, c_i, x_i) \in Y$, $j \neq i$, and $c_j \in \mathbb{R}_{++}$, define the function:

$$G\left(\left(i, c_i, x_i\right), j, c_j\right) \equiv \inf\left\{E : S_i\left(\left(c_i, c_j\right), E\right) \geq x_i\right\}.$$

Figure 6 illustrates.

Observe that for continuous rules, inf can be replaced with min. Hence $G\left(\left(i, c_i, x_i\right), j, c_j\right)$ is the smallest endowment needed for S to award x_i to agent i (when i's and j's respective claims are c_i and c_j).

Define the binary relation R_1 over Y as follows.

Definition 1 $(i, c_i, x_i)\, R_1\, (j, c_j, x_j)$ *if* $i \neq j$ *and* $G\left(\left(i, c_i, x_i\right), j, c_j\right) \leq G\left(\left(j, c_j, x_j\right), i, c_i\right)$.

Let I_1 and P_1 denote the symmetric and asymmetric parts of R_1 respectively. Obviously, $(i, c_i, x_i)\, I_1\, (j, c_j, x_j)$ if and only if $G\left(\left(i, c_i, x_i\right), j, c_j\right) = G\left(\left(j, c_j, x_j\right), i, c_i\right)$ and $(i, c_i, x_i)\, P_1\, (j, c_j, x_j)$ if and only if $G\left(\left(i, c_i, x_i\right), j, c_j\right) < G\left(\left(j, c_j, x_j\right), i, c_i\right)$.

Consider Figure 6 again. As the endowment increases from 0 to $c_1 + c_2$, the amount awarded follows the path of awards from the origin to (c_1, c_2). The path crosses the vertical line representing x_1 before it crosses the horizontal line representing x_2. (Observe that this particular division rule is not resource monotonic, and so the path of awards crosses x_1 multiple times.) This means the rule awards x_1 to 1 before it awards x_2 to 2, or $(1, c_1, x_1)\, P_1\, (2, c_2, x_2)$. In this sense, we think of R_1 as being (part of) an underlying social preference over situations that determines how a rule adjudicates problems. That is, if $(i, c_i, x_i)\, R_1\, (j, c_j, x_j)$, then presumably this is because the rule deems it just to award x_i to i before it awards x_j to j (when their respective claims are c_i and c_j).

[5] Note however that since Young assumes Symmetry, his definition of a situation does not include the individual's identity. That is he defines a situation as the pair (c_0, x_0) where $c_0 > 0$ and $0 < x_0 < c_0$.

However R_1 is not a complete ordering as it requires that $i \neq j$. That is, R_1 does not make intrapersonal comparisons directly. But as illustrated in Example 1, a rule's social preference over intrapersonal situations can be revealed indirectly. That is, if we have $(i, c_i, x_i)P_1(j, c_j, x_j)$ and $(j, c_j, x_j)P_1(i, c_i', x_i')$, then we can interpret that as saying there is a social preference of (i, c_i, x_i) over (i, c_i', x_i') even though there is no claims problem that would reveal that social preference directly. This brings us to our axiom.

Intrapersonal Consistency. *For every* $(i, c_i, x_i), (i, c_i', x_i'), (j, c_j, x_j), \left(j, c_j', x_j'\right) \in Y$ *where* $i \neq j$, *if*

$$(i, c_i, x_i)\, P_1\, (j, c_j, x_j)\, P_1\, (i, c_i', x_i'),$$

then it is not true that

$$(i, c_i', x_i')\, P_1\, \left(j, c_j', x_j'\right) P_1\, (i, c_i, x_i).$$

Intrapersonal Consistency is a restriction on how the rule implicitly allocates between different claims of the same claimant. It states that if there is a social preference for (i, c_i, x_i) over (i, c_i', x_i'), then there cannot also be a social preference for the opposite.[6] Obviously Example 1 violates Intrapersonal Consistency.

To understand how Intrapersonal Consistency relates to Consistency, consider the following variation.

Alternative to Consistency. *For every* $(i, c_i, x_i), (j, c_j, x_j), (k, c_k, x_k), (l, c_l, x_l) \in Y$ *where* i, j, k *are distinct and* i, k, l *are distinct, if*

$$(i, c_i, x_i)\, P_1\, (j, c_j, x_j)\, P_1\, (k, c_k, x_k),$$

then it is not true that

$$(k, c_k, x_k)\, P_1\, (l, c_l, x_l)\, P_1\, (i, c_i, x_i).$$

This alternative to Consistency concerns interpersonal comparisons, as opposed to the intrapersonal comparisons in Intrapersonal Consistency. One can show that Consistency implies the above alternative.[7] Conversely, one could replace Consistency with the above alternative in Theorem 1 and the result would still hold.

We now formalize the implicit intrapersonal ranking of a rule. Define the binary relation R_2 over Y as follows.

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[6] Though stated as a restriction on the ordering R_1, *Intrapersonal Consistency* is a restriction on division rules since R_1 is defined from S. The statement of the axiom purely in terms of (continuous) division rules is as follows: For every $\{i, j\} \subset \mathbb{N}$, $c_i, c_i', c_j, c_j' \in \mathbb{R}_{++}$, and $E, E', \hat{E}, \hat{E}' \in \mathbb{R}_+$, if $S((c_i, c_j), E) = (x_i, x_j)$, $S((c_i', c_j), E') = \left(x_i', x_j'\right)$, $S\left(\left(c_i, c_j'\right), \hat{E}\right) = (x_i, \hat{x}_j)$, $S\left(\left(c_i', c_j'\right), \hat{E}'\right) = \left(x_i', \hat{x}_j'\right)$, $E = G((i, c_i, x_i), j, c_j)$, $E' = G((i, c_i, x_i'), j, c_j)$, $\hat{E} = G\left((i, c_i, x_i), j, c_j'\right)$, $\hat{E}' = G\left((i, c_i, x_i'), j, c_j'\right)$, and $x_j < x_j'$, then $\hat{x}_j \leq \hat{x}_j'$.

[7] This is essentially shown in the first case in the proof of Lemma 10.

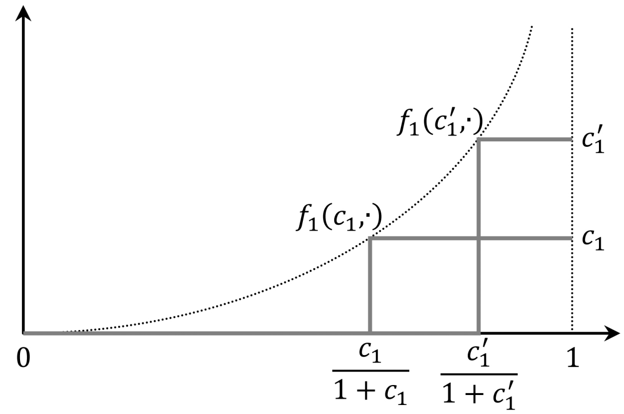

Figure 7: ***"Parametric" representation for Example 2.*** *f_1 is a collection of step correspondences.*

Definition 2 *$(i, c_i, x_i)\, R_2\, (i, c_i', x_i')$ if there exists $(j, c_j, x_j) \in Y$ where $j \neq i$ such that $(i, c_i, x_i)\, R_1\, (j, c_j, x_j)\, R_1\, (i, c_i', x_i')$.*

Let I_2 and P_2 denote the symmetric and asymmetric parts of R_2 respectively. Define the binary relation C over Y as follows.

Definition 3 *$(i, c_i, x_i)\, C\, (i, c_i', x_i')$ if either $(i, c_i, x_i)\, R_2\, (i, c_i', x_i')$ or $(i, c_i', x_i')\, R_2\, (i, c_i, x_i)$.*

So we interpret C to be the "comparable" relation. That is, situations (i, c_i, x_i) and (i, c_i', x_i') are comparable if there exists some (j, c_j, x_j) that separates them. However it may be that such a (j, c_j, x_j) does not exist, in which case (i, c_i, x_i) and (i, c_i', x_i') would not be comparable. Let NC denote the "not comparable" relation.

3.4 Non-comparability Continuity in Claims at Priority Points

Our final axiom concerns when two situations should not be comparable. To motivate this axiom, consider the following example.

Example 2 *We construct a rule that is continuous, consistent, intrapersonally consistent, and resource monotonic, but which does not have a parametric representation. For $i \neq 1$, let $f_i(c_0, \lambda) = \lambda c_0$ be i's parametric function on $[0, 1]$. For $i = 1$, f_1 is not a function, but rather a correspondence on $[0, 1]$, defined by*

$$f_1(c_0, \lambda) = \begin{cases} 0, & \text{for } \lambda < \frac{c_0}{1+c_0} \\ [0, c_0] & \text{for } \lambda = \frac{c_0}{1+c_0} \\ c_0, & \text{for } \lambda > \frac{c_0}{1+c_0} \end{cases} .$$

So for every c_0, $f_1(c_0, \cdot)$ is a step correspondence where $\frac{c_0}{1+c_0}$ is the location of the step and c_0 is its height. Figure 7 illustrates f_1.

Observe that for any (c, E), there exists a unique λ such that $E \in \sum_{i \in N} f_i(c_i, \lambda)$. So define a division rule S as follows. For any (c, E), for $i \neq 1$,

$$S_i(c, E) \equiv f_i(c_i, \lambda),$$

and for $i = 1$,

$$S_1(c, E) \equiv E - \sum_{i \in N \setminus \{1\}} S_i,$$

where λ is chosen so that $E \in \sum_{i \in N} f_i(c_i, \lambda)$. (Hence, $S_1(c, E) \in f_1(c_1, \lambda)$.) One can show that S is continuous, consistent, intrapersonally consistent, and resource monotonic.

We claim that S does not have an asymmetric parametric representation. Suppose it does have a representation, $\hat{f}$. For every $c_0 > 0$, set

$$m(c_0) \equiv \sup \left\{ \lambda \in [a, b] : \hat{f}_1(c_0, \lambda) = 0 \right\}$$

and

$$M(c_0) \equiv \inf \left\{ \lambda \in [a, b] : \hat{f}_1(c_0, \lambda) = c_0 \right\}.$$

Hence, $[m(c_0), M(c_0)]$ is the smallest subinterval of $[a, b]$ such that the range of $\hat{f}_1(c_0, \cdot)$ is $[0, c_0]$. Since $\hat{f}_1$ is continuous in the second argument, $m(c_0) < M(c_0)$ for every $c_0 > 0$. By definition of S, if $c_0 < c_0'$, then $M(c_0) < m(c_0')$. (This is because for claimant 1, S gives priority to smaller claims.) Let $\mathbb{Q}$ denote the set of rational numbers. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, choose $q(c_0) \in [m(c_0), M(c_0)] \cap \mathbb{Q}$ for every c_0. By above, if $c_0 < c_0'$, then $q(c_0) < q(c_0')$. Hence $\{q(c_0) : c_0 > 0\}$ is an uncountable set of distinct rational numbers, which is impossible since $\mathbb{Q}$ is countable.

Before stating the axiom, we need a definition.

Definition 4 *The rule S* ***gives priority to $(i, c_i, x_i) \in Y$*** *if $x_i \in (0, c_i)$ and if there exists $\epsilon > 0$ such that for every $(\hat{c}, E)$ where $i \in N$, $\hat{c}_i = c_i$, and $S_i(\hat{c}, E) = x_i$, for every $a \in (-\epsilon, \epsilon)$,*

$$S_i(\hat{c}, E + a) = x_i + a.$$

Though this definition may seem very restrictive, there are many rules which give priority to some situation. For example, $Dic^{\succ}$ gives priority to every (i, c_i, x_i) where $x_i \in (0, c_i)$. Similarly, in Example 2, the rule gives priority to every $(1, c_1, x_1)$ where $x_1 \in (0, c_1)$.

Non-comparability Continuity in Claims at Priority Points. *If S gives priority to (i, c_i, x_i), then there exists $\epsilon > 0$ such that for every $c_i' \in (c_i - \epsilon, c_i + \epsilon)$, we have $(i, c_i', x_i)\, NC\, (i, c_i, x_i)$.*

We will refer to this axiom as N-Continuity. If S gives priority to situation (i, c_i, x_i), then small perturbations of x_i will not be comparable to (i, c_i, x_i). N-Continuity states that if such is the case, then small perturbations of c_i will also not be comparable to (i, c_i, x_i). Though certainly not intuitive, this technical axiom is only needed to rule out pathological rules such as the one in Example 2. So imposing it should not be considered very costly.

