COMPETITIVE PRICING IN SOCIALLY NETWORKED ECONOMIES

Nizar Allouch

And

Myrna Wooders

No 639

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

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Preliminary Results[1]

by
Nizar Allouch
<N.Allouch@warwick.ac.uk>
and
Myrna Wooders
<M.Wooders@warwick.ac.uk>
http://www.warwick.ac.uk/fac/soc/Economics/wooders/
Department of Economics
University of Warwick

This version: April 24, 2002[2]

Keywords: social networks, competitive pricing, cliques, clubs, Edgeworth equivalence, core.

Abstract: In the context of a socially networked economy, this paper demonstrates an Edgeworth equivalence between the set of competitive allocations and the core. Each participant in the economy may have multiple links with other participants and the equilibrium network may be as large as the entire set of participants. A *clique* is a group of people who are all connected with each other. Large cliques, possibly as large as the entire population, are permitted; this is important since we wish to include in our analysis large, world-wide organizations such as workers in multi-national firms and members of world-wide environmental organizations, for example, as well as small cliques, such as two-person partnerships. A special case of our model is equivalent to a club economy where clubs may be large and individuals may belong to multiple clubs. The features of our model that cliques within a networked economy may be as large as the entire population and individuals may belong to multiple cliques thus allow us to extend the extant decentralisation literature on competitive pricing in economies with clubs and multiple memberships (where club sizes are uniformly bounded, independent of the size of the economy).

[1] The authors are indebted to John Conley, Frank Page and Ben Zissimos for helpful comments and for references to the literature..

[2] This paper was submitted to the Athens General Equilibrium Conference in March of this year and is to be presented there in May 2002.

1 Introduction

There have been few studies of competitive pricing in networked economies. Recent literature, however, suggests that whenever all or almost all gains to collective activities can be realized by groups of participants bounded in size, then diverse economies resemble markets, including economies with indivisibilities, nonconvexities, local public goods, club economies with multiple memberships, and with networks. In particular, approximate cores are nonempty, approximate cores treat similar people similarly and economies with many participants, modelled as games with side payments, generate market games.[3] In this paper, we demonstrate an Edgeworth equivalence theorem for socially networked economies where small groups of participants can realize all or almost all gains to collective activities. The economies allow ever increasing returns to network size.

It seems compelling that gains to cooperation may forever increase as the size of an economy increases. One example, for a private goods exchange economy, is presented in Hammond, Kaneko and Wooders (1989). There, because the percentages of agents of each of two types are rational, as they must be in any finite economy, the set of equal treatment outcomes in utility space may be forever increasing (strictly) as the economy is replicated. In economies with public goods, coordination of activities and decreasing costs of providing public goods may also lead to ever increasing gains to population size. Thus, a general model of network economies (or economies with public goods) should allow both small networks bounded in size of membership, base ball teams or marriages for example, and also permit the possibility of ever-increasing gains to larger and larger networks. A general model should also allow overlapping cliques so that a participant may belong, for example, to a two-person partnership, a dance club, and a world-wide environmental organization.

Define a *clique* as a group of individuals who are all connected to each other and all engage in some clique activity. Cliques may overlap; that is, two distinct cliques may contain some of the same individuals. A *network* has the usual interpretation. In the current paper, a network is restricted to be a clique structure of the total population and a clique is analogous to a club, as in Kovalenkov and Wooders (1997), for example; for the current paper, we use the terms clique and club interchangeably. In the literature on approximate cores of games and economies with collective activities and clubs, there are a number of models in the literature permitting ever-increasing gains to coalition and club sizes. See especially Wooders (1983,1994) and also Kovalenkov and Wooders (1997, 2001a, 2001b). None of these models rule out games derived from economies where individuals may belong to overlapping cliques and where there may be ever-increasing gains to network and clique size. In addition, following Shubik and Wooders (1982), Kovalenkov and Wooders (1997) explicitly allow an individual to belong to overlapping cliques. There have also been a number of papers demonstrating that states of the economy in approximate cores of

