ADVANCES IN THE THEORY OF LARGE COOPERATIVE GAMES AND APPLICATIONS TO CLUB THEORY: THE SIDE PAYMENTS CASE

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Advances in the Theory of Large Cooperative Games and Applications to Club Theory; The Side Payments Case[1]

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Abstract: In a series of papers (Kovalenkov and Wooders 2001a, *Games and Economic Behavior*, 2001b, *Mathematics of Operations Research*, and 1997, *Journal of Economic Theory* to appear), the authors have developed the framework of parameterized collections of games and also that of parameterized collections of economies with clubs. These papers apply to collections of games with nontransferable utility and similarly to economies with clubs and general preferences. The game theoretic framework encompasses the earlier `pregame' framework (cf., Wooders 1994b *Econometrica*) and also earlier models of economies with clubs and with possibly multiple memberships in clubs (cf. Shubik and Wooders 1982). In this paper, we consider the special case of games with side payments and illustrate the application of our more general results in this special, and much simpler but still important, framework. The motivation for this line of research is developed and application to environmental problems is discussed.

1. Introduction

While the notion that individuals will act in their own self interests is fundamental to game theory and also to economics, the notion that, if there are potential gains to be realised from cooperation, then individuals will attempt to capture these gains, is also fundamental. Indeed, the economic notion of the *contract curve* -- the set of outcomes of trade for a two-person, two-commodity private goods exchange economy with the properties that each person's trade is individually rational and the outcome cannot be improved upon by both traders acting together -- already appears in Edgeworth (1881). In his seminal work, Edgeworth showed that the contract curve converges to the set of allocations arising from price-taking economic equilibrium.

The game-theoretic notion of the *core* as the set of payoffs of a game that cannot be improved upon by any coalition, as a solution concept for cooperative games, was developed by Shapley (1952) and Gillies (1953). Since Shubik's (1959) seminal paper highlighting the relationship between the core and the contract curve, the core has generated numerous papers in both economics and in game theory. The most celebrated of these works are Debreu and Scarf (1963), showing that the core of an exchange economy, represented by allocations of commodities to individual traders, converges to the set of outcomes of price-taking economic equilibrium, a formalisation and extension of Edgeworth's result, and Aumann (1964), further extending Edgeworth's insight in the context of a game with a continuum of players. Numerous subsequent papers further investigated the issue of convergence of the core of an economy with private goods to the set of price-taking equilibrium outcomes; an entire chapter of the *Handbook of Game Theory*, Volume 1 is devoted to a survey of these papers.[2]

The idea of convergence of cores to equilibrium outcomes is intriguing. The competitive equilibrium requires that individuals take prices as given (by some unknown source) and individually optimise, while the core is based on the idea that if a group of individuals could be better off by forming a coalition and reallocating resources and activities within that coalition, then they will do so. While the notions themselves of core and of competitive equilibrium are appealing, they are totally different in spirit, in fact, almost opposing. One notion is based on self-interested individualistic behaviour, while the other notion takes as given that individuals will cooperate to achieve the benefits of cooperation whenever such a possibility exists. The convergence of outcomes based on these two quite dissimilar concepts is truly remarkable.

Outside of the highly stylised framework of an exchange economy with infinitely divisible commodities and with quasi-concave preferences, both the notions of the core and of competitive equilibrium are problematic. The core may be empty and competitive equilibrium may not exist. A condition for nonemptiness of cores was demonstrated in Bondareva (1962) and Shapley (1967). Intuitively, the condition, called *balancedness*, dictates that if coalitions were able to operate part-time, the total feasible payoff to the players could not be improved.[3] The balancedness condition was extended to games without side payments (*NTU games*) and nonemptiness of cores of balanced NTU games was demonstrated by Scarf (1967). Balancedness, however, rules out many interesting classes of cooperative games, including games derived from economies with indivisible commodities, with non-monotonicities, with coalition production and economies with other deviations from classic models of exchange economies.

Except under highly stylised conditions or for special classes of situation, balancedness assumptions rule out economies where part of the economic activity is for participants to structure themselves into groups

[1] To appear in an edited volume.
[2] Anderson (1992).

[3] Hildenbrand and Kirman (1976) and Kannai (1992) both provide very clear expositions of balancedness.

for the attainment of benefits of cooperation within groups.[4] This is problematic since, as is being increasingly recognised in economic research and indeed, as recognised in this volume, our social and economic life is carried out within groups -- firms, families, nations, market places, and political jurisdictions, for example, in addition to social and professional networks. In such situations, individuals may cooperate within groups to benefit from increasing returns to group size and coordination of activities. Individuals may compete within groups for shares of the surplus generated by the activities of the group. There may be competition between groups for scarce resources and for group members. In these sorts of situations, the core may well be empty.

To resolve the problem of the emptiness of the core of a large game, a continuing line of research has investigated conditions for nonemptiness of approximate cores of large games and economies. Shapley and Shubik (1966) showed that large replica exchange economies -- economies with a fixed distribution of a finite number of types of players -- with quasi-linear preferences have nonempty approximate cores. Under the assumption of *per capita boundedness* -- finiteness of the supremum of average payoff over all games in the collection -- Wooders (1979, 1983) demonstrated nonemptiness of approximate cores of large replica games with and without side payments.

Since then, there have been a number of further contributions to this literature, including Shubik and Wooders (1983), Kaneko and Wooders (1982,1986,1996), and Wooders and Zame (1984).[5] Recent literature demonstrating that large games/economies with small effective groups are like competitive exchange economies has relied on the condition of *small group effectiveness* (SGE), dictating that all or *almost all* gains to collective activities can be realised by cooperation only within relatively small groups of participants (Wooders 1992,1994a,b). Note that SGE does not rule out situations where cooperation of the entire player set is required to achieve all possible gains to cooperation.

The prior literature on approximate cores of large games all uses the framework of a *pregame*. A pregame consists of a compact metric space of player types, possibly finite, and a worth function ascribing a payoff possibilities set to every possible group of players. The worth function depends continuously on the types of players in a coalition. Note that the pregame framework treats collections of games that can all be described by a *single* worth function. This has hidden consequences; for example, as we will illustrate, the

[4] The exceptions include matching models and assignment games (Gale and Shapley 1962), special classes of more general matching games called 'partitioning games' (Kaneko and Wooders 1982), games with a 'tree structure' on the set of admissible coalitions (Demange 1983, Greenberg and Weber 1986, Le Breton, Owen and Weber 1992, Demange 1994, for example), and games with lotteries (Garratt and Qin 1996, among others).