3.5 Characterizations and Independence of Axioms

We now state the main theorem.

Theorem 1 *S has a continuous parametric representation if and only if S satisfies Continuity, N-Continuity, Bilateral Consistency, Intrapersonal Consistency, and Resource Monotonicity.*

The basic outline of the proof is not hard to understand. Using R_1 and R_2, we define a complete ordering R over Y. Consistency and Intrapersonal Consistency imply that R is also transitive. Continuity and N-Continuity imply that there is a countable R-dense subset of Y. Thus, by the standard representation result, there is a numerical representation $r : Y \to \mathbb{R}$ of R. The function r thus acts as a numerical measure of the fairness of a situation, just as the parameter in the parametric function acts as a measure of fairness. Thus taking the inverse of r with respect to the third argument (the award) gives us the parametric function. The remainder of the proof is to show that this does in fact represent the division rule and that it satisfies the properties of a parametric function.[8]

The following examples demonstrate the extent to which the axioms are independent. (Verification that each example satisfies all the axioms but the one stated is left to the reader.)

- **Continuity**. A division rule that satisfies all the axioms but Continuity is as follows. Let $f \in \mathcal{F}$ be such that f_i is strictly increasing in λ but not jointly continuous for every i. Define S according to (1). Then S is not continuous. However, since f_i is strictly increasing in λ for every i, there is no $(i, c_i, x_i) \in Y$ such that S gives priority to (i, c_i, x_i). So N-Continuity is satisfied vacuously. The other axioms can be easily verified.

- **N-Continuity**. Example 2 satisfies all the axioms but N-Continuity.

- **Bilateral Consistency**. A division rule which allocates according to P for two-person groups, but CEA for groups larger than two, would satisfy all the axioms but Consistency.

- **Intrapersonal Consistency**. Example 1 satisfies all the axioms but Intrapersonal Consistency.

It is an open question whether Resource Monotonicity is independent of the other axioms.

[8] Other papers use similar techniques of using an ordering over an appropriately defined situation space in proving their results. Young (1987) and Moreno-Ternero and Roemer (2006) each derive a numerical ordering directly from a rule. Kaminski (2006) defines a binary relation and shows it has a numerical representation, as we do here. However each of these assumes some version of Symmetry, which imposes quite a bit of richness on their respective orderings. Without Symmetry, the establishment of a numerical representation of the ordering takes considerably more effort here than these other papers.

If Resource Monotonicity is strengthened to Strict Resource Monotonicity, then the parametric representation is strictly increasing in the parameter. However, it is interesting to note that for strictly resource monotonic rules, the axioms Intrapersonal Consistency and N-Continuity are not needed for the characterization.

Theorem 2 *S has a continuous parametric representation with a strictly increasing parametric function if and only if S satisfies Continuity, Bilateral Consistency, and Strict Resource Monotonicity.*

3.6 Parametric Rules and Collective Rationality

Choosing an award that maximizes some measure of social welfare is a natural way of solving a claims problem. In this section, we show that every parametric division rule maximizes a continuous, strictly convex, additively separable social welfare function. Young (1987, Theorem 2) has a similar result, though obviously without symmetry.

In our setting, a *social welfare function* (SWF) is a real-valued function W of claims-awards pairs (c, x). We say a SWF is *additively separable* if there exists $U : Y \to \mathbb{R}$ such that $W(c, x) = \sum_{i\in N} U_i(c_i, x_i)$. Note that if U is strictly concave in the third argument, then for every claims problem (c, E), $\arg\max_{x\in X(c,E)} \sum_{i\in N} U_i(c_i, x_i)$ is a singleton.[9] Hence for such a U, we can define a division rule:

$$S^U(c, E) \equiv \underset{x\in X(c,E)}{\arg\max} \sum_{i\in N} U_i(c_i, x_i).$$

We say a division rule S has a *collectively rational additively separable (CRAS) representation* if there exists $U : Y \to \mathbb{R}$ strictly convex in the third argument such that $S = S^U$. If U is continuous, then we say S has a *continuous* CRAS representation.

Theorem 3 *S has a continuous parametric representation if and only if S has a continuous CRAS representation.*

Together, Theorems 1 and 3 imply that Continuity, N-Continuity, Bilateral Consistency, Intrapersonal Consistency, and Resource Monotonicity characterize the family of continuous CRAS rules.

The intuition behind the proof can provide insight into the connection between these two families of rules. Begin with a CRAS rule. The first order conditions for the maximization problem require that the solution equalize marginal utilities between all claimants who are not constrained by their respective claims. Taking the inverse of each claimants marginal utility function with respect to their award gives us their parametric function. Going the other direction, if a rule has a parametric representation, then taking the inverse of each claimants parametric function with respect to the parameter gives us what will become their marginal utility function. Integrating that gives us each claimant's utility function, which we add together to get the additively separable SWF.

[9]Recall that $X(c, E) = \{x : \sum_N x_i = E \text{ and } 0 \leq x_i \leq c_i \text{ for every i}\}$.

The key insight here is that the marginal utility function plays the same role as the ordering R that we construct for the proof of Theorem 1. Specifically, if (i, c_i, x_i) gives a higher marginal utility than (j, c_j, x_j), then in order to maximize social welfare, a CRAS rule must allocate x_i to i before it allocates x_j to j when their respective claims are c_i and c_j. But this is exactly what $(i, c_i, x_i)R(j, c_j, x_j)$ means. Hence the marginal utility function and r, the numerical representation of R, are essentially interchangeable, and this fact is exploited in the proof for Theorem 3.

This also provides us with a new interpretation of a parametric rule. Namely a parametric rule divides by assigning to each claimant a utility function, and then finds an award so as to equalize marginal utilities across the claimants subject to the constraint that no claimant receive more than his claim. In this interpretation, the parameter – the measure of fairness by which the division rule compares situations – is the marginal utility that each claimant receives at his award. Thus the rule treats the claimants fairly be equalizing their marginal utilities.

This method of solving a claims problem is similar to the one characterized by Lensberg (1987) for bargaining problems. Indeed, one could view the family of claims problems as a special class of bargaining problems. One key difference is that here the SWF depends on the individual's claims, something which is not present in Lensberg's result.

We finish with a result for strictly resource monotonic division rules.

Theorem 4 *S has a continuous parametric representation with a strictly increasing parametric function if and only if S has a continuous CRAS representation that is continuously differentiable in the third argument.*

Together, Theorems 2 and 4 imply that Continuity, Bilateral Consistency, and Strict Resource Monotonicity characterize this family of division rules.

4 Concluding Remarks

We conclude by discussing an alternative characterization of the parametric family of division rules. This alternative characterization requires an expanded definition of a claims problem, but standard (appropriately modified) axioms. To do this, we employ a result due to Kaminski (2006).

As discussed in the introduction, the definition of a problem does not allow a division rule to make direct intrapersonal allocations. We have followed this traditional formulation of a problem, and our two novel axioms, Intrapersonal Consistency and N-Continuity, impose restrictions on how a rule indirectly allocates intrapersonally. An alternative approach would be to expand the definition of a problem to directly allow for such intrapersonal allocations, and then impose standard axioms. We outline how that might be formulated.

For a finite group of claimants $N \subset \mathbb{N}$, a problem is a tuple (m, c, E) where $m \in \mathbb{N}^N$ is the vector of identities of the claimants, $c \in \mathbb{R}^N_{++}$ is the vector of their claims, and $E \in \mathbb{R}_+$ is the endowment to be divided, all satisfying $E \leq \sum_{i \in N} c_i$. To avoid confusion with the traditional formulation of a claims problem, we call (m, c, E) an *expanded claims problems*.

The main difference here is in how a claimant's identity is encoded in a problem. Earlier, $i \in N$ represented the identity of a claimant. Now, $i \in N$ simply enumerates a claimant (we refer to i as the claimant's "number"), while m_i represents the identity of the i^{th} claimant. Hence, if $m_1 = m_2$, then we interpret this to mean that claimant number 1 and claimant number 2 are the same individual. Thus a division rule can make direct intrapersonal allocations.

An expanded claims problem is actually a special case of the problems formulated by Kaminski (2006) (though this special case was not one originally considered by Kaminski). In that paper, there is a type space T, which has no structure other than being a separable topological space. Each individual i has a type $t_i \in T$. There is also a function $\max : T \to \mathbb{R}_{++} \cup \{\infty\}$ which determines the maximum award each type may receive, i.e. an award $x_i \geq 0$ for individual i is valid if $x_i \leq \max(t_i)$. For a group $N \subset \mathbb{N}$, a problem is the tuple $(t, E) \in T^N \times \mathbb{R}_{++}$, where $E \leq \sum_N \max(t_i)$. This generalization allows for many different applications.[10] For our purposes, an individual's type will be his identity m_i and claim c_i, and the maximum an individual can receive will be his claim. Thus we have $T = \mathbb{N} \times \mathbb{R}_{++}$ and $\max(m_i, c_i) = c_i$.

As before, for any $f \in \mathcal{F}$, we can define a division rule S^f as follows. For every (m, c, E) and for every $i \in N$,

$$S_i^f(m, c, E) \equiv f_{m_i}(c_i, \lambda), \tag{2}$$

where λ is chosen so that $\sum_{i \in N} f_{m_i}(c_i, \lambda) = E$. Note that here, though the parametric function f belongs to the same family of functions $\mathcal{F}$ as before, the first input of f is now m_i rather than i.

We impose the same axioms as Young (1987), appropriately modified. Continuity and Bilateral Consistency are essentially the same as before. However Symmetry takes on a different meaning here.

Symmetry. *For every (m, c, E) and $\{i, j\} \subset N$, if $m_i = m_j$ and $c_i = c_j$, then $S_i(m, c, E) = S_j(m, c, E)$.*

Here, Symmetry is considerably weaker than earlier because claimants with the same claim but different identities can receive different awards. It is only to "clones" (i.e. claimants with the same identity and claim) that the division rule must give the same award. However, Symmetry still imposes quite a bit of richness on the rule since it implies, with Continuity and Bilateral Consistency, appropriate adaptations of both N-Continuity and Intrapersonal Consistency.

Applying Kaminski (2006, Theorem 1), we get an alternative characterization of our family of parametric rules: A division rule over expanded claims problems satisfies

[10] Some applications that Kaminski cites: (1) Claims problems: $T = \mathbb{R}_{++}$ and $\max(t_0) = t_0$. (2) Surplus sharing: $T = \mathbb{R}_{++}$ and $\max(t_0) = \infty$. (3) Multiple claims: $T = \mathbb{R}_{++}^K$ where K is the number of different claim types and $\max(t_0) = \sum_{k=1}^K t_0^k$. (4) Welfarist rationing: T is the set of all continuous and strictly increasing utility functions and $\max(t_0) = \infty$.

Continuity, Bilateral Consistency, and Symmetry if and only if it has a continuous parametric representation of the form given in (2).[11]

Though this approach to characterizing parametric rules has the advantage of using simpler, well-known axioms, it has the disadvantage of being arguably unrealistic. What does it mean to have multiple versions of the same claimant in a problem? Also it obscures the intrapersonal properties that parametric rules satisfy under the traditional formulation of a claims problem.

[11]To see that this is a characterization of the same family of rules, first observe that the set of claims problems can be embedded into the set of expanded claims problems with the mapping $(c, E) \mapsto (N, c, E)$. Now start with a function $f \in \mathcal{F}$. Define a division rule S^f over claims problems according to (1) and a division rule T^f over expanded claims problems according to (2). Then S^f and T^f will agree on the set of claims problems. Going the other direction, if a parametric division rule S over claims problems and a parametric division rule T over expanded claims problems agree on the set of claims problems, then it is not hard to see that there exists $f \in \mathcal{F}$ that represents both S and T, namely the parametric representation of T.

Appendix

A Continuing Example 1

Here we show that S, the division rule in Example 1, can be extended to a continuous, consistent, and resource monotonic rule over all problems. First we show that there is a continuous extension of S for problems where $N = \{1, 2\}$. Later we will extend S to the domain of all problems.

