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Dynamic Club Formation with Coordination[1]

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We present a dynamic model of jurisdiction formation in a society of identical people. The process is described by a Markov chain that is defined by myopic optimization on the part of the players. We show that the process will converge to a Nash equilibrium club structure. Next, we allow for coordination between members of the same club, i. e. club members can form coalitions for one period and deviate jointly. We define a Nash club equilibrium (NCE) as a strategy configuration that is immune to such coalitional deviations. We show that, if one exists, this modified process will converge to a NCE configuration with probability one. Finally, we deal with the case where a NCE fails to exist due to indivisibility problems. When the population size is not an integer multiple of the optimal club size, there will be left over players who prevent the process from settling down. We define the concept of an *approximate Nash club equilibrium (ANCE)*, which means that all but k players are playing a Nash club equilibrium, where k is defined by the minimal number of left over players. We show that the modified process converges to an ergodic set of states each of which is ANCE.

Keywords: Club formation, Cooperation, Best–reply dynamics, Learning, Approximate Nash club Equilibrium, Approximate core.

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1 Introduction

We provide a dynamic model of club formation within the framework of a local public good economy. Each period, individuals choose one out of a finite set of locations, and those individuals selecting the same location form a club in order to provide a public good for themselves exclusively. We assume that the public good is financed by a poll tax, or equal cost sharing on the part of the members of the club. In addition to the private good, an individual's utility depends on his consumption of the public good and on the size of the club; that is, crowding is anonymous. In the presence of congestion effects, increasing the number of members of a club may have two opposing effects on the members' utilities: On the one hand, the cost shares are diminishing; on the other hand, congestion may be exacerbated. Thus there may be a trade-off between cost sharing and crowding effects. Note, however, that as in the literature on local public goods economies with anonymous crowding, crowding effects are not necessarily negative. For instance, there might be positive externalities in consumption, or fashion effects (e. g. everybody wants to be a member of the hottest club in town). Finally, crowding effects might be both positive and negative over different ranges of the club size. For instance, being the only member of a tennis club is less satisfactory than having a few partners to play tennis with, but too many members may cause aggravating waiting times for courts etc. In any of these cases, an agent's marginal utility from an additional club member is increasing up to a point, the optimal club size, and then decreasing. Such models now have a long history going back, for example, to Pauly (1972) and Wooders (1978).[2] Our anonymity requirement is closer to that of Pauly (1972), however, in that we require all players to have the same payoff functions.

We define a non–cooperative game where each player's strategy set is the set of all locations, or clubs (each club is identified with its location), and each player's payoff is a function of the number of players choosing the same strategy, i. e. location, as himself. This model is a simple version of the local public good games analysed by, among others, Konishi, Le Breton and Weber (1997a, 1998).[3] The existence of pure strategy Nash equilibria of such local public good games has been shown, for various specifications of the game, by several authors. While Konishi et al. (1997a), and, for an even more general model including external effects of group formation on non–members, Hollard (2000) for example, prove existence for the general case, Holzman and Law–Yone (1997), Konishi et al. (1997b), and Milchtaich (1996) are concerned with the special case of congestion games, where each player's payoff is non–increasing in the number of players choosing the same strategy as himself. The latter two articles also provide conditions for the

[2]See Barham and Wooders (1998) for a survey of literature and some recent results on economies with local public goods and anonymous crowding

[3]See also Demange (1994).

existence of Nash club equilibria. Games with positive externalities are analysed in Konishi et al. (1997c).

The setup of our model is closely related to the model in Konishi, Le Breton and Weber. (1997a). This paper defines a free mobility equilibrium of a local public goods economy as an assignment of players to clubs (locations, facilities, or jurisdictions), that partitions the population and has the property that no individual can gain by either moving to any other of the existing clubs, or create his own club. The partition derived from the players' strategy choices is thus stable against unilateral deviations by individuals. It is easy to see that a free mobility equilibrium corresponds to a Nash equilibrium of the club formation game, as defined in this paper.

While a Nash equilibrium is stable in the sense that rational individuals will adhere to their Nash equilibrium strategy choices, the one–stage game approach fails to explain if and how an equilibrium will be reached. The present paper departs from the literature mentioned above in that player mobility is modelled explicitly. That is, we provide a dynamic model where the club formation game is played repeatedly in time, and players are free to move between existing clubs (or create their own) at each step in time. For instance, if a club gets too crowded, some members might consider leaving that club and joining another club of smaller size. The question is: Under what conditions will the movement of players to their preferred clubs reach a stable club structure or, in other words, a free mobility equilibrium? We show that:

Convergence to equilibrium. If strategy choices are determined by a myopic best–reply rule, the dynamic process defined by individual adaptation rules converges to a pure strategy Nash equilibrium of the club formation game with probability one, as time tends towards infinity.