[5] Unlike the prior literature using only per capita boundedness, Wooders and Zame (1984) require that individual marginal contributions to coalitions be bounded, an assumption that turns out to be unnecessarily restrictive, since it implies small group effectiveness, as demonstrated by Wooders (1994a), Proposition 3.9. Example 3.1 of the same paper shows that small group effectiveness does not imply boundedness of marginal contributions.

equivalence between small group effectiveness and per capita boundedness demonstrated in Wooders (1994b) is not necessarily true outside the pregame context.

We note here two particular further results for games derived from pregames that provide further motivation for the study of large games. The first is that large games with side payments are equivalent to market games, games generated by economies with side payments or, in the economic terminology, quasi-linear utilities (Wooders 1994b).[6] This means that for *any* economy, whether one with local public goods, clubs, indivisible commodities, and so on, with quasi-linear utilities, there is a set of commodities (which may be the types of players) so that, relative to that set of commodities, in an approximate sense the economy has a competitive equilibrium and the competitive outcomes are equivalent to the core. Another important result is that approximate cores converge to outcomes with the *equal treatment property*, that is, similar individuals are assigned similar payoffs (Wooders 1979, 1994a). Since competitive payoffs have the equal-treatment property, this result indicates another way in which economies with small effective groups resemble markets.

The pregame framework is especially useful to examine limiting properties of large games, as demonstrated by research on equivalence of large games and markets for example. The pregame, however, relative to a specific game with a finite number of players, requires substantial information and does not aim to make precise statements about individual games.[7] In examining any particular game derived from a pregame, for example, only a small part of the pregame structure is relevant. Moreover, results concerning one pregame do not provide information about 'similar' pregames or, in fact, about how one might describe pregames or games as 'similar'.

To study properties of individual games, Kovalenkov and Wooders, in a series of papers (*Games and Economic Behavior*, 2001a, *Mathematics of Operations Research*, 2001b and *Journal of Economic Theory*, to appear), introduced the framework of the parameterised collections of games. This framework describes collections of games in terms of a set of parameters. The parameters bound (a) the number of approximate types of players and the size of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. For some results, including those of the current paper, an additional parameter is required to bound per capita payoffs. In the current paper, we treat games with side payments and present versions of some of our recent results for this special case. A game with side payments summarises the possible outcomes to a coalition by one real number, the total 'payoff' achievable by the coalition, often called the 'worth' of the coalition. In contrast, a game without side payments describes the possibilities open to a coalition by a set of outcomes, where each outcome states a payoff for each player in the coalition. The concepts of games with and without side payments are not disjoint; a game with side

[6] Shapley and Shubik (1969a) showed that balanced games with side payments are equivalent to games generated by private goods exchange economies with quasi-linear utilities.
[7] Except for such statements as "For all games with sufficiently many players derived from the pregame…".

payments can be described as a game without side payments. Because of the simplicity of a game with side payments, cooperative game theory has been more extensively developed for games with side payments than for games without side payments. The results to be presented in this paper are in the literature for games without side payments.[8] Our presentation here makes these results more accessible for the case of games with side payments.

Turning to the motivation for our research on economies with clubs, one of the many interesting economic situations for which small group effectiveness may hold is that of an economy with local public goods -- public goods subject to exclusion and congestion. Inspired by the insightful ‘Tiebout Hypothesis’ (Tiebout 1956), that large economies with local public goods are ‘market-like’, a number of papers have shown that in the context of economies where it is optimal or near-optimal to have many jurisdictions or clubs producing local public goods, approximate cores are nonempty and converge to price-taking equilibrium outcomes. Wooders (1999) provides a short, concise survey.

In the context of multijurisdictional economies, it is natural to allow an individual to belong to several jurisdictions and jurisdictions may overlap. For example, an individual could be a member of a marriage, an employee of a University, a citizen of a country and a member of a large, international environmental organization. Such a model was introduced in Shubik and Wooders (1982), where it is shown that with a fixed distribution of player types, under the apparently mild condition of boundedness of per capita payoffs, approximate cores are nonempty and treat most identical players similarly.[9] Through application of their game-theoretic results, Kovalenkov and Wooders (1997) demonstrated that, when individuals may belong to multiple clubs and when small groups are effective, then large club economies have nonempty approximate cores. In this paper, we present a version of this result for economies with side payments.[10]

Further motivation for our research on large games comes from the literature on multijurisdictional economies. Since its beginnings, with the research of Pauly (1970) treating economies with side payments and essentially identical agents,[11] competitive pricing of outcomes in the core of an economy with gains to

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[8] We refer the reader to Kaneko and Wooders (1994b) for a more in-depth discussion of the concepts of games with and without side payments.

[9] These results are in an approximate or asymptotic sense. More precisely, given $\varepsilon > 0$ there is some replication number r_0 so that all larger replica economies (with the same distribution of player types but with possibly improved payoff opportunities due to gains to scale), ε-cores are nonempty and most players of the same type are treated nearly equally.

[10] Actually we apply a strong result for games with side payments from Kovalenkov and Wooders (2001b) to the side-payments variation of the model of economies with clubs from Kovalenkov and Wooders (1997).

[11] The agents are essentially identical in that, from Pauly's assumptions, agents are indistinguishable to other agents -- whether club memberships are homogeneous or heterogeneous is irrelevant. Wooders (1978) begins treatment of more general cases, where it is possible to show, for example, that unless agents of different types have the same demands, core jurisdictions are homogeneous.

collective activities of groups possibly smaller than the entire population has been of continuing interest to economic theorists.[12] The programme, in the view of one of the authors of this paper, has been to show that: (i) large, multijurisdictional economics, where jurisdictions may be as large as the entire population and may overlap, have nonempty approximate cores; (ii) states of the economy in approximate cores can be described as approximate price-taking competitive outcomes; and (iii) states of the economy in approximate cores converge to equal treatment states (where similar players realize similar utility levels) and to competitive outcomes. Only recently, such results have been obtained (see Allouch and Wooders, 2002); the existence of equilibrium result in that paper depends on game-theoretic nonemptiness of approximate cores.

