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# Regulating a Monopolist with unknown costs and unknown quality capacity*

Charles Blackorby and Dezsö Szalay
The University of Warwick

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**Abstract**

We study the regulation of a firm with unknown demand and cost information. In contrast to previous studies, we assume demand is influenced by a quality choice, and the firm has private information about its quality capacity in addition to its cost. Under natural conditions, asymmetric information about the quality capacity is irrelevant. The optimal pricing is weakly above marginal costs for all types and no type is excluded.
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## 1 Introduction

When duplication of fixed costs is wasteful, a service is efficiently provided by a natural monopoly. To keep the service provider from abusing its monopoly power, the pricing of the firm is regulated. If the regulator had access to the firm's information, price regulation would be a trivial matter. As is well known, the firm should follow a marginal cost pricing rule and should be subsidized for the losses it makes on a lump-sum basis. However, the problem is precisely that the firm has better information than the regulator has about payoff relevant circumstances. Baron and Myerson [1982] first analyzed this problem when the firm's costs are unknown to the regulator but known to the firm. Marginal cost

*We thank Paul Beaudry, Claudio Mezzetti, and Benny Moldovanu for helpful conversations. Correspondence can be sent to the authors at the Economics Department, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, United Kingdom, or to c.blackorby@warwick.ac.uk and dezso.szalay@warwick.ac.uk.

pricing is no longer optimal as this rule gives firms with relatively low marginal costs incentives to exaggerate their costs in order to get larger subsidies. To make such exaggeration unattractive, prices are distorted upwards for all but the most efficient firms.

While cost conditions are an important source of firms' private information they are not the only one. Indeed, it is primarily for analytical convenience rather than economic reasoning that we usually model firms as being better informed about a single parameter (marginal costs or fixed costs) rather than a vector of parameters. However, firms being closer to consumers than the regulator, they naturally have a better access to information about demand conditions in addition to their information about costs. Thus, it is quite natural to think of the regulation problem in a world of (at least) two-dimensional asymmetric information. How important is this change of perspective? In other words, how different are the optimal pricing rules arising in the new situation?

Lewis and Sappington [1988] have proposed a model to address this question. They assume that the firm knows the intercept of a linear demand function and the value of its marginal cost parameter, while the regulator does not have any of this information. Under a particular verifiability assumption, they demonstrate that this problem is amenable to techniques developed in Laffont, Maskin and Rochet [1987]. In particular, the regulator has only one instrument - the marginal price - to screen firms, but information has two dimensions. Hence, firms whose demand and cost parameter add up to the same value of a certain statistic (in that case, simply the sum of the parameters) behave the same way; that is, these firms are bunched together. They show that optimal pricing is strikingly different in this world; some firms are induced to set prices below marginal costs. This problem was later reinvestigated by Armstrong [1999], clarifying that the optimal pricing policy in the Lewis and Sappington [1988] model should induce exclusion of certain high cost-low demand firms. Technically, exclusion is optimal because the density of the sum of two random variables goes to zero at the bounds of its support. Armstrong [1996] shows that exclusion is robust in these kind of settings under more general assumptions. However, Armstrong [1999] confirms in a two-by-two-type model that optimal pricing can indeed be below marginal costs for some types in this context. The direction of the distortions relative to the first-best allocation depends on the particular pattern of binding incentive constraints; this is in contrast to the one dimensional case where the binding incentive constraints are always those inhibiting low cost firms from exaggerating their costs.

We study the regulation problem when the firm has demand and cost information, but we depart from the Lewis-Sappington model in an important way. To add asymmetric information about demand in an interesting way, the Lewis-Sappington-Armstrong model assumes that the regulator cannot verify the quantity consumers buy. Otherwise, the regulator could simply infer demand information from observing the price the firm sets and the quantity that is demanded by consumers. On the other hand, the regulator needs to be able to verify whether or not the consumers' demand has been satisfied. We find this verifiability assumption hard to justify in general, and hence that an investigation under different verifiability standards is interesting. In particular, we assume that the realized demand function is observable, but the firm has private information about its ability to shift the demand function upwards. The firm produces a service of observable quantity and quality, but given an observed quality, the regulator does not know whether the firm would have been capable of delivering the good in still higher quality. In addition to that, the firm's profit is unobservable to the regulator, so information about costs poses non-trivial problems as well.

We show that this seemingly small departure from the original model in a natural direction has dramatic effects on the optimal pricing scheme and the participation of firms. Neither is there pricing below marginal costs, nor is there exclusion of firms. The reason for these differences is that the regulator now has two instruments to screen firms (the marginal price and the quality choice) and that the firms ability to produce quality enters the incentive problem in a way that has not been analyzed widely so far. The firms ability puts an upper bound on what the firm is able to do, so it enters the firm's problem through a constraint rather than directly. The most surprising result we are able to establish is that, under reasonable conditions, private information about the ability to produce quality does not affect the solution of the screening problem. If the regulator acts as if he knew this piece of information, and offers a family of price-quality-subsidy menus, each one conditional on the firm's ability parameter, then the firm voluntarily selects into the right menu. The reason firms agree to share this piece of information is that they receive a weakly higher rent for revealing their cost information. This arises at the optimum if higher quality-ability firms produce higher amounts of quantity, which in turn is optimal for the regulator provided that quantity and quality are complements to the consumer and the quality-ability and cost parameters are affiliated. We extend our analysis in a number of directions that depart from these assumptions. Provided that the distribution of types has full support, none of these results feature pricing below marginal costs nor exclusion of firms.

Our study is related to a growing literature on multi-dimensional screening. Most related in terms of the techniques employed is our own taxation paper, Beaudry, Blackorby, and Szalay [2008]. Closest in terms of focus - the Lewis and Sappington [1988] and Armstrong [1999] papers notwithstanding - is Matthews and Moore [1987]. The main difference to Matthews and Moore is that in their paper information is one-dimensional but there are two instruments to choose; in our paper both information and the set of instruments is two-dimensional. Other studies in multi-dimensional screening include McAfee and MacMillan [1988], and Jehiel, Moldovanu, and Stacchetti [1999]. Wilson [1993] offers a general solution (the demand profile approach) to the multi-dimensional pricing problem. The most general results to date have been obtained by Armstrong [1996] and Rochet and Choné [1998]. In addition there are two useful surveys of these multidimensional problems—Armstrong and Rochet [1999] and Rochet and Stole [2003].

The paper is organized as follows. In Section two we lay out the model, explain the regulator's problem and explain its solution for the case where the regulator has perfect information. In Section three, we study the case where the regulator knows the firms' quality capacity, but does not have access to the firm's cost information. We show that our problem is nicely amenable to monotone comparative statics methods, and we use these methods to investigate how the pricing policy depends on the firm's quality capacity. In Section four, we address the full problem when the regulator knows neither the cost parameter nor the firm's quality capacity. Depending on the support of the quality information and on the joint-distribution of quality capacity and marginal costs, we are able to fully characterize the optimal pricing policy. The final section concludes. All proofs have been relegated to the appendix.

# 2 The model

Consumers' valuations for a quantity $x$ of a good whose quality is $q$ are described by the downward sloping inverse demand function $P(x,q)$. We assume that, for any $q$, $P(\cdot,q)$ ranges from zero to infinity, as would be the case for a constant elasticity demand function. Define the gross consumer surplus of a consumer who buys $x$ units of a good of quality $q$ at a constant marginal price as

$$V(x,q) \equiv \int_0^x P(z,q)\,dz.$$

The consumer values higher $x$ and $q$ in the sense that $V_x(x,q) > 0$ for all $x$ and $q$ and $V_q(x,q) > 0$ for all $x > 0$ and $q$. The good is produced by a monopoly firm subject to price regulation. The firm's cost of producing the good in quantity $x$ and quality $q$ is

$$\mathcal{C}(x,q,\theta,\eta) = \begin{cases} C(x,q,\theta) & \text{for } q \leq \eta \\ \infty & \text{for } q > \eta. \end{cases}$$

where $\theta$ and $\eta$ are parameters that are known only to the firm but not to the regulator. $\theta$ shifts the cost function on the relevant domain upwards, $C_\theta(x,q,\theta) > 0$ for all $\theta$ and $x, q > 0$. Costs are strictly increasing in $x$ and weakly increasing in $q$, that is $C_x(x,q,\theta) > 0$ and $C_q(x,q,\theta) \geq 0$ for all $x, q, \theta$ such that $x > 0$. We note that the differentiability of surplus and costs with respect to $x$ and $\theta$ is crucial for our approach; the differentiability with respect to $q$ is not and could be dispensed with at the cost of additional notational clutter.

$\eta$ defines an upper bound on qualities that the firm is capable of producing. $x$ and $q$ are verifiable; hence contracts can be written on these variables. The parameters $\theta$ and $\eta$ are known only to the firm; the regulator knows only the joint distribution of these variables. $\theta, \eta$ are distributed on a product set $\mathbf{\Theta} \times \mathbf{H}$ with probability density function $f(\theta,\eta) > 0$ for all $\theta, \eta$. The set $\mathbf{\Theta}$ is taken as the interval $\left[\underline{\theta}, \overline{\theta}\right]$ throughout the paper, where $\underline{\theta} > 0$. The set $\mathbf{H}$ can either be discrete or continuous. In the latter case we take it as the interval $\left[\underline{\eta}, \overline{\eta}\right]$ where $\underline{\eta} > 0$. Let $G(\eta)$ denote the cdf of $\eta$. Given our full support assumption, for each $\eta$ that has $dG(\eta) > 0$, the conditional distribution of $\theta$ given $\eta$ has full support. The density of this distribution is denoted $f(\theta\,|\eta)$.

The firm is subject to price and quality regulation. If it sets a marginal price $p$ and produces a quality $q$ then it receives a subsidy $t$ and its profit is for $q \leq \eta$

$$t + pX(p,q) - C(x,q,\theta),$$

where $X(p,q)$ is the direct demand function for the good which is assumed to be downward sloping, $X_p(p,q) < 0$. For $q > \eta$, the profit becomes minus infinity.

Define the sum of consumer and producer surplus as

$$S(x,q,\theta) \equiv V(x,q) - C(x,q,\theta) \text{ for } q \leq \eta.$$

We place the following assumptions on the joint surplus function. First, we assume that $S(x,q,\theta)$ satisfies the single crossing condition in $x, \theta$ : for all $x, \theta$

and all $q \leq \eta$

$$S_{x\theta}\left(x,q,\theta\right)=-C_{x\theta}\left(x,q,\theta\right)<0, \tag{1}$$

a standard sorting condition. Second, we assume that $S\left(x,q,\theta\right)$ satisfies the boundary condition, for all $\theta$ and all $q>0$

$$\lim_{x\to 0}S_{x}\left(x,q,\theta\right)=\infty. \tag{2}$$

Fourth, the surplus function is strictly concave in $x$ and $q$. For all $x,\theta$, and $q\leq\eta$

$$S_{xx}\left(x,q,\theta\right),S_{qq}\left(x,q,\theta\right)<0,\text{ and }S_{xx}\left(x,q,\theta\right)S_{qq}\left(x,q,\theta\right)-\left(S_{xq}\left(x,q,\theta\right)\right)^{2}>0. \tag{3}$$

## 2.1 The Regulator's Problem

We think of the regulator's problem in term's of a direct revelation mechanism, which is a triple of functions $\{q\left(\theta,\eta\right),p\left(\theta,\eta\right),t\left(\theta,\eta\right)\}$ for all $\left(\theta,\eta\right)\in\mathbf{\Theta}\times\mathbf{H}$ that satisfy incentive compatibility constraints. Given $p\left(\theta,\eta\right),q\left(\theta,\eta\right)$, consumers make their consumption decisions resulting in demand $X\left(p\left(\theta,\eta\right),q\left(\theta,\eta\right)\right)$.[1] The regulator maximizes a weighted sum of net consumer surplus and producer surplus under incentive constraints. If a firm announces costs $\hat{\theta}$ and quality capacity $\hat{\eta}$, where $q\left(\hat{\theta},\hat{\eta}\right)>\eta$, then its profits are minus infinity; if it announces a type such that $q\left(\hat{\theta},\hat{\eta}\right)\leq\eta$, then its profits are given by

$$\Pi\left(\hat{\theta},\theta,\hat{\eta}\right)\equiv t(\hat{\theta},\hat{\eta})+p(\hat{\theta},\hat{\eta})X\left(p(\hat{\theta},\hat{\eta}),q(\hat{\theta},\hat{\eta})\right)-C\left(X\left(p(\hat{\theta},\hat{\eta}),q(\hat{\theta},\hat{\eta})\right),q(\hat{\theta},\hat{\eta}),\theta\right).$$

Under a truthful mechanism, the weighted joint surplus for a given pair $\left(\theta,\eta\right)$ is equal to

$$\begin{aligned}W\left(\theta,\eta\right)\equiv\;&V\Big(X\big(p\left(\theta,\eta\right),q\left(\theta,\eta\right)\big),q\big(\theta,\eta\big)\Big)-p\left(\theta,\eta\right)X\big(p\left(\theta,\eta\right),q\big(\theta,\eta\big)\big)\\&-t\left(\theta,\eta\right)+\alpha\Pi\left(\theta,\theta,\eta\right)\end{aligned}$$

where $\alpha\in\left(0,1\right)$. Since $\alpha$ is kept constant throughout the paper, we suppress the dependence of the welfare function on $\alpha$ in what follows. We let $\Theta$ and $H$ denote the random variables with typical realizations $\theta$ and $\eta$, respectively, and let $\mathcal{E}_{\Theta H}$ denote the expectation operator taken over the random variables $\Theta$ and

[1] Although it is useful to begin with the instruments $p,q$ and $t$, it will become clear below that one can actually think of the problem as of choosing a triple of functions $\{q\left(\theta,\eta\right),x\left(\theta,\eta\right),t\left(\theta,\eta\right)\}$, where $x\left(\theta,\eta\right)\equiv X\left(p\left(\theta,\eta\right),q\left(\theta,\eta\right)\right)$.

$H$. The regulator's problem is

$$\max_{p(\cdot,\cdot),q(\cdot,\cdot),t(\cdot,\cdot)} \mathcal{E}_{\Theta H}\left[W(\theta,\eta)\right] \tag{4}$$

s.t. for all $\theta, \eta$ and all $\hat{\theta}, \hat{\eta}$ such that $q\left(\hat{\theta}, \hat{\eta}\right) \leq \eta$

$$\Pi(\theta,\theta,\eta) \geq \Pi\left(\hat{\theta},\theta,\hat{\eta}\right), \tag{5}$$

and for all $\theta, \eta$

$$\Pi(\theta,\theta,\eta) \geq 0, \tag{6}$$

and

$$q(\theta,\eta) \leq \eta. \tag{7}$$

(5) is the incentive compatibility condition, requiring that a firm of type $\theta, \eta$ must have no incentive to mimic any other type which itself produces a quality the firm is able to produce too. Which types fall in this category depends on the allocation of qualities, $q(\theta,\eta)$. Of course, the firm must also have no incentive to mimic types such that $q\left(\hat{\theta}, \hat{\eta}\right) > \eta$. But, the profit arising from such a choice is equal to minus infinity, which is obviously never tempting. In fact, the constraint (6) requires that each firm in equilibrium obtain a non-negative profit, which is surely better than imitating a type $\left(\hat{\theta}, \hat{\eta}\right)$ such that $q\left(\hat{\theta}, \hat{\eta}\right) > \eta$.[2] Finally, condition (7) requires that the allocation must be technically feasible.