An issue often addressed in the context of club formation is the possibility of coordinated action on the part of a group of players. For instance, if a club becomes too crowded, a subset of its members might jointly decide to move to other locations, even if every single one of them would prefer not to unilaterally deviate from his current location. To ensure stability against such coordinated deviations, we analyse a variant of Nash club equilibria, i. e. equilibria that are immune to joint deviations by groups of players. While the allocation associated with a Nash club equilibrium is by definition (weakly) Pareto efficient, the problem is that in many cases such equilibria may fail to exist.[4] This problem creates a dilemma. On the one hand, equilibrium notions based on stability against unilateral deviations, like the Nash equilibrium, are too weak in the sense that they allow many inefficient and unintuitive equilibria to exist. On the other hand,

[4]This problem has been addressed in a recent paper by Conley and Konishi (2000).

equilibrium notions based on coalitional deviations are too strong, as they may fail to exist. For instance, suppose the optimal club size is three, and the total number of players is ten. In this case, a strong equilibrium may not exist: Whenever there are three clubs of size three each, there will be one left over player who can gain by joining any of the three clubs, resulting in a club size of four. However, when there is a club of size four, three of its members could gain by jointly deviating and forming a club of their own. Conley and Konishi (2000) solve the problem of non existence due to left over players by analysing migration proof equilibria, which are stable only against credible deviations on the part of a coalition. They show that the migration proof equilibrium exists for the class of games under consideration, is unique, and asymptotically efficient in the sense that payoffs approach the maximum as the number of players goes to infinity.

We pursue a different approach that emphasizes the mobility aspect, and is based on the assumption of myopic optimization on the part of the players. We allow for coalitional deviations only by groups of players within a club. That is, a subgroup of club members may form a coalition for one period, and deviate jointly to some other strategy, e. g. join another club, or distribute themselves evenly across existing clubs. In the next period, there will be a new distribution of individuals within clubs and new coalitions may be formed. It is important to note that the model is truly dynamic: At each time step, a probability distribution determines the state for the next period.[5] Also, the restrictions on joint deviations are appealing; typically clubs are assumed to form so that individuals within the club can interact with each other. Coordination of strategies is one form of interaction. We call a vector of strategy choices that is immune to improving deviations by coalitions contained in any club a *Nash club equilibrium.*

Coalitions last for one period only and, as noted above, coalitions can be formed only within an existing club. This assumption seems reasonable because members of a club can be assumed to know each other, and coordinate their actions. In the context of this particular paper, this assumption is not at all restrictive: All our results would still hold if we allowed for arbitrary coalition formation. In other words, in the context of the model of this paper, a Nash club equilibrium is a strong Nash equilibrium. Nevertheless, the process will converge to a Nash club equilibrium with probability one, if it exists.

If the optimal coalition size, say s^*, defined as that which maximizes the per capita utility of the members of the club, is not an integer divisor of the population size then, provided that the population is larger then the optimal club size, a club Nash equilibrium will fail to exist because some players will be 'left over'. This problem often arises in the literature on club economies and economies with

[5] This is in strong contrast to models of club formation where, once a group of players has agreed upon forming a coalition, this group will drop out of the population and of negotiations, like in Ray and Vohra (1997).

local public goods. Here, we adopt an approach similar to that used in cooperative game theoretic approaches and consider approximate equilibrium outcomes.[6] Moreover, we assume that the number of locations is sufficiently large so that it is possible to form the maximal number of optimal clubs. Let k be a given positive integer. A strategy configuration is an $k -$ *remainderNash club equilibrium* (kNCE) if there are k players such that, if these k players are removed from the population, the remaining $n - k$ players are playing a Nash club equilibrium (on the reduced strategy space). An *approximate Nash club equilibrium* is a k–remainder equilibrium with minimal k. We demonstrate:

Existence of an approximate Nash club equilibrium (ANCE). Let s^* denote the optimal club size and define $k = n - \ell s^*$, where n is the total number of players in the game and ℓ is the largest integer for which k is non-negative. Then an ANCE exists.

Convergence to an ergodic set of strategy configurations. Let k be the integer defined above. Then the dynamic process will converge to an ergodic set of strategy configurations each of which is an ANCE.

Of course when n is large relative to s^*, then the percentage of left-over players, k/n, will be small. Intuitively, the economic motivation for our concept of an approximate Nash club equilibrium is that while the core may be empty and a Nash club equilibrium may not exist, in a large economy with relatively small clubs, 'most' clubs will have an equilibrium number of members most of the time.