[3] Wooders (1994) demonstrates that games with many players and side payments are market games as introduced in Shapley and Shubik (1969).

economies with clubs can be supported as price-taking equilibrium outcomes. None of these papers studying price-taking equilibrium, however, satisfy the criteria that equilibrium clubs may be large – as large as the entire population, for example – and that individuals may belong to overlapping clubs. The model of the current paper allows both of these features. We obtain preliminary results for replication economies with a fixed percentage of players of each of a finite number of types but where cliques and networks may become larger and larger as the economy is replicated. We demonstrate conditions ensuring that feasible equal-treatment payoffs are bounded. This implies that, given $\varepsilon > 0$, for all sufficiently large replications, there are equal-treatment states of the economy in approximate cores.[4] We then show that such states of the economy are approximate price-taking equilibrium outcomes. We stress that while the results of this paper are only for sequences of economies with the same percentages of agents of each type, the techniques to broadly extend the framework and results are all in place, primarily based on results on nonemptiness of approximate cores of large games in Kovalenkov and Wooders (1997, 2001a, 2001b).

In the special setting of this preliminary paper, a clique is analogous to a club and overlapping cliques are analogous to club economies allowing multiple memberships. The technique introduced in this paper can be extended to networks of significant generality and with appropriate assumptions, the results of this paper will also extend.[5] Note that some of these papers allow both directed networks and hierarchical (or super) networks, that is, situations where there may be networks of networks, which we anticipate will be important in subsequent research.

Our approach in this paper is in part based on earlier research, especially Wooders (1983), showing that under apparently mild restrictions – boundedness of per capita payoffs in utility space – approximate cores of replication games are nonempty.[6] The *crucial* innovation in the current paper, which will allow us to obtain significantly more general results, is our construction of the commodity space. Part of this innovation is in extending and further developing the Foley (1970)-Wooders (1985) proof technique of defining 'preferred sets of allocations of private goods' for coalitions. To ensure that the games derived from the economies satisfy per capita boundedness – simply boundedness of the set of equal treatment payoffs – we make an assumption of 'desirability of wealth'. Informally, this assumption dictates that there is some level of wealth, measured in terms of a bundle of private goods, such that an individual would prefer that level of wealth and membership in some bounded number of cliques, all bounded in size, to any feasible equal-treatment outcome in any economy, no matter how

[4] That is, identical agents are assigned consumption bundles that yield the same utility levels (although the bundles themselves may differ).

[5] We have in mind networks as defined in, for example, Goyal (1993), Jackson and Wolinsky (1996), Dutta and Mutuswami (1997), Belleflamme and Bloch (2001), van den Nouweland and Slikker (2001), and Page and Wooders (2001).

[6] Note that in Wooders (1983), the set of players is replicated but the payoff set to any coalition of players may increase as the size of the total population increases. See also Wooders (1988) for a first application of her 1983 results to economies with multiple private goods and with local public goods.

large. Loosely, desirability of wealth implies that private goods can compensate for membership in large networks and cliques. Following conditions laid down in Shubik and Wooders (1982) and Kovalenkov and Wooders (1997), clique structures are required to satisfy some apparently mild properties: individuals may subsist in cliques consisting of themselves alone; and for any two disjoint coalitions, the union of admissible clique structures of the union of the two coalitions is a clique structure of the coalition.

This paper points the way to broader results. In particular, in future research we:

1. Make a distinction between the crowding types and taste types of agents (as in Conley and Wooders 1997, 2001, for example) and show that the outcomes of price-taking equilibrium, with clique participation prices defined to depend only on crowding types of participants, are equivalent to the set of outcomes in the core;

2. Allow endogenous choice of crowding types, skills for example, (as in Conley and Wooders 1996, 2001) and still obtain equivalence of price-taking equilibrium outcomes and the core;

3. Relax the assumption of replication of the set of participants to allow a compact metric space of crowding types and arbitrary player sets. (It is here that the results of Kovalenkov and Wooders will be useful.)