For now, S is only defined over problems where $N = \{1, 2\}$. Here we show that there is a continuous extension of S given $S\left(\left(c_1, c_2\right), c_1\right) = (c_1, 0)$, $S\left(\left(c'_1, c_2\right), c_2\right) = (0, c_2)$, $S\left(\left(c_1, c'_2\right), c'_2\right) = (0, c'_2)$, and $S\left(\left(c'_1, c'_2\right), c'_1\right) = (c'_1, 0)$. We do this by describing the path of awards for any problem. Every path of awards will be piecewise linear of exactly two pieces: the first piece starting at the origin, the second piece ending at the claims vector, and the two meeting somewhere in between at an inflection point. Thus for any claims vector (x, y), the path of awards will be defined by the inflection point $i(x, y)$ lying somewhere in the box formed by the origin and (x, y). This guarantees that the rule is resource monotonic. Also, the inflection point will vary continuously with respect to claims, guaranteeing the rule is continuous.

In what follows, set:

$$\alpha(x) \equiv \begin{cases} 0 & \text{if } x \leq c_1 \\ \dfrac{x - c_1}{c'_1 - c_1} & \text{if } c_1 < x \leq c'_1 \\ 1 & \text{if } x > c'_1 \end{cases}$$

and

$$\beta(y) \equiv \begin{cases} 0 & \text{if } x \leq c_2 \\ \dfrac{y - c_2}{c'_2 - c_2} & \text{if } c_2 < x \leq c'_2 \\ 1 & \text{if } x > c'_2. \end{cases}$$

Now set:

$$\begin{aligned} i(x, y) \equiv \big(&(1 - \alpha(x))(1 - \beta(y)) \min\{x, c_1\} + \alpha(x)\beta(y) \max\{x, c'_1\}\ , \\ &\alpha(x)(1 - \beta(y)) \min\{y, c_2\} + (1 - \alpha(x))\beta(y) \max\{y, c'_2\}\big)\,. \end{aligned}$$

Observe that $i(c_1, c_2) = (c_1, 0)$, which is what is required given that $S\left(\left(c_1, c_2\right), c_1\right) = (c_1, 0)$. Similarly we have $i(c'_1, c_2) = (0, c_2)$, $i(c_1, c'_2) = (0, c'_2)$, and $i(c'_1, c'_2) = (c'_1, 0)$. Also, observe that for $x \in (c_1, c'_1)$ and $y \in (c_2, c'_2)$, we can write $i(x, y)$ in one of two ways: either as the $\alpha(x)$ mixture of $i(c_1, y)$ and $i(c'_1, y)$, or as the $\beta(y)$ mixture of $i(x, c_2)$ and $i(x, c'_2)$. Figure 8 illustrates.

It is straightforward to show that $0 \leq i_1(x, y) \leq x$ for every $x > 0$, $0 \leq i_2(x, y) \leq y$ for every $y > 0$, and that $i(x, y)$ is continuous. Thus S is continuous and resource monotonic.

Now we extend S to the domain of all claims problems. First, for any problem where $1, 2 \notin N$, then $S = Par$ where Par is any parametric rule (e.g. the Proportional

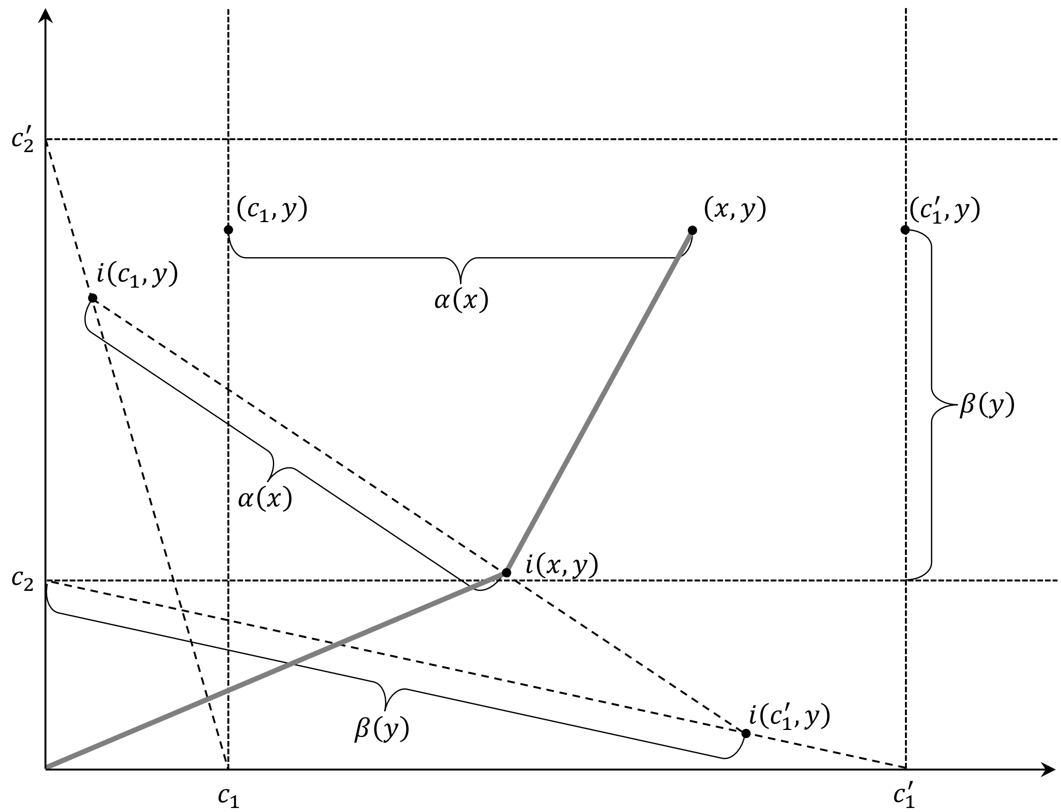

Figure 8: ***Construction of path of awards.*** *The path of awards for claims vector (x, y) is shown. Note that $i(x, y)$ is the convex combination of $i(c_1, y)$ and $i(c_1', y)$.*

rule). For any other problem (where $1 \in N$, $2 \in N$, or $1, 2 \in N$), S divides the endowment by first satisfying the claims of claimants 1 and 2, dividing between them as above if $1, 2 \in N$, and then allocating the remainder of the endowment to the rest of the claimants according to Par. It is not hard to show that S is continuous, consistent, and resource monotonic. However as shown previously, S cannot have a parametric representation.

B Proof of Theorem 1

Verifying that the axioms are necessary is a straightforward exercise. We show now that the axioms are sufficient. The proof proceeds as follows. We define a complete and transitive binary relation R over Y. We show that there exists a countable R-dense subset of Y, implying that R has a numerical representation. Taking the inverse of this representation gives us our parametric function. The proof is concluded by showing this function satisfies the necessary properties.

B.1 Preliminaries

Set

$$m(i, c_i, x_i) \equiv \inf\{x_i' \leq x_i : (i, c_i, x_i')\, NC\, (i, c_i, x_i)\}$$

and

$$M(i, c_i, x_i) \equiv \sup\{x_i' \geq x_i : (i, c_i, x_i')\, NC\, (i, c_i, x_i)\}$$

and

$$\theta(i, c_i, x_i) \equiv \begin{cases} 0 & \text{if } m(i, c_i, x_i) = M(i, c_i, x_i) \\ \frac{x_i - m(i,c_i,x_i)}{M(i,c_i,x_i) - m(i,c_i,x_i)} & \text{if } m(i, c_i, x_i) < M(i, c_i, x_i) \end{cases}.$$

When they are well-defined, $m(i, c_i, x_i) \leq x_i \leq M(i, c_i, x_i)$ and $0 \leq \theta(i, c_i, x_i) \leq 1$.

Lemma 1 *If $(i, c_i, x_i)\, NC\, (i, c_i', x_i')$, then $m(i, c_i, x_i)$, $M(i, c_i, x_i)$, and $\theta(i, c_i, x_i)$ are well-defined.*

Proof. If $(i, c_i, x_i)\, NC\, (i, c_i', x_i')$, then there exists no (j, c_j, x_j) such that $(i, c_i, x_i)\, I_1\, (j, c_j, x_j)$. Hence $(i, c_i, x_i)\, NC\, (i, c_i, x_i)$, which implies

$$x_i \in \{x_i' \leq x_i : (i, c_i, x_i')\, NC\, (i, c_i, x_i)\}$$

and

$$x_i \in \{x_i' \geq x_i : (i, c_i, x_i')\, NC\, (i, c_i, x_i)\}.$$

■

Define the binary relation R_3 over Y as follows.

$$(i, c_i, x_i)\, R_3\, (i, c_i', x_i') \text{ if } (i, c_i, x_i)\, NC\, (i, c_i', x_i') \text{ and } \theta(i, c_i, x_i) \leq \theta(i, c_i', x_i').$$

Define $R \equiv R_1 \cup R_2 \cup R_3$.

Lemma 2 *R is complete.*

Proof. For every (i, c_i, x_i) and (j, c_j, x_j), either (i) $i \neq j$, (ii) $i = j$ and $(i, c_i, x_i)\, C\, (j, c_j, x_j)$, or (iii) $i = j$ and $(i, c_i, x_i)\, NC\, (j, c_j, x_j)$. If (i), then either $G((i, c_i, x_i), j, c_j) \leq G((j, c_j, x_j), i, c_i)$ or $G((i, c_i, x_i), j, c_j) \geq G((j, c_j, x_j), i, c_i)$. If (ii), then $(i, c_i, x_i)\, C\, (j, c_j, x_j)$ implies either $(i, c_i, x_i)\, R_2\, (j, c_j, x_j)$ or $(j, c_j, x_j)\, R_2\, (i, c_i, x_i)$. If (iii), then either $\theta(i, c_i, x_i) \leq \theta(j, c_j, x_j)$ or $\theta(i, c_i, x_i) \geq \theta(j, c_j, x_j)$. ■

B.2 R Is Transitive

Lemma 3 *For every $(i, c_i, x_i) \in Y$, $j \neq i$, and $c_j \in \mathbb{R}_{++}$, we have*

$$S_i((c_i, c_j), G((i, c_i, x_i), j, c_j)) = x_i.$$

Proof. This follows directly from Continuity. ■

Lemma 4 *For every i, j, k distinct, for every $c_i, c_j, c_k \in \mathbb{R}_{++}$, for every $x_i \in [0, c_i]$, there exists E such that*

$$S\left(\left(c_i, c_j, c_k\right), E\right) = \left(x_i, G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i, G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i\right).$$

Proof. Set

$$E \equiv G\left(\left(i, c_i, x_i\right), j, c_j\right) + G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i$$

and

$$\left(x_i', x_j', x_k'\right) \equiv S\left(\left(c_i, c_j, c_k\right), E\right).$$

Hence,

$$x_i' + x_j' + x_k' = G\left(\left(i, c_i, x_i\right), j, c_j\right) + G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i. \tag{3}$$

First we show $x_i' = x_i$. Suppose $x_i' < x_i$. Observe that Bilateral Consistency implies $S\left(\left(c_i, c_j\right), x_i' + x_j'\right) = \left(x_i', x_j'\right)$ and $S\left(\left(c_i, c_k\right), x_i' + x_k'\right) = \left(x_i', x_k'\right)$. Also, by Lemma 3,

$$S\left(\left(c_i, c_j\right), G\left(\left(i, c_i, x_i\right), j, c_j\right)\right) = \left(x_i, G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i\right)$$

and

$$S\left(\left(c_i, c_k\right), G\left(\left(i, c_i, x_i\right), k, c_k\right)\right) = \left(x_i, G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i\right).$$

Since $x_i' < x_i$, Resource Monotonicity implies $x_j' \leq G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i$ and $x_k' \leq G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i$. Hence,

$$x_i' + x_j' + x_k' < G\left(\left(i, c_i, x_i\right), j, c_j\right) + G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i,$$

which contradicts equation (3). The proof is similar if $x_i' > x_i$. Hence $x_i' = x_i$.

Now we show $x_j' = G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i$ and $x_k' = G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i$. By definition of G, $x_i + x_j' \geq G\left(\left(i, c_i, x_i\right), j, c_j\right)$ and $x_i + x_k' \geq G\left(\left(i, c_i, x_i\right), k, c_k\right)$. If either of these inequalities are strict, then

$$x_i + x_j' + x_k' > G\left(\left(i, c_i, x_i\right), j, c_j\right) + G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i,$$

which contradicts equation (3). Hence, $x_i + x_j' = G\left(\left(i, c_i, x_i\right), j, c_j\right)$ and $x_i + x_k' = G\left(\left(i, c_i, x_i\right), k, c_k\right)$. ■

Lemma 5 $\left(i, c_i, x_i\right) P_1 \left(j, c_j, x_j\right)$ *if and only if* $x_i + x_j > G\left(\left(i, c_i, x_i\right), j, c_j\right)$.

