Voting in Legislative Elections Under Plurality Rule

Niall Hughes

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VOTING IN LEGISLATIVE ELECTIONS UNDER PLURALITY RULE*

Niall Hughes
University of Warwick,
Coventry, CV4 7AL,
United Kingdom
n.e.hughes@warwick.ac.uk

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Abstract

Conventional models of single district plurality elections show that with three parties anything can happen - extreme policies can win regardless of voter preferences. I show that when when single district elections are used to fill a legislature we get back to a world where the median voter matters. An extreme policy will generally only come about if it is preferred by the median voter in a majority of districts, while the mere existence of a centrist party can lead to moderate outcomes even if the party itself wins few seats. Furthermore, I show that while standard single district elections always have *misaligned voting* i.e. some voters do not vote for their preferred choice, equilibria of the legislative election exist with no misaligned voting in any district. Finally, I show that when parties are impatient, a fixed rule on how legislative bargaining occurs will lead to more coalition governments, while uncertainty will favour single party governments.

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1   Introduction

Plurality rule (a.k.a. first-past-the-post) is used to elect legislatures in the U.S., U.K., Canada, India, Pakistan, Malaysia as well as a host of other former British colonies - yet we know very little about how it performs in such settings. The literature on single district elections shows that plurality rule performs well when there are only two candidates but poorly when there are more.[1] Indeed, plurality has recently been deemed the worst voting rule by a panel of voting theorists.[2] However, the objectives of voters differ in single district and legislative elections. In a legislative election, many districts hold simultaneous plurality elections and the winner of each district takes a seat in a legislature. Once all seats are filled, the elected politicians bargain over the formation of government and implement policy. If voters only care about which policy is implemented in the legislature, they will cast their ballots to influence the outcome of the legislative bargaining stage. A voter's preferred candidate will therefore depend on the results in other districts. Meanwhile, in a single district election - such as a mayoral election - a voter's preference ordering over candidates is fixed, as only the local result matters. These different objectives are at the heart of this paper. I show that when three parties compete for legislative seats and voters care about national policy, several negative properties of plurality rule are mitigated.

While there has been some key work on voting strategies in legislative elections under proportional representation (PR), notably Austen-Smith and Banks (1988) and Baron and Diermeier (2001), there has been scant attention paid to the question of how voters should act when three parties compete in a legislative election under plurality rule. Studies of plurality rule have either focused on two-party legislative competition or else on three-party single district elections, in which voters only care about the result in that district. In the former case, as voters face a choice of two parties, they have no strategic decision to make - they simply vote for their favourite. However, for almost all countries using plurality rule, with the notable exception of the U.S., politics is not a two-party game: the U.K. has the Conservatives, Labour and the Liberal Democrats; Canada has the Conservatives, Liberals, and New Democrats; India has Congress, BJP and many smaller parties.[3] With a choice of three candidates, voters must consider how others will vote when deciding on their own ballot choice.

In a single plurality election, only one candidate can win. Therefore, when faced with

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[1]See for example Myerson (2000) and Myerson (2002).

[2]See Laslier (2012)

[3]Other countries with plurality rule and multiple parties represented in the legislature include: Bangladesh, Botswana, Kenya, Liberia, Malawi, Malaysia, Mongolia, Pakistan, Trinidad & Tobago, Tanzania, Uganda, and Zambia.

a choice of three options, voters who prefer the candidate expected to come third have an incentive to abandon him and instead vote for their second favourite, so that in equilibrium only two candidates receive votes. These are the only serious candidates. This effect, known as Duverger's law[4], was first stated by Henry Droop in 1869:

> "Each elector has practically only a choice between two candidates or sets of candidates. As success depends upon obtaining a majority of the aggregate votes of all the electors, an election is usually reduced to a contest between the two most popular candidates or sets of candidates. Even if other candidates go to the poll, the electors usually find out that their votes will be thrown away, unless given in favour of one or other of the parties between whom the election really lies." (Droop cited in Riker (1982), p. 756)

A vast literature has pointed out two negative implications of Duverger's law in single district elections.[5] First, *"anything goes"*: the equilibrium is completely driven by voters' beliefs, so *any* of the three candidates could be abandoned, leaving the other two to share the vote. This means that, regardless of voter preferences, there can always be polarisation - where a race between the two extreme choices results in an implemented policy far away from the centre. Second, when each of the three choices is preferred by some voter, there will always be *misaligned voting*. That is, some people will vote for an option which is not their most preferred. Misaligned voting undermines the legitimacy of the elected candidate: one candidate may win a majority simply to "keep out" a more despised opponent, so the winner's policies may actually be preferred by relatively few voters.

In this paper, I model a legislative election in which each voter casts a ballot in a local district but their utility depends on policy determined in the national parliament. I show that the two negative properties of plurality elections - polarisation and misaligned voting - while always present in the stand-alone district elections, need not hold in a legislative election setting.

The intuition for the polarisation result is as follows. For any party to win a majority of seats in the legislature it must be that they are preferred to some alternative by a majority of voters in a majority of districts. I show that the alternative to a left majority will generally not be a right majority but rather a moderate coalition government. Therefore, for an extreme policy to come about, it must be that the median voter in the median district

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[4]The law takes it's name from French sociologist Maurice Duverger who popularised the idea in his book *Political Parties*. While Riker (1982) argues that Duverger's law should be interpreted as the tendency of plurality rule to bring about a two-party system, most scholars use the term to describe the local effect: in any one district only two candidates will receive votes. I also use the local interpretation.

[5]See Palfrey (1989), Myerson and Weber (1993), Cox (1997), Fey (1997), Myerson (2002), Myatt (2007).

prefers this policy to the moderate coalition policy. This contrasts with traditional plurality elections where extreme policy outcomes are always possible, regardless of preferences.

The misaligned voting result stems from the fact that voters condition their ballots on a wider set of events in my setting. In a standard plurality election, voters condition their vote on the likelihood of being pivotal in their district. However, in a legislative election, voters will condition their ballot choice on their vote being pivotal <u>and</u> their district being decisive in determining the government policy. In many cases a district will be decisive between two policies, even though there are three candidates. For example, a district might be decisive in either granting a majority of seats to a non-centrist party, say the left party, or bringing about a coalition by electing one of the other parties. Under many bargaining rules this coalition policy will be the same regardless of which of the weaker parties is elected. So, voters only face a choice between two policies: that of the left party and that of the coalition. When voters have a choice over two policies there can be no misaligned voting - everyone must be voting for their preferred option of the two.

One technical contribution of the paper is to extend the Poisson games framework of Myerson (2000) to a multi-district setting. I show in Lemma 1 that the Magnitude Theorem and it's Corollary can be used to rank the likelihood of various pivotal events across districts. This greatly reduces the complexity of working with multi district pivotal events and makes the problem much more tractable.

I examine the workings of my model under several legislative bargaining settings - varying the scope of bargaining, the patience of politicians, and the bargaining protocol. Government formation processes *do* vary across countries. In some countries potential coalition partners may bargain jointly over policy and perks, while in others perks may be insufficient to overcome ideological differences. The patience of politicians will also differ across countries depending on aspects such as how quickly successive rounds of bargaining occur, and how likely politicians are to be re-elected.[6] A further feature of government formation which has been studied extensively is the protocol for selecting a formateur (a.k.a. proposer). The two standard cases are *random recognition* - where a party's probability of being the formateur in each round of bargaining is equal to its seat share - and *fixed order* - where the largest party makes the first offer, then the second largest, and so on. Diermeier and Merlo (2004) analyse 313 government formations in Western European over the period 1945–1997 and find the data favours random recognition rule. On the other hand, Bandyopadhyay, Chatterjee, and Sjöström (2011) note that a fixed order of bargaining is constitutionally enshrined in Greece and Bulgaria, and is a strong norm in the U.K. and India, where elections are held

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[6]See Austen-Smith and Banks (2005) and Banks and Duggan (2006) for a discussion of discount rates in legislative bargaining.

under plurality rule. Therefore, I examine these various combinations to see how robust the polarisation and misaligned voting results are.

In my benchmark model, parties in the legislature bargain only over policy and they do not discount the future. Here, if no party holds a majority of seats, the moderate party's policy will be implemented. Two clear predictions emerge from this benchmark model. First, when the moderate party wins at least one seat, polarisation is mitigated: the policy of the left or right party can only be implemented if a majority of voters in a majority of districts prefer it to the policy of the moderate party. Second, if either the left or right party is a serious candidate in less than half of the districts, there can be no misaligned voting. These results change somewhat under different bargaining rules, but their flavour remains the same. When parties bargain over perks of office as well as policy, the polarisation result is strengthened - it is even more difficult to have extreme outcomes - while the misaligned voting result is weakened - it can only be ruled out if a non-centrist party is serious in less than a quarter of districts.

Finally, when politicians are impatient, I show that if a country uses a random recognition rule then coalition governments will be more difficult to construct than under a fixed order of recognition, all else equal. The reason is that a fixed order rule gives a significant advantage to the largest party and also makes it easier for voters to predict which government policy will be implemented after the election. As the difference in policy between, say, a left majority government and a coalition led by the left is quite small, voter preferences must be very much skewed in favour of the left party in order for it to win a majority. With a random recognition rule, however, risk averse voters will prefer the certainty of a non-centrist single-party government to the lottery over policies which coalition bargaining would induce.

This paper contributes primarily to the theoretical literature on strategic voting in legislative elections. The bulk of this works has been on PR. Austen-Smith and Banks (1988) find that, with a minimum share of votes required to enter the legislature, the moderate party will receive just enough votes to ensure representation, with the remainder of its supporters choosing to vote for either the left or right party. Baron and Diermeier (2001) show that, with two dimensions of policy, either minimal-winning, surplus, or consensus governments can form depending on the status quo. On plurality legislative elections Morelli (2004) and Bandyopadhyay, Chatterjee, and Sjöström (2011) show that if parties can make pre-electoral pacts, and candidate entry is endogenous, then voters will not need to act strategically. My paper nonetheless focuses on strategic voting because in the main countries of interest, the U.K. and Canada, there are generally no pre-electoral pacts and the three main parties compete in almost every district, so strategic candidacy is not present.[7]

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[7]In the 2010 U.K. General Election, the three main parties contested 631 out of 650 districts (None of

The paper proceeds as follows. In the next section I introduce the benchmark model and define an equilibrium. In Section 3, I solve the model and show conditions which must hold in equilibrium. Section 4 presents the main results on the level of polarisation and misaligned voting in the benchmark model. Section 5 adds perks of office to the bargaining stage of the model, while Section 6 shows how the benchmark results change when parties discount the future. Finally, Section 7 discusses the assumptions of the model and concludes.

2   Model

Parties   There are three parties; l, m, and r, contesting simultaneous elections in D districts, where D is an odd number. The winner of each of the D elections is decided by plurality rule: whichever party receives the most votes in district $d \in D$ is deemed elected and takes a seat in the legislature. The outcome from all districts gives a distribution of seats in the legislature, $S \equiv (s_l, s_m, s_r)$, with party $c \in \{l, m, r\}$ having s_c seats and $\sum_c s_c = D$. Party c has a preferred platform a_c in the unidimensional policy space $\mathbb{X} = [-1, 1]$ on which it must compete in every district. A party cannot announce a different platform to gain votes; voters know that a party will always implement its preferred platform if it gains a legislative majority. Once all the seats in the legislature have been filled, the parties bargain over the formation of government and implement a policy z. As such, a party cannot commit to implement its platform as the policy outcome z depends on bargaining. Party c has the payoff $W_c = b_c - (z - a_c)^2$, linear in its share of government benefits b_c, and negative quadratic in the distance between its platform a_c and the implemented policy z. A feasible allocation of benefits is $b = (b_l, b_m, b_r)$ where each b_c is non-negative and $\sum_c b_c \leq B$.

The benchmark model I use is that of Baron (1991), where $B = 0$ so that bargaining is over policy alone.[8] In Section 5, I consider the more complicated case of $B > 0$ due to Austen-Smith and Banks (1988). If a party has a majority of the seats in the legislature it can form a unitary government and will implement its preferred policy. If no party wins an outright majority we enter a stage of legislative bargaining. I consider two bargaining protocols: one in which the order of bargaining is random and one in which it is fixed. Under

them contest seats in Northern Ireland), while in the 2011 Canadian Federal Election the three major parties contested 307 of the 308 seats.

   [8]A large literature has grown from legislative bargaining model of Baron and Ferejohn (1989), in which legislators bargain over the division of a dollar. See Baron (1991), Banks and Duggan (2000), Baron and Diermeier (2001), Jackson and Moselle (2002), Eraslan, Diermeier, and Merlo (2003), Kalandrakis (2004), Banks and Duggan (2006), and many others. Morelli (1999) introduces a different approach to legislative bargaining whereby potential coalition partners make demands to an endogenously chosen formateur. In contrast with the Baron and Ferejohn (1989) setup, the formateur does not capture a disproportionate share of the payoffs.

the former rule, one party is randomly selected as formateur, where the probability of each party being chosen is equal to its seat share in the legislature. The formateur proposes a policy in $[-1, 1]$, which is implemented if a majority of the legislature support it; if not, a new formateur is selected, under the same random recognition rule, and the process repeats itself until agreement is reached. Under the fixed order rule, the party with the largest number of seats proposes a policy in $[-1, 1]$, which is implemented if a majority of the legislature support it; if not, the second largest party proposes a policy. If this second policy does not gain majority support, the smallest party proposes a policy, and if still there is no agreement, a new round of bargaining begins with the largest party again first to move. I assume for now that parties are perfectly patient, $\delta = 1$, but this is relaxed in Section 6. A party's strategy specifies which policy to propose if formateur, and which policies to accept or reject otherwise.

Voters Individuals are purely policy-motivated with quadratic preferences on $\mathbb{X}$. As such, a voter does not care who wins his district *per se*, nor does he care which parties form government; all that matters is the final policy, z, decided in the legislature. A voter's type, $t \in T \subset \mathbb{X}$, is simply his position on the policy line; his utility is $u_t(z) = -(z - t)^2$. I assume T is sufficiently rich that for any tuple of distinct policies, $\{a_l, a_m, a_r\}$, there is at least one voter type who prefers one of the three over the other two. Furthermore, I assume for simplicity that there is no type which is exactly indifferent between two platforms. Let $V \equiv \{v_l, v_m, v_r\}$ be the set of feasible actions an individual can take, with v_c indicating a vote for party c. Voting is costless; thus, there will be no abstention.

Following Myerson (2000, 2002), the number of voters in each district d is not fully known but rather is a random variable n_d, which follows a Poisson distribution and has mean n. The probability that there are exactly k voters in a district is $Pr[n_d = k] = \frac{e^{-n}n^k}{k!}$. Appendix A summarises several properties of the Poisson model. The use of Poisson games in large election models is now commonplace as it simplifies the calculation of probabilities while still producing the same predictions as models with fixed but large populations.[9]

Each district has a distribution of types from which its voters are drawn, f_d, which has full support over $[-1, 1]$. The probability of drawing a type t is $f_d(t)$. The *actual* population of voters in d consists of n_d independent draws from f_d. A voter knows his own type, the distribution from which he was drawn, and the distribution functions of the other districts,

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[9]Krishna and Morgan (2011) use a Poisson model to show that in large elections, voluntary voting dominates compulsory voting when voting is costly and voters have preferences over ideology and candidate quality. Bouton and Castanheira (2012) use a Poisson model to show that when a divided majority need to aggregate information as well as coordinate their voting behaviour, approval voting serves to bring about the first-best outcome in a large election. Furthermore, Bouton (2013) uses a Poisson model to analyse the properties of runoff elections.