The remainder of the paper is organized as follows. The next section presents the formal framework of our model, i. e. the stage game and the adaptive learning process on the part of each individual player. Section 3 describes the Markov process resulting from the individual adaptation rules. This process will converge to a pure strategy Nash equilibrium with probability one. Section **??** describes the process of coalitional deviations and the modified adaptation process that allows for coalitional deviations. In section 5, we show that absorbing states of this modified process can be identified with Nash club equilibria, and the process converges to a Nash club equilibrium configuration with probability one. Further, we prove that a NCE corresponds to a strong Nash equilibrium in our model. Finally, section 6 defines our notion of k-remainder Nash club equilibrium (kNCE) and approximate Nash club equilibrium (ANCE), and shows that the process will converge with probability one to an ergodic set of states consisting only of ANCE configurations. The final section concludes.

[6]See, for example, Wooders (1980) or Kovalenkov and Wooders (1997) for nonemptiness of approximate cores of economies with local public goods and clubs. A survey of much related literature is provided in Wooders (1999).

2 The Basic Model

We consider a finite set $N = \{1, \ldots, n\}$ of individuals who consume a private good x and a public good y. Each person can choose a location from a finite set G, $|G| > n$. Individuals choosing the same location form a club in order to provide the public good for themselves, and share the cost equally. Since there is no danger of confusion, we will identify a club with the location at which it is formed, e. g. club c is the name of the club formed at location $c \in G$.

Since the focus of this paper is on the formation of clubs, we assume that the locations are identical, i. e. individuals' preferences do not depend on the specific location at which a club is formed, but only on the membership of the club.

The choice of club is each person's strategy. A strategy profile is a vector $g = (g_1, \ldots, g_n)$, indicating a club (location) for each player. We consider only pure strategies.

Each person's utility depends not only on the consumption of the goods but also on the club size. Formally, this crowding effect is captured by a function $h : G^n \times N \to R$, where $h(a, s)$ is the (dis)utility to a member of club a when the total number of members (himself included) is s.

We design a non–cooperative game $\Gamma = \{N, G, (u_i)_{i \in N}\}$ where N is the set of players, G is the common strategy set, i. e. the set of locations the players can choose from, and $u_i : G^n \times N \to R$ is player i's payoff function.

For any given strategy profile $g = (g_1, \ldots, g_n)$, let $n_a(g)$ denote the number of players choosing strategy $a \in G$, and let $c(s)$ denote the cost of providing the optimal amount of public good for s club members. The payoff to player i playing strategy $g_i = a$ in strategy profile g is then given by the indirect utility function

$$u_i(g, x_i) = x_i - \frac{c(n_a(g))}{n_a(g)} + h(a, n_a(g)), \tag{1}$$

Note that crowding affects all players in the same way. This assumption is stronger than anonymity, which requires that the agents' utilities depend only on the number of users of a facility, and not on their identities, but the way in which this number affects an agent's utility may differ across agents.

In what follows, to economize on notation we will write $u_i(g)$ for $u_i(g, x_i)$. A Nash equilibrium of Γ is a strategy profile g with the property $u_i(g) \geq u_i(g_{-i}, b)$ for all $i \in N$ and all $b \in G$, where $(g_{-i}, b) := (g_1, \ldots, g_{i-1}, b, g_{i+1}, \ldots, g_n)$, i. e.

$$x_i - \frac{c(a)}{n_a(g)} + h(a, n_a(g)) \geq x_i - \frac{c(b)}{n_b(g) + 1} + h(b, n_b(g) + 1)$$

for all $i \in N$ and for all $b \in G$, where a is the strategy adopted by player i in strategy profile g.

Konishi et al. (1997a) and Hollard (2000) show the existence of pure strategy Nash equilibrium for the game Γ.[7] We provide a proof of existence of Nash equilibrium for the class of games Γ both for the reader's convenience.

Proposition 1 *(Konishi et al. 1997a) The game Γ admits a Nash equilibrium in pure strategies.*

Proof. For any given strategy profile g, let $N_a(g)$ denote the set of players choosing strategy a in profile g. We construct a function $P : G^n \to R$ as follows:

$$P(g) = \sum_{a \in G} \left[\sum_{i \in N_a(g)} x_i - \sum_{s=1}^{n_a(g)} \left(\frac{c(a)}{s} - h(a,s) \right) \right]. \tag{2}$$

This function P is the potential function associated with Γ (see Monderer and Shapley (1996), Konishi et al. (1997a)). A potential function is a function on the set of all strategy profiles such that, for any given profile g, and any single player i's deviation from g, the resulting change in the potential function equals the change in the deviating player's payoff. Suppose a single player i deviates from the strategy profile g by switching from strategy a to b. The resulting change in the potential function is

$$P(g) - P(g_{-i}, b) = u_i(g) - u_i(g_{-i}, b).$$

Clearly, any local[8] maximum of P is a pure–strategy Nash equilibrium of Γ. Further, as G^n is finite, P assumes a maximum on this set. Thus, a pure–strategy Nash equilibrium exists. □

2.1 Adaptive Learning

We construct a dynamic learning process on the part of the players that converges to a Nash equilibrium of Γ as time tends towards infinity.[9]

[7]The class of games considered in Konishi et al. (1997a) is more general than ours, as their model is not restricted to finite strategy sets. Hollard (2000) allows for external effects on players outside the group.