The pregame framework and the framework of parameterised collections of games both have application to situations with environmental externalities, where the payoff to a coalition depends on the actions of the complementary coalition. Concepts and ideas arising out of research on games derived from pregames have facilitated some studies of economic situations with widespread externalities. We have in mind research by Kaneko and Wooders (1994a) and Hammond, Kaneko, Wooders (1989), Kaneko and Wooders (1989) and Hammond (1999), studying economies where coalitions are constrained to be relatively small, due to coalition formation costs, for example, and externalities are widespread. Since externalities are widespread, the activities of members of a coalition affect the payoffs of members of the complementary coalition. A question that immediately arises in this sort of situation is what a coalition assumes about the behaviour of the complementary coalition. In their papers treating situations with widespread externalities, Hammond, Kaneko and Wooders make a 'Nash' sort of assumption that a permissible coalition takes as given the actions of the complementary coalition. This assumption is most appropriate in economies with many participants and where the effect of any permissible coalition on aggregate outcomes is small. These papers demonstrate that when coalitions are constrained to be small relative to the total player set while externalities are widespread, so that they cannot be 'internalised' within coalitions, then the core is equivalent to the set of competitive equilibrium outcomes but the competitive equilibrium outcome is not optimal. These sorts of results could also be obtained using the framework of parameterised collections of games.

The framework of parameterised collections of games can also treat collections of games where the payoff to a group of players depends on the economy in which it is embedded. For example, in the classic paper by Shapley and Shubik (1969b) on cores of economies with externalities, the authors describe a collection of games, where each game is determined by the number of players. The players are factories, located around a lake. Each factory contributes to the pollution of the lake and thus, the greater the number of factories, the higher the costs of all factories around the lake. Thus, the larger the total player set the smaller the payoff to any coalition. Such a collection of games, with varying numbers of factories (the

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[12] Again Wooders (1999) provides a very brief (three journal pages) but quite comprehensive survey.

players) can be accommodated within the framework of parameterised collections of games. Further examples of situations with widespread externalities are developed in the body of this paper.[13]

While our framework of parameterised collections of games incorporates collections of games with varying numbers of players, as in Shapley and Shubik (1969b), within a given game the choice of activity of a coalition is assumed to have no effect on the payoffs of the complementary coalition. When coalitions are not constrained to be negligible or near negligible relative to the total player set and externalities may spill over from one coalition to another, then the problem that the choice of activities by a coalition affects the payoffs to the members of the complementary set of players might not be reasonably ignored. Suppose, for example, coalitions form to produce pure public goods so that the entire player can consume the public good produced by one coalition set -- a radio programme, for example. Then each coalition is affected by the actions chosen by members of the complimentary coalition. Typically, it is assumed that the complementary coalition does its *worst*; for our example, this means that, for any coalition S, in defining the payoff possibilities for the members of S it would be assumed that members of the complementary set of players produce zero public goods. Indeed, the cooperative game framework may be most applicable to situations where what a coalition can achieve is independent of the actions of the complementary coalition, as is the case in private goods economies without externalities and in our example of a club economy. We note that an important line of research, which takes into account that the opportunities available to a group of players may depend on the activities of the nonmembers of the group (see, for example, Ray and Vohra 1996, Yi 1997, Bloch 1996) and thus may be better suited to treat widespread externalities, is developed further by other papers in this volume.

The paper is organised as follows. The next section introduces the basic definitions and states two results on the nonemptiness of approximate cores of large games: in the framework of a pregame and in the framework of parameterised collections of games. Section 3 includes examples illustrating the advantages of the latter result. Section 4 presents an application to a very general model of economies with clubs. Section 5 concludes the body of the paper.

2. Nonemptiness of e-core of large games: Two frameworks

A *game with side payments* is a pair (N,v) where $N=\{1,...,n\}$ is a finite set and v is a function from the subsets of N to the nonnegative real numbers with the property that $v(\emptyset) = 0$. A game (N,v) is *superadditive* if $v(S)\geq\sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\{ S^k \}$ of S. We will restrict our attention to superadditive games; this is appropriate whenever the union of two coalitions can obtain at least

[13] Kaneko and Wooders (1994a) also contains a number of examples with widespread externalities.

as much total payoff as could be realized by the two coalitions acting separately. Let $|S|$ denote the number of players in a group S.

Let (N,v) be a game with side payments and let $x \in \mathbb{R}^N$, called a *payoff vector*. Given $\varepsilon \geq 0$, the payoff vector x is in the ε-*core* of the game if

$$v(S) - \varepsilon|S| \leq x(S) \text{ for all groups } S \subset N \text{ and}$$
$$x(N) \leq v(N)$$

where $x(S) := \sum_{i \in S} x_i$ .When $\varepsilon = 0$, the 0-core is called simply the *core*.

Nonemptiness of ε-cores of large games is first question to be addressed. Prior to the research of Kovalenkov and Wooders (2001a, 2001b, 1997) this question, as well as the further questions on the properties of the ε-cores of large games, was addressed primarily within the framework of pregames.

2.1. Pregames

For simplicity, we treat here only pregames with a finite number of types of players. The more general setting of pregames with a compact metric space of player attributes and small group effectiveness, SGE, appears in Wooders (1992)[14].

Let T be a positive integer, let $\mathbb{Z}_+^T$ denote the T-fold Cartesian product of the nonnegative integers and let Ψ be a function from $\mathbb{Z}_+^T$ to $\mathbb{R}_+$. Then the pair (N,Ψ) is a *pregame* (with T types of players).

Let (T,Ψ) be a pregame with T types and let $n = (n_1,...,n_T) \in \mathbb{Z}_+^T$. Define $N=\{(t,q)\colon t = 1,...,T$ and $q=1,...,n_t\}$ and, for each coalition $S \subset N$, define the vector *profile* $(S) \in \mathbb{Z}_+^T$ by its components

$$profile_t(S) = |\{(t, q) : q = 1,\dots, n_t\} \cap S|,$$

$t = 1,...,T$. Then *the game induced by* Ψ *and* n is denoted by (N,v), where v maps subsets of N into the nonnegative real numbers,

[14] The side payments pregame framework first appeared in Wooders (1979) and was extended to a compact metric space of player types in Wooders and Zame (1984). The case of games without side payments was addressed in Wooders (1983).