$\mathbf{f} \equiv \{f_1, \ldots, f_d, \ldots, f_D\}$, but he does not know the actual distribution of voters that is drawn in any district.

A voter's strategy is a mapping $\sigma : T \to \Delta(V)$ where $\sigma_{t,d}(v_c)$ is the probability that a type t voter in district d casts ballot v_c. The usual constraints apply: $\sigma_{t,d}(v_c) \geq 0, \forall c$ and $\sum_c \sigma_{t,d}(v_c) = 1, \forall t$. In a Poisson game, all voters of the same type in the same district will follow the same strategy (see Myerson (1998)). Given the various $\sigma_{t,d}$'s, the expected vote share of party c in the district is

$$\tau_d(c) = \sum_{t \in T} f_d(t) \sigma_{t,d}(v_c) \tag{1}$$

which can also be interpreted as the probability of a randomly selected voter playing v_c. The expected distribution of party vote shares in d is $\tau_d \equiv (\tau_d(l), \tau_d(m), \tau_d(r))$. The realised profile of votes is $x_d \equiv (x_d(l), x_d(m), x_d(r))$, but this is uncertain *ex ante*. As the population of voters is made up of n_d independent draws from f_d, where $E(n_d) = n$, the expected number of ballots for candidate c is $E(x_d(c)|\sigma_d) = n\tau_d(c)$. In the extremely unlikely event that nobody votes, I assume that party m wins the seat.[10] Let $\boldsymbol{\sigma} \equiv \{\sigma_1, \ldots, \sigma_d, \ldots, \sigma_D\}$ denote the profile of voter strategies across districts and let $\boldsymbol{\sigma_{-d}}$ be that profile with σ_d omitted. Let $\boldsymbol{\tau} \equiv \{\tau_1, \ldots, \tau_d, \ldots, \tau_D\}$ denote the profile of expected party vote share distributions and let $\boldsymbol{\tau_{-d}}$ be that profile with τ_d omitted. Thus, we have $\boldsymbol{\tau}(\boldsymbol{\sigma}, \mathbf{f})$.

At this point, I could define an equilibrium of the game; however, it is more convenient to define equilibrium in terms of pivotality and decisiveness, so I first introduce these additional concepts below.

Pivotality, Decisiveness and Payoffs A single vote is *pivotal* if it makes or breaks a tie for first place in the district. A district is *decisive* if the policy outcome z depends on which candidate that district elects. When deciding on his strategy, a voter need only consider cases in which his vote affects the policy outcome. Therefore, he will condition his vote choice on being *pivotal* in his district <u>and</u> on the district being *decisive*. The ability to do so is key, as if a voter cannot condition on some event where his vote matters then he does not know how he should vote.

Let $piv_d(c, c')$ denote when, in district d, a vote for party c' is pivotal against c. This occurs when $x_d(c) = x_d(c') \geq x_d(c'')$ – so that an extra vote for c' means it wins the seat – or when $x_d(c) = x_d(c') + 1 \geq x_d(c'')$ – so that an extra vote for c' forces a tie. In the event of a tie, a coin toss determines the winner.

Let λ_d denote an event in which district d is decisive in determining which policy z is

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[10]The probability of zero turnout in a district is e^{-n}.

implemented; and let λ_d^i denote the i-th most likely decisive event for district d. Here the decision of district d will lead to one of three final policy outcomes, thus, we can write each decisive event as $\lambda_d^i(z_l^i, z_m^i, z_r^i)$ where each z_c^i is the policy outcome of the legislative bargaining stage when the decisive district elects party c. Note that these policies need not correspond to the announced platforms of the parties - typically coalition bargaining will lead to compromised policies. Two decisive events λ_d^i and λ_d^j are *distinct* if $(z_l^i, z_m^i, z_r^i) \neq (z_l^j, z_m^j, z_r^j)$. Let Λ be the set of distinct decisive events; this set consists of I elements. As we will see, the number and type of decisive events in the set Λ depends on the legislative bargaining rule used.

It is useful for the following sections to classify decisive events into three categories. Let $\lambda(3)$ be a decisive event where all three policies z_l^i, z_m^i, and z_r^i are different points on the policy line; let $\lambda(2)$ be a case where two of the three policies are identical.[11] Finally, let $\lambda(2')$ be an event where there are three different policies but one of them is the preferred choice of no voter.[12] We can now turn to voter payoffs.

Let $G_{t,d}(v_c|n\boldsymbol{\tau})$ denote the expected gain for a voter of type t in district d of voting for party c, given the strategies of all other players in the game – this includes players in his own district as well as those in the other $D - 1$ districts. The expected gain of voting v_l is given by

$$G_{t,d}(v_l|n\boldsymbol{\tau}) = \sum_{i=1}^{I} Pr[\lambda_d^i]\left(Pr[piv_d(m,l)]\left(u_t(z_l^i) - u_t(z_m^i)\right) + Pr[piv_d(r,l)]\left(u_t(z_l^i) - u_t(z_r^i)\right)\right) \tag{2}$$

with the gain of voting v_m and v_r similarly defined. The probability of being pivotal between two candidates, $Pr[piv_d(c,c')]$, depends on the strategies and distribution of player types in that district, summarised by τ_d, while the probability of district d being decisive depends on the strategies and distributions of player types in the other $D - 1$ districts, $\boldsymbol{\tau_{-d}}$. The best response correspondence of a type t in district d to a strategy profile and distribution of types given by $\boldsymbol{\tau}$ is

$$BR_{t,d}(n\boldsymbol{\tau}) \equiv \operatorname*{argmax}_{\sigma_{t,d}} \sum_{v_c \in V} \sigma_{t,d}(v_c)G_{t,d}(v_c|n\boldsymbol{\tau}) \tag{3}$$

Timing The sequence of play is as follows:

1. In each district, nature draws a population of n_d voters from f_d.

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[11]Obviously, $\lambda(1)$ events cannot exist; if electing any of the three parties gives the same policy, it is not a decisive event.

[12]For this to be the case, the universally disliked policy must be a lottery over two or more policies.

2. Voters observe platforms $\{a_l, a_m, a_r\}$ and cast their vote for one of the three parties. Whichever party wins a plurality in a district takes that seat in the legislature.

3. A government is formed according to a specified bargaining process and a policy outcome, z, is chosen.

Equilibrium Concept   The equilibrium of this game consists of a *voting equilibrium* at stage 2 and a *bargaining equilibrium* at stage 3. In a *bargaining equilibrium*, each party's strategy is a best response to the strategies of the other two parties. I restrict attention to stationary bargaining equilibria, as is standard in such games.[13]

The solution concept for the voting game at stage 2 is strictly perfect equilibrium (Okada (1981)).[14] A strategy profile $\boldsymbol{\sigma^*}$ is a strictly perfect equilibrium if and only if $\exists \epsilon > 0$ such that $\forall \tilde{\tau}_d \in \Delta V : |\tilde{\tau}_d - \tau_d(\boldsymbol{\sigma^*}, \mathbf{f})| < \epsilon$ then $\sigma^*_{t,d} \in BR_{t,d}(n\tilde{\boldsymbol{\tau}})$ for all $(t, d) \in T \times D$. That is, the equilibrium must be robust to epsilon changes in the strategies of players. Bouton and Gratton (2014) argue that restricting attention to such equilibria in multi-candidate Poisson games is appropriate because it rules out unstable and undesirable equilibria identified by Fey (1997). If, instead, Bayesian Nash equilibrium is used there may be knife-edge equilibria in which voters expect two or more candidates to get exactly the same number of votes. Bouton and Gratton (2014) also note that requiring strict perfection is equivalent to robustness to heterogenous beliefs about the distribution of preferences, $\mathbf{f}$. As I am interested in the properties of large national elections, I analyse the limiting properties of such equilibria as $n \to \infty$.

3   Equilibrium

I solve for the equilibrium of the game by backward induction. The bargaining equilibrium at stage 3 follows from Baron (1991). Of greater interest to us is the voting equilibrium at stage 2. While there are multiple voting equilibria for any distribution of voter types, I show that every equilibrium has only two candidates receiving votes in each district and I present several properties which must hold in any equilibrium.

Stage 3: Bargaining Equilibrium   When no party has a majority of seats and $\delta = 1$, in any stationary bargaining equilibrium $z = a_m$ is proposed and eventually passed with

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[13]An equilibrium is stationary if it is a subgame perfect equilibrium and each party's strategy is the same at the beginning of each bargaining period, regardless of the history of play.

[14]The original formulation of strictly perfect equilibrium was for games with a finite number of players; Bouton and Gratton (2014) extend this to Poisson games.

probability one.[15] This is regardless of whether the protocol is fixed order or random. To see this, note that if any other policy is proposed, a majority of legislators will find it worthwhile to wait until a_m is offered (which will occur when party m is eventually chosen as formateur). The equilibrium policy outcome of the legislative bargaining stage is then

$$
z = \begin{cases}
a_l & \text{if } s_l > \frac{D-1}{2} \\
a_r & \text{if } s_r > \frac{D-1}{2} \\
a_m & \text{otherwise}
\end{cases}
\tag{4}
$$

Every feasible seat distribution is mapped into a policy outcome, so, at stage 2, voters can fully anticipate which policy will arise from a given seat distribution. The set of distinct decisive events is given by

$$
\Lambda = \{\lambda(a_l, a_m, a_m), \lambda(a_m, a_m, a_r), \lambda(a_l, a_m, a_r)\}
\tag{5}
$$

Stage 2: Voting Equilibrium A slight detour on how voters optimally cast their vote is in order before describing the voting equilibrium. We know from the Magnitude Theorem (Myerson (2000), see appendix) that as $n \to \infty$ voters in a single district election need only condition their choice on the most likely vote profile in which their vote is pivotal. The following lemma extends this result to the case of multi-district elections considered here.

Lemma 1. *As $n \to \infty$ a voter need only condition his vote on the most likely event in which his vote is both pivotal and decisive over two distinct policy outcomes.*

Proof. See Appendix A. □

An intuitive way of thinking of a voter's decision process is the following. As in stand-alone plurality elections, a voter must consider the relative probabilities of his vote changing the outcome in his district. However, he must also consider what happens in other districts. Given $\boldsymbol{\tau_{-d}}$, the voter will have an expectation about what the seat distribution will be before his district votes, $E(S_{-d})$. If such an expectation means that d is not decisive, then the voter will look to the most likely upset out of all the districts - where an expected winner in a district instead loses. If d is now decisive, he can condition on this event; if not, he must consider the next most likely upset. This processes of continues until the voter has worked out the most likely chain of upsets which must occur in order to make his district decisive. Combining the probability of various upsets in the other districts with the probability of being pivotal in district d, the voter can rank the probabilities of the various cases in which

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[15]For a proof see Jackson and Moselle (2002).

his vote choice will change his utility. Only the relative probabilities of these events matter for the voter. The lemma says that in a large election a voter need only condition his ballot on the most likely of these events.

Now that we have analysed the decision problem of a single voter, we can see what happens in an equilibrium. Voting games where players have three choices typically have multiple equilibria; this game is no exception. However, I show that every voting equilibrium involves only two candidates getting votes in each district.

Proposition 1. *For any majoritarian legislative bargaining rule and any distribution of voter preferences, $\mathbf{f} \equiv \{f_1, \ldots, f_d, \ldots, f_D\}$, there are multiple equilibria; in every equilibrium districts are duvergerian.*

Proof. See Appendix A. □

It is perhaps unsurprising that there are multiple equilibria and districts are always duvergerian, given the findings of the extensive literature on single district plurality elections. The logic as to why races are duvergerian is similar to the single district case; voters condition their ballot choice on the most likely case in which they are pivotal, decisive and not indifferent over outcomes. This greatly simplifies the decision process of voters and means they need only consider the two frontrunners in their district.[16] While we cannot pin down which equilibrium will be played, the following properties will always hold.

1. In each district only two candidates receive votes; call these *serious candidates*.

2. If $\tau_d(c) > \tau_d(c') > 0$, candidate c is the expected winner and his probability of winning goes to one as $n \to \infty$. Let d_c denote such a district.

3. The expected seat distribution is $E(S) = E(s_l, s_m, s_r) = (\#d_l, \#d_m, \#d_r)$.

4. A district with c and c' as serious candidates will condition on the most likely decisive event $\lambda^i \in \Lambda$ such that $z^i(c) \neq z^i(c')$.

The fourth property says: if a district's most likely decisive event, λ_d^1, is of the type $\lambda(3)$ or $\lambda(2')$, then voters must be conditioning on this event; if λ_d^1 is of type $\lambda(2)$, voters will be conditioning on it only if they are not indifferent between the two serious candidates.

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[16]By restricting attention to strictly perfect equilibria we rule out knife edge cases where candidates are expected to get exactly the same share of votes.

4   Analysis of Benchmark Model

This section presents the main results of the paper: the problems associated with standard plurality rule election - polarisation and misaligned voting - can be mitigated in legislative elections. The proofs of results in the rest of the paper rely on graphical arguments, hence, a slight detour is needed to explain this approach.

Recall, from Equation 5, that there are three distinct decisive events a district may condition on:

$$\Lambda = \{\lambda(a_l, a_m, a_m), \lambda(a_m, a_m, a_r), \lambda(a_l, a_m, a_r)\}$$

The first two are $\lambda(2)$ events while the final one is a $\lambda(3)$ event. A $\lambda(a_l, a_m, a_m)$ event occurs when a district is decisive in determining whether l wins a majority of seats and implements $z = a_l$, or it falls just short, allowing a coalition to implement $z = a_m$. Here, voter types $t < \frac{a_l + a_m}{2} \equiv a_{lm}$ will vote v_l while those of type $t > a_{lm}$ will coordinate on either v_m or v_r, as they are indifferent between the two policies offered. Similarly, in a $\lambda(a_m, a_m, a_r)$ event, a district can secure party r a majority of seats or not; those with $t < a_{mr}$ will vote either v_l or v_m, while the rest will choose v_r. Finally, when a district finds itself at a point $\lambda(a_l, a_m, a_r)$, it is conditioning on l and r winning half the seats each before d's result is included: $S_{-d} = (\frac{D-1}{2}, 0, \frac{D-1}{2})$. Electing either l or r would give them a majority, while electing m would bring about a coalition. Therefore, depending on the result in d, any of the three policies a_l, a_m or a_r could be implemented.