Proof. By Lemma 3,

$$S\left(\left(c_i, c_j\right), G\left(\left(i, c_i, x_i\right), j, c_j\right)\right) = \left(x_i, G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i\right)$$

and

$$S\left(\left(c_i, c_j\right), G\left(\left(j, c_j, x_j\right), i, c_i\right)\right) = \left(G\left(\left(j, c_j, x_j\right), i, c_i\right) - x_j, x_j\right).$$

($\Rightarrow$) So $G\left(\left(i, c_i, x_i\right), j, c_j\right) < G\left(\left(j, c_j, x_j\right), i, c_i\right)$. Resource Monotonicity implies $G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i \leq x_j$. If $G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i = x_j$, then

$$S\left(\left(c_i, c_j\right), G\left(\left(i, c_i, x_i\right), j, c_j\right)\right) = \left(x_i, x_j\right).$$

But this would imply $G((i,c_i,x_i),j,c_j) \geq G((j,c_j,x_j),i,c_i)$ by definition of G, a contradiction. Hence, $G((i,c_i,x_i),j,c_j) - x_i < x_j$.

($\Leftarrow$) Suppose $x_j > G((i,c_i,x_i),j,c_j) - x_i$. Then Resource Monotonicity implies $G((i,c_i,x_i),j,c_j) < ((j,c_j,x_j),i,c_i)$, or $(i,c_i,x_i)\,P_1\,(j,c_j,x_j)$. ■

Observe that an alternative statement of Lemma 5 is $(i,c_i,x_i)\,R_1\,(j,c_j,x_j)$ if and only if $x_i + x_j \leq G((j,c_j,x_j),i,c_i)$.

Lemma 6 *If $(i,c_i,x_i)\,NC\,(i,c_i,x_i')$, then $m(i,c_i,x_i) = m(i,c_i,x_i')$ and $M(i,c_i,x_i) = M(i,c_i,x_i')$.*

Proof. Follows immediately from transitivity of NC. ■

Lemma 7 *$(i,c_i,x_i)\,R\,(i,c_i,x_i')$ if and only if $x_i \leq x_i'$.*

Proof. Case 1: $\boldsymbol{(i,c_i,x_i)\,C\,(i,c_i,x_i')}$.

So there exists (j,c_j,x_j) such that

$$(i,c_i,x_i)\,R_1\,(j,c_j,x_j)\,R_1\,(i,c_i,x_i').$$

This is true if and only if $G((i,c_i,x_i),j,c_j) \leq G((j,c_j,x_j),i,c_i) \leq G((i,c_i,x_i'),j,c_j)$. But Resource Monotonicity implies $G((i,c_i,x_i),j,c_j) \leq G((i,c_i,x_i'),j,c_j)$ if and only if $x_i \leq x_i'$.

Case 2: $\boldsymbol{(i,c_i,x_i)\,NC\,(i,c_i,x_i')}$.

By Lemma 6, $m(i,c_i,x_i) = m(i,c_i,x_i')$ and $M(i,c_i,x_i) = M(i,c_i,x_i')$. If $m(i,c_i,x_i) = M(i,c_i,x_i)$, then $0 = \theta(i,c_i,x_i) = \theta(i,c_i,x_i')$ and $m(i,c_i,x_i) = M(i,c_i,x_i) = x_i = x_i'$. If $m(i,c_i,x_i) < M(i,c_i,x_i)$, then the definition of θ implies $\theta(i,c_i,x_i) \leq \theta(i,c_i,x_i')$ if and only if $x_i \leq x_i'$. ■

Lemma 8 *If $i \neq j$ and $(i,c_i,x_i)\,P_1\,(j,c_j,x_j)$, then there exists $\epsilon > 0$ such that for every $x_j' \in (x_j - \epsilon, x_j)$,*

$$(i,c_i,x_i)\,P_1\left(j,c_j,x_j'\right)P\,(j,c_j,x_j).$$

Proof. By Lemma 5, $(i,c_i,x_i)\,P_1\,(j,c_j,x_j)$ implies $x_i + x_j > G((i,c_i,x_i),j,c_j)$. Set

$$\epsilon \equiv x_i + x_j - G((i,c_i,x_i),j,c_j).$$

Then for $x_j' \in (x_j - \epsilon, x_j)$, $x_i + x_j' > G((i,c_i,x_i),j,c_j)$. So $(i,c_i,x_i)\,P_1\left(j,c_j,x_j'\right)$ by Lemma 5, while $\left(j,c_j,x_j'\right)P\,(j,c_j,x_j)$ by Lemma 7. ■

Lemma 9 *If $i \neq j$ and $(i,c_i,x_i)\,P\,(i,c_i,x_i')\,R_1\,(j,c_j,x_j)$, then $(i,c_i,x_i)\,P_1\,(j,c_j,x_j)$.*

Proof. Suppose $(j,c_j,x_j)\,R_1\,(i,c_i,x_i)$. Then by definition, $(i,c_i,x_i')\,R_2\,(i,c_i,x_i)$. Lemma 7 implies $x_i' \leq x_i$. But $(i,c_i,x_i)\,P\,(i,c_i,x_i')$ and Lemma 7 imply $x_i < x_i'$, a contradiction. ■

Lemma 10 *R is transitive.*

Proof. Suppose $(i, c_i, x_i) \, R \, (j, c_j, x_j) \, R \, (k, c_k, x_k)$.

Case 1: $\boldsymbol{i, j, k}$ distinct.

By Lemma 4, there exists E, E', and E'' such that

$$\begin{aligned} S\left(\left(c_i, c_j, c_k\right), E\right) &= \left(x_i, G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i, G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i\right), \\ S\left(\left(c_i, c_j, c_k\right), E'\right) &= \left(G\left(\left(j, c_j, x_j\right), i, c_i\right) - x_j, x_j, G\left(\left(j, c_j, x_j\right), k, c_k\right) - x_j\right), \\ S\left(\left(c_i, c_j, c_k\right), E''\right) &= \left(G\left(\left(k, c_k, x_k\right), i, c_i\right) - x_k, G\left(\left(k, c_k, x_k\right), j, c_j\right) - x_k, x_k\right). \end{aligned}$$

First we show $E \leq E'$. If $E > E'$, then Resource Monotonicity implies

$$x_i \geq G\left(\left(j, c_j, x_j\right), i, c_i\right) - x_j,$$

$$x_j \leq G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i,$$

and

$$G\left(\left(j, c_j, x_j\right), k, c_k\right) - x_j \leq G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i,$$

with one of these strict. But $G\left(\left(i, c_i, x_i\right), j, c_j\right) \leq G\left(\left(j, c_j, x_j\right), i, c_i\right)$. Hence $x_i = G\left(\left(j, c_j, x_j\right), i, c_i\right) - x_j$ and $x_j = G\left(\left(i, c_i, x_i\right), j, c_j\right) - x_i$, which implies

$$G\left(\left(j, c_j, x_j\right), k, c_k\right) - x_j < G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i \tag{4}$$

and

$$S\left(\left(c_i, c_j, c_k\right), E'\right) = \left(x_i, x_j, G\left(\left(j, c_j, x_j\right), k, c_k\right) - x_j\right).$$

Bilateral Consistency implies

$$S\left(\left(c_i, c_k\right), x_i + G\left(\left(j, c_j, x_j\right), k, c_k\right) - x_j\right) = \left(x_i, G\left(\left(j, c_j, x_j\right), k, c_k\right) - x_j\right).$$

By definition,

$$G\left(\left(i, c_i, x_i\right), k, c_k\right) \leq x_i + G\left(\left(j, c_j, x_j\right), k, c_k\right) - x_j.$$

But this contradicts inequality (4).

Similarly, $E' \leq E''$. Hence, $E \leq E''$. Resource Monotonicity implies $x_i \leq G\left(\left(k, c_k, x_k\right), i, c_i\right) - x_k$ and $G\left(\left(i, c_i, x_i\right), k, c_k\right) - x_i \leq x_k$. Hence $G\left(\left(i, c_i, x_i\right), k, c_k\right) \leq G\left(\left(k, c_k, x_k\right), i, c_i\right)$, and so $(i, c_i, x_i) \, R_1 \, (k, c_k, x_k)$ by definition.

Case 2: $\boldsymbol{i = k \neq j}$.

Then $(i, c_i, x_i) \, R_2 \, (k, c_k, x_k)$ by definition.

Case 3: $\boldsymbol{i = j \neq k}$. So let $(i, c_i, x_i) \, R \, (i, c_i', x_i') \, R_1 \, (k, c_k, x_k)$.

3.1: $\boldsymbol{(i, c_i, x_i) \, C \left(i, c_i', x_i'\right)}$.

Then there exists $(j', c_{j'}, x_{j'})$ where $j' \neq i$ such that

$$(i, c_i, x_i) \, R_1 \, (j', c_{j'}, x_{j'}) \, R_1 \, (i, c_i', x_i') \, R_1 \, (k, c_k, x_k).$$

If $j' \neq k$, then Case 1 (applied twice) implies $(i, c_i, x_i) \, R_1 \, (k, c_k, x_k)$.

If $j' = k$, then

$$(i, c_i, x_i) \, R_1 \, (k, c_k', x_k') \, R_1 \, (i, c_i', x_i') \, R_1 \, (k, c_k, x_k).$$

Suppose $(k, c_k, x_k) P_1 (i, c_i, x_i)$. By Lemmas 8 and 9, there exists $\hat{x}_i$, $\hat{x}'_k$, and $\hat{x}'_i$ such that

$$(k, c_k, x_k) P_1 (i, c_i, \hat{x}_i) P_1 (k, c'_k, \hat{x}'_k) P_1 (i, c'_i, \hat{x}'_i) P_1 (k, c_k, x_k).$$

But this contradicts Intrapersonal Consistency. Hence $(i, c_i, x_i) R_1 (k, c_k, x_k)$.

3.2: $\boldsymbol{(i, c_i, x_i) NC (i, c'_i, x'_i)}$.

If $(k, c_k, x_k) P_1 (i, c_i, x_i)$, then $(i, c'_i, x'_i) R_2 (i, c_i, x_i)$ by definition. This contradicts $(i, c_i, x_i) NC (i, c'_i, x'_i)$.

Case 4: $\boldsymbol{i \neq j = k}$.

Similar to Case 3.

Case 5: $\boldsymbol{i = j = k}$. So let $(i, c_i, x_i) R (i, c'_i, x'_i) R (i, c''_i, x''_i)$.

5.1: $\boldsymbol{(i, c_i, x_i) C (i, c'_i, x'_i)}$.

Then there exists $(j', c_{j'}, x_{j'})$ where $j' \neq i$ such that $(i, c_i, x_i) R_1 (j', c_{j'}, x_{j'}) R_1 (i, c'_i, x'_i)$. By Case 4, $(j', c_{j'}, x_{j'}) R (i, c''_i, x''_i)$. By Case 2, $(i, c_i, x_i) R (i, c''_i, x''_i)$.

5.2: $\boldsymbol{(i, c'_i, x'_i) C (i, c''_i, x''_i)}$.

Then there exists $(j', c_{j'}, x_{j'})$ where $j' \neq i$ such that $(i, c'_i, x'_i) R_1 (j', c_{j'}, x_{j'}) R_1 (i, c''_i, x''_i)$. By Case 3, $(i, c_i, x_i) R (j', c_{j'}, x_{j'})$. By Case 2, $(i, c_i, x_i) R (i, c''_i, x''_i)$.

5.3: $\boldsymbol{(i, c_i, x_i) NC (i, c'_i, x'_i) NC (i, c''_i, x''_i)}$.

Then $(i, c_i, x_i) NC (i, c''_i, x''_i)$ by the transitivity of NC. Also $\theta (i, c_i, x_i) \leq \theta (i, c'_i, x'_i) \leq \theta (i, c''_i, x''_i)$. Hence $(i, c_i, x_i) R_3 (i, c''_i, x''_i)$. ■

B.3 Countable R-dense Subset

Lemma 11 *If $(i, c_i, x_i) P (i, \hat{c}_i, \hat{x}_i)$, then there exists $\epsilon > 0$ such that for every $\hat{x}'_i \in (\hat{x}_i - \epsilon, \hat{x}_i)$, we have*

$$(i, c_i, x_i) P (i, \hat{c}_i, \hat{x}'_i) P (i, \hat{c}_i, \hat{x}_i).$$

Proof. Case 1: $\boldsymbol{(i, c_i, x_i) C (i, \hat{c}_i, \hat{x}_i)}$.