4. Incorporate general network structures as in Page and Wooders (2001) and Rockafellar (1984), and allow hierarchical networks.

(Of course the above results hold in general only for approximate cores and approximate equilibrium outcomes since, without special assumptions, we can only ensure nonemptiness of approximate cores.) We also plan to eventually allow widespread externalities, as in Hammond, Kaneko and Wooders (1989) and Kaneko and Wooders (1989). In this case, however, without further assumptions, we will loose the optimality of equilibrium outcomes.

Continuing with the motivation for our paper, the importance of overlapping cliques is perhaps clear. What may not be immediately clear is the importance of allowing optimal or equilibrium cliques to be unbounded in size and possibly as large as the entire economy. Consider, however, questions of global pollution, global harmonization of productive activities and memberships in networks. If we wish our model to describe cliques such as the World Trade Organization, the United Nations, the World Environmental Organization, or Christian religions which wish to embrace all people, then a model with bounded clique sizes, where cliques become infinitesimal as the economy grows large, is not appropriate; clique sizes must be unbounded.

In the following, we first develop the formal model and state and prove the results, except for Theorem 3, the main result, which is proven in an appendix. After stating the results, we relate our research to the literature and then provide some concluding remarks.

2 The Model

2.1 Agents

We consider an economy with $N = \{t = 1, ..., T\}$ agents. For each positive integer r *the set of agents in the* r^{th} *replica economy* is denoted by

$$N_r = \{(t, q) : t = 1, ..., T \text{ and } \quad q = 1, ..., r\},$$

and (t, q) is called the *qth agent of type t.* It will be the case that all agents of the same type are identical in terms of their consumption sets, endowments and preferences.

Given N_r and $t \in \{1, ..., T\}$, let $[t]_r$ denote the set of agents of type t in the r^{th} replica economy. We call every nonempty subset S of N_r a *clique.* Let s be a vector in N^T with t^{th} component given by

$$s_t = |S \cap [t]_r|.$$

The vector s, called the *profile* of S, lists the numbers of agents of each type in S.

The economy has L private goods. A vector of private goods is denoted by $x = (x_1, ..., x_\ell, ..., x_L) \in \mathbf{R}_+^L$.

For given a nonempty subset S of N_r, we call any collection of subsets of S that covers S, denoted by $\mathcal{S}$, a *clique structure* of S. A clique structure of N_r is called simply a *network.* (We re-iterate that this special form of network is adopted here but our techniques will significantly extend, to allow directed and hierarchical networks). There is only one clique good, a public good for the membership of the clique, available to each clique S. The production of the clique good requires $z_S \in -\mathbf{R}_+^L$ *inputs of private goods.*

Given a clique structure $\mathcal{S} = \{S_1, \ldots, S_k, \ldots, S_K\}$ of $S \subset N_r$ and $tq \in S$, let $\mathcal{S}[tq] = \{S_k \in \mathcal{S} \mid tq \in S_k\}$ denote the set of all cliques in $\mathcal{S}$ that contain consumer tq. That is, the set $\mathcal{S}[tq]$ denotes the clique goods consumption of agent t with respect to $\mathcal{S}$. The set $I[t] = \cup_{\{\mathcal{S},q\}}\mathcal{S}[tq]$ is called the *network consumption set* for agents tq. Let $\{tq\}$ denote the clique consisting of consumer tq only.