These three distinct decisive events are represented in Figure 1. The simplex represents the decision problem of voters in district d, holding fixed the strategies of those in the other $D-1$ districts.[17] Each point corresponds to an expected distribution of $D-1$ seats among the three parties: the bottom left point corresponds to $E(S_{-d}) = (D-1, 0, 0)$; the bottom right, $E(S_{-d}) = (0, 0, D-1)$; the apex is $E(S_{-d}) = (0, D-1, 0)$. For any given point, the number of party m seats can be read directly off the y-axis, while the the number of party r seats can be read by moving down the 45 degree line in a southwesterly direction to the x-axis. The number of party l seats is simply the remainder.

Each district will condition on one of the distinct events $\lambda \in \Lambda$ when voting, and this must be consistent with the equilibrium properties given in the previous section. All d_l have $E(S_{-d_l}) = E(s_l - 1, s_m, s_r)$, all d_m have $E(S_{-d_m}) = E(s_l, s_m - 1, s_r)$, and all d_r have $E(S_{-d_r}) = E(s_l, s_m, s_r - 1)$. In Figure 1 this corresponds to the various $E(S_{-d})$ forming an inverted triangle. An example is shown for the case of $E(S) = (16, 2, 7)$.

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[17]While the simplex represents the case of $D = 25$, the same would hold for any odd D. To avoid the case where two parties could share the seats equally, I ignore the case where D is even.
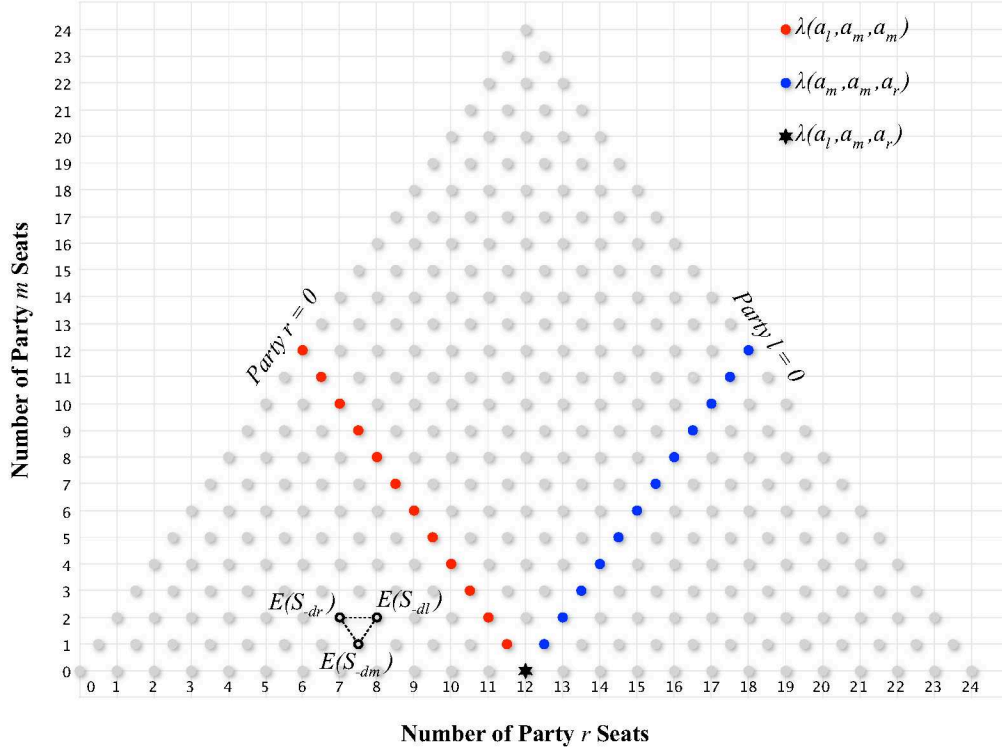
Figure 1: Cases in which d is decisive when $D = 25$

4.1 Polarisation

Much attention in the U.S. has focused on how a system with two polarised parties has led to policies which are far away from the median voter's preferred point.[18] The same problem may also arise if there are three parties but one of them is not considered a serious challenger. An open question is the degree to which policy outcomes reflect the preferences of voters in a legislative election with three parties. Let $\tilde{t}_d$ be the expected position of the median voter in district d, and label the districts so that $\tilde{t}_1 < \ldots < \tilde{t}_{\frac{D+1}{2}} < \ldots < \tilde{t}_D$. Then, $\tilde{t}_{\frac{D+1}{2}}$ is the expected median voter in the median district. The following proposition shows that, while polarisation of outcomes can always occur in stand-alone plurality elections, an extreme policy can only be implemented in my benchmark model if it is preferred by the median voter in the median district.

Proposition 2. *For any distribution of voter preferences, where bargaining occurs over policy and $\delta = 1$, if $E(s_m) > 0$ then the expected outcome can be a_l only if the median voter in the median district prefers policy a_l to policy a_m; that is, if $\tilde{t}_{\frac{D+1}{2}} < a_{lm}$. Similarly, a necessary condition for $E(z) = a_r$ is $\tilde{t}_{\frac{D+1}{2}} > a_{mr}$.*

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[18]See McCarty, Poole, and Rosenthal (2008).

Proof. For the expected policy to be a_l it must be that $E(S_{-d})$ is in the bottom left triangle of Figure 1 for all districts i.e. party l is expected to win a majority. Any d_l district must be conditioning on either a $\lambda(a_l, a_m, a_m)$ event or a $\lambda(a_l, a_m, a_r)$ event. However, given $E(s_m) > 0$, the probability of a $\lambda(a_l, a_m, a_m)$ event is strictly greater than the probability of a $\lambda(a_l, a_m, a_r)$ event, making the former infinitely more likely. This can be seen in Figure 1 by the fact that the upsets needed to get from $E(S_{-d_l})$ to the closest $\lambda(a_l, a_m, a_m)$ event are a subset of the upsets needed to get to the closest $\lambda(a_l, a_m, a_r)$ event. As each of these upsets are independent, a $\lambda(a_l, a_m, a_m)$ event is infinitely more likely by Lemma 1. Therefore, these d_l districts must be conditioning on $\lambda(a_l, a_m, a_m)$ and as l is the expected winner in these districts, they must have $\tilde{t} < a_{lm}$. It follows that for a_l to be the expected policy outcome, the (expected) median of median voters must be $\tilde{t}_{\frac{D+1}{2}} < a_{lm}$. The proof for $E(z) = a_r$ is analogous. $\square$

The focus on expected policies in the proposition is because of the random nature of the model. It is always possible, though immensely unlikely, for the realised population of voters to differ from the expected population sufficiently that $E(z) \neq z$. However, as $n \to \infty$ this probability goes to zero. The proposition stands in stark contrast to single district plurality elections. In a stand-alone plurality election it can always be that l and r are the focal candidates, so either $z = a_l$ or $z = a_r$ will be implemented. A median voter will elect l as long as he prefers a_l to a_r. In a legislative election it take much more for non-moderate policies to come about. Party l will only win a majority of seats if a_l is preferred to a_m by a majority of voters in a majority of districts.

This result gives a novel insight into multiparty legislative elections under plurality. In the U.K., until recently, a vote for the Liberal Democrats (Lib Dems) has typically been considered a "wasted vote".[19] The popular belief was that the Lib Dems were not a legitimate contender for government and so, even if they took a number of seats in parliament, they would not influence policy. As a result, centrist voters instead voted for either the Conservatives or Labour. My model shows that electing a Lib Dem candidate is far from a waste. Electing just one member of the moderate party to the legislature will be enough to bring about that party's preferred policy unless voter preferences favour one of the non-centrist parties very much. Indeed, the result suggests that concerns about the average voter not being adequately represented in the U.K. or Canada are misplaced. If the Conservatives win a majority in parliament it must be that a majority of voters in a majority of districts prefer their policy to that of the centrist Lib Dems/Liberals. On the other hand, a coalition implementing $z = a_m$ can come about for any distribution of voter preferences.[20]

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[19]See "What Future for the Liberal Democrats" by Lord Ashcroft, 2010.

[20]For any **f**, there are always equilibria where $z = a_m$ is the expected outcome. If support is strong

Supporters of the Lib Dems in the U.K. and the Liberals in Canada are therefore hugely advantaged by the current electoral system; it is the supporters of the non-centrist parties who are disadvantaged.

4.2 Misaligned Voting

All voters are strategic: a voter chooses his ballot to maximise his expected utility; which ballot this is depends on how the others vote. In any given situation, an individual may cast the same ballot he would if his vote unilaterally decided the district, or the strategies played by the others in the district may mean his best response is to vote for a less desirable option. Following Kawai and Watanabe (2013), I call the latter misaligned voting.

Definition. *A voter casts a **misaligned vote** if, conditioning on the strategies of voters in other districts, he would prefer a different candidate to win his district than the one he votes for.*

If a voter casts a misaligned vote, he is essentially giving up on his preferred candidate due to the electoral mechanism. In a single plurality election there is only one district - so there is no conditioning on other districts. With candidates l, m and r, whichever candidate is least likely to be pivotal will be abandoned by his supporters, leading to a two-party race. Therefore, either all types with $t < a_{lm}$, all types with $t > a_{mr}$, or those in the interval in between will cast a misaligned vote. In contrast, Proposition 3 below shows that there are many equilibria of the legislative election in which there is no misaligned voting.

Proposition 3. *For any distribution of voter preferences, with bargaining over policy and $\delta = 1$, there always exist equilibria with no misaligned voting in any district. These occur when l or r are serious candidates in fewer than $\frac{D-1}{2}$ districts.*

Proof. By Proposition 1, only two candidates will receive votes in each district. With D districts there will be $2D$ serious candidates. If party r's candidates are serious in less than $\frac{D+1}{2}$ districts, party r can never win a majority of seats. If party r's candidates are serious in less than $\frac{D-1}{2}$ districts, the decisive event in which an extra seat for party r gives them a majority can never come about. Therefore, in any equilibrium where party r is serious in less than $\frac{D-1}{2}$ districts, the only decisive event voters can condition on is $\lambda(a_l, a_m, a_m)$. As this is the only decisive event, in each district, voters with $t < a_{lm}$ will vote v_l while those with $t > a_{lm}$ will vote for whichever of m or r is expected to receive votes. As long as less

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for party r then an equilibrium in which districts focus on a $\lambda(a_l, a_m, a_m)$ decisive event will give $z = a_m$. Likewise, if l is popular then a focus on $\lambda(a_m, a_m, a_r)$ will give $z = a_m$.

than $\frac{D-1}{2}$ of these districts coordinate on r, there will be no misaligned voting. An analogous result holds when party l is serious in less than $\frac{D-1}{2}$ districts. $\qquad\square$

The crux of the proposition is that when one of the non-centrist parties is a serious candidate in less than half the districts, only one distinct decisive event exists. This event is a choice over two policies; with only two policies on the table there is no strategic choice to make - voters simply vote for their preferred option of the two. So, there can be no misaligned voting. A voter with $t > a_{lm}$ facing a $\lambda(a_l, a_m, a_m)$ decisive event is indifferent between electing m or r; he will vote for whichever of the two the other voters are coordinating on.

The proposition gives us a clear prediction on when there will and will not be misaligned voting with three parties competing in a legislative election. It shows that the conventional wisdom - no misaligned with two candidates, always misaligned with three - is wide of the mark; whether there is misaligned voting or not depends on the strength of the non-centrist parties. The proposition also has implications for the study of third-party entry into a two-party system. Suppose, as is plausible, that a newly formed party cannot become focal in many districts - maybe because they have limited resources, or because voters do not yet consider them a serious alternative. Either way, an entering third-party is likely to be weaker than the two established parties. Proposition 3 tells us that if a third party enters on the flanks of the two established parties, then there will be no misaligned voting and no effect on the policy outcome as long as this party is serious in less than half the districts. On the other hand, if a third party enters at a policy point in between the two established parties, this can shake up the political landscape. First of all there will necessarily be misaligned voting, and second of all the policy outcome could be any of a_l, a_m or a_r depending on which equilibrium voters focus on. Success for the new party in just one district can radically change the policy outcome. The implication is that parties in a two-party system should be less concerned about the entry of fringe parties and more concerned about potential centrist parties stealing the middle ground.

5 Legislative Bargaining over Policy and Perks

While the model of bargaining over policy in the previous section is tractable, it lacks one of the key features of the government formation process: parties often bargain over perks of office such as ministerial positions as well as over policy. Here, as parties can trade off losses in the policy dimension for gains in the perks dimension, and vice versa, a larger set of policy outcomes are possible. This section will show that, nonetheless, the results of the

benchmark model extend broadly to the case of bargaining over policy and perks.

The following legislative bargain model with $B > 0$ is due to Austen-Smith and Banks (1988). As usual, if a party wins an overall majority it will implement its preferred policy and keep all of B. Otherwise, the parties enter into a stage of bargaining over government formation. The party winning the most seats of the three begins the process by offering a policy outcome $y^1 \in \mathbb{X}$ and a distribution of a fixed amount of transferable private benefits across the parties, $b^1 = (b_l^1, b_m^1, b_r^1) \in [0, B]^3$. It is assumed that B is large enough so that any possible governments can form, i.e. l can offer enough benefits to party r so as to overcome their ideological differences. If the first proposal is rejected, the party with the second largest number of seats gets to propose (y^2, b^2). If this is rejected, the smallest party proposes (y^3, b^3). If no agreement has been reached after the third period, a caretaker government implements (y^0, b^0), which gives zero utility to all parties. At its turn to make a proposal, party c solves

$$\max_{b_{c'}, y} B - b_{c'} - (y - a_c)^2 \qquad (6)$$
$$\text{subject to} \quad b_{c'} - (y - a_{c'})^2 \geq W_{c'}$$

where $W_{c'}$ is the continuation value of party c' and $W_{c''} + (y - a_{c''})^2 > W_{c'} + (y - a_{c'})^2$, so that the formateur makes the offer to whichever party is cheaper. Solving the game by backward induction, Austen-Smith and Banks (1988) show that a coalition government will always be made up of the largest party and the smallest party. They solve for the equilibrium policy outcome, for any possible distance between a_l, a_m and a_r.

Table 1 shows the policy outcome for each seat distribution and distance between parties, where $\Delta_l \equiv a_m - a_l$ and $\Delta_r \equiv a_r - a_m$. I assume if two parties have exactly the same number of seats, a coin is tossed before the bargaining game begins to decide the order of play. So, if $s_l = s_r > s_m$, then with probability one-half, the game will play out as when $s_l > s_r > s_m$ and otherwise as $s_r > s_l > s_m$.

The set of possible policy outcomes depends on the number of seats in the legislature, and on the distance between party policies. The simplex in Figure 2 shows what the policy will be, for any seat distribution, when there are 25 districts and $\Delta_l < \Delta_r \leq 2\Delta_l$. Notice that there are far more policy possibilities than in the case of $B = 0$. Figure 3 shows the various decisive cases from the perspective of a single district; it is the analogue of Figure 1. While there are many more decisive cases than when $B = 0$, they can be grouped into the three categories defined previously: $\lambda(2), \lambda(2')$ and $\lambda(3)$ events.