$$v(S) = \Psi\ (profile\ (S))$$

for each nonempty subset S of N and

$$v(\emptyset) = 0$$

Given a profile f let us define $\| f \| = \Sigma^{T}_{t=1} f_t$. A *partition* of a profile f is a collection of profiles $\{f^k\}$ satisfying the property that $\Sigma_k f^k = f$. A pregame (T,Ψ) satisfies *small group effectiveness* (SGE) if, for each $\varepsilon>0$, there is an integer $\eta_0(\varepsilon)$ such that for every profile f there is a partition $\{f^k\}$ of f satisfying:

$$\| f^k \| \leq h_0(\varepsilon) \text{ for each profile } f^k, \text{ and}$$

$$\Psi(f) - \Sigma_k \Psi(f^k) \leq \varepsilon \, \| f \|.$$

Informally, as stated in the Introduction, SGE dictates that all or almost all gains to collective activities are realizable by groups of players bounded in size.[15] For the following discussion, think of a profile as a description of a player set or simply as a player set. In the definition of SGE the parameter ε provides a measure of the distance, in per capita terms, between total payoff realisable by a total player set and payoff realisable by a partition of that set into smaller groups. Given $\varepsilon>0$, there is a bound on group size, $\eta_0(\varepsilon)$, such that, no matter how large the total player set there is a partition of that player set into groups, all bounded in size by $\eta_0(\varepsilon)$, with the property what the total payoff achievable by cooperation only within these groups is within ε of the total payoff to the entire player set. Note that as $\| f \|$ becomes large, $\frac{\eta_0(\varepsilon)}{\|f\|}$ tends to zero – near-effective groups become negligible.

The next result summarises nonemptiness of the ε-cores of large games derived from pregames satisfying SGE. Notice that this result holds also in the more general setting of pregames with a compact metric space of player attributes.

Theorem 1. *Nonemptiness of the e-cores of large games satisfying small group effectiveness* (Wooders 1992, 1994a). Let (T, Ψ) be a pregame satisfying small group effectiveness and let $\varepsilon>0$ be given. Then there is an integer $h_1(\varepsilon)$ such that all games (N, v) induced by pregame with $|N| > h_1(\varepsilon)$ have nonempty ε-cores.

[15] *Strict* small group effectiveness, the property that *all* gains to collective activities can be realized by groups of players bounded in size (albeit with a different name) was introduced in Wooders (1979) and has appeared in a number of subsequent papers including Bennett and Wooders (1979) and most recently, Wooders (1994b).

The pregame framework appears to be especially well suited to the study of limiting properties of large games. For example, in Wooders (1994b) it is shown that "in the limit", the per capita payoff function to large games derived from a pregame is concave and can be interpreted as the utility function used in describing the canonical representation of a game as a market (see also Shapley and Shubik 1969a)[16]. From the viewpoint of the analysis of large finite games , a substantial drawback of the pregame framework is that it permits results only for a fixed pregame. Moreover, the bounds obtained, such as $h_1(\varepsilon)$ in the above result, depend on the entire pregame. The pregame approach does not permit *uniform* treatment of large games that are not necessarily derived from the *same pregame*. For example, marriage or matching games and games where any two players are effective both have two-person effective coalitions but they cannot be modelled in one pregame. The pregame framework cannot encompass even situations with small *externalities* (when the payoff to a coalition depends on the game where it is embodied). These considerations motivate the approach introduced in Kovalenkov and Wooders (2001a, 2001b, 1997).

2.2. Parameterized collections of games

Recall that given a game (N, v), players i and j are *substitutes* if, for all coalitions S with $i, j \notin S$, it holds that $v(S \cup \{i\}) = v(S \cup \{j\})$. A fundamental property of games derived from pregames with a compact space of player types, and indeed of replicated player sets, is that in derived games with many players, most players have may close substitutes. There are counterexamples showing the necessity of this condition for the result that games with many players have nonempty approximate cores. Our notion of parameterised collections of games does not require any topological structure on the space of player types but does rely on the notion of approximate substitutes.

Let (N,v) be a game and let d be a non-negative real number. A ***d**-substitute partition* of N is a partition $\{N[t]: t = 1,..,T\}$ of N with the property that, for any type-consistent permutation τ and any coalition S,

$$|v(S) - v(\tau(S))| \leq d|S|.$$

Note that when $d = 0$, two players who are 0-substitutes are substitutes in the usual sense of the term. The notion of d-substitutes extends the notion of a compact metric space of player types. Note that given any

[16] In fact, in the literature of private goods exchange economies, since Kannai (1970) an analogous construct has been used to treat limiting properties of large economies. In particular, a pair consisting of a space of endowments and a space of preferences is taken as given; this pair could be described as a pre-economy. Then an economy is determined by a set of agents and a function assigning each agent a preference relation and an endowment. Our approach here, like Anderson (1978) and Manelli (1991) treats individual games.

game (N,v) and any $d \geq 0$ it is possible to determine the minimum number T so that the player set can be partitioned into a T-member d-substitute partition.

Let b be a given non-negative real number, and let B be a given integer. A game (N,v) has *b-effective B-bounded groups* if for every group $S \subset N$ there is a partition $\{S^k\}$ of S into subgroups with $|S^k| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq b\,|S|.$$

The notion of β-effective B-bounded groups extends the notion of small group effectiveness (SGE) from Wooders (1991, 1994b) and the earlier notion of strict SGE. SGE, defined for pregames, dictates that all or almost all gains to collective activities can be realised by coalitions bounded in size or by relatively small coalitions, where the bound or the size of the required coalitions depends on the distance allowed from achievement of *all* gains to collective activities. The notion of b-effective B-bounded groups makes explicit how close coalitions bounded in size by B are to being able to realise all gains to collective activities for a given game. Then $b=0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups.*

Let C be a positive real number. A game (N,v) has a *per capita bound* of C if $\frac{v(S)}{|S|} \leq C$ for all coalitions $S \subset N$. Within the pregame framework, small group effectiveness implies per capita boundedness and, if the numbers of players of each type (or approximate type) is sufficiently large, then per capita boundedness implies small group effectiveness (Wooders 1994b, Theorem 6). This is no longer true in the context of parameterized collections of games, as we will illustrate in the next section by examples.

Given nonnegative real numbers d and b and positive integers T, C and B, a *parameterized collection of games with side payments*, denoted by $\Gamma((d, T), C, (b, B))$, is a collection of games with the properties that for each game (N,v) in the collection:

a) There is a d-substitute partition of N into no more than T subsets $\{N[t] : t=1,...,T', T' \leq T\}$.

b) The game has a *per capita bound* of C.

c) The game has b-effective B-bounded groups.

Note that given any game (N,v) it is possible to determine d, T, b, B and C so that (N,v) belongs to the collection $\Gamma((d, T), C, (b, B))$. Given d and b one can determine the minimum required values of T and B, although if d and b are both ‘small’ and there are increasing returns to coalition size, it may be that $T = B = |N|$, the worst case.