[8]A local maximum of the potential function is a strategy configuration such that any deviation by a single player leads to a decrease in the potential.

[9]Adaptive learning models have been applied, in a context formally similar to the one of this paper, to both cooperative and non–cooperative games, see e. g. Dieckmann (1999) for non–cooperative games, and e. g. Dieckmann and Schwalbe (2002) for cooperative games.

Time is divided into discrete periods $t = 0, 1, 2, \ldots$. In the initial period $t = 0$, we start with an arbitrary strategy profile $g \in G^n$.[10] Each player's opportunity to revise his strategy occurs at random. In each period t, each player gets the chance to adjust his strategy with probability $\gamma \in (0, 1)$, which is independent across players and periods. If player i gets the opportunity to adjust in period t, he will observe the current strategy profile g^t, and play a best reply to g^t_{-i}. Players who do not get the opportunity to revise maintain their current strategies. This "inertia" can be justified by the assumption that strategy adjustments involve a cost, for example the cost of moving from one location to another.

Formally, this best–reply rule is defined by

$$g_i^{t+1} = \arg\max_{a \in G} u_i(g^t_{-i}, a). \tag{3}$$

If the maximizer of (3) is not unique, we assume that the player randomizes, placing positive probability on each maximizer. Further, we assume that a player switches strategies only if this will strictly improve his payoff. That is, a player who is currently playing a best reply will not change his strategy.

3 The Dynamics

The individual players' best–reply rules define a Markov chain on the finite state space G^n. Let $n_{gg'}$ denote the number of players whose strategy under g differs from their strategy under g'. The transition probability between any two states g and g' is given by

$$p_{gg'} = \Pi_{i \in n_{gg'}} \gamma \beta_i(g'|g)(1-\gamma)^{n-n_{gg'}}, \tag{4}$$

where β_i is defined by the best–reply rule, i. e. $\beta_i(g'|g) > 0$ if and only if g'_i is a best reply for player i given g.

An absorbing state is a strategy profile g with $p_{gg} = 1$. Once an absorbing state is reached, the system will remain in that state forever, i. e. no player will have an incentive to switch strategies. It is obvious that the set of absorbing states of the Markov chain coincides with the set of Nash equilibria of the game Γ: If no player switches strategies, all players must be playing best replies, and vice versa.

Observe that a state $g \in G^n$ is absorbing if and only if g is a Nash equilibrium of Γ. However, this fact does not ensure convergence of the process to an absorbing state. Instead, the process might get trapped in a cycle. We will show that this

[10] We will later see that the choice of the initial strategy profile is irrelevant with respect to the results of the model.

is not the case. Our first result states that the Markov process will converge to an absorbing state, that is a Nash equilibrium of Γ, with probability one.[11]

Proposition 2 *The Markov process defined by the transition probabilities* (4) *converges to a Nash equilibrium of* Γ *with probability one as time goes to infinity.*

Proof. The following definitions are needed. A *path* $g \to g'$ on a set of states $M \subset G^n$ is a sequence of states $g^1, \ldots, g^k$, $g^t \in M$, $g^1 = g$, $g^k = g'$, with positive transition probabilities $p_{g^t, g^{t+1}} > 0$ for $t = 1, \ldots, k-1$. An *ergodic* set is a set of states $E \subset G^n$ such that (i) for any two states $g, g' \in E$ there is a path $g \to g'$, and (ii) for any $g \in E$, $g'' \notin E$, $p_{gg''} = 0$, and no nonempty proper subset of E has this property. That is, any state in an ergodic set can be reached from every other state in that set through a sequence of transitions with positive probability, and an ergodic set cannot be left again once it has been entered. Absorbing states are singleton ergodic sets.

A well known result from the theory of finite Markov chains[12] states that any finite Markov chain reaches an ergodic set with probability one as time goes to infinity, irrespective of the initial state. Thus, to prove the proposition, it suffices to show that all ergodic sets are singletons. To this end, we resort to a result derived by Monderer and Shapley (1996). Some further definitions are in order.

An *improvement path* on the set of strategy profiles G^n is a sequence $g^0, g^1, g^2, \ldots$ with the property that, for every $t \geq 1$, there exists a player $i \in N$ and a strategy $a \in G$ such that $g^t = (g^{t-1}_{-i}, a)$, where $a = g^t_i \neq g^{t-1}_i$, and $u_i(g^t) > u_i(g^{t-1})$. That is, each state in the sequence results from the previous state by a single player's strategy adjustment, and this adjustment is such that it strictly increases this player's payoff.