The following proposition shows that, even when parties can bargain over perks as well as policy, a non-centrist party will only win a majority if the median voter in the median
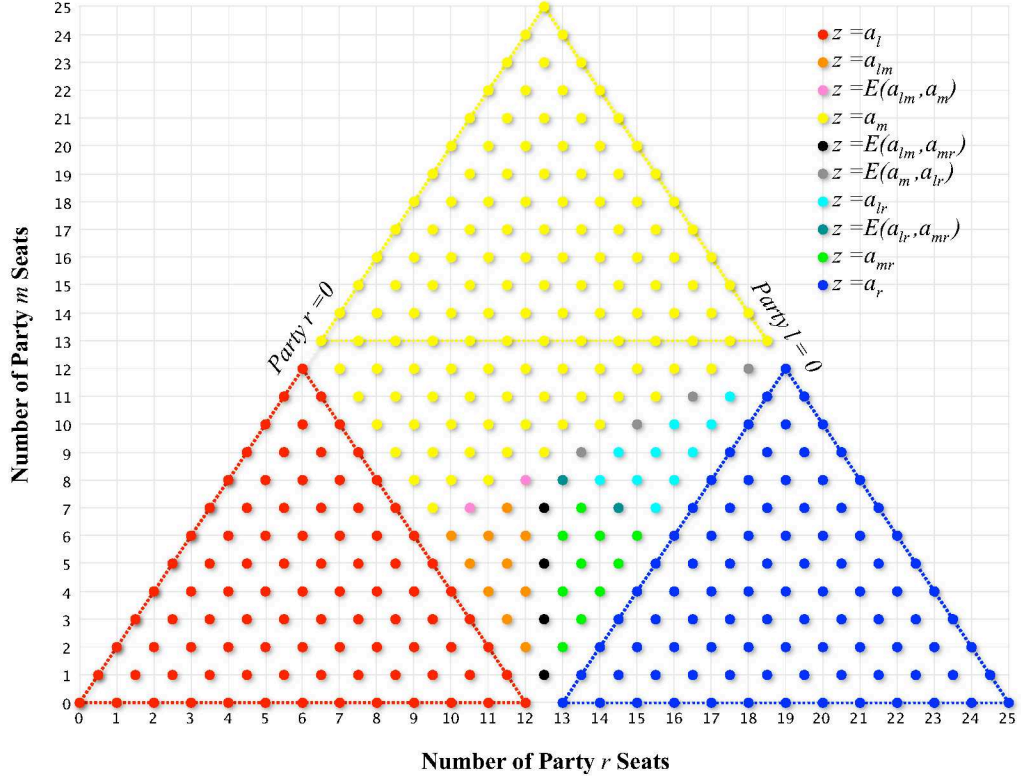
18

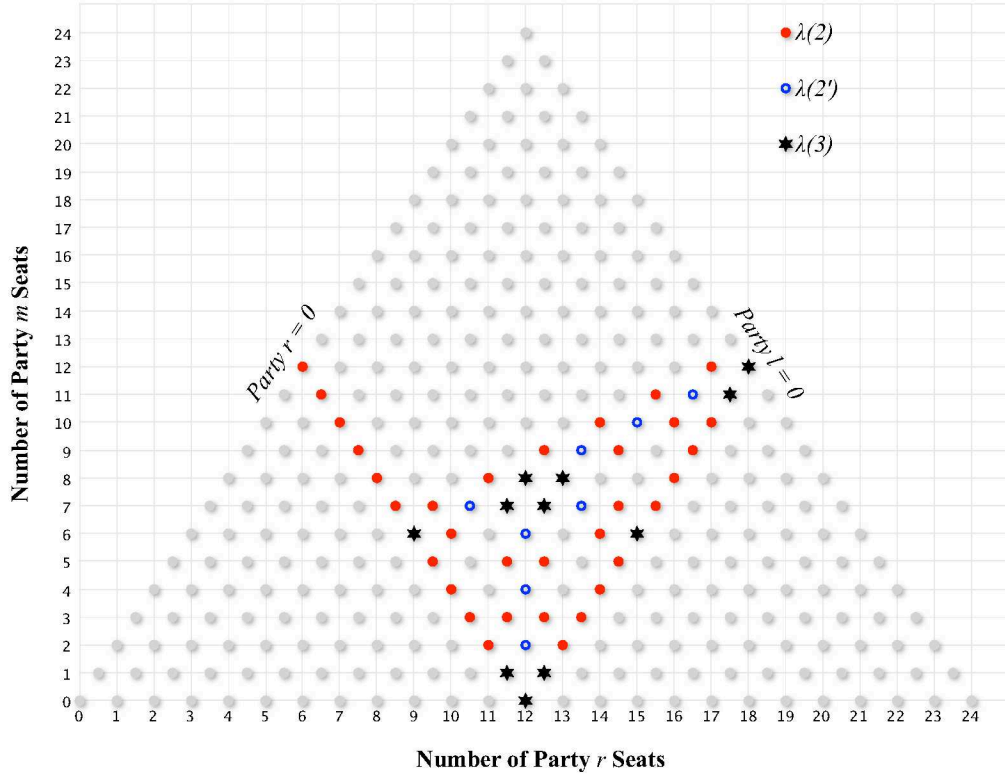Figure 2: Policy outcomes under ASB bargaining, with $D = 25$ and $\Delta_l < \Delta_r \leq 2\Delta_l$

Figure 3: Decisive events under ASB bargaining, with $D = 25$ and $\Delta_l < \Delta_r \leq 2\Delta_l$

Seat Share	$3\Delta_r < \Delta_l$	$2\Delta_r < \Delta_l \leq 3\Delta_r$	$\Delta_r < \Delta_l \leq 2\Delta_r$	$\Delta_l = \Delta_r$	$\Delta_l < \Delta_r \leq 2\Delta_l$	$2\Delta_l < \Delta_r \leq 3\Delta_l$	$3\Delta_l < \Delta_r$
$s_l > (D-1)/2$	a_l	a_l	a_l	a_l	a_l	a_l	a_l
$(D+1)/2 > s_l > s_r > s_m$	a_{lm}	a_{lm}	a_{lm}	a_{lm}	a_{lm}	$2a_m - a_{lr}$	a_l
$(D+1)/2 > s_l > s_m > s_r$	a_{lr}	a_{lr}	a_{lr}	a_m	a_m	a_m	a_m
$s_m > s_l, s_r$	a_m	a_m	a_m	a_m	a_m	a_m	a_m
$(D+1)/2 > s_r > s_m > s_l$	a_m	a_m	a_m	a_m	a_{lr}	a_{lr}	a_{lr}
$(D+1)/2 > s_r > s_l > s_m$	a_r	$2a_m - a_{lr}$	a_{mr}	a_{mr}	a_{mr}	a_{mr}	a_{mr}
$s_r > (D-1)/2$	a_r	a_r	a_r	a_r	a_r	a_r	a_r

Table 1: Policy outcomes in Austen-Smith and Banks (1988) for any seat distribution and distance between parties.

district prefers its policy to that of the centrist party.

Proposition 4. *For any distribution of voter preferences, with a fixed order of bargaining over policy and perks and $E(s_m) > 1$, the expected outcome can be a_l only if $\tilde{t}_{\frac{D+1}{2}} < a_{lm}$. Similarly, a necessary condition for $E(z) = a_r$ is $\tilde{t}_{\frac{D+1}{2}} > a_{mr}$.*

Proof. Suppose we have $\Delta_l < \Delta_r \leq 2\Delta_l$, as in Figure 3. If the expected policy outcome is a_l and the moderate party is expected to win more than one seat, we must be in the bottom left part of the figure and each d_l district must be conditioning on one of the following events: $\lambda(a_l, a_m, a_m)$, $\lambda(a_l, a_{lm}, a_{lm})$, or $\lambda(a_l, a_m, a_{lm})$. If a district is conditioning on a race between policies a_l and a_m, then a_l will win in expectation if $\tilde{t} < a_{lm}$. If a district is conditioning on a race between policies a_l and a_{lm}, then a_l will win in expectation only if $\tilde{t} < \frac{a_l + a_{lm}}{2}$, a stricter condition. Therefore the minimum requirement for a district to elect l is $\tilde{t} < a_{lm}$. For l to be the expected winner in a majority of seats, it must be that this condition is met in at least $\frac{D+1}{2}$ districts.

Next, I show that if the expected policy is a_r, we must have $\tilde{t}_{\frac{D+1}{2}} > a_{mr}$. For $E(z) = a_r$ we must be in the bottom right section of Figure 3 and, given $E(s_m) > 1$, each d_r must be conditioning on one of the following events: $\lambda(a_{mr}, a_{mr}, a_r), \lambda(a_{mr}, a_{lr}, a_r), \lambda(a_{lr}, a_{lr}, a_r), \lambda(a_{lr}, E(a_m, a_{lr}), a_r)$, or $\lambda(E(a_m, a_{lr}), a_m, a_r)$. If a district is conditioning on a race between policies a_r and a_m, then a_r will win in expectation only if $\tilde{t} > a_{mr}$. All of the other possible races in the set of decisive events listed involve stricter conditions on how large $\tilde{t}$ must be in order to elect party r. The minimum requirement for a district to elect r is thus $\tilde{t} > a_{mr}$. For r to be the expected winner in a majority of seats, it must be that this condition is met in at least $\frac{D+1}{2}$ districts. From Table 1, we can see that these bounds of a_{lm} and a_{mr} will apply no matter what the distance between the three party platforms. □

This reaffirms the result of Proposition 2, that moderate coalitions will be the norm in legislative elections unless the population is heavily biased in favour of one of the non-centrist parties. Moreover, bargaining over perks as well as policy can lead to even less

extreme policies than the benchmark case. This can be seen from Figure 2: starting from a point $E(S)$ where $E(s_l) > \frac{D-1}{2}$, $E(s_m) > 1$ and $\frac{D-1}{4} < E(s_r) < \frac{D-1}{2}$, the most likely decisive event for each district must be $\lambda(a_l, a_{lm}, a_{lm})$. Therefore, such a party l majority could only come about if $\tilde{t}_{\frac{D+1}{2}} < \frac{a_l + a_{lm}}{2}$ - an even stricter requirement than that of the benchmark case. This result is noteworthy as in U.K. and Canadian elections party seat shares tend to be in line with this case: one of the non-centrist parties wins a majority, the other wins more than a quarter of the seats, while the centrist party wins much less than a quarter.

On the other dimension of interest bargaining over policy and perks does not perform as well; the restrictions required to completely rule out misaligned voting are more severe than in the benchmark model. However, as Proposition 6 will show, there are many equilibria in which a large subset of districts have no misaligned voting.

Proposition 5. *In a legislative election with a fixed order of bargaining over policy and perks of office, there always exist equilibria with no misaligned voting in any district.*

1. *When a_l and a_r are equidistant from the moderate policy, $\Delta_l = \Delta_r$, there is no misaligned voting if either party l or party r receive votes in fewer than $\frac{D-1}{4}$ districts.*

2. *When a_l is closer than a_r to the moderate policy, $\Delta_l < \Delta_r$, there is no misaligned voting if party r receive votes in fewer than $\frac{D-1}{4}$ districts.*

3. *When a_l is further than a_r to the moderate policy, $\Delta_l > \Delta_r$, there is no misaligned voting if party l receive votes in fewer than $\frac{D-1}{4}$ districts.*

Proof. See Appendix A. $\square$

The intuition is the same as in Proposition 3: when a non-centrist party is not a serious candidate in enough districts, there is no hope of it influencing the order of recognition in the legislative bargaining stage. The threshold for relevance is lower than in the benchmark case because under this bargaining protocol the order of parties matters for the policy outcome. From Figure 3 we see that once it is possible for party r to win $\frac{D-1}{4}$ districts, two distinct decisive events exist: $\lambda(a_l, a_m, a_m)$ and $\lambda(a_l, a_m, a_{lm})$. No matter which of these two events a district focuses on, and which two candidates are serious, some voters in the district will always be casting misaligned votes.

When party l or r have serious candidates in more than $\frac{D-1}{4}$ districts we cannot rule out misaligned voting. However, there are equilibria in which there is no misaligned voting in a subset of districts. The following proposition holds for all bargaining rules.

Proposition 6. *There will be no misaligned voting in a district if either*

1. *The most likely decisive event λ_d^1 is a $\lambda(2')$ event where candidates c and c' are serious and $\underline{z^1(c'')}$ is preferred by no voter.*

2. *The most likely decisive event λ_d^1 is a $\lambda(2)$ event where candidates c and c' are serious, $z^1(c) = z^1(c'')$, $\underline{and}$ all those voting v_c must have $u_t(z^i(c)) > u_t(z^i(c''))$ in the next most likely decisive event $\lambda^i \in \Lambda$ such that $z^i(c) \neq z^i(c'')$.*

Proof. See Appendix A. □

The proposition is best understood by way of example. Take a $\lambda(2')$ event, for example, $S_{-d} = (\frac{D-3}{2}, 2, \frac{D-3}{2})$. Electing l will give $s_l > s_r > s_m$ resulting in $z = a_{lm}$, electing r instead will give $s_r > s_l > s_m$ and bring about $z = a_{mr}$, while electing m will lead to a tie for first place between l and r. A coin toss will decide which of the two policies comes about, but *ex ante* voters' expectation is $E(a_{lm}, a_{mr})$. As voters have concave utility functions, every voter strictly prefers either a_{lm} or a_{mr} to the lottery over the pair. If this decisive event is the most likely (i.e. infinitely more likely than all others) and the district focuses on a race between l and r, nobody in the district is casting a misaligned vote.

To see the second part of the proposition, suppose the most likely decisive event is $S_{-d} = (\frac{D-3}{2}, 3, \frac{D-5}{2})$. Here, electing l or m gives a_{lm} while electing r brings about a coin toss and an expected policy $E(a_{lm}, a_{mr})$. Suppose further that the second most likely decisive event is $S_{-d} = (\frac{D-5}{2}, 3, \frac{D-3}{2})$, where electing m or r gives policy a_{mr} while electing l gives $E(a_{lm}, a_{mr})$. In the most likely event, all voters below a certain threshold will be indifferent between electing l and electing m. However, in the second most likely decisive event, all of these voters would prefer to elect l than m. Given that each decisive event is infinitely more likely to occur than a less likely decisive event, these voters need only consider the top two decisive events. Any voter who is indifferent between l and m in the most likely decisive event strictly prefers l in the second most likely. So, if the district focuses on a race between l and r there will be no misaligned voting.

Proposition 6 is quite useful, as it holds for any bargaining rule. It will allow me to say that in the next section, even though we cannot get results such as Proposition 3 and Proposition 5, we do not return to the single plurality election case of "always misaligned voting". Instead, there are again many equilibria in which a subset of districts have no misaligned voting.