It is assumed that each agent has a positive endowment of each private good and that there are no endowments of clique goods. Let w^{tq} be the *endowment of the* tq^{th} agent *of the private goods*, $w^{tq} \in \mathbf{R}_{++}^L$. It is assumed that all agents of the same type have the same endowments, that is

$$w^{tq} = w^{tq'}, \qquad \forall q, q' \in \{1, \ldots, r\}.$$

The utility function of the tq^{th} agent is denoted by $u^{tq}(\cdot, \cdot)$ and maps $X^t \times I[t]$ into R, where X^t, called the *commodities consumption set* for agents tq, is a subset of R^L. We assume that w^{tq} is in the interior of X^t. The utility functions of all agents of type t are identical, let $u^t(\cdot, \cdot)$ denote the utility function of an arbitrary agent of type t.

It is assumed that for each $\mathcal{S}[tq]$ in $I[t]$ the utility function satisfies the usual properties of monotonicity, continuity and convexity. Specifically, for any given network goods consumption $\mathcal{S}[tq]$ in $I[t]$, the utility function u^t satisfies:

(a) **Monotonicity**: $u^t(\cdot, \mathcal{S}[tq])$ is an increasing function, that is, if $x < x'$ then $u^t(x, \mathcal{S}[tq]) < u^t(x', \mathcal{S}[tq])$.

(b) **Continuity**: $u^t(\cdot, \mathcal{S}[tq])$ is a continuous function.

(c) **Convexity**: $u^t(\cdot, \mathcal{S}[tq])$ is a quasi-concave function.

(d) **Desirability of endowment**: if $u^t(w^{tq}, \{tq\}) \leq u^t(x', \mathcal{S}[tq])\}$ then $x' >> 0$.

With the exception of (d), the conditions above are all standard. Condition (d) incorporates the Hammond-Kaneko-Wooders (1989) and Kaneko-Wooders (1989) condition that the endowment is preferred to any outcome which assigns an agent zero of any of the indivisible (clique) goods. Some such assumption is needed to ensure that for large economies, states of the economy in the core have the equal treatment property.[7]

2.2 States of the Economy

Let S be a nonempty subset of N_r and let $\mathcal{S}$ be a network for S. A *state of the economy for S relative to $\mathcal{S}$* is an ordered pair $(x^S, \mathcal{S})$, where x^S is an allocation for S. The state $(x^S, \mathcal{S})$ is *feasible* if

$$\sum_{tq \in S} (x^{tq} - w^{tq}) \leq \sum_{S \in \mathcal{S}} z_{\mathcal{S}}.$$

The following concept of the core can be interpreted as either a notion of an approximate core or as an exact core subject to communication. If a group of agents is to form an alliance – a coalition – then they must communicate with each other and possibly reallocate goods among the members of the coalition. Such communication is costly. This affects the net resources available to any coalition after it has formed. We suppose that there is a communication cost in order to form any coalition. Let $\bar{z} \in -R^L_{++}$ and $\varepsilon > 0$, we denote by $c(\varepsilon, S) = \varepsilon|S|\bar{z}$ the communication cost for coalition S. Moreover, we assume that agents share the communication cost equally. The state $(x^S, \mathcal{S})$ is c(ε)-*feasible* if

$$\sum_{tq \in S} (x^{tq} - w^{tq}) \leq \sum_{S \in \mathcal{S}} z_{\mathcal{S}} + \varepsilon|S|\bar{z}.$$

[7] In the literature of private goods exchange economies, related, more restrictive conditions go back to Broome (1973). For economies with local public goods/clubs an analogous condition was introduced in Wooders (1978,1980). The Hammond-Kaneko-Wooders (1989) condition is less restrictive.

2.3 The c(ε)-Core

Let $(x^{N_r}, \mathcal{N}_r)$ be a state of the economy relative to the network $\mathcal{N}_r$. A coalition S can $c(\varepsilon)-$*improve upon* the state $(x^{\mathcal{N}_r}, \mathcal{N}_r)$ if there is a network $\mathcal{S}$ of S and a $c(\varepsilon)$-feasible state of the economy $(x'^S, \mathcal{S})$ for S such that for all consumers $t \in S$ it holds that:

$$u^t(x'^{tq}, \mathcal{S}[tq]) > u^t(x^{tq}, \mathcal{N}_r[tq]).$$

A state of the economy $(x^{\mathcal{N}_r}, \mathcal{N}_r)$ is in the $c(\varepsilon)$-*core* (of the economy) if it cannot be $c(\varepsilon)$-improved upon by any coalition S.[8] It is clear that when $\varepsilon = 0$, the c(ε)-core coincides with the usual core.