Given nonnegative real numbers d and b and positive integers T, C and B, let $(N,v)\in\Gamma((d,T), C, (b,B))$, let $\{N[t]\}$ be a partition of N into d-substitutes, and let x be a payoff vector for (N,v). We say that x is an *equal-treatment payoff vector* if, for each $N[t]$ and for all i and j in $N[t]$ it holds that $x_i = x_j$. Note that we could use a weaker definition of an equal treatment payoff; in particular, we could require, for example, only that similar players are treated similarly.

Theorem 2. *(Nonemptiness of the equal-treatment e-core from Kovalenkov and Wooders* 2001*b.)* Let $(N,v)\in\Gamma((d,T), C, (b,B))$ and let ε be a positive real number. Then if

$$\varepsilon \geq \frac{TC(B-1)}{|N|} + \delta + \beta$$

the equal treatment ε-core of (N,v) is non-empty.[17]

The bound presented in this theorem is the exact bound in some cases. First, it is interesting to observe that for inessential games, which always have nonempty cores, Theorem 2 gives a bound on ε of zero. Let (N,v) be a game where all coalitions are inessential (that is, only singleton coalitions matter). Such a game has a nonempty core. Thus, the ε-core is nonempty for any $\varepsilon\geq 0$. To apply Theorem 2 notice that $(N,v)\in\Gamma((0, |N|), C, (0,1))$ for some per capita bound C. The lower bound given by the expression above, $\frac{TC(B-1)}{|N|} + \delta + \beta$ is zero since $d = 0$, $b = 0$ and $(B-1)=0$. Even in this extreme case, the bound works well.

Now let us consider games where, given some positive real number K, within each game, all players are identical, only two-player coalitions are effective, and any two-player coalition can earn a payoff of less than or equal to $2K$. Obviously all these games belong to the class $\Gamma((0, 1), K, (0, 2))$. In this case, $\frac{TC(B-1)}{|N|} + \delta + \beta = \frac{K}{|N|}$. The core is empty if the number of players is odd. Let $|N| = 2m + 1$ for some positive integer m. But we can easily determine the least lower bound on ε so that the ε-core is nonempty for any game (N,v) in the collection with $|N| = 2m+1$. In particular, suppose we assign each of the first $2m$ players the payoff $K-\varepsilon$ and the remaining player the payoff $2m\varepsilon$. Suppose ε^* solves $K - \varepsilon^* = 2m\varepsilon^*$. Then the ε-core is nonempty for any game in the collection for any $\varepsilon \geq \varepsilon^*$, but may be empty otherwise. (Take, for example, K=1 and m=1. Then $\varepsilon^* = 1/3$ and the ε-core is empty for any $\varepsilon<1/3$). Solving for ε^*, we obtain

$$\varepsilon^* = \frac{K}{(2m+1)} = \frac{K}{|N|}.$$

This bound coincides with the bound given by the Theorem. Thus the bound given in the Theorem is the best possible bound for this collection.

In the following section we present further examples illustrating the theorem.

3. Examples and Comparisons

We begin with a very simple and straightforward example of matching models.

[17] A form of this Theorem, with a larger bound, appeared in Wooders (1994c). We note β-effective B-bounded groups permits all coalitions containing no more than B members. Were we to restrict the set of permissible coalitions further, a better bound may be obtained.

Example 1. *Exact types and strictly effective small groups*.

Let us consider a game (N,v) with two types of players. Assume that any player alone can get only 0 units or less, that is $v(\{i\})=0$ for all $i\in N$. Suppose that any coalition of the two players of types i and j can get up to γ_{ij} units of payoff to divide. Let γ_{11}, $\gamma_{12} = \gamma_{21}$, and γ_{22} be some numbers from the interval $[0,1]$. An arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than two players. We leave it to the reader to check that $(N,v)\in\Gamma((0,2), ½, (0,2))$. Thus, we have from Theorem 2 that for $\varepsilon \geq \frac{1}{|N|}$ the equal-treatment ε-core of (N,v) is non-empty. Notice that this result holds uniformly for all possible numbers γ_{11}, $\gamma_{12} = \gamma_{21}$, and γ_{22}.

The following example illustrates how our result can apply to games derived from pregames with a compact metric space of player types. For brevity, our example is somewhat informal. While the example is worded in terms of firms and workers, as in Crawford and Knoer (1981), for example, it could easily be modified to treat the hospital and intern matching problem as in Roth (1984) or any such assignment problem.

Example 2. *Approximate player types*.

Consider a pregame with two sorts of players, firms and workers. The set of possible types of workers is given by the points in the interval $[0,1)$ and the set of possible types of firms is given by the points in the interval $[1, 2]$. Formally, let N be any finite player set and let ξ be an *attribute function*, that is, a function from N into $[0,2]$. If $\xi(i) \in [0, 1)$ then i is a worker and if $\xi(i) \in [1, 2]$ then i is a firm. Firms can profitably hire up to three workers and the payoff to a firm i and a set of workers $W(i) \subset N$, containing no more than 3 members, is given by $v(\{i\} \cup W(i)) = \xi(i) + \Sigma_{j\in W(i)}\, \xi(j)$. Workers and firms can earn positive payoff only by cooperating so $v(\{i\}) = 0$ for all $i \in N$. For any coalition $S \subset N$ define $v(S)$ as the maximum payoff the group S could realize by splitting into coalitions containing either workers only, or 1 firm and no more than 3 workers. This completes the specification of the game. We leave it to the reader to verify that for any positive integer m every game (N, v) derived from the pregame is a member of the class $\Gamma\left(\left(\frac{1}{m},2m\right), 2, (0,4)\right)$. Then Theorem 2 implies that for any $\varepsilon \geq \frac{12m}{|N|}+\frac{1}{m}$ the equal-treatment ε-core of (N, v) is non-empty.

This implies that for any $\varepsilon^0 > 0$ there exist a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ the game (N,v) has a non-empty equal-treatment ε^0-core. (For an explicit bound take an integer $m^0 \geq \frac{2}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \frac{24m^0}{\varepsilon^0}$.)

For completeness, we present a simple but formal example with nearly effective groups. It also presents a collection of games that cannot be described by a single pregame thus demonstrating a broader applicability of our Theorem.

Example 3. *Nearly effective groups*.