Before we study the solution to this problem, we describe the first-best allocation when types are observable.

## 2.2 The first-best

Since $\alpha < 1$, the regulator allocates all surplus to the consumer in the first-best allocation; the participation constraint is binding for each type, $\Pi(\theta,\theta,\eta) = 0$, so

$$t(\theta,\eta) = C\left(X\left(p(\theta,\eta), q(\theta,\eta)\right), q(\theta,\eta), \theta\right) - p(\theta,\eta)\, X\left(p(\theta,\eta), q(\theta,\eta)\right). \tag{8}$$

[2] It is important to stress that we allow each type to imitate every other type, so there is no limit on what a firm can communicate. This technical difference is important for the proof of the revelation principle in this context. For a formal proof that the revelation principle applies under our verifiability assumptions, see Proposition 6 in Beaudry, Blackorby, and Szalay [2008].

Substituting (8) into the regulator's objective function, we obtain

$$\max_{p(\cdot,\cdot),q(\cdot,\cdot)} \mathcal{E}_{\Theta H}\Big[V\Big(X\big(p(\theta,\eta),q(\theta,\eta)\big),q(\theta,\eta)\Big)-C\Big(X\big(p(\theta,\eta),q(\theta,\eta)\big),q(\theta,\eta),\theta\Big)\Big]$$

Given the boundary condition (2) and that the surplus function is concave in $x$ and $q$, (3), the optimal marginal price schedule $p^{fb}(\theta,\eta)$ satisfies for each $\theta,\eta$ the first-order condition

$$V_x\Big(X\big(p^{fb}(\theta,\eta),q(\theta,\eta)\big),q(\theta,\eta)\Big)-C_x\Big(X\big(p^{fb}(\theta,\eta),q(\theta,\eta)\big),q(\theta,\eta),\theta\Big)=0. \tag{9}$$

Moreover, the optimal quality allocation satisfies either $q^{fb}(\theta,\eta)<\eta$ and

$$V_q\Big(X\big(p(\theta,\eta),q^{fb}(\theta,\eta)\big),q^{fb}(\theta,\eta)\Big)-C_q\left(X\left(p(\theta,\eta),q^{fb}(\theta,\eta)\right),q^{fb}(\theta,\eta),\theta\right)=0, \tag{10}$$

or $q^{fb}(\theta,\eta)=\eta$ and

$$V_q\Big(X\big(p(\theta,\eta),q^{fb}(\theta,\eta)\big),q^{fb}(\theta,\eta)\Big)-C_q\Big(X\big(p(\theta,\eta),q^{fb}(\theta,\eta)\big),q^{fb}(\theta,\eta),\theta\Big)\geq 0. \tag{11}$$

Since $V_x(x,q)=P(x,q)$, condition (9) states that at the optimum price should equal marginal cost. Condition (10) states that the marginal cost of providing quality should equal the marginal benefit of quality, unless of course the quality bound is binding.

We now address the principal's problem when $\theta$ and $\eta$ are not observable to him. It is useful to begin the investigation with the case where the firm can only misrepresent its cost parameter $\theta$, but cannot lie about the parameter $\eta$. In this case the regulator faces a family of firms, indexed by their quality capacity $\eta$. For each given $\eta$, the regulator faces a problem that is now familiar from the analysis of Baron and Myerson (1982). Since the full two dimensional problem is analytically most convenient to handle when it has certain additive separability properties, we treat the one dimensional problem under the same assumption. In particular, we assume that the cost function can be written as

$$C(x,q,\theta)=c(x,\theta)+k(x,q), \tag{12}$$

the important feature being that there is no interaction between $\theta$ and $q$.

# 3 The Case of Observable Quality Bounds

If $\eta$ is known, the regulator can condition his instruments on $\eta$. We let $p(\theta;\eta)$, $q(\theta;\eta)$, and $t(\theta;\eta)$ for all $\theta$ denote the price, quality, and transfer schedule conditional on $\eta$ and define

$$\Pi\left(\hat{\theta},\theta;\eta\right) \equiv t\left(\hat{\theta};\eta\right)+p\left(\hat{\theta};\eta\right)X\left(p\left(\hat{\theta};\eta\right),q\left(\hat{\theta};\eta\right)\right)-C\left(X\left(p\left(\hat{\theta};\eta\right),q\left(\hat{\theta};\eta\right)\right),q\left(\hat{\theta};\eta\right),\theta\right)$$

The principal solves, for each given $\eta$, the following problem

$$\max_{p(\cdot;\eta),t(\cdot;\eta),q(\cdot;\eta)} \int_{\underline{\theta}}^{\overline{\theta}} W(\theta;\eta)\, f(\theta|\,\eta)\, d\theta \tag{13}$$

$$\text{subject to, for all } \theta,\eta:$$

$$\Pi(\theta,\theta;\eta) \geq \Pi\left(\hat{\theta},\theta;\eta\right) \text{ for all } \hat{\theta}, \tag{14}$$

$$\Pi(\theta,\theta;\eta) \geq 0, \quad \text{and} \tag{15}$$

$$q(\theta;\eta) \leq \eta. \tag{16}$$

This is a standard problem and is normally solved by reformulating the incentive and participation constraints. We state a more tractable version of these constraints in the following lemma. We call a triple of pricing schedule $p(\theta;\eta)$, quality schedule $q(\theta;\eta)$, and transfer schedule $t(\theta;\eta)$ implementable if they satisfy constraints (14) and (15). Moreover, we let $\pi(\theta;\eta) \equiv \max_{\hat{\theta}} \Pi\left(\hat{\theta},\theta;\eta\right)$.

**Lemma 1** *The price, quality, and payment schedules, $p(\theta;\eta)$, $q(\theta;\eta)$, and $t(\theta;\eta)$, are implementable if and only if*

$$\begin{aligned} t(\theta;\eta) &= C\left(X\left(p(\theta;\eta),q(\theta;\eta)\right),q(\theta;\eta),\theta\right) \qquad (17)\\ &\quad +\int_{\theta}^{\overline{\theta}} c_y\left(X\left(p(y;\eta),q(\theta;\eta)\right),y\right)dy - p(\theta;\eta)\,X\left(p(\theta;\eta),q(\theta;\eta)\right) \end{aligned}$$

*and $X(p(y;\eta),q(\theta;\eta))$ is non-increasing in $\theta$.*

The first part of Lemma 1 is standard, but there are some subtle differences to the standard case. Using envelope arguments, the equilibrium profit of the firm is equal to $\pi(\theta;\eta) = \int_{\theta}^{\overline{\theta}} c_y\left(X\left(p(y;\eta),q(y;\eta)\right),y\right)dy$, which is just another way of writing condition (17). $X$ has to satisfy a monotonicity condition, which amounts— roughly speaking—to a second order condition, which is typical in incentive problems. However, it is interesting to note that incentive compatibility places no restrictions on the function $q(\theta;\eta)$ or on the function $p(\theta;\eta)$, although these functions are jointly constrained to give rise to a monotonic function $X(p(\theta;\eta), q(\theta;\eta))$. Additive separability of the cost function is the reason our problem has this feature; there is no interaction between $q$ and $\theta$ in the cost function.

Using our assumption that the inverse demand function has full range for all $q$, we can simplify the principal's problem. For any given function $q(\theta;\eta)$, we can still induce any desired quantity of consumption schedule $X(p(\theta;\eta), q(\theta;\eta))$ by simply adjusting the price schedule $p(\theta;\eta)$ accordingly. This means we can effectively choose the pair of quantity and quality schedules $(x(\theta;\eta), q(\theta;\eta))$ directly, and worry about the pricing schedule that induces $x(\theta;\eta) \equiv X(p(\theta;\eta), q(\theta;\eta))$ afterwards. Hence, substituting the expression for equilibrium profits into the objective function, and integrating by parts, we can write the regulator's problem as

$$\max_{x(\cdot;\eta),q(\cdot;\eta)} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V(x(\theta;\eta), q(\theta;\eta)) - C(x(\theta;\eta), q(\theta;\eta), \theta)\Big] f(\theta|\eta)\, d\theta \tag{18}$$

$$- \int_{\underline{\theta}}^{\overline{\theta}} \Big[(1-\alpha)\, c_\theta(x(\theta;\eta), \theta) \frac{F(\theta|\eta)}{f(\theta|\eta)}\Big] f(\theta|\eta)\, d\theta$$

$$s.t.\ x(\theta;\eta) \text{ non-increasing in } \theta, \text{ and}$$

$$q(\theta;\eta) \leq \eta \text{ for all } \theta.$$

It is customary to solve this problem imposing regularity conditions that ensure that the monotonicity constraint is automatically satisfied at the solution to this problem. Then, in the absence of this monotonicity constraint, the instruments can effectively be chosen to maximize the integrand in (18) pointwise. Moreover, the constraint set for each $\theta$ is $x(\theta;\eta) \geq 0$ and $q(\theta;\eta) \in [0,\eta]$, a lattice. Let the virtual surplus be

$$B(x,q,\theta) \equiv V(x,q) - C(x,q,\theta) - (1-\alpha)\, c_\theta(x,\theta) \frac{F(\theta|\eta)}{f(\theta|\eta)}. \tag{19}$$

Notice that $B(x,q,\theta)$ is supermodular in $(x,q)$ for each $\theta$ if and only if $S(x,q,\theta)$ is supermodular in $(x,q)$. Since $B(x,q,\theta)$ is differentiable to the desired degree, this is equivalent to $B_{xq}(x,q,\theta) \geq 0$ for all $x,q,\theta$. Moreover, if $B(x,q,\theta)$ is submodular in $(x,q)$ for each $\theta$, then, with $\hat{q} \equiv q^{-1}$, $B\left(x, \frac{1}{\hat{q}}, \theta\right)$ is supermodular in $(x,\hat{q})$ for each $\theta$, because $B_{x\hat{q}}\left(x, \frac{1}{\hat{q}}, \theta\right) < 0$ if and only if $B_{xq}(x,q,\theta) > 0$. Next consider changes in $\theta$. We characterize their effect on $B(x,q,\theta)$ in the following lemma:

**Lemma 2** *i)* $B(x,q,\theta)$ *has non-increasing differences in* $(x,q)$ *and* $\theta$ *if and only if* $c(x,\theta)+(1-\alpha)\,c_\theta(x,\theta)\,\frac{F(\theta|\eta)}{f(\theta|\eta)}$ *has non-decreasing differences in* $x$ *and* $\theta$.
*ii) If* $c_{x\theta\theta}(x,\theta)\geq 0$, *and* $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ *is non-decreasing in* $\theta$ *(and the single-crossing condition* $c_{x\theta}(x,\theta)\geq 0$ *holds) then* $c(x,\theta)+(1-\alpha)\,c_\theta(x,\theta)\,\frac{F(\theta|\eta)}{f(\theta|\eta)}$ *has non-decreasing differences in* $x$ *and* $\theta$.

Complementarities in the virtual surplus function determine the comparative statics properties of the maximizer set. We also want to have a unique maximizer, as is standard in incentive theory. To this end, we will impose

$$B(x,q,\theta) \text{ is strictly concave in } (x,q) \text{ for all } \theta. \tag{20}$$

Notice that, given $S(x,q,\theta)$ is strictly concave in $(x,q)$, it suffices to assume that $c_\theta(x,\theta)$ is strictly convex in $x$ to guarantee that $B(x,q,\theta)$ is strictly concave in $(x,q)$ for all $\theta$.

As usual we will characterize the solution to problem (18) assuming such a solution exists. Note that given (20) (in conjunction with the conditions mentioned in the next proposition) the solution is unique and let it be denoted $x^*(\theta;\eta)$ and $q^*(\theta;\eta)$, respectively. We can now characterize this solution:

**Proposition 1** *Suppose that* $c(x,\theta)+(1-\alpha)\,c_\theta(x,\theta)\,\frac{F(\theta|\eta)}{f(\theta|\eta)}$ *has non-decreasing differences in* $x$ *and* $\theta$. *Then,*
*i) the optimal quantity schedule* $x^*(\theta;\eta)$ *satisfies the first-order condition*

$$V_x(x^*(\theta;\eta),q(\theta;\eta))-C_x(x^*(\theta;\eta),q(\theta;\eta),\theta)-(1-\alpha)\,c_{x\theta}(x^*(\theta;\eta),\theta)\,\frac{F(\theta\,|\eta)}{f(\theta\,|\eta)}=0, \tag{21}$$

*and the optimal quality schedule satisfies either*

$$q^*(\theta;\eta)<\eta \text{ and } V_q(x(\theta;\eta),q^*(\theta;\eta))-C_q(x(\theta;\eta),q^*(\theta;\eta),\theta)=0, \text{ or} \tag{22}$$

$$q^*(\theta;\eta)=\eta \text{ and } V_q(x(\theta;\eta),\eta)-C_q(x(\theta;\eta),\eta,\theta)\geq 0; \tag{23}$$

*ii) if* $B(x,q,\theta)$ *is supermodular in* $(x,q)$ *for each* $\theta$, $x^*(\theta;\eta)$ *and* $q^*(\theta;\eta)$ *are both non-increasing in* $\theta$ *and the quality capacity constraint is binding for low values of* $\theta$ *if any;*
*iii) if* $B(x,q,\theta)$ *is submodular in* $(x,q)$ *for each* $\theta$, $x^*(\theta;\eta)$ *is non-increasing and* $q^*(\theta;\eta)$ *is non-decreasing in* $\theta$ *and the quality capacity constraint is binding for high values of* $\eta$ *if any.*

Using $V_x(x^*(\theta;\eta), q(\theta;\eta)) = P(x^*(\theta;\eta), q(\theta;\eta))$ and the definition $P(X(p(\theta;\eta), q(\theta;\eta)), q(\theta;\eta)) \equiv p(\theta;\eta)$, we can rewrite the first-order condition (21) in terms of the optimal price-schedule $p^*(\theta;\eta)$ :

$$\begin{aligned} &p^*(\theta;\eta) - C_x(X(p^*(\theta;\eta), q(\theta;\eta)), q(\theta;\eta), \theta) \\ &\qquad - (1-\alpha)\, c_{x\theta}(X(p^*(\theta;\eta), q(\theta;\eta)), \theta) \frac{F(\theta|\eta)}{f(\theta|\eta)} = 0. \end{aligned} \tag{24}$$

The solution has the remarkable feature that for a *given* price schedule, the quality schedule satisfies the same first-order condition as in the first-best. Thus, the only distortions relative to the first-best case arise from the familiar trade-off that causes the price schedule to be distorted away from first-best. We can write the first-order condition as

$$(p(\theta;\eta) - C_x(X(p(\theta;\eta), \eta), \eta, \theta))\, f(\theta|\eta) = (1-\alpha)\, c_{x\theta}(X(p(\theta;\eta), \eta), \eta, \theta)\, F(\theta|\eta).$$

To understand our results that follow, it is useful to review the trade-off behind the optimal pricing in some detail. On the left side we have the principal's desire to implement an efficient solution, which requires that price equals marginal cost. The weight given to this motive is $f(\theta|\eta)$, the likelihood of type $\theta|\eta$. On the right side appears the principal's desire to limit the firm's rents. A decrease in $p(\theta;\eta)$ increases the rents that have to be given to all types that are more efficient at producing than type $\theta|\eta$. Since there is a mass $F(\theta|\eta)$ of firms of these types, the weight attached to this motive is $F(\theta|\eta)$. The trade-off is optimally resolved by having all firms except the most efficient one price above marginal costs; the most efficient one prices exactly at marginal costs. Moreover, given our standard assumptions that the type distribution has full support and the surplus function satisfies the boundary condition (2), all types produce a strictly positive quantity at the optimum; that is, excluding firms is not optimal.