A game has the finite improvement property (FIP) if every improvement path is finite. Monderer and Shapley (1996) show that every finite game that possesses a potential function has the FIP. This is because the deviating player's payoff at each step equals the difference in the potential function. Thus, the potential is increasing with each step along the path. Since the potential function is defined on a finite set, the process must reach a maximum in a finite number of steps. This terminates the improvement path. As the potential function for our game Γ is given by (2), Γ has the FIP. Obviously, every finite improvement path of maximal length must terminate in a Nash equilibrium.

We will now show that all ergodic sets of the Markov chain are singletons. Suppose not. Then there exists an ergodic set E with $|E| \geq 2$. We derive a contradiction by constructing a path $g \to g'$ with $g \in E$ and g' being a Nash equilibrium.

[11] A similar result is derived by Milchtaich (1996) for the special case of congestion games, where each player's payoff is non–increasing in the number of players choosing the same strategy as himself.

[12] E.g. Kemeny and Snell (1976), Theorem 3.1.1 on p. 43.

Start from a state $g^0 \in E$. This state cannot be a Nash equilibrium, or else g^0 would have to be an absorbing state. Therefore, there exists a player i who is not playing a best reply. There is a positive probability that exactly one such player i gets the opportunity to revise his strategy. Suppose this happens. Then player i will play a best reply to g^0_{-i}, and strictly increase his payoff. The resulting state g^1 is either a Nash equilibrium, in which case we are done, or there exists a player j who is not playing a best reply. Repeating this argument yields an improvement path. As Γ possesses the FIP, this improvement path will terminate in a Nash equilibrium after a finite number of steps. □

4 Coordination of Coalitions within Clubs

We now turn to coalitional deviations and Nash club equilibria. In each period, people within the same club can form coalitions, or syndicates, for one period. A *coalition* is thus a subset of the set of members of a club. Note that the set of admissible coalitions may change from period to period. If a coalition is formed, its members will jointly decide which location each of them will choose. A coalition will form whenever it is in the interest of every single coalition member to do so. That is, each member must strictly benefit from forming the coalition. In the next period, all coalitions dissolve, and new ones can be formed. Formally:

Given any strategy profile g, define the resulting partition of the player set by

$$K(g) = \{N_a(g) \ldots , N_m(g)\},$$

where $N_a(g)$ denotes the club of all players choosing location a under the strategy profile g.

Definition 1 *Given any strategy profile g, a* coalition *is a group of players C that form a non–empty subset of any existing club, i. e. $C \subseteq N_a(g)$ for any $N_a(g)$.*

To incorporate the idea of coalition formation into our dynamic model, we modify the adaptation process to allow for joint decisions on the part of the members of a coalition, at each time step.

4.1 The Modified Adaptation Process

In any period t, given any strategy profile g and the resulting club structure $N(g)$, the adaptation process involves the following steps.

1. For every club a under the strategy profile g, the members of a can form coalitions. A coalition $C \subseteq N_a(g)$ will form only if there exists a strategy profile $y = (y_C, g_{-C})$ such that $u_i(y) > u_i(g)$ for all $i \in C$.

   If there is more than one coalition in $N_a(g)$ satisfying this condition, say $C_1, C_2 \in N_a(g)$, where $C_1 \cap C_2 \neq \emptyset$, only one of these coalitions can form. We assume that it is decided at random which of these coalitions will form, each having strictly positive probability.

2. Coalition formation gives rise to a partition of each club into coalitions: $N_a(g) = \{C_{1a}, \ldots, C_{ka}\}$, some or all of which may be singleton coalitions, for all $a \in G$.

3. The set of decision makers in period t is the set of coalitions formed in t (including the singleton coalitions).

4. Each decision maker's opportunity to revise their strategy occurs at random with probability $\eta \in (0, 1)$, which is independent across all decision makers and all periods.

5. If a player i who is in a singleton coalition gets the opportunity to revise, he will employ the best–reply rule defined in the unmodified adaptation process:

   $$g_i^{t+1} = \arg\max_{a \in G} u_i(g_{-i}^t, a).$$

6. If coalition $C \subseteq N_a(g)$ gets the opportunity to revise, and there exists a strategy profile $y = (y_C, g_{-C})$ such that $u_i(y) > u_i(g)$ for all $i \in C$, then the players in C will adopt this strategy vector y_C. If there are several such vectors that improve each members utility, the coalition will randomize, placing positive probability on each vector.

7. As in the unmodified process, decision makers (coalitions) will switch strategies only if they can strictly improve their payoffs.

Note that we do not model coalition formation within clubs explicitly.[13] We simply assume that a coalition forms if it is in the interest of all its members, i. e. if each of the members can increase their payoff for the next period. That is, players form a coalition if this coalition has a joint strategy such that employing this strategy will increase each member's payoff in the next period. If overlapping coalitions are possible, the coalition that actually forms will be chosen randomly.