Then there exists (k, c_k, x_k) where $k \neq i$ such that $(i, c_i, x_i) R_1 (k, c_k, x_k) R_1 (i, \hat{c}_i, \hat{x}_i)$, where one of these is strict. By Lemma 8, we can assume without loss of generality that

$$(i, c_i, x_i) R_1 (k, c_k, x_k) P_1 (i, \hat{c}_i, \hat{x}_i).$$

Also by Lemma 8, there exists $\epsilon > 0$ such that for every $\hat{x}'_i \in (\hat{x}_i - \epsilon, \hat{x}_i)$,

$$(k, c_k, x_k) P_1 (i, \hat{c}_i, \hat{x}'_i) P (i, \hat{c}_i, \hat{x}_i).$$

Transitivity of R implies $(i, c_i, x_i) P (i, \hat{c}_i, \hat{x}'_i) P (i, \hat{c}_i, \hat{x}_i)$ for any such $\hat{x}'_i$.

Case 2: $\boldsymbol{(i, c_i, x_i) NC (i, \hat{c}_i, \hat{x}_i)}$.

So $(i, c_i, x_i) P_3 (i, \hat{c}_i, \hat{x}_i)$ implies $\theta (i, c_i, x_i) < \theta (i, \hat{c}_i, \hat{x}_i)$. Thus $\theta (i, \hat{c}_i, \hat{x}_i) > 0$, which implies $m (i, \hat{c}_i, \hat{x}_i) < \hat{x}_i \leq M (i, \hat{c}_i, \hat{x}_i)$. So for every $x_0 \in (m (i, \hat{c}_i, \hat{x}_i), \hat{x}_i)$, $(i, \hat{c}_i, x_0) NC (i, \hat{c}_i, \hat{x}_i)$. This implies $(i, \hat{c}_i, x_0) NC (i, c_i, x_i)$ for every such x_0 since NC is transitive.

Set

$$\epsilon \equiv \hat{x}_i - (1 - \theta (i, c_i, x_i)) m (i, \hat{c}_i, \hat{x}_i) - \theta (i, c_i, x_i) M (i, \hat{c}_i, \hat{x}_i).$$

Observe $\epsilon > 0$ since $m(i, \hat{c}_i, \hat{x}_i) < M(i, \hat{c}_i, \hat{x}_i)$ and

$$\begin{aligned}
\theta(i, c_i, x_i) &< \theta(i, \hat{c}_i, \hat{x}_i) \\
&= \frac{x_i' - m(i, \hat{c}_i, \hat{x}_i)}{M(i, \hat{c}_i, \hat{x}_i) - m(i, \hat{c}_i, \hat{x}_i)}.
\end{aligned}$$

Let $\hat{x}_i' \in (\hat{x}_i - \epsilon, \hat{x}_i)$. Then by above $(i, \hat{c}_i, \hat{x}_i')\, NC\, (i, \hat{c}_i, \hat{x}_i)\, NC\, (i, c_i, x_i)$. Lemma 6 implies $m(i, \hat{c}_i, \hat{x}_i') = m(i, \hat{c}_i, \hat{x}_i)$ and $M(i, \hat{c}_i, \hat{x}_i') = M(i, \hat{c}_i, \hat{x}_i)$. Hence

$$\begin{aligned}
\hat{x}_i' &> \hat{x}_i - \epsilon \\
&= (1 - \theta(i, c_i, x_i))\, m(i, \hat{c}_i, \hat{x}_i) + \theta(i, c_i, x_i)\, M(i, \hat{c}_i, \hat{x}_i) \\
&= (1 - \theta(i, c_i, x_i))\, m(i, \hat{c}_i, \hat{x}_i') + \theta(i, c_i, x_i)\, M(i, \hat{c}_i, \hat{x}_i')
\end{aligned}$$

which implies

$$\frac{\hat{x}_i - m(i, \hat{c}_i, \hat{x}_i')}{M(i, \hat{c}_i, \hat{x}_i') - m(i, \hat{c}_i, \hat{x}_i')} > \theta(i, c_i, x_i),$$

or $\theta(i, \hat{c}_i, \hat{x}_i') > \theta(i, c_i, x_i)$. Similarly, one can show $\theta(i, \hat{c}_i, \hat{x}_i') < \theta(i, \hat{c}_i, \hat{x}_i)$. Hence $(i, c_i, x_i)\, P_3\, (i, \hat{c}_i, \hat{x}_i')\, P_3\, (i, \hat{c}_i, \hat{x}_i)$. ■

Definition 5 *The rule S* ***gives left priority to (i, c_i, x_i)*** *if $x_i \neq 0$ and if there exists $\epsilon > 0$ such that for every $(\hat{c}, E)$ where $i \in N$, $\hat{c}_i = c_i$, and $S_i(\hat{c}, E) = x_i$, for every $a \in (0, \epsilon)$, we have $S_i(\hat{c}, E - a) = x_i - a$.*

Lemma 12 *If S gives left priority to (i, c_i, x_i), then there exists $\epsilon > 0$ such that for every $x_i' \in (x_i - \epsilon, x_i)$, we have $(i, c_i, x_i')\, NC\, (i, c_i, x_i)$.*

Proof. Take ϵ from the definition of left priority. Fix $a \in (0, \epsilon)$ and (j, c_j, x_j). Suppose x_j' satisfies $S((c_i, c_j), x_i + x_j') = (x_i, x_j')$. Then by the definition of left priority, for every $a' \in (0, \epsilon)$,

$$S((c_i, c_j), x_i + x_j' - a') = (x_i - a', x_j'),$$

which implies $G((j, c_j, x_j'), i, c_i) < G((i, c_i, x_i - a), j, c_j)$, or $(j, c_j, x_j')\, P_1\, (i, c_i, x_i - a)$. Hence if $x_j \leq x_j'$ then $(j, c_j, x_j)\, P_1\, (i, c_i, x_i - a)$ by Lemma 7 and transitivity. However, if $x_j > x_j'$, then since $S((c_i, c_j), x_i + x_j') = (x_i, x_j')$, it must be that $G((i, c_i, x_i), j, c_j) < G((j, c_j, x_j), i, c_i)$, or $(i, c_i, x_i)\, P_1\, (j, c_j, x_j)$. Hence, (j, c_j, x_j) cannot separate $(i, c_i, x_i - a)$ and (i, c_i, x_i). ■

Lemma 13 *If S gives left priority to (i, c_i, x_i), then there exists $\epsilon > 0$ such that for every $\hat{c}_i \in (c_i - \epsilon, c_i + \epsilon)$, there exists $\hat{x}_i \leq \hat{c}_i$ such that $(i, c_i, x_i)\, I_3\, (i, \hat{c}_i, \hat{x}_i)$.*

Proof. Since S gives left priority to (i, c_i, x_i), then Lemma 12 implies that there exists $x_i'' < x_i' < x_i$ such that $(i, c_i, x_i'')\, NC\, (i, c_i, x_i')\, NC\, (i, c_i, x_i)$. Also, it must be that S gives priority to (i, c_i, x_i'') and (i, c_i, x_i'). Lemma 7 implies $\theta(i, c_i, x_i) > \theta(i, c_i, x_i') > \theta(i, c_i, x_i'') \geq 0$. N-Continuity implies that there exists $\epsilon > 0$ such that for every $\hat{c}_i \in (c_i - \epsilon, c_i + \epsilon)$, $(i, c_i, x_i'')\, NC\, (i, \hat{c}_i, x_i'')$ and $(i, c_i, x_i')\, NC\, (i, \hat{c}_i, x_i')$.

Fix $\hat{c}_i \in (c_i - \epsilon, c_i + \epsilon)$. Hence $(i, \hat{c}_i, x_i'') \, NC \, (i, \hat{c}_i, x_i')$ by transitivity of NC. Lemma 6 implies $m(i, \hat{c}_i, x_i'') = m(i, \hat{c}_i, x_i')$ and $M(i, \hat{c}_i, x_i'') = M(i, \hat{c}_i, x_i')$. Hence $m(i, \hat{c}_i, x_i'') \leq x_i'' < x_i' \leq M(i, \hat{c}_i, x_i')$ implies $m(i, \hat{c}_i, x_i'') < M(i, \hat{c}_i, x_i'')$. Set

$$\hat{x}_i \equiv (1 - \theta(i, c_i, x_i)) \, m(i, \hat{c}_i, x_i'') + \theta(i, c_i, x_i) \, M(i, \hat{c}_i, x_i'').$$

Observe $m(i, \hat{c}_i, x_i'') < \hat{x}_i \leq M(i, \hat{c}_i, x_i'')$ since $\theta(i, c_i, x_i) > 0$.

Claim: $(i, \hat{c}_i, \hat{x}_i) \, NC \, (i, \hat{c}_i, x_i'')$. Obviously if $\hat{x}_i < M(i, \hat{c}_i, x_i'')$ then $(i, \hat{c}_i, \hat{x}_i) \, NC \, (i, \hat{c}_i, x_i'')$. So suppose $\hat{x}_i = M(i, \hat{c}_i, x_i'')$ but $(i, \hat{c}_i, \hat{x}_i) \, C \, (i, \hat{c}_i, x_i'')$. Observe then that $\hat{x}_i \geq x_i' > x_i''$. Hence Lemma 7 implies $(i, \hat{c}_i, x_i'') \, P_2 \, (i, \hat{c}_i, \hat{x}_i)$. So there exists (j, c_j, x_j) such that $(i, \hat{c}_i, x_i'') \, R_1 \, (j, c_j, x_j) \, R_1 \, (i, \hat{c}_i, \hat{x}_i)$, with one of these strict. Lemma 8 implies that it is without loss of generality that $(j, c_j, x_j) \, P_1 \, (i, \hat{c}_i, \hat{x}_i)$. But then Lemma 8 implies that there exists $x_0 < \hat{x}_i$ such that $(j, c_j, x_j) \, P_1 \, (i, \hat{c}_i, x_0) \, P \, (i, \hat{c}_i, \hat{x}_i)$. This implies $(i, \hat{c}_i, x_i'') \, R_1 \, (j, c_j, x_j) \, P_1 \, (i, \hat{c}_i, x_0)$ which implies $(i, \hat{c}_i, x_0) \, C \, (i, \hat{c}_i, x_i'')$, which implies $x_0 \geq M(i, \hat{c}_i, x_i'') = \hat{x}_i$, a contradiction. Hence $(i, \hat{c}_i, \hat{x}_i) \, NC \, (i, \hat{c}_i, x_i'')$.

So by Lemma 6, $m(i, \hat{c}_i, \hat{x}_i) = m(i, \hat{c}_i, x_i'')$ and $M(i, \hat{c}_i, \hat{x}_i) = M(i, \hat{c}_i, x_i'')$. Then

$$\begin{aligned} \theta(i, \hat{c}_i, \hat{x}_i) &= \frac{\hat{x}_i - m(i, \hat{c}_i, \hat{x}_i)}{M(i, \hat{c}_i, \hat{x}_i) - m(i, \hat{c}_i, \hat{x}_i)} \\ &= \frac{\hat{x}_i - m(i, \hat{c}_i, x_i'')}{M(i, \hat{c}_i, x_i'') - m(i, \hat{c}_i, x_i'')} \\ &= \theta(i, c_i, x_i). \end{aligned}$$

This implies $(i, \hat{c}_i, \hat{x}_i) \, I_3 \, (i, c_i, x_i)$ since $(i, \hat{c}_i, \hat{x}_i) \, NC \, (i, \hat{c}_i, x_i'') \, NC \, (i, c_i, x_i'') \, NC \, (i, c_i, x_i)$. ■

Lemma 14 *If $(i, c_i, x_i) \, P \, (i, c_i, x_i') \, P_1 \, (j, c_j, x_j)$, then $x_i < S_i((c_i, c_j), x_i + x_j)$ and $x_j > S_j((c_i, c_j), x_i + x_j)$.*

Proof. So $x_i < x_i'$ by Lemma 7 and $(i, c_i, x_i) \, P_1 \, (j, c_j, x_j)$ by transitivity of R. By Lemma 5, $G((i, c_i, x_i), j, c_j) < x_i + x_j$ and $x_i' + x_j \leq G((j, c_j, x_j), i, c_i)$. Since $x_i < x_i'$, this implies

$$G((i, c_i, x_i), j, c_j) < x_i + x_j < G((j, c_j, x_j), i, c_i).$$

Hence Resource Monotonicity implies

$$\begin{aligned} x_i &= S_i((c_i, c_j), G((i, c_i, x_i), j, c_j)) \\ &\leq S_i((c_i, c_j), x_i + x_j). \end{aligned}$$

Observe

$$S_i((c_i, c_j), x_i + x_j) + S_j((c_i, c_j), x_i + x_j) = x_i + x_j.$$

So if $x_i = S_i((c_i, c_j), x_i + x_j)$, then $x_j = S_j((c_i, c_j), x_i + x_j)$. But then the definition of G implies $G((j, c_j, x_j), i, c_i) \leq x_i + x_j$, a contradiction. Hence $x_i < S_i((c_i, c_j), x_i + x_j)$. This implies $x_j > S_j((c_i, c_j), x_i + x_j)$. ■

Lemma 15 *If $(i, c_i, x_i) \, R_1 \, (j, c_j, x_j)$, then $x_i \leq S_i((c_i, c_j), x_i + x_j)$ and $x_j \geq S_j((c_i, c_j), x_i + x_j)$.*

Proof. By Lemma 5, $x_i + x_j \leq G((j, c_j, x_j), i, c_i)$. Resource Monotonicity implies

$$\begin{aligned} x_j &= S_j((c_i, c_j), G((j, c_j, x_j), i, c_i)) \\ &\geq S_j((c_i, c_j), x_i + x_j). \end{aligned}$$

Hence $x_i \leq S_i((c_i, c_j), x_i + x_j)$. ■

Lemma 16 *$Y' \equiv \{(i, c_i, x_i) \in \mathbb{N} \times \mathbb{Q}_{++} \times \mathbb{Q}_+ : 0 \leq x_i \leq c_i\}$ is a countable R-dense subset of Y.*

Proof. Obviously, Y' is countable. Let $(i, c_i, x_i), (j, c_j, x_j) \in Y$ satisfy $(i, c_i, x_i) P (j, c_j, x_j)$.