## 3.1 Comparative Statics

We now establish some important comparative statics properties of the solution. In particular, it is interesting to see how the quality and quantity schedules depend on $\eta$. To this end we write virtual surplus as $B(x, q, \theta, \eta)$, to emphasize its dependence on $\eta$. The following lemma gathers again some useful facts.

**Lemma 3** *i) $B(x, q, \theta, \eta)$ has non-decreasing (non-increasing) differences in $(x, q)$ and $\eta$ for given $\theta$ if and only if $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is non-increasing (non-decreasing)*

*in $\eta$.*
*ii) If $(\eta,\theta)$ are affiliated, then $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is non-increasing in $\eta$. If $(-\eta,\theta)$ are affiliated, then $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is non-decreasing in $\eta$.*

We can now state how the solution of the regulation problem depends on $\eta$ :

**Proposition 2** *i) Suppose that $(\eta,\theta)$ are affiliated, so that $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is non-increasing in $\eta$. Then, if $B(x,q,\theta,\eta)$ is supermodular in $(x,q)$ for each $\theta,\eta$, $x^*(\theta;\eta)$ and $q^*(\theta;\eta)$ are both non-decreasing in $\eta$;*
*ii) Suppose that $(-\eta,\theta)$ are affiliated, so that $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is non-decreasing in $\eta$. Then, if $B(x,q,\theta,\eta)$ is submodular in $(x,q)$ for each $\theta,\eta$, $x^*(\theta;\eta)$ is non-increasing in $\eta$ and $q^*(\theta;\eta)$ is non-decreasing in $\eta$.*

The intuition for these results is as follows. If $(\eta,\theta)$ are affiliated, then, the higher is $\eta$, the lower is $\frac{F(\theta|\eta)}{f(\theta|\eta)}$. Hence, the rent extraction motive receives a smaller weight in the government's objective function. Hence, the induced quantity that consumers purchase is higher at the optimum. When in addition $x$ and $q$ are complements, then this implies that the quality should also be weakly higher the higher is $\eta$. The intuition for the second case is reversed, as far as the quantity of consumption is concerned. But since $x$ and $q$ are substitutes in this case, again the quality is non-decreasing in $\eta$.

Obviously, there are four possible combinations of affiliation assumptions and supermodularity assumptions on $B(x,q,\theta,\eta)$. However, we can deal in general only with those cases where the implied change in quality as a response to an increase in $\eta$ is non-negative. The reason is that the constraint set depends on $\eta$. In particular, it is expanding with an increase in $\eta$, and hence monotone comparative statics methods cannot be used to establish results where $q^*(\theta;\eta)$ is non-increasing in $\eta$. In particular, if $q^*(\theta;\eta)$ is at a corner solution for both a low and a high value of $\eta$, then $q^*(\theta;\eta)$ must obviously be increasing in $\eta$ for some $\eta$ even though the marginal incentive to increase quality may be lower for the higher value of $\eta$.

However, interesting exceptions to this negative conclusion are when the constraint set is either "high enough" or "low enough"for any value of $\eta$ in a sense we make precise below. In particular, the former case arises when $\underline{\eta}$ is sufficiently high.[3] In this case, we can apply direct methods to the pair of first-order conditions (21) and (22). In particular, if the inverse hazard rate $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is differentiable in $\eta$, the solution schedules $x^*(\theta;\eta)$ and $q^*(\theta;\eta)$ are both differentiable in $\eta$.

[3]We note that while the differentiability of surplus with respect to $q$ was not essential for our results to this point, it is essential for the remaining ones in this section.

**Proposition 3** *Let $\underline{\eta}$ be sufficiently high so that the solution satisfies $q^*(\theta;\eta) < \eta$ for all $\theta,\eta$. Then,*
*i) If $\eta$ and $\theta$ are affiliated, then $x^*(\theta;\eta)$ is non-decreasing in $\eta$ for each $\theta$. If $-\eta$ and $\theta$ are affiliated, then $x^*(\theta;\eta)$ is non-increasing in $\eta$ for each $\theta$.*
*ii) If $B(x,q,\theta,\eta)$ is supermodular in $q$ and $x$ for all $\theta,\eta$, then $sign\frac{dq^*}{d\eta} = sign\frac{dx^*}{d\eta}$. If $B(x,q,\theta,\eta)$ is submodular in $q$ and $x$ for all $\theta,\eta$, then $sign\frac{dq^*}{d\eta} = -sign\frac{dx^*}{d\eta}$.*

Another interesting case is when $\overline{\eta}$ is sufficiently small. In that case, the capacity constraint is binding for all types, so that $q(\theta,\eta) = \eta$ for all $\theta,\eta$. We can again apply direct differentiability methods to find that:

**Proposition 4** *Let $\overline{\eta}$ be sufficiently small so that the solution satisfies $q^*(\theta;\eta) = \eta$ for all $\theta,\eta$. If $B(x,q,\theta,\eta)$ is supermodular in $q$ and $x$ for all $\theta,\eta$ and $\eta$ and $\theta$ are affiliated, then $x^*(\theta;\eta)$ is non-decreasing in $\eta$ for each $\theta$; if $B(x,q,\theta,\eta)$ is submodular in $q$ and $x$ for all $\theta,\eta$ and $-\eta$ and $\theta$ are affiliated, then $x^*(\theta;\eta)$ is non-increasing in $\eta$ for each $\theta$.*

Thus, our model allows for clear comparative statics predictions in a wide range of settings. These comparative statics results are important building blocks of the multi-dimensional regulation problem, to which we now turn.

# 4 The Two-dimensional Problem

We now address the problem when both $\theta$ and $\eta$ are unobservable to the regulator. Remarkably, our problem allows for a complete characterization of the optimum in a variety of cases, despite there being two dimensions of asymmetric information. Even more remarkable, the solution coincides with the one found in the one-dimensional case under natural assumptions. In these cases, asymmetric information about $\eta$ is irrelevant as there are no misaligned interests with respect to this dimension. We start by making some basic observations about the implications of incentive compatibility in our model

## 4.1 Incentive Compatibility

The basic obstacle multi-dimensional screening problems face is that the number of deviations to consider is simply too large to deal with. This is a striking difference to our problem where the number of constraints is simply the "sum" of constraints in each dimension alone. To derive this fundamental result, we first state a technical preliminary needed in the proof of the result.

**Lemma 4** *Let* $\eta' < \eta''$ *and* $\theta' < \theta'' < \theta'''$ *and suppose* $q(\theta, \eta'') \leq \eta'$ *for* $\theta \in [\theta', \theta''']$. *Then, for all* $(\theta, \eta) \in \{\eta', \eta''\} \times [\theta', \theta''']$ *and all* $\left(\hat{\theta}, \hat{\eta}\right) \in \{\eta', \eta''\} \times [\theta', \theta''']$ :
*i)* $\left(x\left(\hat{\theta}, \hat{\eta}\right) - x(\theta, \eta)\right)\left(\hat{\theta} - \theta\right) \leq 0$; *and*
*ii) if* $x(\theta, \eta')$ *is continuous in* $\theta$ *at* $\theta''$, *then* $x(\theta'', \eta'') = x(\theta'', \eta')$; *moreover*
*iii)* $\Pi(\theta, \theta, \eta') = \Pi(\theta, \theta, \eta'')$

Part i) of the Lemma is a straightforward generalization of the monotonicity properties of incentive compatible solutions in one dimension. The difference is that $\eta$ is allowed to vary as well. As is well known, monotonic functions are continuous almost everywhere. Hence, part ii) applies almost everywhere, stating that bunching of quantity schedules will arise if two types $(\theta, \eta')$ and $(\theta, \eta'')$ both produce qualities that are feasible for both types. Finally, part iii) says that if types $(\theta, \eta')$ and $(\theta, \eta'')$ can mimic each other, then their profits must be the same. Using these properties, we can now show that a mechanism is incentive compatible if and only if no type has any incentive to misrepresent his type in one dimension at a time.

**Lemma 5** *The incentive constraint* (5) *is satisfied if and only if the constraints*

$$\Pi(\theta, \theta, \eta) \geq \Pi\left(\hat{\theta}, \theta, \eta\right) \text{ for all } \hat{\theta} \tag{25}$$

*and*

$$\Pi(\theta, \theta, \eta) \geq \Pi(\theta, \theta, \hat{\eta}) \text{ for all } \hat{\eta} \text{ such that } q(\theta, \hat{\eta}) \leq \eta \tag{26}$$

*are satisfied.*

The intuition for the result is quite straightforward. Since costs are additively separable in a part related to quality and quantity, and in a part related to quantity and the firm's cost parameter $\theta$, the firm's incentive to misrepresent its cost parameter $\theta$ depends solely on the quantity allocation. The intuition why only the one-dimensional constraints are relevant is best seen in the case where a type $(\theta, \eta)$ considers mimicking a type $\left(\hat{\theta}, \hat{\eta}\right)$ with $\hat{\eta} < \eta$. The profit type $(\theta, \eta)$ obtains this way is $\Pi\left(\hat{\theta}, \theta, \hat{\eta}\right)$, just the same profit type $(\theta, \hat{\eta})$ obtains when it mimics type $\left(\hat{\theta}, \hat{\eta}\right)$. But by (25) applied to type $(\theta, \hat{\eta})$, we know that it would be better to state the true cost parameter $\theta$, so $\Pi(\theta, \theta, \hat{\eta}) \geq \Pi\left(\hat{\theta}, \theta, \hat{\eta}\right)$. But then, by constraint (26), being truthful about the quality capacity would result in an even higher profit. Using Lemma 4 above, we can generalize this insight also to the case where the type mimicked, $\left(\hat{\theta}, \hat{\eta}\right)$, is such that $\hat{\eta} > \eta$. If the deviation is feasible in the first place, then the quantity allocation and the profit must be exactly the same as if type $\left(\hat{\theta}, \eta\right)$ was mimicked. But then, by (25), this deviation gives rise to a profit that is weakly lower than the profit obtained by being truthful.

The crucial property of the problem that allows us to reduce the dimensionality of the problem in this way is that the profit $\Pi\left(\hat{\theta}, \theta, \hat{\eta}\right)$ depends only on messages and the true cost parameter, but not on the true quality capacity parameter.

Building on the Lemmas four and five, we can bring the incentive constraints into a more tractable form. Let $\Lambda\left(\hat{\theta}, \theta, \eta\right) \equiv \left\{\hat{\theta}, \hat{\eta} : q\left(\hat{\theta}, \hat{\eta}\right) \leq \eta\right\}$ and let $\pi(\theta, \eta) \equiv \max_{\hat{\theta}, \hat{\eta} \in \Lambda\left(\hat{\theta}, \theta, \eta\right)} \Pi\left(\hat{\theta}, \theta, \hat{\eta}\right)$.

**Lemma 6** *The incentive and participation constraints are satisfied if and only if i)*

$$t(\theta, \eta) = C(X(p(\theta, \eta), q(\theta, \eta)), q(\theta, \eta), \theta) + \pi(\theta, \eta) - p(\theta, \eta) X(p(\theta, \eta), q(\theta, \eta)), \tag{27}$$

*where*

$$\pi(\theta, \eta) = \pi\left(\overline{\theta}, \eta\right) + \int_{\theta}^{\overline{\theta}} c_y(X(p(y, \eta), q(\theta, \eta)), y)\, dy \tag{28}$$

*and* $X(p(\theta, \eta), q(\theta, \eta))$ *is non-increasing in* $\theta$ *for all* $\eta$ *and* $\pi\left(\overline{\theta}, \eta\right) \geq 0$ *for all* $\eta$,
*ii)* $\pi(\theta, \eta)$ *is non-decreasing in* $\eta$ *for all* $\theta$ *and*
*iii) if for* $\eta' < \eta''$ $q(\theta, \eta'') \leq \eta'$ *then* $\pi(\theta, \eta') = \pi(\theta, \eta'')$ *and* $X(p(\theta, \eta'), q(\theta, \eta')) = X(p(\theta, \eta''), q(\theta, \eta''))$.

We can compute the rent of a firm of type $(\theta, \eta)$ by summing the rent of the most inefficient cost type for a given quality capacity, $\pi\left(\overline{\theta}, \eta\right)$, and the marginal changes of the firm's rent with respect to changes in its cost parameter $\theta$. Notice that (28) allows for the case where $\pi\left(\overline{\theta}, \eta\right) > 0$, so some high cost types may receive rents. Apart from this, we can essentially use the same procedure as in Lemma 1. There are additional conditions, ii) and iii) in the lemma, that are introduced through incentive compatibility conditions in the $\eta$ dimension. ii) requires that the firm's rent must be non-decreasing in $\eta$; this is required because mimicking a firm with a lower $\eta$ is - by technical feasibility of the quality allocation - always possible. Condition iii) summarizes the essence of Lemma 4; if two types with the same cost parameter $\theta$ but different quality capacities can mimic each other, then they must receive the same rent and they must produce the same quantities. This final condition in the Lemma is somewhat difficult to treat analytically. The problem is that the regulator's ability to implement distinct quantity schedules for firms of the same cost but with different quality capacities depends on the allocation of quality for these firms. Thus, to solve the problem, we need to have a good guess about the quality allocation. Before we turn to this question, it is worth discussing some important differences between this problem and other problems of multi-dimensional screening.

A crucial obstacle the multi-dimensional screening problem faces is that there is no natural order of types (see Rochet and Choné (2003). In their problem it would not make sense to compute the rent of a firm simply by integrating up from the most inefficient type for each given $\eta$. A deviation of a type $(\theta, \eta)$ to some statement $\left(\hat{\theta}, \hat{\eta}\right)$ must be ruled out irrespective of the particular path connecting $(\theta, \eta)$ and $\left(\hat{\theta}, \hat{\eta}\right)$. In our problem, only orthogonal incentive constraints play a role, and only those in the $\theta$-dimension are necessarily binding at the optimum. Using these binding incentive constraints, we obtain conditions (27) and (28). Incentive compatibility in the $\eta$ dimension is then a relatively simple monotonicity condition in the $\eta$ dimension.