[13]That is, we do not model a negotiation process like e. g. Ray and Vohra (1997). We simply assume that players coordinate their moves, if it is advantageous. The process by which they arrive at their mutually beneficial strategies is not modelled.

Note that, in our model, the formation of both coalitions (for one period) and clubs is always reversible: Clubs may form and dissolve again, since at each time step, decision makers are free to choose their strategies, regardless of their current situation.

Further, we do not have to redefine the state space of the Markov process. It is still the set of all possible strategy vectors $g \in G^n$. Only the transition probabilities have changed.

We define a Nash club equilibrium as a strategy configuration that is stable against deviations by coalitions, i. e. subsets of clubs.

Definition 2 *A strategy profile g is a* Nash club equilibrium (NCE) *if there is no club $c \in G$, no coalition $C \subset N_c(g)$, and no strategy configuration $y = (y_C, g_{-C})$ such that $u_i(y) > u_i(g)$ for all $i \in C$.*

5 Convergence to a Nash Club Equilibrium

To simplify notation, we express utility as a function of coalition size, rather than as a function of strategies, as usual in non cooperative game theory. Define

$$u(s) = u_i(g) \text{ where } n_a(g_i) = s.$$

That is, the payoff to a player i who is a member of a club of size s under the strategy profile g will get the payoff $u(s)$.

For the remainder of this paper we assume, like Conley and Konishi (2000), that the preferences are single peaked. This implies that there exists an optimal coalition size, which might be any number between 1 (singleton coalitions) and n (the grand coalition). Denote this number by s^*.

Assumption 1 *Preferences are single peaked if there exists an integer $s^* \in \{1, \ldots, n\}$ such that*

1. *for any clubs a, b with $n_a < n_b \leq s^*$ we have $u(n_a) < u(n_b)$, and*
2. *for any clubs a, b with $n_a > n_b \geq s^*$ we have $u(n_a) < u(n_b)$.*

Under assumption 1, providing $n > s^*$, as we assume, a necessary and sufficient condition for the existence of a Nash club equilibrium is that $n/s^* \in \mathbb{N}$.

Observe that, once a Nash club equilibrium configuration is reached, no player (or coalition) will switch clubs anymore. A Nash club equilibrium configuration is thus an absorbing state of the process. Conversely, any strategy configuration that is not a Nash club equilibrium cannot be an absorbing state, as the following proposition shows.

Proposition 3 *Let g be a strategy profile that is an absorbing state of the modified adaptation process. Then g is a Nash club equilibrium.*

Proof. Suppose a Nash club equilibrium (NCE) exists, and g is a strategy profile that is not NCE. Two cases are possible:

(i) There exists a club a of size $n_a < s^*$, or
(ii) there exists a club b of size $n_b > s^*$.

In either case, there are coalitions of players in a or b that can gain by deviating.

(i) Suppose all clubs are smaller than s^*. Then there are two clubs a and b with $n_a \leq n_b < s^*$. This cannot be an absorbing state since any member of a, i. e. any singleton coalition, will switch to b whenever they get the chance.

Now suppose that $n_a \leq s^*$ for all $a \in G$. It cannot be that all clubs but one are of optimal size, or else no NCE would exist. Therefore, again, there must be two clubs a and b with $n_a \leq n_b < s^*$, which cannot be an absorbing state.

Finally, suppose there are two clubs a and b with $n_a < s^* < n_b$. This cannot be an absorbing state since any member of b will switch to a if they get the chance.

(ii) Suppose there is a coalition a with $n_a > s^*$. Then there exists a coalition $C \subset N_a$ of size s^* all of whose members can gain by jointly switching to some other (unoccupied) club b. This coalition has a positive probability of forming, and of getting the chance to adjust its strategy. In this case, the coalition will move, and the state cannot be absorbing. □

We will now show that, if an NCE exists, the modified adaptation process will converge to an NCE configuration with probability one.

Proposition 4 *If the set of Nash club equilibria of the game Γ is nonempty, the modified adaptation process will converge to a Nash club equilibrium configuration with probability one.*

Proof. Proposition 3 implies that, if the process converges at all, it will converge to a NCE configuration. It remains to show that an NCE configuration can be reached with positive probability, starting from any other state.

Suppose $g \in G^N$ is a club structure that is not a NCE configuration. Two cases may arise:

(i) There exists a club a of size $n_a < s^*$, or
(ii) there exists a club b of size $n_b > s^*$.