6 Impatient Parties

In this section, I examine how the results of the benchmark model change when $\delta < 1$, so that parties are no longer perfectly patient. It is likely that the discount rates of politicians vary across countries depending on things such as constitutional constraints of bargaining, the status quo, and the propensity of politicians to be reelected.[21] In the benchmark model it didn't matter whether the bargaining protocol was random or had a fixed order; a coalition would always implement $z = a_m$. However, once parties discount the future, we get vasty different results depending on which bargaining protocol is used. The scope for policy polarisation and misaligned voting not only depends on how the formateur is selected but also on the location of the status quo policy, Q. I assume the status quo is neither too extreme, $Q \in (a_l, a_r)$, nor too central $Q \neq a_m$.[22] In each period where no agreement is reached, the status quo policy remains and enters party's payoff functions. All parties discount the future at the same rate of $\delta \in (0, 1)$. Therefore, if a proposal y is passed in period t, the payoff of party c is

$$W_c = -(1 - \delta^{t-1})(Q - a_c)^2 - \delta^{t-1}(y - a_c)^2 \tag{7}$$

For ease of analysis I assume, without loss of generality, that $a_m = 0$.[23] Banks and Duggan (2000) show that all stationary equilibria are no-delay equilibria and are in pure strategies when the policy space is one-dimensional and $\delta < 1$.

6.1 Fixed Order Bargaining

The order of recognition is fixed and follows the ranking of parties' seat shares. In Appendix A, I derive the policy outcomes for any ordering of parties; these are presented in Table 2 below. From the table we see that the further party m moves down the ranking of seat shares, the further the policy moves away from a_m. Figure 4 shows the various policy outcomes for any seat distribution in the legislature. Figure 5 shows the frequency of the three categories of decisive events.

The proposition below shows that when the bargaining protocol is fixed, parties discount the future, and the status quo is not exactly a_m, it is even more difficult for a non-centrist party to win a majority of seats and implement its preferred policy than is the case in the benchmark model.

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[21] After the 2010 Belgian elections, legislative bargaining lasted for a record-breaking 541 days, suggesting high values of δ. Conversely, after the 2010 U.K. elections, a coalition government was formed within five days.

[22] If $Q = a_m$ the result is the same as the benchmark case of $\delta = 1$.

[23] Taking any original positions $(a_l, a_m \neq 0, a_r)$, we can always alter $\mathbf{f}$ so that the preferences of all voter types are the same when $(a'_l, a'_m = 0, a'_r)$.
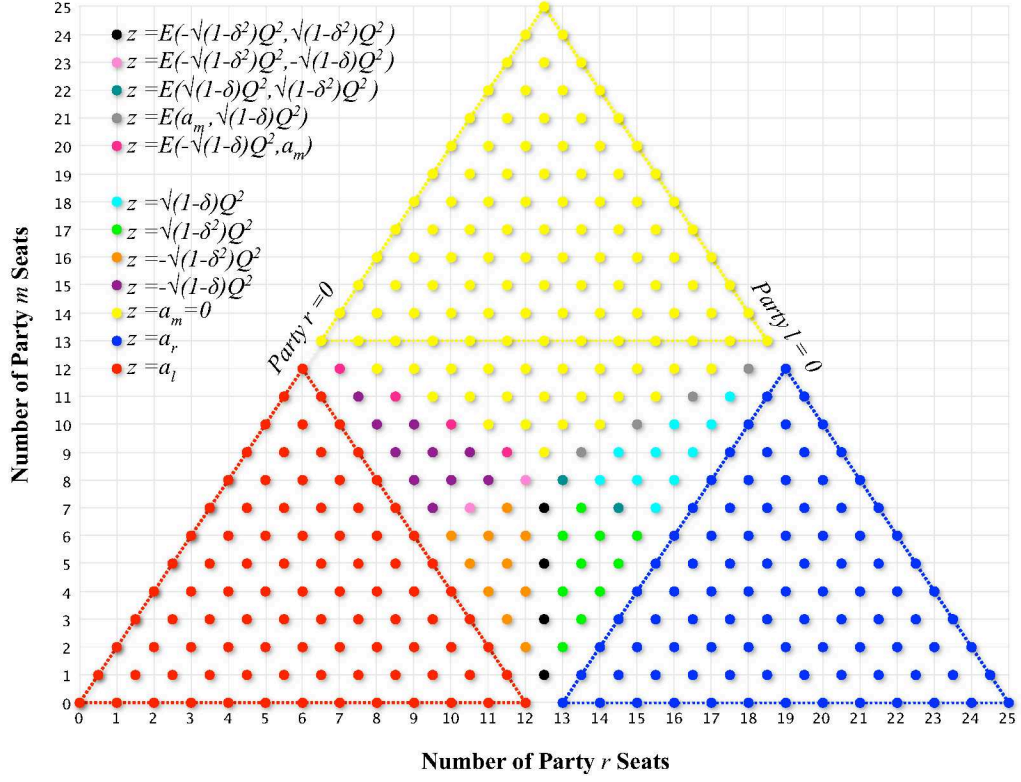
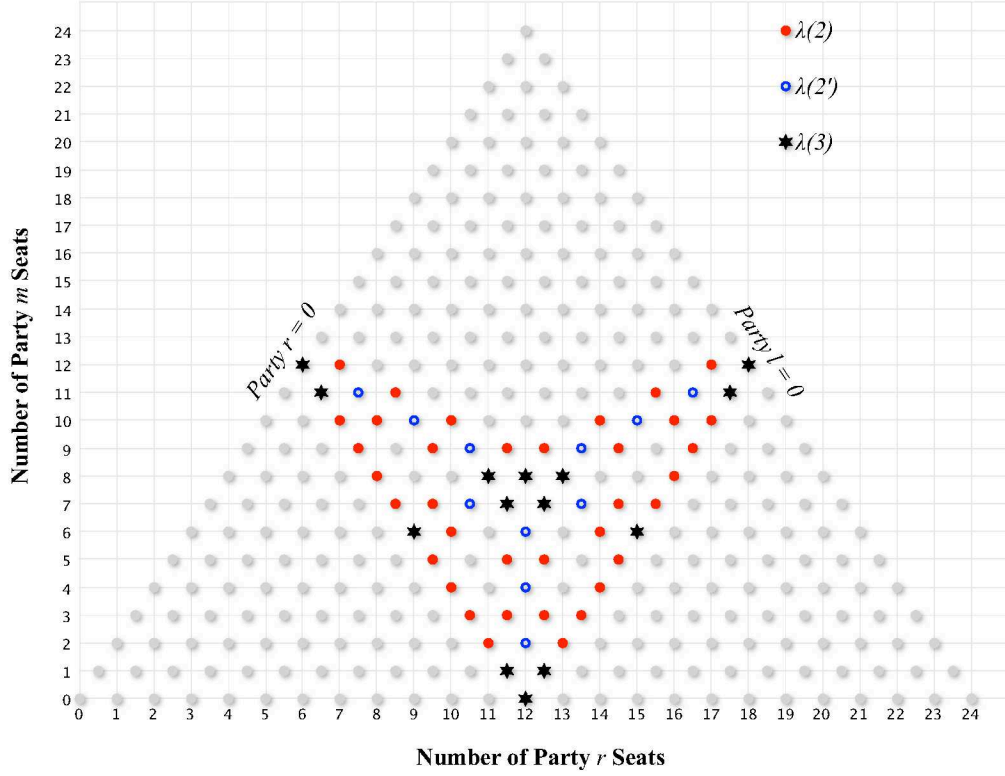Figure 4: Policy outcomes under fixed order bargaining, with $D = 25$ and $\delta < 1$

Figure 5: Decisive events under fixed order bargaining, with $D = 25$ and $\delta < 1$

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Seat Share	Policy
$s_l > (D-1)/2$	a_l
$(D+1)/2 > s_l > s_r > s_m$	$-\sqrt{(1-\delta^2)Q^2}$
$(D+1)/2 > s_l > s_m > s_r$	$-\sqrt{(1-\delta)Q^2}$
$s_m > s_l, s_r$	$a_m = 0$
$(D+1)/2 > s_r > s_m > s_l$	$\sqrt{(1-\delta)Q^2}$
$(D+1)/2 > s_r > s_l > s_m$	$\sqrt{(1-\delta^2)Q^2}$
$s_r > (D-1)/2$	a_r

Table 2: Policy outcomes with fixed order bargaining over policy and $\delta < 1$.

Proposition 7. *For any distribution of voter preferences, with a fixed order of bargaining over policy, $\delta < 1$ and $E(s_m), E(s_r) > 1$; the expected outcome can be a_l only if $\tilde{t}_{\frac{D+1}{2}} < \frac{a_l - \sqrt{(1-\delta)Q^2}}{2} < a_{lm}$. Similarly, when $E(s_l), E(s_m) > 1$; then a necessary condition for $E(z) = a_r$ is $\tilde{t}_{\frac{D+1}{2}} > \frac{a_r + \sqrt{(1-\delta)Q^2}}{2} > a_{mr}$.*

Proof. See Appendix A. ∎

As the difference in policy between, say, an l majority government and a coalition led by party l is quite small, the majority government will only come about if the electorate is sufficiently biased in favour of policy a_l - even more so than in the benchmark case.[24] The reason is that in the benchmark case every coalition implements $z = a_m$, while with discounting and a fixed order protocol, the largest party has a significant advantage in coalition negotiations and so can use this to get an alternative policy passed. Voters anticipate the power that the leading party l will have in coalition formation and so will only vote to bring about a party l majority if they prefer it to the l led coalition. What the proposition also shows is that the further the status quo policy, Q, is from the moderate party policy, a_m, the more likely we are to have coalition governments, all else equal; a more distant status quo gives the formateur even more bargaining power over the moderate party.

Figure 5 shows the frequency of the three types of decisive event for this bargaining rule. While it is quite similar to Figure 3, the difference is that now there is no condition we can impose so as to ensure there is no misaligned voting. The corner decisive events of Figure 5 are $\lambda(3)$ events, so at least one of them can always be conditioned on. If a district is conditioning on a $\lambda(3)$ event there must be misaligned voting in that district. On the other hand, Proposition 6 also holds here - so there are equilibria with misaligned voting in only

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[24]It is worth mentioning that without the restriction to $E(s_r) > 1$ in the proposition, the threshold becomes $\tilde{t}_{\frac{D+1}{2}} < a_{lm}$ as in the benchmark case. This is because some districts may then condition on $(\frac{D-1}{2}, \frac{D-1}{2}, 0)$, and have l and m as serious candidates. In such a case l will win the district only if $\tilde{t} < a_{lm}$.

a subset of districts. The following proposition summaries the state of misaligned voting under this bargaining rule.

Proposition 8. *In a legislative election with a fixed order of bargaining over policy and $\delta < 1$, there always exist equilibria with misaligned voting. However, equilibria exist with no misaligned voting in a subset of districts.*

6.2 Random Recognition Bargaining

In each period one party is randomly selected as formateur, where the probability of each party being chosen is equal to its seat share in the legislature, $\frac{s_c}{D}$. Party payoffs are again given by Equation 7. As usual if a party has a majority of seats it will implement its preferred policy. Following Banks and Duggan (2006), when no party has a majority I look for an equilibrium of the form $y_l = a_m - \Omega$, $y_m = a_m$, $y_r = a_m + \Omega$. Cho and Duggan (2003) show that this stationary equilibrium is unique. As bargaining is only over policy, any minimum winning coalition will include party m. When there was no discounting this meant party m could always achieve $z = a_m$. Now, however, the presence of discounting allows parties l and r to offer policies further away from a_m, which party m will nonetheless support. The moderate party will be indifferent between accepting and rejecting an offer y when

$$W_m(y) = -(\Omega)^2 = -(1-\delta)(Q)^2 - \frac{\delta(D - s_m)}{D}(\Omega)^2 \tag{8}$$

which, when rearranged gives

$$\Omega = \pm\sqrt{\frac{(1-\delta)Q^2}{1 - \delta\frac{D - s_m}{D}}} \tag{9}$$

Table 3 shows the equilibrium offer each party will make when chosen as formateur. Notice that the policies offered by l and r depend on the seat share of party m. The more seats party m has, the closer these offers get to zero.

Formateur	Policy
y_l	$-\sqrt{\frac{1-\delta}{1 - \delta\frac{D - s_m}{D}}Q^2}$
y_m	$a_m = 0$
y_r	$\sqrt{\frac{1-\delta}{1 - \delta\frac{D - s_m}{D}}Q^2}$

Table 3: Policy proposals with random order bargaining over policy, $\delta < 1$.

For a seat distribution such that no party has a majority, the expected policy outcome

from bargaining is

$$E(z) = -\frac{s_l}{D}\left(\sqrt{\frac{1-\delta}{1-\delta\frac{D-s_m}{D}}Q^2}\right) + \frac{s_m}{D}(0) + \frac{s_r}{D}\left(\sqrt{\frac{1-\delta}{1-\delta\frac{D-s_m}{D}}Q^2}\right) \qquad (10)$$

An extra seat for any of the three parties will increase their respective probabilities of being the formateur and so affect the expected policy outcome. Thus, every district always faces a choice between three distinct (expected) policies. We also see that as s_m increases, the expected policy moves closer and closer to zero. This occurs for two reasons; firstly because there is a higher probability of party m being the formateur, and secondly because s_m enters the policy offers of l and r; as s_m increases the absolute value of these policies shrink.

The proposition below shows that when the bargaining protocol is random, parties discount the future, and the status quo is not exactly a_m, it is easier for a non-centrist party to win a majority of seats and implement its preferred policy than is the case in the benchmark model.

Proposition 9. *For any distribution of voter preferences, with a random order of bargaining over policy, $\delta < 1$ and $E(s_m) > 0$; the expected outcome can be a_l only if $\tilde{t}_{\frac{D+1}{2}} < z_l^*$, where $z_l^* > a_{lm}$. Similarly, a neccesary condition for $E(z) = a_r$ is $\tilde{t}_{\frac{D+1}{2}} > z_r^*$, where $z_r^* < a_{mr}$.*

Proof. See Appendix A. □

The proposition implies that we should witness more majority governments than coalition governments when the bargaining protocol is random. The reason is that with a random recognition rule voters face vast uncertainty if they choose to elect a coalition. The implemented policy will vary greatly depending on which party is randomly chosen as formateur. As voters are risk averse, they find the certainty of policy provided by a majority government appealing. The median voter in the median district need not prefer the policy of a non-centrist party to that of party m in order for the former to win a majority of seats. Along with the previous propositions on polarisation, this proposition shows that no matter which of the bargaining rules is used, there is less scope for polarisation in legislative elections using plurality rule than there is in stand-alone plurality elections such as mayoral or presidential elections.

Figure 6 below shows the various decisive cases when the random recognition rule is used. Almost all points are $\lambda(3)$ events, and as we know, if a district is conditioning on such an event it must have misaligned voting.[25] There are however, a selection of $\lambda(2')$ events when

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[25]The picture changes somewhat depending on the values of Q, D and δ: for certain values, decisive events where s_m is small may be $\lambda(2')$ events or may even be events where all voters would like to elect m. However this does not alter Proposition 10. Figure 6 shows the case of $D = 25$, $\delta = 0.99$ and $|Q| < 0.33$.