## 4.2 The optimal allocation of quality

Recall from our analysis of the first-best allocation that the first-best optimal quality allocation, $q^{fb}(\theta, \eta)$, satisfies either condition (10) or (11) given the optimal pricing schedule $p^{fb}(\theta, \eta)$. Define the quality allocation schedule $q^{fb}(\theta, \eta; p(\theta, \eta))$ as one satisfying either condition (10) or (11), but for an arbitrary pricing schedule $p(\theta, \eta)$. The schedule $q^{fb}(\theta, \eta; p(\theta, \eta))$ provides a lower bound on the second best allocation:

**Proposition 5** *An optimal second-best allocation entails* $q^*(\theta, \eta) \geq q^{fb}(\theta, \eta; p(\theta, \eta))$ *for all* $\theta, \eta$.

The intuition for this result is as follows. Given additive separability of the cost function, the regulator can just compensate the firm for its additional cost if it is asked to provide higher quality. The regulator does not have to pay more than the increase in pure economic costs, precisely because there is no interaction between $q$ and $\theta$ in the firm's cost function; hence the firm's informational rent with respect to its information $\theta$ is not affected by changes in the quality allocation. In addition to that, one needs to be sure that changing the quality allocation does not open new deviation possibilities for the firm. Clearly, that is true if we raise the quality allocation pointwise, starting from an allocation below the first-best one for a given pricing rule. The reason is that such a change makes deviations for the firm more difficult, so if the initial allocation is incentive compatible, the new one is as well. Additive separability of the cost function is obviously crucial for this result. If the cost function were not additively separable in $\theta$ and $q$, then changing the allocation would change the agent's incentive to mimic other types.

We are now ready to characterize the complete solution to the regulation problem in a number of cases.

# 5 Tractable Cases

Interesting cases that can be studied are the extreme cases where the optimal quality schedule satisfies either $q^*(\theta,\eta) \leq \underline{\eta}$ for all $(\theta,\eta)$, or $q^*(\theta,\eta) = \eta$ for all $(\theta,\eta)$. The former case arises as part of an optimum if $\underline{\eta}$ is sufficiently large; the latter case arises as part of an optimum when $\overline{\eta}$ is sufficiently small.

## 5.1 The Case of $\overline{\eta}$ small

Proposition five suggests a simple solution procedure for the case where the highest capacity for quality is small. In particular, let $\overline{\eta}$ be so small that all types are constrained for some pricing schedule $p(\theta,\eta)$, that is $q^{fb}(\theta,\eta;p(\theta,\eta)) = \eta$ for all $\theta,\eta$. Of course, this argument involves some circularity, because we really need $q^{fb}(\theta,\eta;p^*(\theta,\eta)) = \eta$ for all $\theta,\eta$, where $p^*(\theta,\eta)$ is the optimal pricing schedule, which is not known to begin with. However, for $\overline{\eta}$ small enough, the capacity constraint must become binding for all types at the optimal pricing schedule. To show this, we solve the problem assuming that $q^*(\theta,\eta) = \eta$ for all $\theta,\eta$. Using the pricing schedule that is optimal for this quality allocation rule, we can verify from the solution that it satisfies $q^{fb}(\theta,\eta;p^*(\theta,\eta)) = \eta$ for all $\theta,\eta$.

A binding capacity constraint simplifies the problem dramatically. By definition, the feasibility constraint is always met. Moreover, types will only be tempted to mimic downwards in the $\eta$ dimension, but not upwards. Hence, $\pi(\theta,\eta)$ must be non-decreasing in $\eta$ to rule out downward deviations. With $q(\theta,\eta) = \eta$ for all $\theta,\eta$, problem (4) subject to (5), (6), and (7), is equivalent to the problem

$$\max_{x(\cdot;\cdot),\pi(\overline{\theta},\cdot)} \mathcal{E}_{\mathcal{H}}\left[\int_{\underline{\theta}}^{\overline{\theta}}\Big[V\left(x\left(\theta,\eta\right),\eta\right)-C\left(x\left(\theta,\eta\right),\eta,\theta\right)\Big]f\left(\theta\left|\eta\right.\right)d\theta\right] \tag{29}$$

$$-\mathcal{E}_{\mathcal{H}}\left[\int_{\underline{\theta}}^{\overline{\theta}}\left[\left(1-\alpha\right)\left(\pi\left(\overline{\theta},\eta\right)+c_{\theta}\left(x\left(\theta,\eta\right),\theta\right)\frac{F\left(\theta\left|\eta\right.\right)}{f\left(\theta\left|\eta\right.\right)}\right)\right]f\left(\theta\left|\eta\right.\right)d\theta\right]$$

$$s.t.\ x\left(\theta,\eta\right) \text{ non-increasing in } \theta \text{ for all } \eta \text{ and} \tag{30}$$

$$\pi\left(\overline{\theta},\eta\right)+\int_{\theta}^{\overline{\theta}}c_{y}\left(x\left(y,\eta\right),y\right)dy \text{ non-decreasing in } \eta \text{ for all } \theta. \tag{31}$$

We now show that, given certain restrictions on the joint distribution of characteristics, the agent has no incentive to mimic another type who produces a lower quality level. Formally, we have the following result:

**Proposition 6** *If $\overline{\eta}$ is sufficiently small and $\eta$ and $\theta$ are affiliated and $B\left(x,q,\theta,\eta\right)$ is supermodular in $q$ and $x$ for all $\theta,\eta$ then unobservability of $\eta$ does not affect the solution to the principal's problem. Formally, the solution is given the quality schedule $q\left(\theta,\eta\right)=\eta$ for all $\left(\theta,\eta\right)$, by the pricing schedule* (24) *and the transfer schedule* (17) *.*

The formal proof of this statement is omitted, since the argument is obvious. Indeed, suppose one solves problem (29) under the assumptions in the Proposition, neglecting both constraints (30) and (31) and setting $\pi\left(\overline{\theta},\eta\right)=0$ for all $\eta$. By Proposition 4, the optimal quantity schedule arising from this problem, $x^{*}\left(\theta,\eta\right)$, will be non-decreasing in $\eta$ for each $\theta$. But then, constraint (31) will automatically be satisfied at $x^{*}\left(\theta,\eta\right)$. Moreover, $x^{*}\left(\theta,\eta\right)$ is monotonic in $\theta$ as required by (30). Hence, the procedure picks up the solution to the full problem including its constraints (30) and (31).

The intuition is quite simple: there is no conflict of interest with respect to $\eta$. It is optimal from the regulator's perspective to let higher $\eta$ types produce more, because $x$ and $q$ are complements in the social surplus function, and because the higher is $\eta$ the higher is $\frac{f(\theta|\eta)}{F(\theta|\eta)}$, so the greater is the weight given to the principal's efficiency motive as opposed to the motive to limit the firm's rents. Hence, higher $\eta$ types receive larger rents so they have no incentive to report a lower value of $\eta$.

The case where $-\eta$ and $\theta$ are affiliated and $B\left(x,q,\theta,\eta\right)$ is submodular in

$q$ and $x$ for all $\theta, \eta$ is considerably harder to tackle analytically. However, a natural conjecture is the extreme opposite of the result in Proposition 6: we expect that types with the same $\theta$ but different $\eta$ will be bunched together for all $\theta$. Showing this for the general case requires quite heavy use of variational methods. However, the intuition can be gained by looking at a special case of our model, assuming that the cost function takes the form $c(x, \theta) = \theta x$ and that there are only two levels of $\eta$, i.e., $\eta \in \{\underline{\eta}, \overline{\eta}\}$.

Constraint (31) takes the form

$$\pi + \int_{\theta}^{\overline{\theta}} x(y, \overline{\eta})\, dy \geq \int_{\theta}^{\overline{\theta}} x(y, \underline{\eta})\, dy \text{ for all } \theta, \tag{32}$$

where $\pi \equiv \pi(\overline{\theta}, \overline{\eta})$ and where we have set $\pi(\overline{\theta}, \underline{\eta}) = 0$ as there is no reason to leave a rent to the firm of type $(\overline{\theta}, \underline{\eta})$. It is less clear that $\pi = 0$ is also optimal; when setting $\pi$, the regulator faces a trade-off between separating firms with different quality capacities and extracting rents from high quality capacity producers. To see this, it is useful to think of a two-step procedure where we compute the optimal quantity schedules first for any given $\pi$ and search for the optimal $\pi$ in a second step. Let $x^*(\theta; \underline{\eta})$ and $x^*(\theta; \overline{\eta})$ denote the optimal schedules conditional on $\eta$. Given submodularity and affiliation of $-\eta, \theta$, Proposition 4 shows that $x^*(\theta; \underline{\eta}) \geq x^*(\theta; \overline{\eta})$ for all $\theta$. Define

$$\overline{\pi} \equiv \int_{\underline{\theta}}^{\overline{\theta}} x^*(y; \underline{\eta})\, dy - \int_{\underline{\theta}}^{\overline{\theta}} x^*(y; \overline{\eta})\, dy.$$

For $\pi \geq \overline{\pi}$, offering $x^*(\theta; \underline{\eta})$ and $x^*(\theta; \overline{\eta})$ is incentive compatible, and thus optimal. Hence, offering a rent of $\overline{\pi}$ to firm $(\overline{\theta}, \overline{\eta})$ allows the regulator to tailor the quantity schedule to the firm's $\eta$-type, because (31) is slack at the optimum. For $0 < \pi < \overline{\pi}$, the constraint (32) must be binding for some $\theta$. If it were not binding for any $\theta$, then the solution would be the two schedules $x^*(\theta; \underline{\eta})$ and $x^*(\theta; \overline{\eta})$; but since $\pi < \overline{\pi}$, this leads to a contradiction. Take now any two schedules $x(\theta, \underline{\eta}) \geq x(\theta, \overline{\eta})$, where the inequality is strict for some $\theta$. Suppose there is $\widehat{\theta} > \underline{\theta}$ such that (32) holds as an equality at $\theta = \widehat{\theta}$. For any $\theta < \widehat{\theta}$, the incentive constraint can be written as

$$\int_{\theta}^{\widehat{\theta}} x(y, \overline{\eta})\, dy \geq \int_{\theta}^{\widehat{\theta}} x(y, \underline{\eta})\, dy$$

Since $x\left(\theta,\underline{\eta}\right) \geq x\left(\theta,\overline{\eta}\right)$, this inequality can only hold if $x\left(\theta,\underline{\eta}\right) = x\left(\theta,\overline{\eta}\right)$ for $\theta < \widehat{\theta}$. That is, there must be bunching of types $\left(\theta,\underline{\eta}\right)$ and $\left(\theta,\overline{\eta}\right)$ for all $\theta < \widehat{\theta}$. Thus, heuristically speaking, the regulator faces a trade-off between rent extraction and separation. Increasing $\pi$ allows the regulator to solve the rents-versus-efficiency trade-off within firms of the same quality capacity but with different marginal costs in a better way, in the sense that constraint (32) is binding on a smaller set of types.

Even though looking at the special case simplifies our problem a great deal, the resulting control problem is still hard to solve. In particular, the problem involves two state variables and derivatives of first and second order. Nevertheless we are able to characterize a local optimum, which entails $\pi^* = 0$. In turn, this implies that there is bunching of all $\left(\theta,\underline{\eta}\right)$ and $\left(\theta,\overline{\eta}\right)$ types for all $\theta$. Let $f(\theta) \equiv \mathcal{E}_H\left[f(\theta\,|\eta)\right]$ and let $F(\theta) = \int_{\underline{\theta}}^{\theta} f(y)\,dy$.

**Proposition 7** *Suppose that $\overline{\eta}$ is sufficiently small and $-\eta$ and $\theta$ are affiliated and $B(x,q,\theta,\eta)$ is submodular in $q$ and $x$ for all $\theta,\eta$ and let $c(x,\theta) = \theta x$ and $\eta \in \left\{\underline{\eta},\overline{\eta}\right\}$. Then, a local solution to the regulation problem involves the quality schedule $q^*(\theta,\eta) = \eta$ for all $(\theta,\eta)$ and $\pi^* = 0$. If in addition $\frac{F(\theta)}{f(\theta)}$ is non-decreasing, then the quantity schedule satisfies*

$$\mathcal{E}_{H|\theta}\left[V_x\left(x^*(\theta),\eta\right) - C_x\left(x^*(\theta),\eta,\theta\right) \mid \Theta = \theta\right] - (1-\alpha)\frac{F(\theta)}{f(\theta)} = 0. \tag{33}$$

Around $\pi = 0$, a marginal increase in $\pi$ has two effects. On the one hand, all types $(\theta,\overline{\eta})$ receive a higher rent. Letting $\beta$ denote the probability that $\eta = \overline{\eta}$, the expected welfare cost of increasing $\pi$ is $\beta(1-\alpha)$. On the other hand, increasing $\pi$ allows the regulator to tailor the quantity schedules to the conditional distributions of $\theta$ given $\eta$ on a small interval of high cost types. However, the length of this interval goes to zero as $\pi$ goes to zero, which means that the measure of types that can be separated this way goes to zero as well. Hence, the costs of increasing $\pi$ outweigh the costs for small values of $\pi$. As a result the quantity schedule is optimized against an average distribution of types; more precisely, the marginal distribution of $\theta$.

We emphasize that - in contrast to all other results in this paper - Proposition 7 characterizes a local optimum instead of a global optimum. To show this, we merely have to demonstrate that increasing $\pi$ away from zero results in a welfare loss to the principal. This can be done because we can completely characterize the solution to the regulation problem for $\pi = 0$ by the quantity and quality schedules given in the Proposition 6. In contrast, for a positive $\pi$ smaller than $\overline{\pi}$, there is necessarily bunching in the classical sense, that is across some types with different $\theta$ but the same $\eta$. This makes it hard to characterize the gain from a further increase in $\pi$.