Call a game (N,v) a k-quota game if any coalition $S \subset N$ of size less than k can realize only 0 units (that is, $v(S) = 0$ if $|S| < k$), any coalition of size k can realise 1 unit (that is, $v(S) = 1$ if $|S| = k$), and an arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than k players. Let Q be a collection, across all k, of all k-quota games with player set N. We leave it to the reader to verify that for any positive integer $m>1$, the class Q is contained in the class $\Gamma\left((0,1),1,\left(\frac{1}{m},m-1\right)\right)$. Hence Theorem 2 implies that for any $\varepsilon \geq \frac{(m-2)}{|N|}+\frac{1}{m}$ and for any $(N,v)\in Q$ the equal-treatment ε-core of (N,v) is non-empty. This implies that for any $\varepsilon^0>0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N|\geq N(\varepsilon^0)$ any game $(N,v)\in Q$ has a non-empty equal-treatment ε^0-core. (For an explicit bound take an integer $m^0 \geq \frac{2}{e^0}$ and define $N(\varepsilon^0) \geq \frac{2(m^0-2)}{\varepsilon^0}$.)

This example also illustrates some differences between parameterized collections of games and games derived from pregames. A pregame takes as given a topological space of player types and a *single* worth function determining payoff sets for groups of players described by their types. It is immediate that the payoff structure of a pregame is invariant in the sense that only the size and composition of player sets can vary, not the payoff to a given set of players described by their types. Given the player set N, the class Q consists of games generated by varying the payoff structure of the games. Thus, the collection Q cannot be described as a collection of games generated by a pregame.

Theorem 2 requires both per capita boundedness and small group effectiveness. As noted previously, in the context of pregames with side payments, when arbitrarily small percentages of players of any particular type is ruled out, then these two conditions are equivalent. But in important economic contexts, neither condition implies the other. The next example illustrates voting games satisfying per capita boundedness. There is only one player type in each game so the “thickness” condition of the equivalence result of Wooders (1994b) is satisfied. But small group effectiveness does not hold and Theorem 2 does not apply.

Example 4. *Voting games.*

Consider a sequence of games $(N^m, v^m)^{¥}_{m=1}$ where the m^{th} game has $3m$ players. Suppose that there are widespread positive externalities so that in the m^{th} game, any group S consisting of at least $2m$ players can get up to $2m$ units of payoff to divide among its members, that is, $v^m(S) = 2m$. Assume that if $|S| < 2m$, then $v^m(S) = 0$. We can think of the games as a sequence of voting games where a winning group must contain 2/3 of the population, for example, impeachment of a President of the United States or ratification of a treaty in some parliaments.

Observe that each game in the sequence has one exact player type and a per capita bound of 1. That is, T=1, C=1, and δ=0. However, the 1/7-core of the game is empty for arbitrarily large values of m.

To see that the 1/7-core is empty, observe that for any feasible payoff vector there are m players that are assigned, in total, no more than $\frac{2m}{3m}m = \frac{2}{3}m$. There are another m players that get in total no more than $\frac{2\mathrm{m}}{2\mathrm{m}}\mathrm{m} = \mathrm{m}$. These $2m$ players can form a group and receive $2m$ in total. This group can improve upon the given payoff vector for each of its members by 1/6, since $((2m – (5/3)m)/(2m) = 1/6$.

The following example, of matching games with widespread positive externalities, illustrates economic situations where per capita boundedness does not hold and Theorem 2 does not apply.

Example 5. *A matching game with widespread positive externalities*.

The economic situation we have in mind is one where any two players can carry out some job but their reward from the job depends on the size of the economy in which they live. (It would be easy to modify the example to become a two or many-sided matching game.) Consider a sequence of games $(N^m, v^m)^{\infty}_{m=1}$ where the m^{th} game has $2m + 1$ players. Assume that any player alone can get only 0 units or less, that is $v^m(\{i\}) = 0$ for all $i \in N$. Also assume that any two-player group can get up to $2m$ units of payoff to divide; $v^m(S)$

= $2m$ if $|S| =2$. An arbitrary group can gain only what it can obtain in partitions where no member of the partition contains more than two players.

The games $(N^m, v^m)^{\infty}_{m=1}$ are members of the collection of games with one exact player type and strictly effective small groups of two. That is, $T=1$, $B=2$, and $d = b = 0$. However, the 1/7-core of the game is empty for arbitrarily large values of m.

To see that the 1/7-core is empty, observe that for any feasible payoff vector there is a player whose payoff is no more than $\frac{2m^2}{2m+1}$. There is another player whose payoff must be no more than $\frac{2m^2}{2m} = m$. These two players may form a group and realize $2m$. Thus they gain $m - \frac{2m^2}{2m+1} = \frac{m}{2m+1} \geq \frac{m}{3m} = \frac{1}{3}$. Obviously, together this two-player group can improve upon the given payoff by 1/6 for each member of the group.

In the next section we present the main example on the broad applicability of Theorem 2 - a very general model on the economies with clubs.

4. Economies with clubs and multiple memberships

In Kovalenkov and Wooders (1997), by application of Theorems on nonemptiness of cores and approximate cores, it is shown that a class of economies with apparently mild restrictions on club formation has nonempty approximate cores. In particular, following Shubik and Wooders (1982), individuals may belong to multiple clubs -- that is, multiple memberships are permitted. Here, we illustrate the Kovalenkov and Wooders' model of club economies for situations with side payments. This highlights the fact that our results for large games and economies apply even with multiple memberships.

We define admissible club structures in terms of natural properties and take as given the set of all admissible club structures for each group of players. We remark that it would be possible to separate crowding types of players (those observable characteristics that affect the utilities of others, or, in other words, their external characteristics) from taste types, as in Conley and Wooders (1996, 1997, 2001) and have players' roles as club members depend on their crowding types. In those papers, however, the separation of crowding type and taste type has an important role; the authors show that prices for public goods -- or club membership prices -- need only depend on observable characteristics of players and not on their preferences. The current paper treats only the core; at this point separation of taste and crowding type would have no essential role.

agents. There are T "types" of agents. Let $\rho = (\rho_1,...,\rho_T)$ be a given profile, called the *population profile*. The set of agents is given by

$$N_\rho = \{(t, q) \ : q = 1,\ldots,\rho_t \text{ and } t = 1,\ldots,T\},$$

and (t, q) is called the q^{th} *agent of type t.* It will later be required that all agents of the same type may play similar role in club structures. For example, in a traditional marriage model, all females could have the role of "wife". Define $N_\rho[t] := \{(t, q): q = 1,\ldots,\rho_t\}$. For our Proposition members of $N_\rho[t]$ will be approximate substitutes for each other.

commodities. The economy has L private goods. A vector of private goods is denoted by $y = (y_1,\ldots, y_l ,\ldots, y_L) \in \mathbf{R}^L_+$.

clubs. A *club* is a nonempty subset of players. For each club $S \subset N_\rho$, a *club structure of S*, denoted by $\mathcal{S}$, is a set of clubs whose union coincides with S. The set of *admissible club structures for S*, denoted by $C(S)$, is assumed to be nonempty for any $S \neq \emptyset$. This assumption ensures that a club consisting of only one player has a unique admissible club structure - a singleton set. The sets $C(S)$ are also required to satisfy the following two properties:

1. If S and S' are nonempty disjoint subsets of players and $\mathcal{S}$ and $\mathcal{S}'$ are admissible club structures of S and S' respectively, then $\{C : C \in \mathcal{S} \cup \mathcal{S}'\}$ is an admissible club structure of $S \cup S'$ (unions of admissible club structures of disjoint groups are admissible club structures of the unions of the groups).