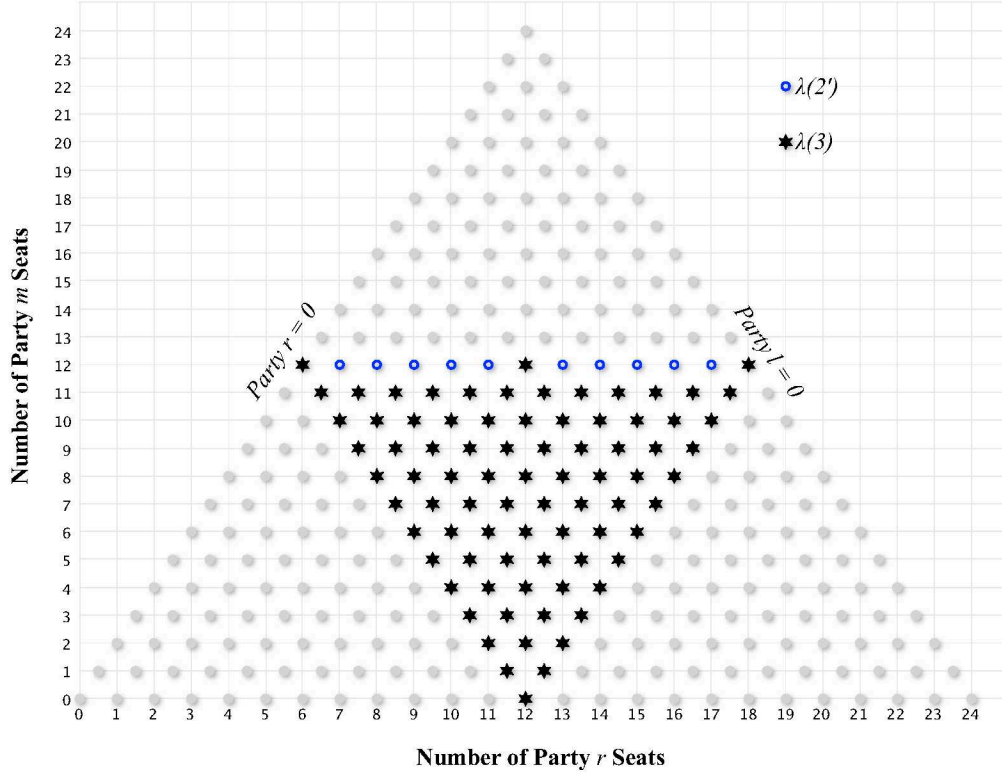Figure 6: Decisive events under random recognition bargaining, with $D = 25$ and $\delta < 1$

an extra seat for party m gives them a majority. If $E(s_m) > \frac{D-1}{2}$, then there are many equilibria in which there is misaligned voting in only a subset of districts.

Proposition 10. *For any distribution of voter preferences, in a legislative election with a random order of bargaining over policy and $\delta < 1$, there are no equilibria without misaligned voting. However, equilibria exist with only misaligned voting in a subset of districts.*

Proof. See Appendix A. □

At any decisive event, voters may be conditioning on, they face a choice between three distinct (expected) policies. In almost all cases this means that there will be misaligned voting in the districts. However, as I show in the proof, and as can be seen from Figure 6, when m is the largest party then the expected policy which comes about by electing the smallest party in the legislature is actually not preferred by any voter type. So, in this case, if a district focuses on the two national frontrunners as the two serious candidates in their district, there will be no misaligned voting. In fact, there may only be misaligned voting in one district. If party m has a majority of seats, party l has one seat, and r has the rest, then as long as all d_m and d_r districts have m and r as serious candidates, only the one d_l district will have misaligned voting. This gives a fresh insight into the idea of "wasted votes": if

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party l or r is expected to be the smallest of the parties in the legislature, and party m is expected to have a majority, then the least popular national party should optimally be abandoned by voters. Any district which actually elects the weakest national party does so due to a coordination failure; a majority there would instead prefer to elect one of the other two parties. Notice however, that for this to be the case, the moderate party must be expected to win an overall majority. So, while the idea of a wasted vote does carry some weight, it clearly does not apply to the case of the Liberal Democrats.

7 Discussion

In this paper, I introduced and analysed a model of three-party competition in legislative elections under plurality rule. I showed that two negative aspects of plurality rule - polarisation and misaligned voting - are significantly reduced when the rule is used to elect a legislature. The degree to which these phenomena are reduced depends on the institutional setup - specifically, on how legislative bargaining occurs.

In the benchmark model, parties are perfectly patient and bargain only over policy. Two clear results emerged from this model. First, while an extreme policy can always come about in standard plurality elections, in my setting a non-centrist policy needs broad support in the electorate in order to be implemented; specifically, the median voter in the median district must prefer the extreme policy to the moderate party's policy. Second, while standard plurality elections with three distinct choices always have misaligned voting, in my benchmark model this is the case only if the non-centrist parties are serious candidates in more than half the districts - otherwise there is no misaligned voting in any district.

The results of the benchmark model largely hold up under the other bargaining rules considered: the non-centrist parties cannot win for any voter preferences (unlike in standard plurality elections), and there are always equilibria in which there is no misaligned voting (at least in a subset of districts). Moreover, if parties are impatient we gain an additional insight: with a fixed order of formateur recognition we should see more coalitions while when the order is random we should see more single-party governments, all else equal.

In the remainder of this section I discuss the robustness of my modelling assumptions. First, if utility functions are concave rather than specifically quadratic, the benchmark model is unchanged. When bargaining also involves perks, the same is true as long as there are enough perks to allow a coalition between the left and right parties to form.[26] Second, if parties bargain by making demands rather than offers, as in Morelli (1999), the results will

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[26]If the perks are not large enough or parties don't value perks enough, a coalition will always involve the moderate party and we return to the simpler bargaining over policy case.

be the same as in the benchmark model.[27] Third, if instead of a Poisson model I assumed a fixed population size drawn from a multinomial distribution, the results of my model would still go through.[28]

A key assumption is that voters only care about the policy implemented in the legislature. If they also have preferences over who wins their local district, the results of the model no longer hold: the probability of being pivotal locally would outweigh any possible utility gain at the national level so that voters will only consider the local dimension. However, in Westminster systems, a Member of Parliament has no power to implement policy at a local level; he merely serves as an agent of his constituents: bringing up local issues in parliament, helping constituents with housing authority claims, etc. So, if voters *do* have preferences over their local winner, it should only be on a common-value, valence dimension. If this were indeed the case, party policies would be irrelevant for how voters cast their ballots.

Finally, the assumption of perfect information is unrealistic in a real world election. The asymptotic elements of the model means voters can perfectly rank the probabilities of certain events. In real life we are never that confident: polls may not be accurate, or more often, polls may not exist at the district level. Myatt (2007) and Fisher and Myatt (2014) have analysed single plurality elections with aggregate uncertainty over voters' intentions. While this paper abstracts from aggregate uncertainty for the sake of comparison with standard models, including greater uncertainty in a multi-district model is an important path for future research.

A   Appendix

A.1   Poisson Properties

The number of voters in a district is a Poisson random variable n_d with mean n. The probability of having exactly k voters is $Pr[n_d = k] = \frac{e^{-n}n^k}{k!}$. Poisson Voting games exhibit some useful properties. By *environmental equivalence*, from the perspective of player in the game, the number of other players is also a Poisson random variable n_d with mean n. By the *decomposition property*, the number of voters of type t is Poisson distributed with mean $nf_d(t)$, and is independent of the number of other players types. For simplicity here I drop the district subscript. The probability of a vote profile $x = (x(l), x(m), x(r))$ given voter

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[27]Whenever there is no clear majority, the head of state selects party m as the first mover, so the coalition policy will be $z = a_m$.

[28]Myerson (2000) shows that the magnitude of an event with a multinomial distribution is a simple transformation of its magnitude with a Poisson distribution. This transformation preserves the ordering of events and so this means that the ordering of sets of pivotal and decisive events in my model would remain unchanged, and therefore so would the equilibria.

strategies is

$$Pr[x|n\tau] = \prod_{c \in \{l,m,r\}} \frac{e^{-n\tau(c)}(n\tau(c))^{x(c)}}{x(c)!} \tag{11}$$

Magnitude Theorem Let an event E be a subset of all possible vote profiles. The magnitude theorem (Myerson (2000)) states that for a large population of size n, the magnitude of an event, $\mu(E)$, is:

$$\mu(E) \equiv \lim_{n \to \infty} \frac{\log(Pr[E])}{n} = \max_{x \in E} \sum_{c \in \{l,m,r\}} \tau_d(c)\psi\left(\frac{x_d(c)}{n\tau_d(c)}\right) \tag{12}$$

where $\psi(\theta) = \theta(1 - \log(\theta)) - 1$. That is, as $n \to \infty$, the magnitude of an event E is simply the magnitude of the most likely vote profile $x \in E$. The magnitude $\mu(E) \in [-1, 0]$ represents the speed at which the probability of the event goes to zero as $n \to \infty$; the more negative its magnitude, the faster that event's probability converges to zero.

Corollary to the Magnitude Theorem If two events E and E' have $\mu(E) < \mu(E')$, then their probability ratio converges to zero as $n \to \infty$.

$$\mu(E) < \mu(E') \Rightarrow \lim_{n \to \infty} \frac{Pr[E]}{Pr[E']} = 0 \tag{13}$$

It is possible that two distinct events have the same magnitude. In this case, we must use the offset theorem to compare their relative probabilities.

Offset Theorem Take two distinct events, $E \neq E'$ with the same magnitude, then

$$\mu(E) = \mu(E') \Rightarrow \lim_{n \to \infty} \frac{Pr[E]}{Pr[E']} = \phi \quad 0 < \phi < \infty \tag{14}$$

Suppose we have $\tau(c_1) > \tau(c_2) > \tau(c_3)$, so that the subscript denotes a party's expected ranking in terms of vote share. Maximising Equation 12 subject to the appropriate constraints we get

$$
\begin{aligned}
\mu(piv(i,j)) &= \mu(piv(j,i)) \quad \forall i,j \in C \\
\mu(c_1\text{-win}) &= 0 \\
\mu(c_2\text{-win}) = \mu(piv(c_1,c_2)) &= 2\sqrt{\tau(c_1)\tau(c_2)} - \tau(c_3) \\
\mu(c_3\text{-win}) = \mu(piv(c_2,c_3)) = \mu(piv(c_1,c_3)) &= 3\sqrt[3]{\tau(c_1)\tau(c_2)\tau(c_3)} - 1 \quad \text{if } \tau(c_1)\tau(c_3) < \tau(c_2)^2 \\
\mu(c_3\text{-win}) = \mu(piv(c_1,c_3)) &= 2\sqrt{\tau(c_1)\tau(c_3)} - \tau(c_2) \quad \text{if } \tau(c_1)\tau(c_3) > \tau(c_2)^2
\end{aligned}
\tag{15}
$$

With a magnitude of zero, by the corollary, the probability of candidate c_1 winning goes to 1 as n gets large. Also, as the magnitude of a pivotal event between c_1 and c_2 is greater than all other pivotal events, a pivotal event between c_1 and c_2 is infinitely more likely than a pivotal event between any other pair as n gets large.

A.2 Proof of Lemma 1

Let $\boldsymbol{\tau}$ be the expected vote shares in each district given the strategies of all player types. A voter of type t in district d, decides how to vote by comparing his expected gain from each ballot, given by Equation 2. In order to exactly compute his expected gain for each ballot, the voter would need to work out the probability of each combination of being both pivotal and decisive, $Pr[piv_d(c, c')]Pr[\lambda_d^i]$ for all $c, c' \in C$ and $i \in I$. The probability of a particular profile of votes across all districts is

$$Pr[x|n\boldsymbol{\tau}] = \prod_{\substack{d \in D \\ c \in \{l,m,r\}}} \left(\frac{e^{-n\tau_d(c)}(n\tau_d(c))^{x_d(c)}}{x_d(c)!} \right) \tag{16}$$

After some manipulation, taking the log of both sides, and taking the limit as $n \to \infty$ we get the magnitude of this profile of votes

$$\mu(x) \equiv \lim_{n \to \infty} \frac{log(Pr[x|n\boldsymbol{\tau}])}{n} = \lim_{n \to \infty} \sum_{c \in \{l,m,r\}} \sum_{d \in D} \tau_d(c)\psi\left(\frac{x_d(c)}{n\tau_d(c)} \right) \tag{17}$$

Notice that the magnitude of a particular profile of votes across districts, is simply the sum of the magnitudes in each district. This is because each district's realised profile of votes is independent of all other districts. So while $\mu(x_d) \in (-1, 0)$ in a single district, when considering the profile of votes in all districts, x, we have $\mu(x) \in (-D, 0)$.

The magnitude theorem (see Appendix A.1) shows that the magnitude of an event occuring in a given district (such as a tie for first) is simply to equal the magnitude of the most likely vote profile in the set of vote profiles comprising that event. The fact that district event probabilities and magnitudes are independent across districts means that the magnitude theorem can be applied across districts. This is evident from the linearity of Equation 17. That is, the magnitude of an cross-district event (such as one party winning half the districts) is simply the magnitude of the most likely set of district level vote profiles which bring that cross-district event about. Specifically,

$$\mu(E) \equiv \lim_{n \to \infty} \frac{log(Pr[E|n\boldsymbol{\tau}])}{n} = \max_{x \in E} \sum_{c \in \{l,m,r\}} \sum_{d \in D} \tau_d(c)\psi\left(\frac{x_d(c)}{n\tau_d(c)} \right) \tag{18}$$

Now, a voter can calculate the magnitude of various jointly pivotal and decisive events $piv_d(c, c')\lambda_d^i$ simply as the magnitude of their most likely vote profile. We know from the corollary to the magnitude theorem (see Appendix A.1) that if that magnitude of one event is greater than another, the former is infinitely more likely to occur as $n \to \infty$. As districts are independent, the same applies when looking at cross-district events - if the magnitude of some particular distribution of seats in the legislature is larger than the magnitude associated with some other seat distribution, then the former seat distribution will be infinitely more likely to occur as $n \to \infty$.

After applying the magnitude theorem and its corollary a voter knows that the event $piv_d(c_1, c_2)\lambda_d^1$ is infinitely more likely than the other pivotal and decisive events. That is, the most likely event in which district d is decisive and the voter is pivotal between the two front runners in his district. Voter t will choose to vote for c_1 if $u_t(z_{c_1}^1) > u_t(z_{c_2}^1)$ and for c_2 if the inequality is reversed. Importantly, if $u_t(z_{c_1}^1) = u_t(z_{c_2}^1)$ then voter t is indifferent between the two candidates in the most likely pivotal and decisive case. She therefore considers the event with the next largest magnitude in which there is a utility differential. $\qquad \square$

A.3 Proof of Proposition 1

I will first show that in all strictly perfect equilibria, the pivotal and decisive events voters condition on are unique i.e. races are duvergerian. First, note that if the decisive event with the largest magnitude λ_d^1 is a $\lambda(3)$ or $\lambda(2')$ event then it is immediate that the most likely pivotal and decisive event is $piv_d(c_1, c_2)\lambda_d^1$. I now show that this event is unique in a strictly perfect equilibrium.

- If in a given district $\tau(c_1) = \tau(c_2) = \tau(c_3) = \frac{1}{3}$, that is all candidates have the same expected vote share, then the most likely pivotal and decisive event would not be unique, and we could support non-duvergerian results. However, these equilibria are knife edge and do not survive when we adjust vote strategies by ϵ.