## 5.2 The case of $\underline{\eta}$ large

When $\underline{\eta}$ is large enough, the parameter $\eta$ loses its relevance altogether: the solution schedules for quality, price and transfer all become independent of $\eta$. To see this formally, simply observe that when $q(\theta,\eta) \leq \underline{\eta}$, we can apply Lemma 4 to all types $(\theta,\eta)$. Hence, it follows that for each $\theta$, $x(\theta,\eta)$ and $\pi(\theta,\eta)$ must be independent of $\eta$. Consider now the quality schedule. It is easy to see that the schedule must satisfy the condition $q^*(\theta,\eta) = q^{fb}(\theta,\eta;p(\theta,\eta))$ for all $\theta,\eta$. While Proposition 5 only provided a lower bound on the quality allocation, this bound must be tight if $q^*(\theta,\eta) < \eta$ for all $\theta,\eta$. The reason is as follows. In Proposition 5 we can only derive a lower bound on the quality allocation, because we must make sure that we do not introduce additional possibilities for deviations for any type. If we increase the quality schedule pointwise, we will never increase these possibilities, but will at most reduce them or leave them the same. But among allocations where every type can mimic any other type, this caveat is irrelevant, and we can extend the argument in the proof of Proposition 5 to show that $q^*(\theta,\eta) > q^{fb}(\theta,\eta;p(\theta,\eta))$ cannot arise at an optimum either. Hence, it follows that $q^*(\theta,\eta) = q^{fb}(\theta,\eta;p(\theta,\eta))$ for all $\theta,\eta$. We can now again solve our problem by conjecturing that $q^*(\theta,\eta) = q^{fb}(\theta,\eta;p^*(\theta,\eta)) < \eta$ for all $\theta,\eta$ and verify the conjecture from the optimal pricing schedule that arises as a solution to the problem this way. If $q^*(\theta,\eta) = q^{fb}(\theta,\eta;p^*(\theta,\eta)) < \eta$ for all $\theta,\eta$, then, by part iii of Lemma 6, $x^*(\theta,\eta)$ must be independent of $\eta$. But if $x^*(\theta,\eta)$ is independent of $\eta$, $q^*(\theta,\eta)$ must be independent of $\eta$ as well. In other words, the optimal allocation involves complete bunching in the $\eta$ dimension for each $\theta$. We can then state:

**Proposition 8** *For the case where $\underline{\eta}$ is large, suppose $\frac{F(\theta)}{f(\theta)}$ is non-decreasing in $\theta$. Then, the optimal quantity schedule satisfies*

$$\left(V_x\left(x^*(\theta),q(\theta)\right) - C_x\left(x^*(\theta),q(\theta),\theta\right) - (1-\alpha)\,c_{x\theta}\left(x^*(\theta),\theta\right)\frac{F(\theta)}{f(\theta)}\right) = 0, \tag{34}$$

*and the optimal quality schedule satisfies*

$$V_q\left(x(\theta),q^*(\theta)\right) - C_q\left(x(\theta),q^*(\theta),\theta\right) = 0. \tag{35}$$

We omit a formal proof of the statement, since it follows in an obvious way from our previous results. To complete the argument one can verify that $q^*(\theta)$ as defined by (35) satisfies $q^*(\theta) < \eta$ for all $\theta$ when $\underline{\eta}$ is large enough.

The intuition for the result is straightforward. If the capacity constraint on $q$ is never binding for any type, the regulator does not need to have this information, because the optimal allocation cannot depend on it. All types can mimic others both by exaggerating their quality capacity and by understating it. But then, to keep firms from misrepresenting their quality capacity, the firms' profits must be independent of $\eta$ for each $\theta$. Hence, the total cost of serving customers does not depend on $\eta$. But if all firms with the same $\theta$ receive the same profits, then it is also optimal to offer them the same allocation, that is to make them produce identical quality, quantity bundles.

The quantity schedules in Propositions 6 through 8 demonstrate that pricing is always weakly above marginal costs, despite the fact that the firm is better informed about two parameters rather than one. Since the crucial difference between the Lewis and Sappington [1988] model and ours is the verifiability assumption on realized demand, we conclude this verifiability assumption is what ultimately drives their result. Moreover, since Armstrong [1999] must maintain the same verifiability assumption to demonstrate that exclusion of some low-demand-high-cost firms is optimal in the Lewis and Sappington model, and we do not find exclusion at the optimum of our model, we conclude that the verifiability assumption is also at the heart of the exclusion result in the present context.[4]

## 6 Concluding Remarks

We have solved a regulation problem featuring two dimensional asymmetric information in some detail. We have shown that the qualitative features of optimal allocations in previous studies by Lewis and Sappington [1988] and Armstrong [1999] depend crucially on verifiability assumptions that might not hold in practice. Under our alternative verifiability conditions, pricing is always weakly above marginal costs and no firm is excluded. We have solved our problem completely for the case of extreme quality allocations, where firms are either always producing at capacity or never constrained by their capacity bounds. We leave an exploration of less extreme cases to future work.

One of our results, that under some conditions asymmetric information

[4]We want to emphasize that our critique of verifiability assumptions does not apply to Armstrong [1996], where exclusion is a robust phenomenon.

about one dimension of the problem is irrelevant, has also been found in other contexts. Malakhov and Vohra [2005a,2005b] and Iyengar and Kumar [2006] have studied auction problems where bidders' valuations and capacities for consumption are unknown. They show that the solution to the problem when only valuations are private information remains incentive compatible when the second dimension of private information is added. Beaudry, Blackorby, and Szalay [2008] have first obtained a result in the spirit of Proposition 6 of the present paper. The main difference between the Malakhov/Vohra and Iyengar/Kumar approaches to our own is as follows. In their problem the principal has two choice variables, and one of them - the quantity allocated to an agent - interacts non-trivially with the agent's types. In contrast, our results apply to problems where the principal has three choices to make and two of them - the quantity and the quality allocated to the agent - interact non-trivially with the agent's types. Taken together, these results demonstrate the usefulness of the model structures to obtain insights into the problem of multi-dimensional screening. In ongoing work, we apply our approach to a problem proposed by Che and Gale [2000] where the seller faces possibly budget-constrained buyers.

# 7 Appendix

**Proof of Lemma 1.** We begin showing that $X(p(y;\eta), q(\theta;\eta))$ must be non-increasing. Incentive compatibility requires that type $(\theta;\eta)$ has no incentive to mimic type $\left(\hat{\theta};\eta\right)$

$$\begin{aligned}&t(\theta;\eta) + p(\theta;\eta) X(p(\theta;\eta), q(\theta;\eta)) - C(X(p(\theta;\eta), q(\theta;\eta)), q(\theta;\eta), \theta)\\ &\geq t\left(\hat{\theta};\eta\right) + p\left(\hat{\theta};\eta\right) X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right) - C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \theta\right)\end{aligned}$$

and that type $\left(\hat{\theta};\eta\right)$ has no incentive to mimic type $(\theta;\eta)$

$$\begin{aligned}&t\left(\hat{\theta};\eta\right) + p\left(\hat{\theta};\eta\right) X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right) - C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \hat{\theta}\right)\\ &\geq t(\theta;\eta) + p(\theta;\eta) X(p(\theta;\eta), q(\theta;\eta)) - C\left(X(p(\theta;\eta), q(\theta;\eta)), q(\theta;\eta), \hat{\theta}\right).\end{aligned}$$

Summing these inequalities gives

$$\begin{aligned}&C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \theta\right) - C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \hat{\theta}\right)\\ &\geq C(X(p(\theta;\eta), q(\theta;\eta)), q(\theta;\eta), \theta) - C\left(X(p(\theta;\eta), q(\theta;\eta)), q(\theta;\eta), \hat{\theta}\right).\end{aligned}$$

Writing as integrals, we have

$$\int_{\hat{\theta}}^{\theta} C_z\left(X\left(p\left(\hat{\theta};\eta\right),q\left(\hat{\theta};\eta\right)\right),q\left(\hat{\theta};\eta\right),z\right)dz \geq \int_{\hat{\theta}}^{\theta} C_z\left(X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right),q\left(\theta;\eta\right),z\right)dz.$$

Since $C_{q\theta}=0$, this is equivalent to

$$\int_{\hat{\theta}}^{\theta} c_z\left(X\left(p\left(\hat{\theta};\eta\right),q\left(\hat{\theta};\eta\right)\right),z\right)dz \geq \int_{\hat{\theta}}^{\theta} c_z\left(X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right),z\right)dz.$$

Given that $c_{x\theta} \geq 0$ this requires for $\theta > \hat{\theta}$ that $X\left(p\left(\hat{\theta};\eta\right),q\left(\hat{\theta};\eta\right)\right) \geq X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right).$

Since $X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right)$ is monotonic in $\theta$ it is differentiable almost everywhere in $\theta$. Then, by the envelope theorem, almost everywhere

$$\pi_\theta\left(\theta;\eta\right) = -C_\theta\left(X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right),q\left(\theta;\eta\right),\theta\right) = -c_\theta\left(X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right),\theta\right).$$

Since $\pi_\theta\left(\theta;\eta\right)<0$, the participation constraint must be binding for type $\overline{\theta}$, so $\pi\left(\overline{\theta};\eta\right)=0$. Thus, we can write

$$\pi\left(\theta;\eta\right) = \int_{\theta}^{\overline{\theta}} c_y\left(X\left(p\left(y;q\left(y;\eta\right)\right),q\left(y;\eta\right)\right),y\right)dy.$$

Since $\pi\left(\theta;\eta\right) = t\left(\theta;\eta\right)+p\left(\theta;\eta\right)X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right)-C\left(X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right),q\left(\theta;\eta\right),\theta\right),$ we can write

$$\begin{aligned} t\left(\theta;\eta\right) &= C\left(X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right),q\left(\theta;\eta\right),\theta\right) \\ &\quad + \int_{\theta}^{\overline{\theta}} c_y\left(X\left(p\left(y;\eta\right),q\left(y;\eta\right)\right),y\right)dy - p\left(\theta;\eta\right)X\left(p\left(\theta;\eta\right),q\left(\theta;\eta\right)\right), \end{aligned}$$

which is the expression (17) given in the Lemma.

Consider now the sufficiency part. Almost everywhere $\Pi_{\hat{\theta}}\left(\hat{\theta},\theta;\eta\right)\Big|_{\hat{\theta}=\theta}=0.$ Differentiating totally with respect to $\hat{\theta}$ and $\theta$, and equating $d\hat{\theta}=d\theta$, we find

$$\Pi_{\hat{\theta}\hat{\theta}}\left(\hat{\theta},\theta;\eta\right)\Big|_{\hat{\theta}=\theta} = -\,\Pi_{\hat{\theta}\theta}\left(\hat{\theta},\theta;\eta\right)\Big|_{\hat{\theta}=\theta}.$$

Since

$$- \Pi_{\hat{\theta}\theta}\left(\hat{\theta}, \theta; \eta\right)\Big|_{\hat{\theta}=\theta} = c_{\theta x}\left(X\left(p\left(\theta;\eta\right), q\left(\theta;\eta\right)\right), \theta\right)\frac{dX}{d\theta} \leq 0,$$

we find that the single crossing condition in conjunction with $\frac{dX}{d\theta} \leq 0$ implies that the local second order condition is satisfied. Finally, consider non-local deviations to $\hat{\theta}$. We can write

$$\Pi\left(\hat{\theta}, \theta; \eta\right) = \Pi\left(\hat{\theta}, \hat{\theta}; \eta\right) + C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \hat{\theta}\right) - C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \theta\right).$$

Hence, incentive compatibility requires that

$$\Pi\left(\theta, \theta; \eta\right) \geq \Pi\left(\hat{\theta}, \hat{\theta}; \eta\right) + C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \hat{\theta}\right) - C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \theta\right).$$

Subtracting $\Pi\left(\hat{\theta}, \hat{\theta}; \eta\right)$ from both sides, substituting for $\pi\left(\theta;\eta\right)$ and $\pi\left(\hat{\theta};\eta\right)$, respectively, and using

$$\int_{\hat{\theta}}^{\theta} \pi_y\left(y;\eta\right) dy = -\int_{\hat{\theta}}^{\theta} c_y\left(X\left(p\left(y;\eta\right), q\left(y;\eta\right)\right), y\right) dy,$$

we can write this as

$$\begin{aligned} & -\int_{\hat{\theta}}^{\theta} c_y\left(X\left(p\left(y;\eta\right), q\left(y;\eta\right)\right), y\right) dy \\ \geq \; & C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \hat{\theta}\right) - C\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), q\left(\hat{\theta};\eta\right), \theta\right). \end{aligned}$$

Using additive separability of the cost function, we can substitute $-\int_{\hat{\theta}}^{\theta} c_y\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), y\right) dy$ for the difference on the right hand side. Rearranging the resulting inequality, we have

$$\int_{\hat{\theta}}^{\theta} c_y\left(X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right), y\right) dy \geq \int_{\hat{\theta}}^{\theta} c_y\left(X\left(p\left(y;\eta\right), q\left(y;\eta\right)\right), y\right) dy.$$

For $\hat{\theta} > \theta$, $X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right) \geq X\left(p\left(y;\eta\right), q\left(y;\eta\right)\right)$ for all $y \leq \hat{\theta}$, and hence the inequality is satisfied. For $\hat{\theta} < \theta$, $X\left(p\left(\hat{\theta};\eta\right), q\left(\hat{\theta};\eta\right)\right) \leq X\left(p\left(y;\eta\right), q\left(y;\eta\right)\right)$ for all $y \geq \hat{\theta}$. Switching the lower and upper bound of integration yields the desired insight. ■

**Proof of Lemma 2.** i) Recall that $B(x,q,\theta)$ is said to have non-decreasing differences in $(x,q)$ and $\theta$ if $B(x,q,\theta) - B(x,q,\theta')$ is non-decreasing in $(x,q)$ for every $\theta \geq \theta'$. Given additive separability of the cost function, $C(x,q,\theta) = c(x,\theta) + k(x,q)$, we have

$$\begin{aligned} & B(x,q,\theta) - B(x,q,\theta') \\ = \; & c(x,\theta') + (1-\alpha) c_\theta (x,\theta') \frac{F(\theta' | \eta)}{f(\theta' | \eta)} - \left( c(x,\theta) + (1-\alpha) c_\theta (x,\theta) \frac{F(\theta | \eta)}{f(\theta | \eta)} \right), \end{aligned}$$

which proves the result.

ii) We can write the difference in terms of integrals, as

$$\int_{\theta'}^{\theta} \left( c_\tau (x,\tau) + (1-\alpha) c_{\tau\tau} (x,\tau) \frac{F(\tau | \eta)}{f(\tau | \eta)} + (1-\alpha) c_\tau (x,\tau) \frac{\partial}{\partial \tau} \frac{F(\tau | \eta)}{f(\tau | \eta)} \right) d\tau.$$

Differentiating the integrand with respect to $x$, we find that

$$c_{\tau x} (x,\tau) + (1-\alpha) c_{\tau\tau x} (x,\tau) \frac{F(\tau | \eta)}{f(\tau | \eta)} + (1-\alpha) c_{\tau x} (x,\tau) \frac{\partial}{\partial \tau} \frac{F(\tau | \eta)}{f(\tau | \eta)} \geq 0$$

implies that $c(x,\theta) + (1-\alpha) c_\theta (x,\theta) \frac{F(\theta | \eta)}{f(\theta | \eta)}$ has non-decreasing differences in $x$ and $\theta$. ■

**Proof of Proposition 1.** i) are the first-order conditions resulting from pointwise maximization of the integrand in (18). ii) follows Topkis (1978); see also Theorem 2.8.1 in Topkis (1998). iii) follows also from Theorem 2.8.1 in Topkis (1998) when we maximize over $x$ and $\hat{q}$. Finally, by Lemma 1, these conditions characterize a monotonic quantity schedule $x^* (\theta; \eta)$; hence the resulting schedules are incentive compatible. ■

**Proof of Lemma 3.** i) We can write

$$\begin{aligned} B(x,q,\theta,\eta) - B(x,q,\theta,\eta') &= -(1-\alpha) c_\theta (x,\theta) \frac{F(\theta | \eta)}{f(\theta | \eta)} - \left( -(1-\alpha) c_\theta (x,\theta) \frac{F(\theta | \eta')}{f(\theta | \eta')} \right) \\ &= (1-\alpha) c_\theta (x,\theta) \left( \frac{F(\theta | \eta')}{f(\theta | \eta')} - \frac{F(\theta | \eta)}{f(\theta | \eta)} \right), \end{aligned}$$