Case 1: $\boldsymbol{S}$ gives left priority to $\boldsymbol{(j, c_j, x_j)}$.

By Lemma 13, there exists $\hat{c}_j \in \mathbb{Q}_{++}$ (because $\mathbb{Q}$ is dense in $\mathbb{R}$) and $\hat{x}_j \leq \hat{c}_j$ such that $(j, \hat{c}_j, \hat{x}_j) I_3 (j, c_j, x_j)$. By either Lemma 8 (if $i \neq j$) or Lemma 11 (if $i = j$), there exists $\hat{x}'_j \in \mathbb{Q}_+$ where $\hat{x}'_j < \hat{x}_j$ such that

$$(i, c_i, x_i) P \left(j, \hat{c}_j, \hat{x}'_j\right) P (j, \hat{c}_j, \hat{x}_j).$$

Together this implies $\left(j, \hat{c}_j, \hat{x}'_j\right) \in Y'$ and $(i, c_i, x_i) P \left(j, \hat{c}_j, \hat{x}'_j\right) P (j, c_j, x_j)$.

Case 2: $\boldsymbol{S}$ does not give left priority to $\boldsymbol{(j, c_j, x_j)}$.

By either Lemma 8 or Lemma 11, there exists x'_j and x''_j such that

$$(i, c_i, x_i) P \left(j, c_j, x''_j\right) P \left(j, c_j, x'_j\right) P (j, c_j, x_j).$$

Since S does not give left priority to (j, c_j, x_j), there exists (k, c_k, x_k) where $k \neq j$ such that $\left(j, c_j, x'_j\right) R_1 (k, c_k, x_k) R_1 (j, c_j, x_j)$ with one of these strict. Without loss of generality, assume $\left(j, c_j, x'_j\right) P_1 (k, c_k, x_k) P_1 (j, c_j, x_j)$.[12] Lemma 8 implies that there exists x'_k such that $\left(j, c_j, x'_j\right) P_1 (k, c_k, x'_k) P (k, c_k, x_k)$. Hence

$$(i, c_i, x_i) P \left(j, c_j, x''_j\right) P \left(j, c_j, x'_j\right) P_1 (k, c_k, x'_k) P (k, c_k, x_k) P_1 (j, c_j, x_j).$$

Claim: There exists $\epsilon > 0$ such that for every $\hat{c}_k \in (c_k, c_k + \epsilon)$, $(k, \hat{c}_k, x'_k) P_1 (j, c_j, x_j)$. Suppose not. Then for every $n \in \mathbb{N}$, there exists $c_k^n \in \left(c_k, c_k + \frac{1}{n}\right)$ such that $(j, c_j, x_j) R_1 (k, c_k^n, x'_k)$. Observe that $c_k^n \to c_k$. Lemma 15 implies that for every n, $x_j \leq S_j((c_j, c_k^n), x_j + x'_k)$ and $x'_k \geq S_k((c_j, c_k^n), x_j + x'_k)$. Since $S((c_j, c_k^n), x_j + x'_k) \to S((c_j, c_k), x_j + x'_k)$ by Continuity, $x_j \leq S_j((c_j, c_k), x_j + x'_k)$ and $x'_k \geq S_k((c_j, c_k), x_j + x'_k)$. But since $(k, c_k, x'_k) P (k, c_k, x_k) P_1 (j, c_j, x_j)$, Lemma 14 implies $x_j > S_j((c_j, c_k), x_j + x'_k)$ and $x'_k < S_k((c_j, c_k), x_j + x'_k)$, a contradiction.

Claim: There exists $\epsilon > 0$ such that for every $\hat{c}_k \in (c_k, c_k + \epsilon)$, $\left(j, c_j, x''_j\right) P_1 (k, \hat{c}_k, x'_k)$. The proof for this claim is similar to the claim above.

These two claims imply that there exists $\epsilon > 0$ such that for every $\hat{c}_k \in (c_k, c_k + \epsilon)$, $\left(j, c_j, x''_j\right) P_1 (k, \hat{c}_k, x'_k) P_1 (j, c_j, x_j)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, without loss of generality, there exists $\hat{c}_k \in \mathbb{Q}_{++}$ such that $\left(j, c_j, x''_j\right) P_1 (k, \hat{c}_k, x'_k) P_1 (j, c_j, x_j)$. Lemma 8 implies that there exists $\hat{x}_k \in \mathbb{Q}_+$ such that $\left(j, c_j, x''_j\right) P_1 (k, \hat{c}_k, \hat{x}_k) P (k, \hat{c}_k, x'_k)$. Hence, $(k, \hat{c}_k, \hat{x}_k) \in Y'$ and $(i, c_i, x_i) P (k, \hat{c}_k, \hat{x}_k) P_1 (j, c_j, x_j)$. ■

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[12] If $\left(j, c_j, x'_j\right) I_1 (k, c_k, x_k)$, then Lemma 8 implies there exists x''_j such that $(i, c_i, x_i) P_1 \left(j, c_j, x''_j\right) P \left(j, c_j, x'_j\right) I_1 (k, c_k, x_k)$. If $(k, c_k, x_k) I_1 (j, c_j, x_j)$, then Lemma 8 implies there exists $\hat{x}_k$ such that $\left(j, c_j, x'_j\right) P_1 (k, c_k, \hat{x}_k) P (k, c_k, x_k) I_1 (j, c_j, x_j)$. Hence without loss of generality, $\left(j, c_j, x'_j\right) P_1 (k, c_k, x_k) P_1 (j, c_j, x_j)$.

B.4 Finishing the Proof

Lemma 17 *There exists* $r : Y \to \mathbb{R}$ *such that*

$$(i, c_i, x_i)\, R\, (j, c_j, x_j) \Leftrightarrow r\,(i, c_i, x_i) \leq r\,(j, c_j, x_j)\,.$$

Proof. This is a standard result following from Lemmas 2, 10, and 16. ■

Lemma 18 *For every* (c, E), *for every* $i, j \in N$, *and for every* $\epsilon \in (0, c_j - S_j\,(c, E))$, *we have*

$$r\,(i, c_i, S_i\,(c, E)) < r\,(j, c_j, S_j\,(c, E) + \epsilon)\,.$$

Proof. Fix the problem (c, E), claimants $i, j \in N$, and $\epsilon \in (0, c_j - S_j\,(c, E)) \neq \emptyset$. Set $x \equiv S\,(c, E)$. Bilateral Consistency implies $S\,((c_i, c_j)\,, x_i + x_j) = (x_i, x_j)$. By the definition of G, we have $G\,((i, c_i, x_i)\,, j, c_j) \leq x_i + x_j$. Also, Resource Monotonicity implies $x_i + x_j < G\,((j, c_j, x_j + \epsilon)\,, i, c_i)$. Hence, $G\,((i, c_i, x_i)\,, j, c_j) < G\,((j, c_j, x_j + \epsilon)\,, i, c_i)$, which implies $(i, c_i, x_i)\, P_1\, (j, c_j, x_j + \epsilon)$, which implies $r\,(i, c_i, x_i) < r\,(j, c_j, x_j + \epsilon)$. ■

Define $a \equiv \inf\{r\,(i, c_i, x_i) : (i, c_i, x_i) \in Y\}$, $b \equiv \sup\{r\,(i, c_i, x_i) : (i, c_i, x_i) \in Y\}$, and

$$f_i\,(c_i, \lambda) \equiv \sup\{x_i' : r\,(i, c_i, x_i') \leq \lambda\}\,.$$

Lemma 19 *For every* $(i, c_i, x_i) \in Y$, *we have* $f_i\,(c_i, r\,(i, c_i, x_i)) = x_i$.

Proof. By Lemma 7, $r\,(i, c_i, x_i)$ is strictly increasing in x_i. So

$$\{x_i' : r\,(i, c_i, x_i') \leq r\,(i, c_i, x_i)\} = \{x_i' : x_i' \leq x_i\}\,,$$

which implies

$$\begin{aligned} f_i\,(c_i, r\,(i, c_i, x_i)) &= \sup\{x_i' : x_i' \leq x_i\} \\ &= x_i. \end{aligned}$$

■

Lemma 20 $f \equiv \{f_i\}_{\mathbb{N}} \in \mathcal{F}$.

Proof. Fix $i \in \mathbb{N}$ and $c_i > 0$.

First we show $f_i\,(c_i, \lambda)$ is weakly increasing in λ. Let $\lambda_1, \lambda_2 \in [a, b]$ satisfy $\lambda_1 < \lambda_2$. Observe $\{x_i : r\,(i, c_i, x_i) \leq \lambda_1\} \subset \{x_i : r\,(i, c_i, x_i) \leq \lambda_2\}$. Hence, $f_i\,(c_i, \lambda_1) \leq f_i\,(c_i, \lambda_2)$.

Next we show $f_i\,(c_i, \lambda)$ is continuous in λ. Suppose $f_i\,(c_i, \cdot)$ is discontinuous at λ_0. Set $x_0 \equiv f_i\,(c_i, \lambda_0)$, $x_1 \equiv \lim_{\lambda \to \lambda_0^-} f_i\,(c_i, \lambda)$, and $x_2 \equiv \lim_{\lambda \to \lambda_0^+} f_i\,(c_i, \lambda)$. Hence $x_1 < x_2$ and $x_0 \in [x_1, x_2]$. Let $x_3 \in (x_1, x_0) \cup (x_0, x_2)$. Since $f_i\,(c_i, \lambda)$ is weakly increasing in λ, there is no λ such that $f_i\,(c_i, \lambda) = x_3$. However, by Lemma 19, $f_i\,(c_i, r\,(i, c_i, x_3)) = x_3$, a contradiction.