- Similarly, any equilibrium in which λ_d^1 is a $\lambda(3)$ or $\lambda(2')$ event and two candidates in district d get the same expected vote shares is not robust to trembles and is thus ruled out by strict perfection.

- If $Pr[piv_d(c_1, c_2)\lambda_d^1] = Pr[piv_d(c_1, c_2)\lambda_d^2]$, so that two distinct decisive events are equally likely, then again a tremble on strategies in a district will ensure one decisive event is more likely than the other.

Now let us consider when λ_d^1 is a $\lambda(2)$ event, where voters are indifferent between the top two candidates, c_1 and c_2. Might there be non-duvergerian equilibria here?

- Suppose $Pr[piv_d(c_1, c_3)\lambda_d^1] = Pr[piv_d(c_1, c_2)\lambda_d^2]$ so that some voters are conditioning on the second most likely decisive event, while other voters in the district are conditioning on the most likely decisive event and the imbalance is exactly offset by the fact that the probability of a tie between c_1 and c_2 is higher than that between c_1 and c_3. Here, once again a tremble on players strategies in a district will break this equality, and the race will become duvergerian.

- Suppose $Pr[piv_d(c_2, c_3)\lambda_d^1] = Pr[piv_d(c_1, c_3)\lambda_d^1] > Pr[piv_d(c_1, c_2)\lambda_d^2]$. We see from Equation 15 that this is the only case in which two different pivotal events have the same magnitude. That is when $\tau(c_1)\tau(c_3) < \tau(c_2)^2$ so that $\mu(piv(c_2, c_3)) = \mu(piv(c_1, c_3)) = 3\sqrt[3]{\tau(c_1)\tau(c_2)\tau(c_3)} - 1$. This occurs when the least popular candidate has much less support than the other two. For these two events to be the revenant ones for a voter's decision it must be that λ_d^1 is of type $\lambda(2)$. Trembles will do no good here for two reasons. Firstly, it may not effect the behavior of other districts if district d is a relatively safe seat i.e. other districts are not conditioning on an upset occurring in d. Secondly, and more importantly, the magnitudes of the pivotal events are not affected by such trembles here; any tremble will leave $3\sqrt[3]{\tau(c_1)\tau(c_2)\tau(c_3)} - 1$ unchanged, so we have a candidate for a strictly perfect equilibrium which is non-duvergerian. However, while these magnitudes are the same, we can use the offset theorem to see that their probabilities are not. Specifically $piv_d(c_1, c_3) \neq piv_d(c_2, c_3)$. We know from the offset theorem that

$$\frac{Pr[piv_d(c_1, c_3)]}{Pr[piv_d(c_2, c_3)]} = \phi > 0 \tag{19}$$

The magnitudes of these events are the same because the most likely event in which they occur is when all three candidates get exactly the same number of votes. However, the events consist of more than just this event. In an event $piv_d(c_1, c_3)$, candidate c_2 must have the same or fewer votes than the others. Similarly, in an event $piv_d(c_2, c_3)$, candidate c_1 must have the same or fewer votes than the others. By the decomposition property, for any given number of votes c_3 has, c_1 is always more likely to have more votes than c_2. Therefore, it must be that $\phi > 1$. Returning to the decision of a voter facing

$$\mu(piv_d(c_1, c_3)\lambda_d^1) = \mu(piv_d(c_2, c_3)\lambda_d^1)$$

and given that λ_d^1 is a $\lambda(2)$ event, voters are indifferent between having c_1 or c_2 elected. As I have just shown, $Pr[piv_d(c_1, c_3)] > Pr[piv_d(c_2, c_3)]$, therefore all voters should focus on this event, which will mean the previous second placed candidates loses all her support to the leading candidate, thus ensuring a duvergerian equilibrium.

Therefore, all strictly perfect equilibria involve duvergerian races in every district.

A simple example proves the existence of multiple pure strategy equilibria for any bargaining rule where a majority is needed to implement a policy z. For any $\mathbf{f}$, suppose the right party is never serious in any district. All races will be between l and m. We will have $E(S_{-d_l}) = (k-1, D-k, 0)$ and $E(S_{-d_m}) = (k, D-k-1, 0)$ for some $k \in (0, D)$. As party r receives no votes, every district must be conditioning on the same decisive event $(\frac{D-1}{2}, \frac{D-1}{2}, 0)$. When conditioning on this λ, voters will face a choice between a_l, a_m, and a third policy which would come about if r wins a seat. All the districts may focus on races between l and m, and so it is indeed an equilibrium. Similarly, it is possible that all districts ignore the left party, and so an equilibrium will have every district conditioning on $(0, \frac{D-1}{2}, \frac{D-1}{2})$, or that all districts ignore the moderate party and all condition on $(\frac{D-1}{2}, 0, \frac{D-1}{2})$. These three equilibria always exist for any majoritarian bargaining rule. $\qquad\square$

A.4 Proof of Proposition 5

Case 1: From Table 1 we see that when $l_l = l_r$, the policy outcome will be a_m if no party has a majority and m is not that smallest party. Given that there are D districts and all equilibria are duvergerian, there will be $2D$ serious candidates. Any one party can be serious in at most D districts. Suppose party r is a serious candidate in less than $\frac{D-1}{4}$ districts; then it can win at most that many seats. Meanwhile, l and m must each be serious candidates in more than $\frac{D+1}{4}$ districts. If party m wins less than $\frac{D-1}{4}$ districts and party r comes second, it must be that party l has an overall majority - thus there will be no decisive events. Similarly if party l wins less than $\frac{D-1}{4}$ districts and party r comes second, it must be that party m will have an overall majority; again no decisive events. Conditional on r winning less than $\frac{D-1}{4}$ districts, the only decisive event is $\lambda(a_l, a_m, a_m)$, when l wins $\frac{D-1}{2}$ seats and m is the second largest party. Given that only this distinct decisive event exists, all voters must be conditioning on it. As electing m or r in this decisive event brings about the same policy, voters are indifferent between the two. In each district, those with $t < a_{lm}$ will vote v_l while those with $t > a_{lm}$ will coordinate on either v_m or v_r. As long as the latter group coordinate on v_r in less than $\frac{D-1}{4}$ districts, there is no misaligned voting for any voter in any district. The case of l being a serious candidate in less than $\frac{D-1}{4}$ districts is analogous.

Case 2: From Table 1 we see that when $l_l < l_r$, if $\frac{D+1}{2} > s_l > s_m > s_r$, then the policy outcome will be a_m. As shown above, when r is serious in less than $\frac{D-1}{4}$ districts, the only distinct decisive case is $\lambda(a_l, a_m, a_m)$. All districts will condition on this event and, as before, there is no misaligned voting as long as less than $\frac{D-1}{4}$ districts coordinate on a race between l and r.

Case 3: From Table 1 we see that when $l_r < l_l$, if $\frac{D+1}{2} > s_r > s_m > s_l$, then the policy outcome will be a_m. When l is serious in less than $\frac{D-1}{4}$ districts, the only distinct decisive event is $\lambda(a_m, a_m, a_r)$. All districts will condition on this event and there is no misaligned voting as long as less than $\frac{D-1}{4}$ districts coordinate on a race between l and r. $\qquad\square$

A.5 Proof of Proposition 6

Case 1: Recall that in all $\lambda(2')$ events under this bargaining rule, one of the outcomes is a lottery over the other two and is thus preferred by no voter. The frequency of these events can be seen in Figure 3. If λ_d^1 is a $\lambda(2')$ event, it must be infinitely more likely to occur than any other decisive event, and given that voters are not indifferent between any of the options, whichever option a voter prefers in this case will also be his preferred over all possible decisive events. Every voter prefers one of the two "pure" policies to the lottery over them, so if acting unilaterally would never choose the lottery candidate. Therefore, as long as the lottery candidate is not one of the serious candidates, there is no misaligned voting in that district.

Case 2: Let the most likely decisive event λ_d^1 be a $\lambda(2)$ event where candidates c and c' are serious. If voters are conditioning on λ_d^1 it must be that $z^1(c) = z^1(c'')$ or $z^1(c') = z^1(c'')$; Here, I take it to be the former. Without loss of generality let $z^1(c) < z^1(c')$. Any voter type with $t > \frac{z^1(c)+z^1(c')}{2}$ will vote $v_{c'}$, while any voter type will vote v_c. The former group cannot be casting misaligned votes as they have $u_t(z^1(c')) > u_t(z^1(c))$, and decisive event λ_d^1 is infinitely more likely than all others. Next, we need to consider whether any of the voters choosing v_c might be misaligned. All of these voters have $u_t(z^1(c)) = u_t(z^1(c'')) > u_t(z^1(c'))$, so that they want to beat c' but are indifferent between c and c'' in this most likely decisive event. If one of these voters could unilaterally decide which candidate coordination takes place on, he would decide by looking at the most likely pivotal event in which $z(c) \neq z(c'')$, call this event λ^i. If $u_t(z^i(c)) > u_t(z^i(c''))$ then voter type t would prefer coordination to take place on candidate c, while if $u_t(z^i(c)) < u_t(z^i(c''))$ she'd want coordination on c''. Therefore, if there exists no type such that $u_t(z^i(c)) < u_t(z^i(c''))$ and $u_t(z^1(c)) = u_t(z^1(c'')) > u_t(z^1(c'))$ when c and c' are the serious candidates, then there is no misaligned voting in the district. $\quad\square$

A.6 Bargaining Equilibrium for Fixed Order Protocol and $\delta < 1$

As equilibria are stationary we need only consider two orderings: $l > r > m > l > r > \ldots$ and $r > l > m > r > l > \ldots$. I will derive the equilibrium offers for the case of $l > r > m$, the other is almost identical. I solve the game by backward induction. At stage 3, party m will make an offer y_m which maximises its payoff subject to the proposal being accepted by

either party l or r.

At stage 2, party r will either make an offer $y_r(m)$ to attract party m, or an offer $y_r(l)$ to attract party l. For these proposals to be accepted by m and l respectively requires

$$-y_r(m)^2 \geq -(1-\delta)Q^2 - \delta y_m^2$$
$$-(a_l - y_r(l))^2 \geq -(1-\delta)(a_l - Q)^2 - \delta(a_l - y_m)^2$$

If $y_r(m)$ is chosen then the first inequality will bind and we have $y_r(m) = \sqrt{(1-\delta)Q^2 + \delta y_m^2}$. We can now compare the payoff of party l when $y_r(m)$ and $y_r(l)$ are implemented.

$$-(a_l - \sqrt{(1-\delta)Q^2 + \delta y_m^2})^2 = -a_l^2 - (1-\delta)Q^2 - \delta y_m^2 + 2a_l\sqrt{(1-\delta Q^2 + \delta y_m^2)}$$
$$-(a_l - y_r(l))^2 = -a_l^2 - (1-\delta)Q^2 - \delta y_m^2 + (1-\delta)2a_lQ + \delta 2a_l y_m$$

Party l prefers policy $y_r(l)$ when

$$(1-\delta)2a_lQ + \delta 2a_l y_m > 2a_l\sqrt{(1-\delta Q^2 + \delta y_m^2)}$$
$$2a_l((1-\delta)Q + \delta y_m) > 2a_l\sqrt{(1-\delta Q^2 + \delta y_m^2)}$$
$$(1-\delta)Q + \delta y_m < \sqrt{(1-\delta Q^2 + \delta y_m^2)}$$

the final inequality always holds. As party l gets a higher payoff from $y_r(l)$ than $y_r(m)$, the former must be closer to a_l on the policy line, and therefore further away from a_r. Clearly then, party r maximises its utility by choosing $y_r = \sqrt{(1-\delta)Q^2 + \delta y_m^2}$.

At stage 1, party l will either make an offer $y_l(m)$ to attract party m, or an offer $y_l(r)$ to attract party r. For these proposals to be accepted by m and r respectively requires

$$-y_l(m)^2 \geq -(1-\delta)Q^2 - \delta(-\sqrt{(1-\delta)Q^2 + \delta y_m^2})^2$$
$$-(a_r - y_l(r))^2 \geq -(1-\delta)(a_r - Q)^2 - \delta(a_r - \sqrt{(1-\delta)Q^2 + \delta y_m^2})^2$$

If $y_l(m)$ is chosen then the first inequality will bind and we have $y_l(m) = -\sqrt{(1-\delta^2)Q^2 + \delta^2 y_m^2}$. We can now compare the payoff of party r when $y_l(m)$ and $y_l(r)$

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are implemented.

$$-(a_r + \sqrt{(1-\delta^2)Q^2 + \delta^2 y_m^2})^2 = -a_r^2 - (1-\delta^2)Q^2 - \delta^2 y_m^2 - 2a_r\sqrt{(1-\delta^2 Q^2 + \delta^2 y_m^2)}$$
$$-(a_r - y_l(r))^2 = -(1-\delta)(a_r^2 + Q^2 - 2a_r Q) - \delta a_r^2 - \delta(1-\delta)Q^2 - \delta^2 y_m^2 + \delta 2a_r\sqrt{(1-\delta)Q^2 + \delta y_m^2}$$

Party r prefers policy $y_l(r)$ when

$$(1-\delta)2a_r Q + \delta 2a_r\sqrt{(1-\delta)Q^2 + \delta y_m} > -2a_r\sqrt{(1-\delta^2 Q^2 + \delta^2 y_m^2)}$$
$$(1-\delta)Q + \delta\sqrt{(1-\delta)Q^2 + \delta y_m} > -\sqrt{(1-\delta^2 Q^2 + \delta^2 y_m^2)}$$

the final inequality always holds. As party r gets a higher payoff from $y_l(r)$ than $y_l(m)$, the former must be closer to a_r on the policy line, and therefore further away from a_l. Clearly then, party l maximises its utility by choosing $y_l = -\sqrt{(1-\delta^2)Q^2 + \delta^2 y_m^2}$.

Now, we can return to stage 3 to show that $y_m = 0$. By stationarity, if y_m is rejected at stage 3, then in stage 4 $y_l = -\sqrt{(1-\delta^2)Q^2 + \delta^2 y_m^2}$ will be proposed and accepted. Parties l and r will accept proposal y_m if

$$-(a_l - y_m)^2 \geq -(1-\delta)(a_l - Q)^2 - \delta(a_l + \sqrt{(1-\delta^2)Q^2 + \delta^2 y_m^2})^2$$
$$-(a_r - y_m)^2 \geq -(1-\delta)(a_r - Q)^2 - \delta(a_r + \sqrt{(1-\delta^2)Q^2 + \delta^2 y_m^2})^2$$

Party m's payoff is maximised when $y_m = 0$ (because $a_m = 0$), so we want to check whether this is an implementable proposal. Letting $y_m = 0$ and rearranging, the two inequalities above become

$$0 \leq (1-\delta^3)Q^2 + 2a_l[\delta\sqrt{(1-\delta^2)}Q - (1-\delta)Q]$$
$$0 \leq (1-\delta^3)Q^2 + 2a_r[\delta\sqrt{(1-\delta^2)}Q - (1-\delta)Q]$$

The term in square brackets may be positive or negative. If it is positive then, party r will accept $y_m = 0$, if the term is negative then party l will accept $y_m = 0$. Whenever $\delta > 0.543689$ then the term is positive. Given that we mostly care about values of δ close to one, we can say that it is generally party r who accepts m's offer.