Differentiating with respect to $x$, we have

$$\frac{\partial}{\partial x} \left( B(x,q,\theta,\eta) - B(x,q,\theta,\eta') \right) = (1-\alpha) c_{x\theta} (x,\theta) \left( \frac{F(\theta | \eta')}{f(\theta | \eta')} - \frac{F(\theta | \eta)}{f(\theta | \eta)} \right)$$

ii) Recall that two random variables are affiliated if for $\theta \geq \theta'$ and $\eta \geq \eta'$

$$f(\theta,\eta)\, f(\theta',\eta') \geq f(\theta',\eta)\, f(\theta,\eta').$$

Dividing on both sides by $g(\eta)\, g(\eta')$, where $g(\eta) = \int_{\underline{\theta}}^{\overline{\theta}} f(\theta,\eta)\, d\theta$ and $g(\eta') = \int_{\underline{\theta}}^{\overline{\theta}} f(\theta,\eta')\, d\theta$ are the marginal densities, we can write

$$f(\theta\,|\eta)\, f(\theta'\,|\eta') \geq f(\theta'\,|\eta)\, f(\theta\,|\eta').$$

Integrating over $\theta'$ between $\underline{\theta}$ and $\theta$ we find

$$f(\theta\,|\eta)\, F(\theta\,|\eta') \geq F(\theta\,|\eta)\, f(\theta\,|\eta').$$

Rearranging, we have

$$\frac{F(\theta\,|\eta')}{f(\theta\,|\eta')} \geq \frac{F(\theta\,|\eta)}{f(\theta\,|\eta)}$$

which is, if $\frac{F(\theta|\eta)}{f(\theta|\eta)}$ is differentiable in $\eta$, equivalent to $\frac{\partial}{\partial\eta}\left(\frac{F(\theta|\eta)}{f(\theta|\eta)}\right) \leq 0$. These inequalities are reversed for the case where $(-\eta,\theta)$ are affiliated. ■

**Proof of Proposition 2.** The proof is a direct application of Theorem 2.3 part ii) and remark 10 in Vives [1999]. i) is direct; to see ii) recall that $B(x,\hat{q},\theta,\eta)$ is supermodular in this case, which implies the result follows again from Vives theorem. ■

**Proof of Proposition 3.** We split the proof into two parts. We demonstrate our results first for the case where $\mathbf{H} = \left[\underline{\eta},\overline{\eta}\right]$, and extend them later to the case where $\mathbf{H}$ is discrete.

The case where $\mathbf{H} = \left[\underline{\eta},\overline{\eta}\right]$ : Differentiating (21) totally with respect to $x$, $q$, and $\eta$, we get

$$\begin{aligned} & \left(V_{xx}(x^*,q^*) - C_{xx}(x^*,q^*,\theta) - (1-\alpha)\, c_{xx\theta}(x^*,\theta)\frac{F(\theta\,|\eta)}{f(\theta\,|\eta)}\right) dx^* \\ & + \left(V_{xq}(x^*,q^*) - C_{xq}(x^*,q^*,\theta)\right) dq^* \\ = \; & (1-\alpha)\, c_{x\theta}(x^*,\theta)\frac{\partial}{\partial\eta}\frac{F(\theta\,|\eta)}{f(\theta\,|\eta)} d\eta. \end{aligned} \tag{36}$$

Differentiating (22) totally with respect to $x$ and $q$, we get

$$\left(V_{qx}(x^*,q^*) - C_{qx}(x^*,q^*,\theta)\right) dx^* + \left(V_{qq}(x^*,q^*) - C_{qq}(x^*,q^*,\theta)\right) dq^* = 0.$$

Rearranging, we have

$$dq^* = -\frac{(V_{qx}(x^*, q^*) - C_{qx}(x^*, q^*, \theta))}{V_{qq}(x^*, q^*) - C_{qq}(x^*, q^*, \theta)} dx^*. \tag{37}$$

Substituting from (37) for $dq$, using Young's theorem, the additive separability of the cost function, and rearranging, we get

$$\frac{dx^*}{d\eta} = \frac{(1-\alpha) c_{x\theta}(x^*, \theta) \frac{\partial}{\partial \eta} \frac{F(\theta|\eta)}{f(\theta|\eta)}}{\left(V_{xx}(x^*, q^*) - C_{xx}(x^*, q^*, \theta) - (1-\alpha) c_{xx\theta}(x^*, \theta) \frac{F(\theta|\eta)}{f(\theta|\eta)} - \frac{(V_{qx}(x^*,q^*) - k_{qx}(x^*,q^*))^2}{(V_{qq}(x^*,q^*) - k_{qq}(x^*,q^*))}\right)}.$$

The denominator is negative by concavity of $B(x, q, \theta, \eta)$ in $x, q$. Hence, $\frac{dx^*}{d\eta}$ is of the same sign as $-\frac{\partial}{\partial \eta} \frac{F(\theta|\eta)}{f(\theta|\eta)}$ is. Hence, if $\frac{\partial}{\partial \eta} \frac{F(\theta|\eta)}{f(\theta|\eta)} \leq 0$ ($\theta, \eta$ are affiliated), then $\frac{dx^*}{d\eta} \geq 0$. Dividing both sides of (37) by $d\eta$ we find that $\frac{dq^*}{d\eta}$ has the same sign as $\frac{dx^*}{d\eta}$ has if $B(x, q, \theta, \eta)$ is supermodular in $x, q$, and has the opposite sign if $B(x, q, \theta, \eta)$ is submodular in $x, q$.

The case where $\mathbf{H}$ is discrete: Consider $\eta', \eta'' \in \mathbf{H}$ with $\eta'' > \eta'$. Let

$$z(\gamma) \equiv \frac{F(\theta|\eta')}{f(\theta|\eta')} + \gamma \left( \frac{F(\theta|\eta'')}{f(\theta|\eta'')} - \frac{F(\theta|\eta')}{f(\theta|\eta')} \right) \text{ for } \gamma \in [0, 1].$$

Since $\theta, \eta'$, and $\eta''$ are fixed, we write $z(\gamma)$ for $z(\theta, \eta', \eta'', \gamma)$. Observe that $z(0) \equiv \frac{F(\theta|\eta')}{f(\theta|\eta')}$ and $z(1) \equiv \frac{F(\theta|\eta'')}{f(\theta|\eta'')}$. However, we define $z(\gamma)$ for all $\gamma \in [0, 1]$, so that we can apply differentiability methods, and hence the implicit function theorem. Define $x^*(\theta; \gamma)$ as the solution to the equation

$$V_x(x^*(\theta;\gamma), q(\theta;\gamma)) - C_x(x^*(\theta;\gamma), q(\theta;\gamma), \theta) - (1-\alpha) c_{x\theta}(x^*(\theta;\gamma), \theta) z(\gamma) = 0. \tag{38}$$

Notice that (38) is identically equal to (21) for $\gamma \in \{0, 1\}$, but (38) is well defined and differentiable in $\gamma$ for all $\gamma \in [0, 1]$. Similarly, define $q^*(\theta; \gamma)$ as the solution to the equation

$$V_q(x(\theta;\gamma), q^*(\theta;\gamma)) - C_q(x(\theta;\gamma), q^*(\theta;\gamma), \theta) = 0. \tag{39}$$

Notice again that (22) is identically equal to (22), for $\gamma \in \{0, 1\}$, and is well defined and differentiable in $\gamma$ for all $\gamma \in [0, 1]$. Using the same steps as above we have

$$\frac{dx^*}{d\gamma} = \frac{(1-\alpha) c_{x\theta}(x^*, \theta) z_\gamma(\gamma)}{\left(V_{xx}(x^*, q^*) - C_{xx}(x^*, q^*, \theta) - (1-\alpha) c_{xx\theta}(x^*, \theta) z(\gamma) - \frac{(V_{qx}(x^*,q^*) - k_{qx}(x^*,q^*))^2}{(V_{qq}(x^*,q^*) - k_{qq}(x^*,q^*))}\right)}.$$

Noting that

$$\int_0^1 \frac{dx^*}{d\gamma} d\gamma = x^*(\theta, \eta'') - x^*(\theta, \eta'),$$

we find

$$\begin{aligned} & x^*(\theta, \eta'') - x^*(\theta, \eta') \\ = & \int_0^1 \frac{(1-\alpha) c_{x\theta}(x^*, \theta) z_\gamma(\gamma)}{V_{xx}(x^*, q^*) - C_{xx}(x^*, q^*, \theta) - (1-\alpha) c_{xx\theta}(x^*, \theta) z(\gamma) - \frac{(V_{qx}(x^*, q^*) - k_{qx}(x^*, q^*))^2}{(V_{qq}(x^*, q^*) - k_{qq}(x^*, q^*))}} d\gamma. \end{aligned}$$

Since the denominator is negative, we have

$$sign(x^*(\theta, \eta'') - x^*(\theta, \eta')) = sign\left(\frac{F(\theta|\eta')}{f(\theta|\eta')} - \frac{F(\theta|\eta'')}{f(\theta|\eta'')}\right).$$

The sign of $\left(\frac{F(\theta|\eta')}{f(\theta|\eta')} - \frac{F(\theta|\eta'')}{f(\theta|\eta'')}\right)$ is given by Lemma 3, part ii). Using $(37)$, we can write

$$q^*(\theta, \eta'') - q^*(\theta, \eta') = \int_0^1 \frac{dq^*}{d\gamma} d\gamma = \int_0^1 \left(-\frac{(V_{qx}(x^*, q^*) - C_{qx}(x^*, q^*, \theta))}{V_{qq}(x^*, q^*) - C_{qq}(x^*, q^*, \theta)} \frac{dx^*}{d\gamma}\right) d\gamma$$

and the result follows. ■

**Proof of Proposition 4.** We consider the case where $\mathbf{H} = [\underline{\eta}, \overline{\eta}]$. From (21) for $q^*(\theta; \eta) = \eta$ we have the first-order condition for $x^*(\theta; \eta)$:

$$\left(V_x(x^*(\theta; \eta), \eta) - C_x(x^*(\theta; \eta), \eta, \theta) - (1-\alpha) c_{x\theta}(x^*(\theta; \eta), \theta) \frac{F(\theta|\eta)}{f(\theta|\eta)}\right) = 0$$

Differentiating totally with respect to $x$ and $\eta$ and rearranging, we get

$$\frac{dx^*}{d\eta} = \frac{\left(V_{xq}(x^*(\theta; \eta), \eta) - C_{xq}(x^*(\theta; \eta), \eta, \theta) - (1-\alpha) c_{x\theta}(x^*(\theta; \eta), \theta) \frac{\partial}{\partial \eta} \frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}{-\left(V_{xx}(x^*(\theta; \eta), \eta) - C_{xx}(x^*(\theta; \eta), \eta, \theta) - (1-\alpha) c_{xx\theta}(x^*(\theta; \eta), \theta) \frac{F(\theta|\eta)}{f(\theta|\eta)}\right)}.$$

The proof for the case where $\mathbf{H}$ is discrete using the same procedure as the proof for Proposition 3, and is therefore omitted. ■

**Proof of Lemma 4.** i) From the fact that $q(\theta, \eta'') \leq \eta'$ for $\theta \in [\theta', \theta''']$ we know that any type $(\theta, \eta')$ for $\theta \in [\theta', \theta''']$ can mimic any type $(\theta, \eta'')$ for

$\theta \in [\theta', \theta''']$. Hence, incentive compatibility requires that

$$t(\theta,\eta) + p(\theta,\eta) X(p(\theta,\eta), q(\theta,\eta)) - C(X(p(\theta,\eta), q(\theta,\eta)), q(\theta,\eta), \theta)$$
$$\geq t\left(\hat{\theta},\hat{\eta}\right) + p\left(\hat{\theta},\hat{\eta}\right) X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right) - C\left(X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right), q\left(\hat{\theta},\hat{\eta}\right), \theta\right)$$

and that type $\left(\hat{\theta},\hat{\eta}\right)$ has no incentive to mimic type $(\theta,\eta)$

$$t\left(\hat{\theta},\hat{\eta}\right) + p\left(\hat{\theta},\hat{\eta}\right) X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right) - C\left(X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right), q\left(\hat{\theta},\hat{\eta}\right), \hat{\theta}\right)$$
$$\geq t(\theta,\eta) + p(\theta,\eta) X(p(\theta,\eta), q(\theta,\eta)) - C\left(X(p(\theta,\eta), q(\theta,\eta)), q(\theta,\eta), \hat{\theta}\right).$$

Summing these inequalities gives

$$C\left(X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right), q\left(\hat{\theta},\hat{\eta}\right), \theta\right) - C\left(X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right), q\left(\hat{\theta},\hat{\eta}\right), \hat{\theta}\right)$$
$$\geq C(X(p(\theta,\eta), q(\theta,\eta)), q(\theta,\eta), \theta) - C\left(X(p(\theta,\eta), q(\theta,\eta)), q(\theta,\eta), \hat{\theta}\right).$$

Writing as integrals, we have (using $C_{q\theta} = 0$)

$$\int_{\hat{\theta}}^{\theta} c_z\left(X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right), z\right) dz \geq \int_{\hat{\theta}}^{\theta} c_z(X(p(\theta,\eta), q(\theta,\eta)), z)\, dz.$$

Given that $c_{x\theta} \geq 0$ this requires for $\theta > \hat{\theta}$ that $X\left(p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)\right) \geq X(p(\theta,\eta), q(\theta,\eta))$. Switching to $x(\theta,\eta)$ notation, we have proven the first part of the Lemma.

ii) Suppose $x(\theta,\eta')$ is continuous in $\theta$ at $\theta''$. From our analysis of the one-dimensional problem, we know that $x(\theta',\eta') \geq x(\theta'',\eta') \geq x(\theta''',\eta')$. Taking limits, we have, by continuity, $\lim_{\theta' \nearrow \theta''} x(\theta',\eta') = \lim_{\theta''' \searrow} x(\theta''',\eta') = x(\theta'',\eta')$. We have just shown in part i) that $x(\theta',\eta') \geq x(\theta'',\eta'') \geq x(\theta''',\eta')$. But, this can be true for all $\theta' \leq \theta'' \leq \theta'''$ only if $x(\theta'',\eta') = x(\theta'',\eta'')$.

iii) For any $\theta \in [\theta',\theta''']$, $q(\theta,\eta'') \leq \eta'$, and hence incentive compatibility for type $(\theta,\eta')$ requires that he has no incentive to mimic type $(\theta,\eta'')$, so

$$\Pi(\theta,\theta,\eta') \geq \Pi(\theta,\theta,\eta'').$$

For type $(\theta,\eta'')$, incentive compatibility requires that

$$\Pi(\theta,\theta,\eta'') \geq \Pi(\theta,\theta,\eta').$$

Both inequalities can hold only if

$$\Pi(\theta,\theta,\eta'') = \Pi(\theta,\theta,\eta').$$

■

**Proof of Lemma 5.** It is easy to see that the two one dimensional constraints are necessary for incentive compatibility. We now show they are also sufficient for incentive compatibility. Suppose a type $(\theta,\eta)$ mimics a type $\left(\hat{\theta},\hat{\eta}\right)$ with $q\left(\hat{\theta},\hat{\eta}\right) \leq \eta$, so that the deviation is feasible for type $(\theta,\eta)$. We will show that (25), (26), imply that such a deviation is suboptimal. There are two types of deviations to consider. i) $\hat{\eta} \leq \eta$; ii) $\hat{\eta} > \eta$.