Finally we show $f_i\,(c_i, a) = 0$ and $f_i\,(c_i, b) = c_i$. By definition of Y, for every $\lambda \in [a, b]$, $0 \leq f_i\,(c_i, \lambda) \leq c_i$. Hence, $f_i\,(c_i, a) \geq 0$. Also, $a \leq r\,(i, c_i, 0)$ by definition and $f_i\,(c_i, r\,(i, c_i, 0)) = 0$ by Lemma 19. But since $f_i\,(c_i, \lambda)$ is weakly increasing in λ, $f_i\,(c_i, a) \leq 0$. Hence, $f_i\,(c_i, a) = 0$. We can similarly show that $f_i\,(c_i, b) = c_i$. ■

Lemma 21 *f is a parametric representation of S.*

Proof. Fix (c,E). Set $x \equiv S(c,E)$. Set $j \equiv \arg\max_{i\in N}\{r(i,c_i,x_i)\}$ and $\lambda^* \equiv r(j,c_j,x_j)$. Hence, for every $i \in N$, we have $\lambda^* \geq r(i,c_i,x_i)$, so $f_i(c_i,\lambda^*) \geq f_i(c_i,r(i,c_i,x_i)) = x_i$. If $x_i = c_i$, then $f_i(c_i,\lambda^*) = c_i$. If $x_i < c_i$, then by Lemma 18, for any $\epsilon \in (0, c_j - S_j(c,E))$, $\lambda^* = r(j,c_j,x_j) < r(i,c_i,x_i+\epsilon)$. Hence, $f_i(c_i,\lambda^*) \leq x_i + \epsilon$ by definition of f_i. This implies $f_i(c_i,\lambda^*) \leq x_i$. Hence $f_i(c_i,\lambda^*) = x_i$. ■

Lemma 22 *If S gives priority to (i,c_i,x_i) and if $c_i^n \to c_i$, then $m(i,c_i^n,x_i) \to m(i,c_i,x_i)$ and $M(i,c_i^n,x_i) \to M(i,c_i,x_i)$.*

Proof. Suppose $m(i,c_i^n,x_i) \to \hat{m} < m(i,c_i,x_i)$. Let $x_0 \in (\hat{m}, m(i,c_i,x_i))$. Then $(i,c_i,\hat{m})\, C\, (i,c_i,x_i)$ and $(i,c_i,x_0)\, C\, (i,c_i,x_i)$. Without loss of generality, there exists (j,c_j,x_j) where $j \neq i$ such that[13]

$$(i,c_i,\hat{m})\, P\, (i,c_i,x_0)\, P_1\, (j,c_j,x_j)\, P_1\, (i,c_i,x_i)\,.$$

By Lemma 14, $\hat{m} < S_i((c_i,c_j), \hat{m}+x_j)$ and $x_j > S_j((c_i,c_j),\hat{m}+x_j)$.

By N-Continuity, there exists N such that for every $n \geq N$, we have $(i,c_i,x_i)\, NC\, (i,c_i^n,x_i)$. For every $n \geq N$, set $m^n \equiv m(i,c_i^n,x_i) + \frac{1}{n}$. (For $n < N$, set m^n to any sequence.) Obviously $m^n \to \hat{m}$. Also $(i,c_i^n,m^n)\, NC\, (i,c_i^n,x_i)\, NC\, (i,c_i,x_i)$ for every $n \geq N$. Hence $(j,c_j,x_j)\, P_1\, (i,c_i^n,m^n)$.[14] Lemma 15 implies $m^n \geq S_i((c_i^n,c_j), m^n+x_j)$ and $x_j \leq S_j((c_i^n,c_j),m^n+x_j)$ for every $n \geq N$. Continuity implies $S((c_i^n,c_j),m^n+x_j) \to S((c_i,c_j),\hat{m}+x_j)$. Hence $\hat{m} \geq S_i((c_i,c_j),\hat{m}+x_j)$ and $x_j \leq S_j((c_i,c_j),\hat{m}+x_j)$, a contradiction.

Now suppose $m(i,c_i^n,x_i) \to \hat{m} > m(i,c_i,x_i)$. Let $x_0 \in (m(i,c_i,x_i), \hat{m})$. Then S gives priority to (i,c_i,x_0) and

$$(i,c_i,x_0)\, NC\, (i,c_i,x_i)\,. \tag{5}$$

By N-Continuity, there exists N such that for every $n \geq N$,

$$(i,c_i,x_0)\, NC\, (i,c_i^n,x_0)\,. \tag{6}$$

Similarly, there exists N' such that for every $n \geq N'$,

$$(i,c_i,x_i)\, NC\, (i,c_i^n,x_i)\,. \tag{7}$$

Since $m(i,c_i^n,x_i) \to \hat{m}$, there exists N'' such that for every $n \geq N''$, $m(i,c_i^n,x_i) > x_0$, which implies

$$(i,c_i^n,x_0)\, C\, (i,c_i^n,x_i)\,. \tag{8}$$

Let $n \geq \max\{N,N',N''\}$. Then (5), (6), (7), and the transitivity of NC imply $(i,c_i^n,x_0)\, NC\, (i,c_i^n,x_i)$, which contradicts (8).

The proof for M is similar. ■

[13] Since $(i,c_i,x_0)\, C\, (i,c_i,x_i)$ and $x_0 < x_i$, Lemma 7 implies there exists (j,c_j,x_j) where $j \neq i$ such that $(i,c_i,x_0)\, R_1\, (j,c_j,x_j)\, R_1\, (i,c_i,x_i)$, where one of these is strict. Lemma 8 implies that it is without loss of generality that $(i,c_i,x_0)\, R_1\, (j,c_j,x_j)\, P_1\, (i,c_i,x_i)$. If $(i,c_i,x_0)\, I_1\, (j,c_j,x_j)$, choose $x_0' \in (\hat{m},x_0)$. Then Lemma 7 implies $(i,c_i,\hat{m})\, P\, (i,c_i,x_0')\, P\, (i,c_i,x_0)$. Transitivity implies $(i,c_i,\hat{m})\, P\, (i,c_i,x_0')\, P_1\, (j,c_j,x_j)\, P_1\, (i,c_i,x_i)$.

[14] If $(i,c_i^n,m^n)\, R_1\, (j,c_j,x_j)$, then $(i,c_i^n,m^n)\, R\, (i,c_i,x_i)$, which contradicts $(i,c_i^n,m^n)\, NC\, (i,c_i,x_i)$.

Lemma 23 *Let $f_i(c_i,\lambda) = x_i$. If S gives priority to (i, c_i, x_i) and if $c_i^n \to c_i$, then $f_i(c_i^n, \lambda) \to f_i(c_i, \lambda)$.*

Proof. Since S gives priority to (i, c_i, x_i), there exists $\hat{x}_i$ where $\hat{x}_i < x_i$ such that S gives priority to $(i, c_i, \hat{x}_i)$ and $(i, c_i, \hat{x}_i)\, NC\, (i, c_i, x_i)$. Observe that since S gives priority to x_i, Lemma 7 implies $\theta(i, c_i, x_i) \in (0,1)$. By N-Continuity, there exists N such that for every $n \geq N$, $(i, c_i, \hat{x}_i)\, NC\, (i, c_i^n, \hat{x}_i)$ and $(i, c_i, x_i)\, NC\, (i, c_i^n, x_i)$.

Fix $n \geq N$. Hence $(i, c_i^n, \hat{x}_i)\, NC\, (i, c_i^n, x_i)$ by transitivity of NC. Lemma 6 implies $m(i, c_i^n, \hat{x}_i) = m(i, c_i^n, x_i)$ and $M(i, c_i^n, \hat{x}_i) = M(i, c_i^n, x_i)$. Hence $m(i, c_i^n, \hat{x}_i) \leq \hat{x}_i < x_i \leq M(i, c_i^n, x_i)$ implies $m(i, c_i^n, \hat{x}_i) < M(i, c_i^n, x_i)$. Set

$$x_i^n \equiv (1 - \theta(i, c_i, x_i))\, m(i, c_i^n, x_i) + \theta(i, c_i, x_i)\, M(i, c_i^n, x_i).$$

Observe $m(i, c_i^n, x_i) < x_i^n < M(i, c_i^n, x_i)$ since $\theta(i, c_i, x_i) \in (0,1)$. Hence $(i, c_i^n, x_i^n)\, NC\, (i, c_i^n, x_i)$. So by Lemma 6, $m(i, c_i^n, x_i^n) = m(i, c_i^n, x_i)$ and $M(i, c_i^n, x_i^n) = M(i, c_i^n, x_i)$. Then

$$\begin{aligned}
\theta(i, c_i^n, x_i^n) &= \frac{x_i^n - m(i, c_i^n, x_i^n)}{M(i, c_i^n, x_i^n) - m(i, c_i^n, x_i^n)} \\
&= \frac{x_i^n - m(i, c_i^n, x_i)}{M(i, c_i^n, x_i) - m(i, c_i^n, x_i)} \\
&= \theta(i, c_i, x_i).
\end{aligned}$$

Since $(i, c_i^n, x_i^n)\, NC\, (i, c_i^n, x_i)\, NC\, (i, c_i, x_i)$, then $r(i, c_i, x_i) = r(i, c_i^n, x_i^n)$ by definition of R_3.

Claim: $f_i(c_i^n, \lambda) = x_i^n$. Since $r(i, c_i^n, x_i^n) = r(i, c_i, x_i)$, then obviously $f_i(c_i^n, \lambda) \geq x_i^n$. If $f_i(c_i^n, \lambda) > x_i^n$, then following an argument similar to the one above, we could find $x_0 > x_i$ such that $r(i, c_i, x_0) = r(i, c_i^n, f_i(c_i^n, \lambda))$. But $r(i, c_i^n, f_i(c_i^n, \lambda)) \leq \lambda$. So this would imply $r(i, c_i, x_0) \leq \lambda$, which implies $f_i(c_i, \lambda) \geq x_0 > x_i$, a contradiction. Hence $f_i(c_i^n, \lambda) = x_i^n$.

For $n < N$, let $x_i^n = x_i$. By Lemma 22 and the definition of θ,

$$\begin{aligned}
x_i^n &\to (1 - \theta(i, c_i, x_i))\, m(i, c_i, x_i) + \theta(i, c_i, x_i)\, M(i, c_i, x_i) \\
&= x_i.
\end{aligned}$$

■

Lemma 24 *f is continuous.*

Proof. Fix $i \in \mathbb{N}$ and $(c_i^n, \lambda^n) \to (\hat{c}_i, \hat{\lambda})$. Let $f_i(c_i^n, \lambda^n) \to x^*$ and set $\hat{x} \equiv f_i(\hat{c}_i, \hat{\lambda})$.

By way of contradiction, suppose $x^* < \hat{x}$. Set

$$\lambda^* \equiv \sup \{\lambda : f_i(\hat{c}_i, \lambda) = x^*\}.$$

Observe that $f_i(\hat{c}_i, \lambda^*) = x^*$ since f_i is continuous in λ. Also, since f_i is monotone in λ, $\lambda^* < \hat{\lambda}$.

Claim: For every j and $c_j > 0$, $f_j(c_j, \lambda^*) = f_j(c_j, \hat{\lambda})$. Fix j and $c_j > 0$. For every n, set $E^n \equiv f_i(c_i^n, \lambda^n) + f_j(c_j, \lambda^n)$. Set $\hat{E} \equiv x^* + f_j(c_j, \hat{\lambda})$. Observe that since f_j is continuous in λ, we have $f_j(c_j, \lambda^n) \to f_j(c_j, \hat{\lambda})$. Hence $((c_i^n, c_j), E^n) \to ((\hat{c}_i, c_j), \hat{E})$. For every n,

$$S((c_i^n, c_j), E^n) = (f_i(c_i^n, \lambda^n), f_j(c_j, \lambda^n))$$

since $\{f_i\}_{\mathbb{N}}$ is an asymmetric parametric representation of S. Observe that $S((c_i^n, c_j), E^n) = (f_i(c_i^n, \lambda^n), f_j(c_j, \lambda^n)) \to (x^*, f_j(c_j, \hat{\lambda}))$. By Continuity, $S((c_i^n, c_j), E^n) \to S((\hat{c}_i, c_j), \hat{E})$. So $S((\hat{c}_i, c_j), \hat{E}) = (x^*, f_j(c_j, \hat{\lambda}))$. Also, there exists λ' such that

$$S((\hat{c}_i, c_j), \hat{E}) = (f_i(\hat{c}_i, \lambda'), f_j(c_j, \lambda')).$$

Hence $f_i(\hat{c}_i, \lambda') = x^*$ and $f_j(c_j, \lambda') = f_j(c_j, \hat{\lambda})$. But by definition of λ^*, $\lambda' \leq \lambda^*$. Hence $f_j(c_j, \lambda^*) = f_j(c_j, \hat{\lambda})$ since f_j is monotone in λ.