2. Let S and S' be subsets of players with the same profiles, let $\mathcal{S}$ be an admissible club structure of S and let φ be a type-preserving 1-1 mapping from S onto S' (that is, if $(t, q) \in S$ then $\varphi((t, q)) = (t, q')$ for some $q' = 1,\ldots,\rho_t$). Then

$$\mathcal{S}' = \{C \subset S' : \varphi^{-1}(C) \in \mathcal{S}\}$$

is an admissible club structure of S' (admissible club structures depend only on profiles, that is, all players of the same type have the same roles in clubs).

The first property is necessary to ensure that the game derived from the economy is superadditive. It corresponds to economic situations where one option open to a group is to form smaller groups. Since the singletons are always admissible club structures for clubs of one player, this property implies that the partition of any set $S \subset N_\rho$, $S \neq \emptyset$, into singletons is an admissible club structure for S. The second property corresponds to the idea that the opportunities open to a group depend on the profile of the group.

club activities. For each club C there is a given set of *club activities* $\mathcal{A}(C)$. These club activities could include provision of morning coffee for the club members, or a swimming pool, or a purely social club. An element α of $\mathcal{A}(C)$ requires input $x(C, \alpha) \in \mathbf{R}^L$ of private goods. For any two clubs C and C' with the same profile we require that if $\alpha \in \mathcal{A}(C)$, then $\alpha \in \mathcal{A}(C')$ and $x(C, \alpha) = x(C', \alpha)$. In other words, the club activities feasible for a set of players depend only on the profile of the club membership. For 1-player clubs $\{(t,q)\}$, we assume that there is an activity α_0 with $x(\{(t, q)\}, \alpha_0) = 0$, that is, there is an activity requiring no use of inputs.

preferences and endowments. Only private goods and money are endowed. Let $\omega^{tq} \in \mathbf{R}^L_+$ be the *endowment of the* $(t, q)^{th}$ *participant of private goods* and let $\overline{\xi}^{tq}$ be his endowment of money.

Given $S \subset N_\rho$, $(t, q) \in S$, and a club structure $\mathcal{S} \in C(S)$, *the consumption set of the* $(t, q)^{th}$ player (relative to $\mathcal{S}$) is given by

$$\Phi^{tq}(\mathcal{S}) := X^{tq}(\mathcal{S}) \times \mathbb{R} \times \prod_{C \in \mathcal{S}} \mathcal{A}(C),$$

where $X^{tq}(\mathcal{S}) \subset \mathbf{R}^L$ is the *private goods consumption set relative to* $\mathcal{S}$, assumed to be closed and $\mathbb{R}$ is the real line. Note that the private goods part of the consumption set of a player, $X^{tq}(\mathcal{S})$, may depend on the club structure. To illustrate the motivation for this feature, notice that some clubs may provide food for their members. Whether an individual belonged to such a club would affect the private goods required for subsistence. The entire consumption set of the $(t,q)^{th}$ player is given by

$$\Phi^{tq} := \bigcup_{S \subset N\rho:\, (t,q) \in S} \; \bigcup_{\mathcal{S} \in C(S)} \Phi^{tq}(\mathcal{S}) .$$

We assume that the $(t,q)^{th}$ player can subsist in isolation. That is

$$\left(w^{tq}, \overline{x}^{tq}, a_0\right) \in \Phi^{tq}\left(\{(t,q)\}\right).$$

It is also assumed that for each (t, q), each $S \subset N_\rho$, $(t, q) \in S$, and each club structure $\mathcal{S}$ of S, the preferences of the $(t, q)^{th}$ agent are represented by a continuous utility function $u^{tq}(\bullet\, ; \mathcal{S})$ defined on $\Phi^{tq}(\mathcal{S})$.

states of the economy. Let S be a nonempty subset of N_ρ and let $\mathcal{S} \in C(S)$. A *feasible state of the economy S relative to* $\mathcal{S}$, or simply *a state for* $\mathcal{S}$, is a triple $(y^{\mathcal{S}}, \xi^{\mathcal{S}}, \alpha^{\mathcal{S}})$ where:

a) $y^{\mathcal{S}} = \{ y^{tq}\}_{(t,q)\in S}$ with $y^{tq} \in X^{tq}(\mathcal{S})$ for $(t, q) \in S$;

b) $\xi^{\mathcal{S}} = \{ \xi^{tq}\}_{(t,q)\in S}$ with $\xi^{tq} \in \mathbb{R}$ for $(t, q) \in S$;

c) $\alpha^{\mathcal{S}} = \{ a^{C}\}_{C\in \mathcal{S}}$ with $a^{C} \in \mathcal{A}(C)$ for $C \in \mathcal{S}$ and

d) the allocations of private goods and money are feasible, that is,

$$\sum_{C\in \mathcal{S}} x(C, \alpha^{C}) + \sum_{(t,q)\in S} y^{tq} = \sum_{(t,q)\in S} \omega^{tq} \text{ and}$$

$$\sum_{C\in \mathcal{S}} \xi^{tq} = \sum_{C\in \mathcal{S}} \bar{\xi}^{tq}.$$

feasible payoffs vectors. A payoff vector $(\bar{u}^{tq})_{(t,q)\in S}$ is *feasible for a group S* if there is club structure $\mathcal{S} \in C(S)$ and a feasible state of the economy for S relative to $\mathcal{S}$, $(y^{\mathcal{S}}, \xi^{\mathcal{S}}, \alpha^{\mathcal{S}})$, such that $\bar{u}^{tq} = u^{tq}(y^{tq}, \xi^{tq}, \alpha^{\mathcal{S}} ; \mathcal{S})$ for each $(t, q) \in S$.

the game induced by the economy. For each group $S \subset N_\rho$, define

$$v(S) = \max \{\textstyle\sum_{(t,q)\in S} \bar{u}^{tq} : \text{a payoff vector } (\bar{u}^{tq})_{(t,q)\in S} \text{ is feasible for } S\}.$$

It is immediate that the player set N_ρ and function v determine a game with side payments (N_ρ , v).