The proof for case i) is very simple. By mimicking type $\left(\hat{\theta},\hat{\eta}\right)$, type $(\theta,\eta)$ he obtains profit $\Pi\left(\hat{\theta},\theta,\hat{\eta}\right) \equiv t\left(\hat{\theta},\hat{\eta}\right) + p\left(\hat{\theta},\hat{\eta}\right) X\left(p\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right)\right)$ $-C\left(X\left(p\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right)\right),q\left(\hat{\theta},\hat{\eta}\right),\theta\right)$. But by incentive compatibility for type $(\theta,\hat{\eta})$ in the $\theta$ dimension, type $(\theta,\eta)$ could obtain more by mimicking type $(\theta,\hat{\eta})$, since

$$\Pi(\theta,\theta,\hat{\eta}) \geq \Pi\left(\hat{\theta},\theta,\hat{\eta}\right).$$

But then, by incentive compatibility for type $(\theta,\eta)$ in the $\eta$ dimension, we have

$$\Pi(\theta,\theta,\eta) \geq \Pi(\theta,\theta,\hat{\eta}).$$

Hence, the deviation gives a weakly lower profit than announcing the true type.

ii) From the Lemma above $x\left(\hat{\theta},\hat{\eta}\right) = x\left(\hat{\theta},\eta\right)$ and $\Pi\left(\hat{\theta},\hat{\theta},\eta\right) = \Pi\left(\hat{\theta},\hat{\theta},\hat{\eta}\right)$. From (25) for type $(\theta,\eta)$, we have

$$\Pi(\theta,\theta,\eta) \geq \Pi\left(\hat{\theta},\theta,\eta\right)$$

Now, using the additive separability of the cost function,

$$\begin{aligned}\Pi\left(\hat{\theta},\theta,\eta\right) &= \Pi\left(\hat{\theta},\hat{\theta},\eta\right) + c\left(X\left(p\left(\hat{\theta},\eta\right),q\left(\hat{\theta},\eta\right)\right),\hat{\theta}\right) - c\left(X\left(p\left(\hat{\theta},\eta\right),q\left(\hat{\theta},\eta\right)\right),\theta\right) \\ &= \Pi\left(\hat{\theta},\hat{\theta},\eta\right) + c\left(x\left(\hat{\theta},\eta\right),\hat{\theta}\right) - c\left(x\left(\hat{\theta},\eta\right),\theta\right).\end{aligned}$$

Similarly, we can write

$$\Pi\left(\hat{\theta},\theta,\hat{\eta}\right) = \Pi\left(\hat{\theta},\hat{\theta},\hat{\eta}\right) + c\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\theta}\right) - c\left(x\left(\hat{\theta},\hat{\eta}\right),\theta\right)$$

Since $\Pi\left(\hat{\theta},\hat{\theta},\hat{\eta}\right)=\Pi\left(\hat{\theta},\hat{\theta},\eta\right)$ and $x\left(\hat{\theta},\hat{\eta}\right)=x\left(\hat{\theta},\eta\right)$ it follows that

$$\Pi\left(\hat{\theta},\theta,\hat{\eta}\right)=\Pi\left(\hat{\theta},\theta,\eta\right).$$

Hence, we have shown that $\Pi(\theta,\theta,\eta)\geq\Pi\left(\hat{\theta},\theta,\eta\right)$ implies also $\Pi(\theta,\theta,\eta)\geq\Pi\left(\hat{\theta},\theta,\hat{\eta}\right)$. ■

**Proof of Lemma 6.** By Lemma 5, the one dimensional constraints are necessary and sufficient for the constraint ruling out two-dimensional deviations. Applying the same procedure as in Lemma 1 to (25), we get conditions (27) and (28). By Lemma 1 (27) and (28) in conjunction with monotonicity of $X$ are necessary and sufficient for (25). ii) follows directly from (26), because mimicking a firm with a lower quality capacity is always feasible. iii) is simply restating conditions ii) and iii) from Lemma 4. ■

**Proof of Proposition 6.** Take any incentive compatible allocation given by the triple of schedules $p\left(\hat{\theta},\hat{\eta}\right), q\left(\hat{\theta},\hat{\eta}\right)$, and $t\left(\hat{\theta},\hat{\eta}\right)$ for all $\hat{\theta},\hat{\eta}$. Suppose for some $\hat{\theta},\hat{\eta}$, we have $q\left(\hat{\theta},\hat{\eta}\right)<q^{fb}\left(\hat{\theta},\hat{\eta};p\left(\hat{\theta},\hat{\eta}\right)\right)$. Let $\Phi$ denote the set of $\left(\hat{\theta},\hat{\eta}\right)$ such that $q\left(\hat{\theta},\hat{\eta}\right)<q^{fb}\left(\hat{\theta},\hat{\eta};p\left(\hat{\theta},\hat{\eta}\right)\right)$. Then, we can change the allocation to the new triple of schedules $\tilde{p}\left(\hat{\theta},\hat{\eta}\right), \tilde{q}\left(\hat{\theta},\hat{\eta}\right)$, and $\tilde{t}\left(\hat{\theta},\hat{\eta}\right)$ for all $\hat{\theta},\hat{\eta}$ as follows. We set $\tilde{q}\left(\hat{\theta},\hat{\eta}\right)=q^{fb}\left(\hat{\theta},\hat{\eta};p\left(\hat{\theta},\hat{\eta}\right)\right)$ for all $\left(\hat{\theta},\hat{\eta}\right)$, and set $\tilde{p}\left(\hat{\theta},\hat{\eta}\right)$ such that $X\left(\tilde{p}\left(\hat{\theta},\hat{\eta}\right),\tilde{q}\left(\hat{\theta},\hat{\eta}\right)\right)=X\left(p\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right)\right)$ for all $\hat{\theta},\hat{\eta}$. For all $\left(\hat{\theta},\hat{\eta}\right)\in\Phi$ we adjust the transfers from the initial transfers $t\left(\hat{\theta},\hat{\eta}\right)$, to the new transfers

$$\begin{aligned}\tilde{t}\left(\hat{\theta},\hat{\eta}\right) &= t\left(\hat{\theta},\hat{\eta}\right)+p\left(\hat{\theta},\hat{\eta}\right)X\left(p\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right)\right)-\tilde{p}\left(\hat{\theta},\hat{\eta}\right)X\left(\tilde{p}\left(\hat{\theta},\hat{\eta}\right),\tilde{q}\left(\hat{\theta},\hat{\eta}\right)\right)\\ &\quad +C\left(X\left(\tilde{p}\left(\hat{\theta},\hat{\eta}\right),\tilde{q}\left(\hat{\theta},\hat{\eta}\right)\right),\tilde{q}\left(\hat{\theta},\hat{\eta}\right),\hat{\theta}\right)-C\left(X\left(p\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right)\right),q\left(\hat{\theta},\hat{\eta}\right),\hat{\theta}\right).\end{aligned}$$

We first show that the new allocation with the new transfers is incentive compatible. Then, we show that the surplus to the principal has increased under the new allocation.

By assumption, the functions $p,q$ and $t$ are incentive compatible, that is for all $\theta, \eta$ and all $\hat{\theta}, \hat{\eta}$ such that $q\left(\hat{\theta}, \hat{\eta}\right) \leq \eta$

$$\begin{aligned} & t(\theta, \eta) + p(\theta, \eta) X(p(\theta, \eta), q(\theta, \eta)) - C(X(p(\theta, \eta), q(\theta, \eta)), q(\theta, \eta), \theta) \quad (40) \\ & \geq t\left(\hat{\theta}, \hat{\eta}\right) + p\left(\hat{\theta}, \hat{\eta}\right) X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right) - C\left(X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right), q\left(\hat{\theta}, \hat{\eta}\right), \theta\right). \end{aligned}$$

Incentive compatibility of the new functions $\tilde{p}, \tilde{q}$, and $\tilde{t}$ is given if and only if for all $\theta, \eta$ and all $\hat{\theta}$ and all $\tilde{q}\left(\hat{\theta}, \hat{\eta}\right) \leq \eta$

$$\begin{aligned} & \tilde{t}(\theta, \eta) + \tilde{p}(\theta, \eta) X(\tilde{p}(\theta, \eta), \tilde{q}(\theta, \eta)) - C(X(\tilde{p}(\theta, \eta), \tilde{q}(\theta, \eta)), \tilde{q}(\theta, \eta), \theta) \quad (41) \\ & \geq \tilde{t}\left(\hat{\theta}, \hat{\eta}\right) + \tilde{p}\left(\hat{\theta}, \hat{\eta}\right) X\left(\tilde{p}\left(\hat{\theta}, \hat{\eta}\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right)\right) - C\left(X\left(\tilde{p}\left(\hat{\theta}, \hat{\eta}\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right)\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right), \theta\right). \end{aligned}$$

Substituting for $\tilde{t}\left(\hat{\theta}, \hat{\eta}\right)$, and simplifying, this amounts to

$$\begin{aligned} & t(\theta, \eta) + p(\theta, \eta) X(p(\theta, \eta), q(\theta, \eta)) \\ & + C\left(X(\tilde{p}(\theta, \eta), \tilde{q}(\theta, \eta)), \tilde{q}(\theta, \eta), \theta\right) - C\left(X(p(\theta, \eta), q(\theta, \eta)), q(\theta, \eta), \theta\right) \\ & - C(X(\tilde{p}(\theta, \eta), \tilde{q}(\theta, \eta)), \tilde{q}(\theta, \eta), \theta) \\ & \geq t\left(\hat{\theta}, \hat{\eta}\right) + p\left(\hat{\theta}, \hat{\eta}\right) X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right) \\ & + C\left(X\left(\tilde{p}\left(\hat{\theta}, \hat{\eta}\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right)\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right), \hat{\theta}\right) - C\left(X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right), q\left(\hat{\theta}, \hat{\eta}\right), \hat{\theta}\right) \\ & - C\left(X\left(\tilde{p}\left(\hat{\theta}, \hat{\eta}\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right)\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right), \theta\right). \end{aligned}$$

Adding and subtracting $C\left(X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right), q\left(\hat{\theta}, \hat{\eta}\right), \theta\right)$ on the right-hand side, and noting (40), the new incentive constraint is satisfied if

$$\begin{aligned} 0 \geq\ & C\left(X\left(\tilde{p}\left(\hat{\theta}, \hat{\eta}\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right)\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right), \hat{\theta}\right) - C\left(X\left(\tilde{p}\left(\hat{\theta}, \hat{\eta}\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right)\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right), \theta\right) \\ & + C\left(X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right), q\left(\hat{\theta}, \hat{\eta}\right), \theta\right) - C\left(X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right), q\left(\hat{\theta}, \hat{\eta}\right), \hat{\theta}\right). \end{aligned}$$

Using $C(x, q, \theta) = c(x, \theta) + k(x, q)$, and $X\left(p\left(\hat{\theta}, \hat{\eta}\right), q\left(\hat{\theta}, \hat{\eta}\right)\right) = X\left(\tilde{p}\left(\hat{\theta}, \hat{\eta}\right), \tilde{q}\left(\hat{\theta}, \hat{\eta}\right)\right) \equiv$

$x\left(\hat{\theta},\hat{\eta}\right)$, we can write this as

$$
\begin{aligned}
0 \geq\ & c\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\theta}\right)-c\left(x\left(\hat{\theta},\hat{\eta}\right),\theta\right)+c\left(x\left(\hat{\theta},\hat{\eta}\right),\theta\right)-c\left(x\left(\hat{\theta},\hat{\eta}\right),\hat{\theta}\right) \\
& +k\left(x\left(\hat{\theta},\hat{\eta}\right),\tilde{q}\left(\hat{\theta},\hat{\eta}\right)\right)-k\left(x\left(\hat{\theta},\hat{\eta}\right),\tilde{q}\left(\hat{\theta},\hat{\eta}\right)\right)+k\left(x\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right)\right)-k\left(x\left(\hat{\theta},\hat{\eta}\right),q\left(\hat{\theta},\hat{\eta}\right)\right) \\
=\ & 0.
\end{aligned}
$$

Hence (41) is satisfied.

Finally, the system is overall incentive compatible. This follows, because the schedule $\tilde{q}(\theta,\eta)$ is weakly higher than the schedule $q(\theta,\eta)$, which implies that the set of messages $\left(\hat{\theta},\hat{\eta}\right)$ such that $\tilde{q}\left(\hat{\theta},\hat{\eta}\right)\leq\eta$ is a subset of the set of messages $\left(\hat{\theta},\hat{\eta}\right)$ such that $q\left(\hat{\theta},\hat{\eta}\right)\leq\eta$. Hence, for each type at most as many deviations are feasible in the new system as in the old system.

Consider now the government's surplus. By construction, the firm's equilibrium payoffs are unchanged, as

$$
\begin{aligned}
& t(\theta,\eta)+p(\theta,\eta)X(p(\theta,\eta),q(\theta,\eta))-\tilde{p}(\theta,\eta)X(\tilde{p}(\theta,\eta),\tilde{q}(\theta,\eta)) \\
& +C\Bigg(X(\tilde{p}(\theta,\eta),\tilde{q}(\theta,\eta)),\tilde{q}(\theta,\eta),\theta\Bigg)-C\Bigg(X(p(\theta,\eta),q(\theta,\eta)),q(\theta,\eta),\theta\Bigg) \\
& +\tilde{p}(\theta,\eta)X(\tilde{p}(\theta,\eta),\tilde{q}(\theta,\eta))-C(X(\tilde{p}(\theta,\eta),\tilde{q}(\theta,\eta)),\tilde{q}(\theta,\eta),\theta) \\
& =t(\theta,\eta)+p(\theta,\eta)X(p(\theta,\eta),q(\theta,\eta))-C\Bigg(X(p(\theta,\eta),q(\theta,\eta)),q(\theta,\eta),\theta\Bigg).
\end{aligned}
$$

In other words, the firm is just compensated for the increase in his cost of production, and the entire additional surplus goes to the principal. Hence, holding the pricing schedule $p(\theta,\eta)$ constant, any allocation that satisfies

$$
V_q\Bigg(X\Big(p(\theta,\eta),q(\theta,\eta)\Big),q(\theta,\eta)\Bigg)-C_q(X(p(\theta,\eta),q(\theta,\eta)),q(\theta,\eta),\theta)>0
$$

and $q(\theta,\eta)<\eta$ can be improved by raising $q(\theta,\eta)$. ■

**Proof of Proposition 7.** Let $\beta=\Pr[\eta=\overline{\eta}]$. We will solve our problem in two steps. In the first step we take $\pi$ as given and characterize optimal quantity and quality schedules. In the second step we maximize with respect to the choice $\pi$.