Claim: For every $\tilde{\lambda} \in (\lambda^*, \hat{\lambda})$, either $f_i(\hat{c}_i, \tilde{\lambda}) = x^*$ or $f_i(\hat{c}_i, \tilde{\lambda}) = \hat{x}$. Fix $\tilde{\lambda} \in (\lambda^*, \hat{\lambda})$. Set $\tilde{x} \equiv f_i(\hat{c}_i, \tilde{\lambda})$. Then $x^* \leq \tilde{x} \leq \hat{x}$ since f_i is weakly increasing in λ. By way of contradiction, suppose $x^* < \tilde{x} < \hat{x}$. Then the claim above implies S gives priority to $(i, \hat{c}_i, \tilde{x})$. By Lemma 23, $f_i(c_i^n, \tilde{\lambda}) \to \tilde{x}$. Then there exists N such that for every $n \geq N$, $f_i(c_i^n, \tilde{\lambda}) > \frac{x^*+\tilde{x}}{2}$. Since $f_i(c_i^n, \lambda^n) \to x^*$, there exists N' such that for every $n \geq N'$, $f_i(c_i^n, \lambda^n) < \frac{x^*+\tilde{x}}{2}$. Since $\lambda^n \to \hat{\lambda}$, there exists N'' such that for every $n \geq N''$, $\lambda^n > \tilde{\lambda}$. So for $n \geq \max\{N, N', N''\}$, $f_i(c_i^n, \tilde{\lambda}) > \frac{x^*+\tilde{x}}{2} > f_i(c_i^n, \lambda^n)$. But since $\tilde{\lambda} < \lambda^n$ and f_i is increasing in λ, $f_i(c_i^n, \tilde{\lambda}) < f_i(c_i^n, \lambda^n)$, a contradiction.

But this claim contradicts the fact that f_i is continuous in λ. Hence $x^* \geq \hat{x}$.

Similarly one can show $x^* \leq \hat{x}$. Hence $x^* = \hat{x}$. ■

C Proof of Theorem 2

The necessity of the axioms is straightforward. We show now that the axioms are sufficient.

Lemma 25 *For every (i, c_i, x_i), $j \neq i$, and c_j, there exists x_j such that*

$$(i, c_i, x_i)\, I_1\, (j, c_j, x_j)\,.$$

Proof. Set $x_j \equiv S\left((c_i, c_j), G\left((i, c_i, x_i), j, c_j\right)\right)$. Then Strict Resource Monotonicity implies $G\left((i, c_i, x_i), j, c_j\right) = G\left((j, c_j, x_j), i, c_i\right)$. ■

Lemma 26 *S satisfies Intrapersonal Consistency and N-Continuity.*

Proof. Suppose Intrapersonal Consistency was not satisfied. Then there exists (i, c_i, x_i), (j, c_j, x_j), (i, c_i', x_i'), and $\left(j, c_j', x_j'\right)$ such that

$$(i, c_i, x_i)\, P_1\, (j, c_j, x_j)\, P_1\, (i, c_i', x_i')\, P_1\, \left(j, c_j', x_j'\right) P_1\, (i, c_i, x_i)\,.$$

Fix k and c_k where k is distinct from i and j. By Lemma 25, there exist x_k and x_k' where $(j, c_j, x_j)\, I_1\, (k, c_k, x_k)$ and $\left(j, c_j', x_j'\right) I_1\, (k, c_k, x_k')$. Case 1 of Lemma 10 implies

$$(i, c_i, x_i)\, P_1\, (k, c_k, x_k)\, P_1\, (i, c_i', x_i')\, P_1\, (k, c_k, x_k')\, P_1\, (i, c_i, x_i)\,.$$

Observe that $(k, c_k, x_k)\, P_1\, (i, c_i', x_i')\, P_1\, (k, c_k, x_k')$ and Lemma 7 imply $x_k \leq x_k'$. Similarly, $(k, c_k, x_k')\, P_1\, (i, c_i, x_i)\, P_1\, (k, c_k, x_k)$ and Lemma 7 imply $x_k' \leq x_k$. Hence $x_k' = x_k$. But this is impossible since $(k, c_k, x_k)\, P_1\, (i, c_i', x_i')$ and $(i, c_i', x_i')\, P_1\, (k, c_k, x_k')$.

N-Continuity is satisfied vacuously because there is no (i, c_i, x_i) to which S gives priority because S is strictly resource monotonic. ■

Hence, the proof follows that of Theorem 1 to derive the parametric function f. All there is left to prove is that, without loss of generality, f_i is strictly increasing in λ for every i.

Lemma 27 *For any i, c_i, λ, and λ' where $\lambda < \lambda'$, if $f_i(c_i, \lambda) = f_i(c_i, \lambda')$, then for any j and c_j, $f_j(c_i, \lambda) = f_j(c_i, \lambda')$.*

Proof. Fix $j \neq i$ and c_j. Suppose $f_j(c_i, \lambda) < f_j(c_i, \lambda')$. Since f is a parametric representation of S,

$$S\left((c_i, c_j), f_i(c_i, \lambda) + f_j(c_i, \lambda)\right) = \left(f_i(c_i, \lambda), f_j(c_i, \lambda)\right),$$

and

$$S\left((c_i, c_j), f_i(c_i, \lambda') + f_j(c_i, \lambda')\right) = \left(f_i(c_i, \lambda'), f_j(c_i, \lambda')\right).$$

But $f_i(c_i, \lambda) = f_i(c_i, \lambda')$ and $f_j(c_i, \lambda) < f_j(c_i, \lambda')$, which contradicts Strict Resource Monotonicity. Hence $f_j(c_i, \lambda) = f_j(c_i, \lambda')$.

Using the fact that $f_j(c_i, \lambda) = f_j(c_i, \lambda')$, one can show $f_i(c_i', \lambda) = f_i(c_i', \lambda')$ for any c_i' in a like manner. ■

D Proofs of Theorems 3 and 4

The proof for Theorem 3 is given below. The proof for Theorem 4 is then obvious given the fact that every continuous and strictly monotonic function has an inverse which is continuous and strictly monotonic.

($\Rightarrow$) First, let $\hat{r} : Y \to \mathbb{R}$ be any quasi-inverse of f (e.g. r from Lemma 17). Define U as follows:

$$U_i(c_i, x_i) \equiv \int_0^{x_i} \hat{r}(i, c_i, x_i')dx_i'.$$

Because r is strictly increasing in the third argument, U is strictly concave in the third argument. The proof that U is continuous and that $S^U = S$ follows that of Young (1987, Theorem 2).

($\Leftarrow$) Let S have a continuous CRAS representation U, i.e. $S = S^U$. For fixed i and c_i, let $u_i(c_i, \cdot)$ denote the superdifferential of $U_i(c_i, \cdot)$:

$$u_i(c_i, x_i) \equiv \{\lambda : \lambda(x_0 - x_i) \geq U_i(c_i, x_0) - U_i(c_i, x_i) \text{ for every } x_0 \in [0, c_i]\}.$$

Because $U_i(c_i, \cdot)$ is strictly concave, the weak inequality can be replaced with a strict inequality and the condition that $x_0 \neq x_i$. Also, strict concavity implies that $u_i(c_i, \cdot)$ is strictly decreasing in the sense that if $x_i < x_i'$, $\lambda \in u_i(c_i, x_i)$, and $\lambda' \in u_i(c_i, x_i')$, then $\lambda > \lambda'$. Furthermore, we have $u_i(c_i, x_i) = [d_+u_i(c_i, x_i), d_-u_i(c_i, x_i)]$, where $d_+u_i(c_i, x_i)$

and $d_-u_i(c_i, x_i)$ denote the right and left derivatives of $u_i(c_i, \cdot)$ at x_i respectively.[15] (Set $d_-u_i(c_i, 0) \equiv \infty$ and $d_+u_i(c_i, c_i) \equiv -\infty$.)

Now define f as follows. For $i \in \mathbb{N}$, $c_i \in \mathbb{R}_{++}$, and $\lambda \in \mathbb{R}$, we have $f_i(c_i, \lambda) = x_i$ if $-\lambda \in u_i(c_i, x_i)$. The following lemmas complete the proof.

Lemma 28 $f \in \mathcal{F}$

Proof. First we show f is well-defined; that is, for every $i \in \mathbb{N}$, $c_i \in \mathbb{R}_{++}$, and $\lambda \in \mathbb{R}$, there exists a unique x_i such that $-\lambda \in u_i(c_i, x_i)$. So fix i, c_i, and λ. If $-\lambda \geq d_+u_i(c_i, 0)$, then $-\lambda \in u_i(c_i, 0)$. If $-\lambda \leq d_-u_i(c_i, c_i)$, then $-\lambda \in u_i(c_i, c_i)$. If $d_-u_i(c_i, c_i) < -\lambda < d_+u_i(c_i, 0)$, then we must have

$$\sup\{x_0 : d_+u_i(c_i, x_0) \geq -\lambda\} = \inf\{x_0 : d_-u_i(c_i, x_0) \leq -\lambda\}.$$

Call this x^*. However, Rockafellar (1970, Theorem 24.1) implies that

$$\lim_{x_0 \uparrow x^*} d_+u_i(c_i, x_0) = d_-u_i(c_i, x^*)$$

and

$$\lim_{x_0 \downarrow x^*} d_-u_i(c_i, x_0) = d_+u_i(c_i, x^*).$$

Hence, $d_+u_i(c_i, x^*) \leq -\lambda \leq d_-u_i(c_i, x^*)$, which implies $-\lambda \in u_i(c_i, x^*)$. The uniqueness of x^* is evident from the fact that $u_i(c_i, \cdot)$ is strictly decreasing.

Now we show that f is weakly increasing in the third argument. Fix i and c_i. Suppose $\lambda < \lambda'$ and by way of contradiction suppose $f_i(c_i, \lambda) = x_i > x_i' = f_i(c_i, \lambda)$. Hence

$$-\lambda(x_i' - x_i) > U_i(c_i, x_i') - U_i(c_i, x_i)$$

and

$$-\lambda'(x_i - x_i') > U_i(c_i, x_i) - U_i(c_i, x_i').$$

But together this implies

$$\lambda' < \frac{U_i(c_i, x_i') - U_i(c_i, x_i)}{x_i - x_i'} < \lambda.$$

The continuity of f in the third argument is implied by Lemma 29.

Finally, observe that for every i and c_i, we have $\lim_{\lambda \to -\infty} f_i(c_i, \lambda) = 0$ and $\lim_{\lambda \to \infty} f_i(c_i, \lambda) = c_i$. ■

Lemma 29 *f is continuous.*

Proof. Fix i. Let $(c_i^n, \lambda^n) \to (c_i, \lambda)$. Let $x_i^n = f_i(c_i^n, \lambda^n)$ and $x_i = f_i(c_i, \lambda)$. By way of contradiction, suppose there exists $x_i^* \neq x_i$ a limit point of x_i^n. Without loss of generality, assume the subsequence converging to x_i^* is the sequence itself.

[15]See Rockafellar (1970, Theorem 23.2) for the case of a convex function.

Consider the case when $x_i^* < x_i$. Since $u_i(c_i, \cdot)$ is strictly decreasing and $-\lambda \in u_i(c_i, x_i)$ by definition, we have $-\lambda \notin u_i(c_i, x_i^*)$. This implies that there exists $x_i' \neq x_i^*$ such that

$$-\lambda(x_i' - x_i^*) < U_i(c_i, x_i') - U_i(c_i, x_i^*). \tag{9}$$

For every n large enough, $x_i^n \neq x_i'$. Hence

$$-\lambda^n(x_i' - x_i^n) > U_i(c_i, x_i') - U_i(c_i, x_i^n) \tag{10}$$

for every n large enough. But the left-hand side of (10) converges to $\lambda(x_i' - x_i^*)$ while the right-hand side converges to $U_i(c_i, x_i') - U_i(c_i, x_i^*)$. Hence

$$-\lambda(x_i' - x_i^*) \geq U_i(c_i, x_i') - U_i(c_i, x_i^*)$$

which contradicts (9).

The case when $x_i^* > x_i$ is similar. ■

Lemma 30 *f is a parametric representation of S.*

Proof. For every $(i, c_i, x_i) \in Y$, choose $\hat{r} \in u_i(c_i, x_i)$. Let $\hat{r}(i, c_i, x_i)$ denote this choice. Note $\hat{r}$ is a quasi-inverse of f with respect to its third argument, and that for fixed i and c_i, $\hat{r}(i, c_i, \cdot)$ has only countable points of discontinuity. Define $\hat{U}$ as follows:

$$\hat{U}_i(c_i, x_i) \equiv \int_0^{x_i} \hat{r}(i, c_i, x_i') dx_i'.$$

Using the same argument we used previously, we can show $S^{\hat{U}} = S^f$. Also, it is not hard to see that for every problem (c, E), we have

$$\arg\max_{x \in X(c,E)} \sum_{i \in N} \hat{U}_i(c_i, x_i) = \arg\max_{x \in X(c,E)} \sum_{i \in N} U_i(c_i, x_i).$$

Hence $S^{\hat{U}} = S^U$. Since $S = S^U$, we have thus shown that $S^f = S$. ■

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