(i) First, suppose all clubs are not larger than s^*, and some clubs are smaller. Then, there exist two clubs $a, b \in G$ such that $n_a \leq n_b < s^*$. In this case, all players would benefit from increased club size. This could be achieved by, for example, the members of a forming the grand coalition at a and, if they get the chance to move (which happens with positive probability), distribute themselves over all clubs whose size is smaller than s^*. Note that, as long as there exists a club a with $n_a < s^*$, there must exist another club b with $n_a \leq n_b$, and we can repeat the above argument, until all clubs are of size s^*.

(ii) Consider a club b with $n_b > s^*$. There are two integers l and m such that

$$n_b = ms^* + l \quad \text{with} \quad 0 \leq l < s^*.$$

The members of b form m coalitions of size s^* (there is a positive probability of this happening). Any of these coalitions, if they get the chance to move, will move to an unoccupied location and form a club. The resulting clubs will either be of size s^* or of size $l < s^*$ (the left over players from coalition b). Note that, since $l < s^*$, there cannot be a club c with $n_c > s^*$size $n_c \leq s^*$, and case (i) applies. □

We will now show that, in our model, an NCE corresponds to a strong Nash equilibrium. That is, if a strategy profile is NCE, no group of players (not even from different clubs) can gain by jointly deviating.

Proposition 5 *A Nash club equilibrium is a strong Nash equilibrium of the game* Γ.

Proof. We show that, if a strategy profile g is not a strong Nash equilibrium, then it is not NCE either. Suppose g is not a strong Nash equilibrium. Then, there exists a coalition $S \subset N$ and a strategy vector $y = (y_S, g_{-S})$ such that $u_i(y) > u_i(S)$ for all $i \in S$. We show that any such profitable deviation yC for S can also be achieved by coalitions, i. e. subsets of the members of one club.

Since g is not a strong Nash equilibrium, some members of S must be in a club c that is
(i) either smaller than s^*: $n_c(g) < s^*$,
(ii) or larger than s^*: $n_c(g) > s^*$, for $S \cap N_c(g) \neq \emptyset$.

Case (i): $n_c(g) < s^*$. There must be a club b with either $n_b > s^*$ or $n_b \leq n_c < s^*$. In either case, g cannot be NCE. If $n_b > s^*$, s^* members of b can gain by joint deviation. If $n_b \leq n_c < s^*$ any member of b can gain by switching to c. Thus g is not NCE.

Case (ii): $n_c(g) > s^*$. In this case, s^* members of c can gain by jointly deviating to an unoccupied location. Therefore, g is not NCE. □

6 Approximate Strong Nash Equilibrium

When the optimal club size is such that $n/s^* \notin N$, a NCE might not exist, as the following example shows.

Example.[14] $G = \{a, b, \ldots, f\}$, $N = \{1, 2, \ldots, 5\}$, and $u_i(g) = 1 + \phi(s)$ where

$$\phi(s) = \begin{cases} 0 & \text{for} \quad s = 1 \\ \frac{2}{s} & \text{for} \quad s \geq 2. \end{cases}$$

The table shows each club member's payoff for each possible club size s:

s	$u_i(\cdot)$
1	1
2	2
3	1.66
4	1.5
5	1.4

In this game there is an optimal club size, namely $s^* = 2$. But at most two clubs of size 2 can be formed. The left over player can then gain by joining any of these two clubs, since this increases his payoff from 1 to 1.66. However, in a club of size 3, any two players can gain by forming a coalition and deviating to an unoccupied location. Thus, no Nash club equilibrium exists.

Since the non existence of a Nash club equilibrium is simply due to a sort if 'nonbalancedness' or indivisibility problem,[15] we define a notion of Nash club equilibrium that takes indivisibility into account.

Definition 3 *A strategy configuration g is a* k–remainder NCE *if there exist k players such that, if these players are removed from the population, the strategies of the remaining $n - k$ players will form a NCE (on the reduced strategy space G^{n-k}).*

In the example, for $k = 1$, the strategy configurations $g' = \{a, a, b, b, c\}$ and $g'' = \{a, a, a, b, b\}$ both form a 1–remainder Nash club equilibrium: removing

[14] This example is taken from Dieckmann and Schwalbe (2002).

[15] The indivisibility problem is that the optimal club is indivisible. This would be solved if there were constant per capita benefits to club formation – in which case, clubs containing more than one member would be redundant. An alternative approach, following Wooders (1978), would be to allow a range of optimal club sizes containing two relatively prime integers, for example, s^* and $s^* + 1$. Then, since any sufficiently large population size n can be written as the sum of nonnegative integer multiples of s^* and $s^* + 1$, for all sufficiently large populations, a Nash club equilibrium would exist.

player 5 from g' and player 3 from g'' yields a Nash club equilibrium in both cases. In contrast, the configurations $\{a, a, a, a, b\}$ and $\{a, b, c, d, d\}$ are not 1–remainder Nash club equilibrium.