2.4 The ε-equilibrium

In this section we define an ε-equilibrium and demonstrate that an ε-equilibrium state of the economy is in the $c(\varepsilon)$-core. Our notion of $\varepsilon-$equilibrium allows at least some agents to spend more than their income at the given prices.

A *price system for private goods* is a vector $p \in \mathbf{R}^L_+$. A *participation price system* is a set

$$\Pi = \{\pi^{tq}(S) \in \mathbf{R} : S \subset N_r \text{ and } tq \in S\},$$

stating a participation price, positive, negative, or zero, for each agent in each clique. An ε-*equilibrium* (for an economy with clique goods) is an ordered triple $((x^{N_r}, \mathcal{N}_r), p, \Pi)$ consisting of a state of the economy $(x^{N_r}, \mathcal{N}_r)$, where $\mathcal{N}_r = \{J_1, ..., J_g, ..., J_G\}$, a price system $p \in R^L_+ \setminus \{0\}$ for private goods, and a participation price system Π, such that:

(i) $\sum_{tq \in N_r}(x^{tq} - w^{tq}) \leq \sum\limits_{S \in \mathcal{N}_r} z_S$;

(ii) for each clique $S \subset N_r$,

$$p \cdot z_S + \sum_{tq \in S} \pi^{tq}(S) \leq 0$$

(iii) for any agent $tq \in N_r$ and any network $\mathcal{S}$ of $S \subset N_r$ such that $tq \in S$, if

$$u^t(x'^{tq}, \mathcal{S}[tq]) > u^t(x^{tq}, \mathcal{N}_r[tq])$$

then

$$p \cdot x'^{tq} + \sum_{S \in \mathcal{S}[tq]} \pi^{tq}(S) > p \cdot w^{tq} + \varepsilon p \cdot \bar{z}.$$

[8] A similar notion of the core with some costs of communication was introduced in Wooders (1988).

$$(iv) \qquad \sum_g p \cdot z_{J_g} + \sum_g \sum_{tq \in J_g} \pi^{tq}(J_g) \geq \sum_{tq \in N_r} \varepsilon p \cdot \bar{z}, \quad \text{and}$$

$$\sum_{tq \in N_r} p \cdot x^{tq} + \sum_{tq \in N_r} \sum_g \pi^{tq}(J_g) \leq \sum_{tq \in N_r} p \cdot w^{tq} + \varepsilon p \cdot \bar{z}.$$

Intuitively, our notion of an approximate equilibrium involves bounded rationality. We suppose that there is a small cost of changing ones consumption bundle so that individuals may be slightly in the interior of their budgets sets.

Theorem 1: An ε-equilibrium state of the economy is in the c(ε)-core.

Proof: Suppose the Theorem is false. Then there is an ε-equilibrium $((x^{N_r}, \mathcal{N}_r), p, \Pi)$ with the property that the state of the economy $(x^{N_r}, \mathcal{N}_r)$ is not in the c(ε)-core of the economy. This means that there is a coalition $S \subset N_r$, a network $\mathcal{S}$ of S and an allocation $(x'^S, \mathcal{S})$ such that

$$\sum_{tq \in S} (x'^{tq} - w^{tq}) \leq \sum_{S \in \mathcal{S}} z_S + \varepsilon |S| \bar{z}$$

and

$$u^t(x'^{tq}, \mathcal{S}[tq]) > u^t(x^{tq}, \mathcal{N}_r[tq]).$$

From (ii) of the definition of an ε-equilibrium it follows that

$$p \cdot z_S + \sum_{tq \in S} \pi^{tq}(S) \leq 0$$

and from utility maximization, it holds that

$$p \cdot x'^{tq} + \sum_{S \in \mathcal{S}[tq]} \pi^{tq}(S) > p \cdot w^{tq} + \varepsilon p \cdot \bar{z}.$$