Given $y_m = 0$, we can now characterise the accepted policy proposals (and therefore

policy outcomes) for the fixed order protocol when $l > r > m > l > r > \ldots$.

$$
\begin{aligned}
y_l &= -\sqrt{(1-\delta^2)Q^2} \\
y_r &= \sqrt{(1-\delta)Q^2} \\
y_m &= 0
\end{aligned}
$$

Instead when $r > l > m > r > l > \ldots$, the same process gives:

$$
\begin{aligned}
y_r &= \sqrt{(1-\delta^2)Q^2} \\
y_l &= -\sqrt{(1-\delta)Q^2} \\
y_m &= 0
\end{aligned}
$$

A.7 Proof of Proposition 7

For $z = a_l$ to be the expected outcome it must be that $E(s_l) > \frac{D-1}{2}$. Given the restriction that $E(s_m), E(s_r) > 1$, the set of distinct decisive events which d_l districts can be conditioning on is reduced to

$$
\begin{aligned}
\Lambda = \{ &\lambda(a_l, -\sqrt{(1-\delta^2)Q^2}, -\sqrt{(1-\delta^2)Q^2}), \\
&\lambda(a_l, -\sqrt{(1-\delta^2)Q^2}, -\sqrt{(1-\delta)Q^2}), \\
&\lambda(a_l, -\sqrt{(1-\delta)Q^2}, -\sqrt{(1-\delta)Q^2})\}
\end{aligned}
$$

Any race between a_l and $-\sqrt{(1-\delta)Q^2}$, where the former is the expected winner, must have $\tilde{t} < \frac{a_l - \sqrt{(1-\delta)Q^2}}{2}$. Any race between a_l and $-\sqrt{(1-\delta^2)Q^2}$, where the former is the expected winner, must have $\tilde{t} < \frac{a_l - \sqrt{(1-\delta^2)Q^2}}{2}$, a stricter condition. Therefore in order to for a party l to win a majority in expectation when $E(s_m), E(s_r) > 1$ it must be at least that $\tilde{t}_{\frac{D+1}{2}} < \frac{a_l - \sqrt{(1-\delta)Q^2}}{2}$. Notice that since $-\sqrt{(1-\delta)Q^2} < a_m$, then $\frac{a_l - \sqrt{(1-\delta)Q^2}}{2} < a_{lm}$. Similarly, for party r to win a majority in expectation when $E(s_m), E(s_l) > 1$ it must be that $\tilde{t}_{\frac{D+1}{2}} > \frac{a_r + \sqrt{(1-\delta)Q^2}}{2} > a_{mr}$. $\qquad\square$

A.8 Proof of Proposition 9

Suppose party l is expected to win a majority. Then $E(S)$ must be in the bottom left section of Figure 6. Each of the decisive events are distinct as by increasing a party's seat share by one, it alters the expected policy outcome $E(z)$. Each district must be conditioning on a decisive event where $s_l = \frac{D-1}{2}$. Given this, the policy choice of the district depends on the number of seats m and r have. A larger number of r seats implies a smaller number of m seats. The larger number of r seats means a higher likelihood of a policy to the right of a_m, and on top of this a smaller number of m seats means the policies offered by parties l and r are further from a_m. Therefore, given $s_m > 0$, the furthest expected policy from a_l must be at the point $(\frac{D-1}{2}, 1, \frac{D-3}{2})$. At this point, electing party l gives them a majority and brings about $z = a_l$, while electing party r leads to a coalition with an ex ante expected policy of

$$E(z) = -\frac{D-1}{2D}\left(\sqrt{\frac{1-\delta}{1-\delta\frac{D-1}{D}}Q^2}\right) + \frac{2}{2D}(0) + \frac{D-1}{2D}\left(\sqrt{\frac{1-\delta}{1-\delta\frac{D-1}{D}}Q^2}\right)$$

A voter will prefer the former if

$$-(a_l - t)^2 > -\frac{D-1}{2D}\left(-\sqrt{\frac{1-\delta}{1-\delta\frac{D-1}{D}}Q^2} - t\right)^2 - \frac{2}{2D}(-t)^2 - \frac{D-1}{2D}\left(\sqrt{\frac{1-\delta}{1-\delta\frac{D-1}{D}}Q^2} - t\right)^2$$

rearranging this we get that a voter prefers a_l if

$$t < \frac{a_l}{2} - \frac{D-1}{2Da_l}\left(\frac{1-\delta}{1-\delta(\frac{D-1}{D})}Q^2\right)$$

As $a_l < 0$, the right hand side is greater than $\frac{a_l}{2}$, which is the cutoff point in the benchmark case (recalling that $a_{lm} = \frac{a_l}{2}$ when $a_m = 0$). The cutoff for a party l majority is thus given by

$$\tilde{t}_{\frac{D+1}{2}} < z_l^* \equiv \frac{a_l}{2} - \frac{D-1}{2Da_l}\left(\frac{1-\delta}{1-\delta(\frac{D-1}{D})}Q^2\right) > a_{lm}$$

while the cutoff for a party r majority is given by

$$\tilde{t}_{\frac{D+1}{2}} > z_r^* \equiv \frac{a_r}{2} - \frac{D-1}{2Da_r}\left(\frac{1-\delta}{1-\delta(\frac{D-1}{D})}Q^2\right) < a_{mr}$$

$\square$

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A.9 Proof of Proposition 10

I first show that when $s_m < \frac{D-1}{2}$, there will always be misaligned voting in some, if not all, districts. Then I show that when $s_m > \frac{D-1}{2}$ there are equilibria with no misaligned voting in a subset of districts. Finally, I show that in any equilibrium there must be misaligned voting in at least some districts.

Case 1: When $s_m < \frac{D-1}{2}$ there will always be misaligned voting in equilibrium.

Case 1a: When one of the non-centrist parties is expected to have a majority, there will be misaligned voting.

I examine the case where l is expected to win a majority; the other case is identical. Voters must be conditioning on a decisive event where $s_l = \frac{D-1}{2}$; the expected utility of electing the three different candidates is

$$
\begin{aligned}
u_t(l) &= -(a_l - t)^2 \\
u_t(m) &= -\frac{D-1}{2D}\left(-\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-(s_m+1)}{D}}}}Q^2 - t\right)^2 - \frac{s_m+1}{D}(t)^2 - \frac{D-1-2s_m}{2D}\left(\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-(s_m+1)}{D}}}}Q^2 - t\right) \\
u_t(r) &= -\frac{D-1}{2D}\left(-\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-s_m}{D}}}}Q^2 - t\right)^2 - \frac{s_m}{D}(t)^2 - \frac{D+1-2s_m}{2D}\left(\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-s_m}{D}}}}Q^2 - t\right)^2
\end{aligned}
$$

It suffices to consider the most extreme types $t = -1$, $t = 0$ and $t = 1$. Subbing these values in we see that a type $t = -1$ will always want to elect l and a type $t = 0$ will always want to elect m. For any case $E(s_l) > \frac{D-1}{2}$, $E(s_r) > 0$, there must therefore be misaligned voting as in the districts where r is expected to win, the other voters coordinate on either l or m. The supporters of that candidate which is not serious must be casting misaligned votes.

Case 1b: For any expected seat distribution where no party has a majority, the expected utility of electing the three different candidates is

$$
\begin{aligned}
u_t(l) &= -\frac{s_l+1}{D}\left(-\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-s_m}{D}}}}Q^2 - t\right)^2 - \frac{s_m}{D}(t)^2 - \frac{s_r}{D}\left(\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-s_m}{D}}}}Q^2 - t\right)^2 \\
u_t(m) &= -\frac{s_l}{D}\left(-\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-(s_m+1)}{D}}}}Q^2 - t\right)^2 - \frac{s_m+1}{D}(t)^2 - \frac{s_r}{D}\left(\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-(s_m+1)}{D}}}}Q^2 - t\right)^2 \\
u_t(r) &= -\frac{s_l}{D}\left(-\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-s_m}{D}}}}Q^2 - t\right)^2 - \frac{s_m}{D}(t)^2 - \frac{s_r+1}{D}\left(\sqrt{\frac{1-\delta}{1-\delta^{\frac{D-s_m}{D}}}}Q^2 - t\right)^2
\end{aligned}
$$

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Where I abuse notation slightly to let s_c to be the expected number of seats of party c before district d votes, so that $s_l + s_m + s_r = D - 1$. By subbing in $t = 0$, we see that this type will always want m elected. Whether a type $t = 1$ wants to elect m or r depends on parameters (similarly $t = -1$). Specifically, some algebra shows that a type $t = 1$ prefers to elect r over m when

$$s_r - s_l > \frac{\sqrt{(1-\delta)DQ^2}D - 2(D - \delta D + \delta s_m + \delta)\sqrt{D - \delta D + \delta s_m}}{2[\sqrt{D - \delta D + \delta s_m + \delta} - \sqrt{D - \delta D + \delta s_m}]}$$

A type $t = -1$ will prefer to elect l than m if

$$s_l - s_r > \frac{\sqrt{(1-\delta)DQ^2}D - 2(D - \delta D + \delta s_m + \delta)\sqrt{D - \delta D + \delta s_m}}{2[\sqrt{D - \delta D + \delta s_m + \delta} - \sqrt{D - \delta D + \delta s_m}]}$$

These inequalities will generally hold (they may not hold if Q and D are sufficiently large, and s_m is sufficiently small). If these conditions hold, the three types have three different preferred candidates, so will be misaligned voting. However, even if they do not hold, there cannot be an equilibrium without misaligned voting, for the following reason: a d_r district conditions on r having less seats than a d_l district conditions on (Graphically, $E(S_{-d_r})$ is one point to the left of $E(S_{-d_l})$). So, if at point $E(S_{-d_r})$ a $t = 1$ voter prefers r to m, then for sure the same type at $E(S_{-d_l})$ would also prefer r to m. Hence, at at least one of the three $E(S_{-d})$ points which make up an equilibrium there will be voters who prefer each of the three parties. Therefore, there will be misaligned voting.

Case 2: When $s_m > \frac{D-1}{2}$ a subset of districts may have no misaligned voting, but there will be misaligned voting in at least one district.

If $s_m > \frac{D-1}{2}$, all districts must be conditioning on decisive events where $s_l = \frac{D-1}{2}$. In such cases the expected utility of a type t voter is

$$u_t(l) = -\frac{s_l + 1}{D}\left(-\sqrt{\frac{1-\delta}{1 - \delta\frac{D+1}{2D}}Q^2} - t\right)^2 - \frac{D-1}{2D}(t)^2 - \frac{D - 1 - 2s_l}{2D}\left(\sqrt{\frac{1-\delta}{1 - \delta\frac{D+1}{2D}}Q^2} - t\right)^2$$

$$u_t(m) = -(t)^2$$

$$u_t(r) = -\frac{s_l}{D}\left(-\sqrt{\frac{1-\delta}{1 - \delta\frac{D+1}{2D}}Q^2} - t\right)^2 - \frac{D-1}{2D}(t)^2 - \frac{D + 1 - 2s_l}{2D}\left(\sqrt{\frac{1-\delta}{1 - \delta\frac{D-s_m}{D}}Q^2} - t\right)^2$$

Note that for $t = 0$ we have $u_t(m) > u_t(l) = u_t(r)$. Any voter type with $t < 0$ has $u_t(l) > u_t(r)$, while any voter with $t > 0$ has $u_t(l) < u_t(r)$. However, it could be that some

of these types prefer $u_t(m)$ to either of the other two. In order to check this I calculate the derivative of each of the expected utilities with respect to t.

$$\frac{d[u(l)]}{dt} = -2t + \frac{D-3-4s_l}{D}\left(\sqrt{\frac{1-\delta}{1-\delta\frac{D+1}{2D}}}Q^2\right)$$

$$\frac{d[u(m)]}{dt} = -2t$$

$$\frac{d[u(r)]}{dt} = -2t + \frac{D+1-4s_l}{D}\left(\sqrt{\frac{1-\delta}{1-\delta\frac{D+1}{2D}}}Q^2\right)$$

When $s_l < \frac{D-3}{4}$ then for any $t < 0$ we have $\frac{d[u(m)]}{dt} < \frac{d[u(l)]}{dt} < \frac{d[u(r)]}{dt}$. Combined, with the fact that we have $u_t(m) > u_t(l) = u_t(r)$ for $t = 0$, this means that for $s_l < \frac{D-3}{4}$ there is no type with $u_t(l) > u_t(m), u_t(r)$. When $s_l > \frac{D+1}{4}$ then for any $t > 0$ we have $\frac{d[u(m)]}{dt} < \frac{d[u(r)]}{dt} < \frac{d[u(l)]}{dt}$. Combined, with the fact that we have $u_t(m) > u_t(l) = u_t(r)$ for $t = 0$, this means that for $s_l > \frac{D+1}{4}$ there is no type with $u_t(r) > u_t(m), u_t(l)$.

What this means is that, conditional on $s_m = \frac{D-1}{2}$, if $s_l < \frac{D-3}{4}$ then a district in which m and r are the serious candidates will have no misaligned voting; and if $s_l > \frac{D+1}{4}$ then a district in which m and l are the serious candidates will have no misaligned voting.

However, each equilibrium has misaligned voting in a least one district. Recall that $E(S_{-d_l}) = (s_l - 1, s_m, s_r), E(S_{-d_m}) = (s_l, s_m - 1, s_r)$ and $E(S_{-d_r}) = (s_l, s_m, s_r - 1)$. As all the relevant decisive events occur at $s_m = \frac{D-1}{2}$, d_l and d_r districts will have the same "route" to being decisive. That is, in any equilibrium if d_l districts are conditioning on $(k, \frac{D-1}{2}, \frac{D-1}{2} - k)$, then d_r districts must be conditioning on $(k+1, \frac{D-1}{2}, \frac{D-1}{2} - (k+1))$.

When $0 < s_l < \frac{D-3}{4}$ then all d_m and d_r districts are conditioning on $\lambda(2')$ events. In any of these districts if the serious candidates are m and r, there is no misaligned voting. However, we know that d_l districts must either be conditioning on a $\lambda(2')$ event or else a $\lambda(3)$ event (if it conditions on $S_{-d} = (0, \frac{D-1}{2}, \frac{D-1}{2})$). Whichever one of these is the case, there will always be misaligned voting in these d_l districts. Indeed, if it conditions on $S_{-d} = (0, \frac{D-1}{2}, \frac{D-1}{2})$, and the other districts are all races between m and r, it must be that there is only misaligned voting in this single d_l district. Examining the $\frac{D+1}{4} < s_l < \frac{D-1}{2}$ case gives the same insight for the mirror case; there'll be no misaligned voting in d_l or d_m districts if they focus on races between l and m, but there will always be misaligned voting in the d_r districts. $\qquad\square$

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