Obviously, if preferences are single peaked, an k–remainder Nash club equilibrium always exists. The special case of $k = 0$ is simply the Nash club equilibrium. However, we want k to be as small as possible. Therefore, we define an approximate Nash club equilibrium as a k–remainder Nash club equilibrium where k is minimized, i. e. corresponds to the minimal number of left over players.

Definition 4 *Let $i(x)$ denote the largest integer smaller than or equal to x. An* approximate Nash club equilibrium (ANCE) *is a k–remainder Nash club equilibrium with $k = n - s^* i(n/s^*)$.*

For instance, suppose $n = 17$ and the optimal coalition size is $s^* = 3$. Then, $k = 17 - 3 \cdot 5 = 2$ players will be 'left over'.

However, an ANCE is not necessarily an absorbing state since in each state, there are at least k players who might want to switch when they get the opportunity to adjust their strategies. That is, the process will always converge to an ergodic set of states, where some players keep switching strategies. In the example, for instance, in the strategy configuration $g' = \{a, a, b, b, c\}$, player 5 would switch to either a or b, and in state $g'' = \{a, a, a, b, b\}$, a coalition of players 1 and 2 (or 1 and 3, or 2 and 3) would switch to an unoccupied location. In order to describe the sort of cycle to which the process will converge, we introduce the concept of cyclic approximate NCE.

Definition 5 *A* cyclic approximate Nash club equilibrium (CANCE) *is a set of states $M \subset G^N$ with the following properties:*

1. *Every state $g \in M$ is an ANCE, and*
2. *M is an ergodic set.*

The second condition states that, first, each state in M can be reached from every other state in M in a finite number of steps, and second, once the set M is reached, it cannot be left again, i. e. the probability of the system's going from some state $g \in M$ to some other state $g' \notin M$ is equal to zero.

Note that, even though we call the equilibrium cyclic, this does not imply that all states within the ergodic set will be visited in a fixed order, or at fixed time intervals. This is due to the stochastic nature of the process, and the fact that a coalition's opportunity to adjust its strategy occurs at random.

The following proposition states our main result.

Proposition 6 *The modified adaptation process will converge to a CANCE with probability one as time goes to infinity.*

Proof. Suppose $g \in G^N$ is a strategy profile that is not ANCE. We will show that there is a positive probability of an ANCE being reached, starting from g. Once the process has reached an ANCE, we will show that all states that can be reached from there will also be ANCE, that is, the process will have reached a CANCE.

Start from g. Freeze $k = n - s^* i(n/s^*)$ players to their locations (i. e. assume that they do not get the chance to move for a finite number of periods, which happens with positive probability). For the remaining $n - k$ players, there exists a NCE.

¿From the proof of proposition 4, we know that there is a positive probability of a NCE being reached in a finite number of steps. Suppose this happens. Then k players remain, where k must be smaller than s^* (or else a subset of the k players could deviate by forming a coalition of size s^*). Suppose now that only these k players get the chance to move. There are two possibilities:

1. Either $u(k) > u(s^* + 1)$ or $u(1) > u(s^* + 1)$. In the first case, a club of size k is the next best thing to one of size s^* (in the circumstances), and the k players end up forming a club of size k. In the second case, the k players end up forming singleton coalitions. In either case, the resulting state is an ANCE, and an absorbing state.

2. $u(k) < u(s^* + 1)$. In this case, a ANCE can be reached in one time step: Each of the k players (as a singleton coalition) will join one of the clubs of size s^*, whenever they get the chance to move. This happens with positive probability. The result will be a state where all clubs are either of size s^* or $s^* + 1$. This state will be an ANCE. Call this state $\hat{g}$.

   Now consider a club a with $n_a > s^*$. Any s^* members of a can form a coalition, and deviate to an unoccupied location. The resulting state involves clubs of sizes 1, s^*, and $s^* + 1$. Again, any such state is ANCE. Thus, any state that can be reached from $\hat{g}$ is ANCE, which concludes the proof.

□

Note that the size of the set M, or rather the number of different club structures it contains, is small compared to the size of the set of possible club structures. The set M comprises only states with at most three different club sizes: s^*, $s^* + 1$, and 1.

7 Concluding Discussion

Our process of coalition formation is myopic. A possible interesting direction of research would be to allow long term coalition formation, so that, for example, two players who meet in a club may decide to act jointly for several periods.

The model provided in this paper has the advantage of simplicity and clarity. We plan to continue the investigation in this paper in several directions. First, we plan to consider the case where the total number of players n is smaller than the optimal club size s^*. We then propose to introduce crowding types (that is, external effects of players on each other, independent of their preferences), as in Conley and Wooders (1997). A particularly interesting extension may be to situations where players choose their crowding types, their skills, or educational levels, for example, as in Conley and Wooders (1996,2001). In this case, the discounted sum of expected utilities may be the appropriate measure of payoffs.

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