ε-domination. Let $\mathcal{N}$ be a club structure of the total player set N_ρ and let $(y^{\mathcal{N}}, \xi^{\mathcal{N}}, \alpha^{\mathcal{N}})$ be a feasible state of the economy N_r relative to $\mathcal{N}$. A group S can ε-*dominate* the state $(y^{\mathcal{N}}, \xi^{\mathcal{N}}, \alpha^{\mathcal{N}})$ if there is a club structure $\mathcal{S} = \{S_1,..., S_K\} \in C(S)$ and a feasible state $(y'^{\mathcal{S}}, \xi'^{\mathcal{S}}, \alpha'^{\mathcal{S}})$ for the economy S such that for all consumers $(t,q) \in S$ it holds that

$$u^{tq}(y'^{tq}, \xi'^{tq}, \alpha'^{\mathcal{S}} ; \mathcal{S}) > (y^{tq}, \xi^{tq}, \alpha^{\mathcal{N}}; \mathcal{N}) + \varepsilon .$$

the core of the economy and ε-cores. A state $(y^{\mathcal{N}}, \xi^{\mathcal{N}}, \alpha^{\mathcal{N}})$ of the economy N_ρ relative to $\mathcal{N}$ is in the *core of the economy* if it cannot be 0-dominated by any group S. Notice that the core of the economy may include states of the economy relative to different club structures of the set N_ρ . It is clear, that if $(y^{\mathcal{N}}, \xi^{\mathcal{N}}, \alpha^{\mathcal{N}})$ is a state in the core of the economy then the utility vector induced by that state is in the core of the induced game. Similarly, if $(\bar{u}^{tq})_{(t,q)\in N\rho}$ is in the core of the game induced by an economy then there exist a club structure $\mathcal{N}$ of the total player set N_ρ and a state in the core of the economy $(y^{\mathcal{N}}, \xi^{\mathcal{N}}, \alpha^{\mathcal{N}})$ such that the utility vector induced by that state is $(\bar{u}^{tq})_{(t,q)\in N\rho}$. Notion of the ε-core of the economy is defined in the obvious way.

To obtain our results we require few restrictions on the economy:

(A.0) For some $d \geq 0$ the players in the set $N_r[t]=\{(t,q):q=1,...,r_t\}$ are d-substitutes for each other in the game induced by the economy.

(A.1) There are $b \geq 0$ and an integer B so that the game derived from the economy has b-effective B-bounded groups.

(A.2) There is a constant C such that the condition of per capita boundedness of C is satisfied by the game derived from the economy.

Then the next result follows from Theorem 2.

Proposition 1. *(Nonemptiness of the equal-treatment e-core)*. Let T and B be positive integers. Let *d, b* and C be nonnegative real numbers. Assume that (A.0), (A.1), and (A.2) hold and let ε be a positive real number. Then if

$$\varepsilon \geq \frac{TC(B-1)}{|N|} + \delta + \beta$$

the equal-treatment ε-core of the economy is non-empty.

In subsequent research, Allouch and Wooders (2002 and research in progress) demonstrate existence of approximate competitive equilibrium and convergence result for economies with networks, including club economies with overlapping memberships.[18] Their condition of *desirability of wealth* on primitives ensures

[18] The first decentralization results for situations where individuals may belong to multiple clubs appear in Ellickson et al (1999). Their model requires that club sizes be bounded. Thus, clubs must become infinitesimal in large economies. In contrast, the Kovalenkov and Wooders (2001a,b, 1997) and Allouch and Wooders (2002) formulations allow clubs as large as the entire player set.

that per capita boundedness (and indeed, small group effectiveness) is satisfied. Informally, desirability of wealth is simply that, for each player, enough wealth, in terms of private goods, and a few friends can dominate any feasible equal treatment state of the economy, no matter how large the total player set. Note that this condition allows ever-increasing returns to club sizes and memberships since no feasible equal treatment state of the economy may permit anyone to be so wealthy.

5. Final remarks

Remark 1. *Absolute or relative sizes*?

It is possible to obtain similar results with bounds on *relative* sizes of effective groups. In a finite game with a given number of players, assumptions on absolute sizes and on relative sizes of effective groups are equivalent. We have chosen to develop our results using bounds on absolute sizes of near-effective groups since this seems to reflect typical economic and social situations. Examples include: marriage and matching models (see Kelso and Crawford (1982) and Roth and Sotomayor (1990); models of economies with shared goods and crowding (see Conley and Wooders 1998 for a survey); and private goods exchange economies (see Mas-Colell 1979, Hammond, Kaneko and Wooders 1989 and Kaneko and Wooders 1989 for example). In fact, assumptions on proportions of economic agents typically occur only when there is a continuum of players, cf. Ostroy 1984).

Remark 2. *Further properties of* ε*-cores*.

In the subsequent research of the authors additional interesting properties of the ε-cores of games without side payments have been studied using the framework of the parameterized collection of games. However, some of these papers are incomplete and none is published. Thus we decided to concentrate in this review on the nonemptiness issue.

Remark 3. *Immediate applications*.

The class of economies defined above is very broad. The Proposition can be applied to extend results already in the literature on economies with group structures, such as those with local public goods (called club economies by some authors), cf., Shubik and Wooders (1982). For example, there are a number of papers showing core-equilibrium equivalence in finite economies with local public goods and one private good and satisfying strict effectiveness of small groups, cf., Conley and Wooders (2001) and references therein. In these economies, from the results of Wooders (1983) and Shubik and Wooders (1983), existence of approximate equilibrium where an exceptional set of agents is ignored is immediate. (Just take the largest subgame having a nonempty core and consider the equilibria for that subeconomy; ignore the remainder of the consumers.) Our results in case of the private good being a money allow the immediate extension of these results to results for all sufficiently large economies -- no restriction to replication sequences is required.

Remark 4. *Other applications.*

In principle, it appears that our results on nonemptiness of approximate cores of large games may have other applications. As suggested already by the title of Allouch and Wooders (2002), they can be applied to an interesting class of socially networked economies. They may also have application to economies with asymmetric information, as in Forges, Heifetz and Minelli (2001) for example.

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