For a given, non-negative value of $\pi$ smaller than $\overline{\pi}$, we can write our problem as

$$\Gamma(\pi) = \max_{x(\theta,\overline{\eta}),x\left(\theta,\underline{\eta}\right)} \beta\left[\int\limits_{\underline{\theta}}^{\overline{\theta}}\Big[V\left(x\left(\theta,\overline{\eta}\right),\overline{\eta}\right) - C\left(x\left(\theta,\overline{\eta}\right),\overline{\eta},\theta\right)\Big] f\left(\theta\left|\overline{\eta}\right.\right)d\theta\right]$$

$$-\beta\left[\int\limits_{\underline{\theta}}^{\overline{\theta}}\left[(1-\alpha)\left(\pi + x\left(\theta,\overline{\eta}\right)\frac{F\left(\theta\left|\overline{\eta}\right.\right)}{f\left(\theta\left|\overline{\eta}\right.\right)}\right)\right] f\left(\theta\left|\overline{\eta}\right.\right)d\theta\right]$$

$$+(1-\beta)\left[\int\limits_{\underline{\theta}}^{\overline{\theta}}\left[V\left(x\left(\theta,\underline{\eta}\right),\underline{\eta}\right) - C\left(x\left(\theta,\underline{\eta}\right),\underline{\eta},\theta\right)\right] f\left(\theta\left|\underline{\eta}\right.\right)d\theta\right]$$

$$-(1-\beta)\left[\int\limits_{\underline{\theta}}^{\overline{\theta}}\left[(1-\alpha)\,x\left(\theta,\underline{\eta}\right)\frac{F\left(\theta\left|\underline{\eta}\right.\right)}{f\left(\theta\left|\underline{\eta}\right.\right)}\right] f\left(\theta\left|\underline{\eta}\right.\right)d\theta\right]$$

$$s.t.\ x_\theta\left(\theta,\overline{\eta}\right) \leq 0 \text{ and } x_\theta\left(\theta,\underline{\eta}\right) \leq 0$$

$$\int\limits_{\theta}^{\overline{\theta}} x\left(y,\underline{\eta}\right)dy - \int\limits_{\theta}^{\overline{\theta}} x\left(y,\overline{\eta}\right)dy \leq \pi$$

Let $\underline{z} \equiv -\int_\theta^{\overline{\theta}} x\left(y,\underline{\eta}\right)dy$, $\overline{z} \equiv -\int_\theta^{\overline{\theta}} x\left(y,\overline{\eta}\right)dy$, $x\left(\theta,\underline{\eta}\right) = \underline{x} = \underline{z}_\theta$, and $x\left(\theta,\overline{\eta}\right) = \overline{x} = \overline{z}_\theta$. Moreover, let $\overline{y} = x_\theta\left(\theta,\overline{\eta}\right)$ and $\underline{y} = x_\theta\left(\theta,\underline{\eta}\right)$. We can view this as a control problem with Hamiltonian of the following form:

$$\begin{aligned}
H &= \left(V\left(\overline{x},\overline{\eta}\right) - C\left(\overline{x},\overline{\eta},\theta\right) - (1-\alpha)\,\overline{x}\frac{F\left(\theta\left|\overline{\eta}\right.\right)}{f\left(\theta\left|\overline{\eta}\right.\right)}\right)\beta f\left(\theta\left|\overline{\eta}\right.\right) \\
&\quad + \left(V\left(\underline{x},\underline{\eta}\right) - C\left(\underline{x},\underline{\eta},\theta\right) - (1-\alpha)\,\underline{x}\frac{F\left(\theta\left|\underline{\eta}\right.\right)}{f\left(\theta\left|\underline{\eta}\right.\right)}\right)(1-\beta)\,f\left(\theta\left|\underline{\eta}\right.\right) \\
&\quad + \overline{\lambda}\overline{y} + \underline{\lambda}\underline{y} + \overline{\kappa}\overline{x} + \underline{\kappa}\underline{x} \\
&\quad \mu\left(\pi - \left(\overline{z} - \underline{z}\right)\right) \\
&\quad - \overline{\omega}\overline{y} - \underline{\omega}\underline{y}
\end{aligned}$$

Differentiating with respect to state variables, we get the conditions of optimality

$$\begin{aligned}
\frac{\partial H}{\partial \overline{z}} &= -\mu = -\overline{\kappa}_\theta \\
\frac{\partial H}{\partial \underline{z}} &= \mu = -\underline{\kappa}_\theta
\end{aligned}$$

and

$$\frac{\partial H}{\partial \overline{x}} = \left( V_{\overline{x}}\left(\overline{x},\overline{\eta}\right) - C_{\overline{x}}\left(\overline{x},\overline{\eta},\theta\right) - (1-\alpha)\frac{F\left(\theta\left|\overline{\eta}\right.\right)}{f\left(\theta\left|\overline{\eta}\right.\right)}\right)\beta f\left(\theta\left|\overline{\eta}\right.\right) + \overline{\kappa} = -\overline{\lambda}_{\theta}$$

$$\frac{\partial H}{\partial \underline{x}} = \left( V_{\underline{x}}\left(\underline{x},\underline{\eta}\right) - C_{\underline{x}}\left(\underline{x},\underline{\eta},\theta\right) - (1-\alpha)\frac{F\left(\theta\left|\underline{\eta}\right.\right)}{f\left(\theta\left|\underline{\eta}\right.\right)}\right)(1-\beta) f\left(\theta\left|\underline{\eta}\right.\right) + \underline{\kappa} = -\underline{\lambda}_{\theta}$$

Moreover, the conditions of optimality with respect $\overline{y}$ and $\underline{y}$ are

$$\frac{\partial H}{\partial \overline{y}} = \overline{\lambda} - \overline{\omega} = 0$$

and

$$\frac{\partial H}{\partial \underline{y}} = \underline{\lambda} - \underline{\omega} = 0.$$

The Kuhn-Tucker conditions are $\overline{\omega}\overline{y} = 0, \overline{\omega} \geq 0, \overline{y} \leq 0$; and $\underline{\omega}\underline{y} = 0, \underline{\omega} \geq 0, \underline{y} \leq 0$. Moreover, $\mu\left(\pi - \left(\overline{z} - \underline{z}\right)\right) = 0$ $\mu \geq 0$ and $\pi - \left(\overline{z} - \underline{z}\right) \leq 0$. The transversality conditions are (since $\overline{z}\left(\underline{\theta}\right)$ and $\underline{z}\left(\underline{\theta}\right)$ are both free) $\overline{\kappa}\left(\underline{\theta}\right) = \underline{\kappa}\left(\underline{\theta}\right) = 0$, and $\overline{\lambda}\left(\underline{\theta}\right) = \overline{\lambda}\left(\overline{\theta}\right) = \underline{\lambda}\left(\underline{\theta}\right) = \underline{\lambda}\left(\overline{\theta}\right) = 0$ (since both $\overline{x}$ and $\underline{x}$ are free at both bounds).

It follows directly from the transversality conditions for $\overline{\lambda}$ and $\underline{\lambda}$, that

$$\int_{\underline{\theta}}^{\overline{\theta}} \overline{\lambda}_{\theta} d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \underline{\lambda}_{\theta} d\theta = 0.$$

Moreover, when $\overline{\omega}\left(\underline{\omega}\right) = 0$, $\overline{\lambda}\left(\underline{\lambda}\right) = 0$ as well.

Differentiating $\Gamma\left(\pi\right)$ with respect to $\pi$, making use of the conditions above (including $\int_{\underline{\theta}}^{\overline{\theta}} \overline{\lambda}_{\theta} d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \underline{\lambda}_{\theta} d\theta = 0$), we can write

$$\begin{aligned}
\Gamma_{\pi}\left(\pi\right) &= -\beta\left(1-\alpha\right) + \int_{\underline{\theta}}^{\overline{\theta}} \left( \begin{array}{c} \left(\overline{\lambda}_{\pi} - \overline{\omega}_{\pi}\right)\overline{y} + \left(\underline{\lambda}_{\pi} - \underline{\omega}_{\pi}\right)\underline{y} + \overline{\kappa}_{\pi}\overline{x} + \underline{\kappa}_{\pi}\underline{x} \\ +\mu_{\pi}\left(\pi - \left(\overline{z} - \underline{z}\right)\right) + \mu\left(1 - \left(\overline{z}_{\pi} - \underline{z}_{\pi}\right)\right) \end{array} \right) d\theta \\
&= -\beta\left(1-\alpha\right) + \int_{\underline{\theta}}^{\overline{\theta}} \left(\overline{\kappa}_{\pi}\overline{x} + \underline{\kappa}_{\pi}\underline{x}\right) d\theta
\end{aligned}$$

Using the transversality conditions for $\overline{\kappa}\left(\theta\right)$ and $\underline{\kappa}\left(\theta\right)$ and the equations of motion for these costate variables, we get

$$\overline{\kappa}(\theta) = \overline{\kappa}(\underline{\theta}) + \int\limits_{\underline{\theta}}^{\theta} \overline{\kappa}_{\tau} d\tau = \int\limits_{\underline{\theta}}^{\theta} \mu(\tau)\, d\tau$$

and

$$\underline{\kappa}(\theta) = -\int\limits_{\underline{\theta}}^{\theta} \mu(\tau)\, d\tau$$

Hence,

$$\overline{\kappa}_{\pi} = \int\limits_{\underline{\theta}}^{\theta} \mu_{\pi}(\tau)\, d\tau = -\underline{\kappa}_{\pi}.$$

When $\mu > 0$, we have $\overline{x} = \underline{x}$. When $\mu = 0$, we have $\overline{x} < \underline{x}$. (One can show this easily, because the reverse would lead into a contradiction.) So, we can write

$$\Gamma_{\pi}(\pi) = -\beta(1-\alpha) + \int\limits_{\underline{\theta}}^{\overline{\theta}} \int\limits_{\underline{\theta}}^{\theta} \mu_{\pi}(\tau)\, d\tau \left(\overline{x} - \underline{x}\right) d\theta$$

For $\pi = 0$, we have $\overline{x} = \underline{x}$ for all $\theta$. Hence,

$$\Gamma_{\pi}(0) = -\beta(1-\alpha) < 0.$$

With $\pi = 0$, we can simplify the Hamiltonian to

$$\begin{aligned} H \quad = \quad & \left( V(x, \overline{\eta}) - C(x, \overline{\eta}, \theta) - (1-\alpha)\, x \frac{F(\theta\,|\overline{\eta})}{f(\theta\,|\overline{\eta})} \right) \beta f(\theta\,|\overline{\eta}) \\ & + \left( V\left(x, \underline{\eta}\right) - C\left(x, \underline{\eta}, \theta\right) - (1-\alpha)\, x \frac{F\left(\theta\,|\underline{\eta}\right)}{f\left(\theta\,|\underline{\eta}\right)} \right) (1-\beta)\, f\left(\theta\,|\underline{\eta}\right) \\ & + \lambda y - \omega y \end{aligned}$$

where we have set $\overline{x} = \underline{x} = x$, and $\underline{y} = \overline{y} = y$. Moreover, since by definition for $\overline{x} = \underline{x} = x$, $\int_{\theta}^{\overline{\theta}} \underline{x} dy - \int_{\theta}^{\overline{\theta}} \overline{x} dy = 0 \leq \pi = 0$, we can drop the $z$ variables from the problem. If $\frac{F(\theta)}{f(\theta)}$ is non-decreasing in $\theta$, the monotonicity condition on $x$ is automatically satisfied. Neglecting this constraint, we get the first-order condition in the proposition. Using the fact that concavity is preserved under summation (and hence when taking expectations) it is easy to verify that the quantity schedule characterized by the first-order condition is monotonic in $\theta$, and hence incentive compatible. ■

**Proof of Proposition 8.** Totally differentiating (34) with respect to $x$, $q$ and $\theta$, and using $C(x,q,\theta) = c(x,\theta) + k(x,q)$, we get

$$\begin{aligned} &\left(V_{xx}\left(x^*\left(\theta\right),q\left(\theta\right)\right) - C_{xx}\left(x^*\left(\theta\right),q\left(\theta\right),\theta\right) - (1-\alpha)\,c_{xx\theta}\left(x^*\left(\theta\right),\theta\right)\frac{F\left(\theta\right)}{f\left(\theta\right)}\right)dx^* \\ &+\left(V_{xq}\left(x^*\left(\theta\right),q\left(\theta\right)\right) - k_{xq}\left(x^*\left(\theta\right),q\left(\theta\right)\right)\right)dq^* \\ = &\left(c_{x\theta}\left(x^*\left(\theta\right),\theta\right) + (1-\alpha)\,c_{x\theta\theta}\left(x^*\left(\theta\right),\theta\right)\frac{F\left(\theta\right)}{f\left(\theta\right)} + (1-\alpha)\,c_{x\theta}\left(x^*\left(\theta\right),\theta\right)\frac{\partial}{\partial\theta}\frac{F\left(\theta\right)}{f\left(\theta\right)}\right)d\theta. \end{aligned}$$

Totally differentiating (35) with respect to $x$ and $q$, we get

$$\left(V_{qx}\left(x^*,q^*\right) - C_{qx}\left(x^*,q^*,\theta\right)\right)dx^* + \left(V_{qq}\left(x^*,q^*\right) - C_{qq}\left(x^*,q^*,\theta\right)\right)dq^* = 0.$$

Rearranging, we have

$$dq^* = -\frac{\left(V_{qx}\left(x^*,q^*\right) - C_{qx}\left(x^*,q^*,\theta\right)\right)}{V_{qq}\left(x^*,q^*\right) - C_{qq}\left(x^*,q^*,\theta\right)}dx^*. \tag{42}$$

Substituting from (42) for $dq$, using Young's theorem, the additive separability of the cost function, and rearranging, we get

$$\frac{dx^*}{d\theta} = \frac{c_{x\theta}\left(x^*\left(\theta\right),\theta\right) + (1-\alpha)\,c_{x\theta\theta}\left(x^*\left(\theta\right),\theta\right)\frac{F(\theta)}{f(\theta)} + (1-\alpha)\,c_{x\theta}\left(x^*\left(\theta\right),\theta\right)\frac{\partial}{\partial\theta}\frac{F(\theta)}{f(\theta)}}{\left(V_{xx}\left(x^*,q^*\right) - C_{xx}\left(x^*,q^*,\theta\right) - (1-\alpha)\,c_{xx\theta}\left(x^*,\theta\right)\frac{F(\theta|\eta)}{f(\theta|\eta)} - \frac{\left(V_{qx}(x^*,q^*)-k_{qx}(x^*,q^*)\right)^2}{\left(V_{qq}(x^*,q^*)-k_{qq}(x^*,q^*)\right)}\right)}.$$

Hence, the solution is incentive compatible. ■

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