Summing up these above inequalities, one will have

$$\sum_{tq \in S} p \cdot (x'^{tq} - w^{tq}) > \sum_{S \in \mathcal{S}} p \cdot z_S + p \cdot \varepsilon |S| \bar{z},$$

which is a contradiction.

3 Convergence of the Core to Equilibrium States of the Economy

Before stating our convergence theorem, we define replications of a state of the economy. To replicate a state of the economy, in addition to replicating the consumer set we also replicate the network and consumptions so that all replicas of an individual consumer are in cliques with identical profiles, and are allocated identical consumptions.

For given r', let $\mathcal{N}_{r'} = \{J_1, ..., J_g, ..., J_G\}$ be a network of $N_{r'}$ and let $r = nr'$, where n is a positive integer. Let $\mathcal{N}_r$ be a network of N_r containing nG cliques and denoted by:

$$\mathcal{N}_r = \{J_{gj} : j = 1, ..., n \text{ and } \quad g = 1, ..., G\},$$

where for each $j = 1, ..., n$ and each $g = 1, ..., G$ the profile of J_{gj} equals the profile of J_g. Then $\mathcal{N}_r$ is the n^{th} *replication of* $\mathcal{N}_{r'}$.

Let $(x^{N_{r'}}, \mathcal{N}_{r'})$ be a state of the economy. Given a positive integer n, let $\mathcal{N}_{nr'}$ be the n^{th} replication of $\mathcal{N}_{r'}$. A state of the economy $(x^{N_r}, \mathcal{N}_r)$ is an n^{th} *replication of* $(x^{N_{r'}}, \mathcal{N}_{r'})$ if

(a) for each $g = 1, ..., G$ and each $n = 1, ..., r$,

$$z_{gj} = z_g;$$

(b) for each consumer tq there are n consumers tq' in the replication having the same allocation as tq.

A state of the economy $(x^{N_{r'}}, \mathcal{N}_{r'})$ is in the $c(\varepsilon)$-*core for all replications* if, for each positive integer n, it holds that an n^{th} replication of $(x^{N_{r'}}, \mathcal{N}_{r'})$ is in the c(ε)-core of the n^{th} replication of the economy.

4 The Game Derived from an Economy and Per Capita Boundedness

First, let us select ε_0 such that $w^{tq} + \varepsilon_0 \bar{z} \in X^t$ and if $u^t(w^{tq} + \varepsilon_0 \bar{z}, \{tq\}) \leq u^t(x', \mathcal{S}[tq])\}$ then $x' >> 0$. By continuity with respect to the private good such an ε_0 exists.

Given r and $\varepsilon \in [0, \varepsilon_0]$ we associate a correspondence V_r^ε with the r^{th} economy where V_r^ε maps subsets S of N_r into R^{N_r}. For each subset S of N_r, define V_r^ε as the set of vectors $\alpha \in R^{N_r}$ with the property that for some network $\mathcal{S}$ of S and some $c(\varepsilon)$-feasible state with associated allocation $(x^S, \mathcal{S})$ we have $\alpha^{tq} \leq u^t(x^{tq}, \mathcal{S}[tq])$ for each $tq \in S$. When $\varepsilon = 0$, we denote V_r^ε simply by V_r. It is straightforward to verify the sequences of derived games $(N_r, V_r^\varepsilon)_{r=1}^\infty$ is a sequence of superadditive replica. Moreover the games are comprehensive.

The set of equal treatment payoffs of $(N_r, V_r^\varepsilon)_{r=1}^\infty$ is given by

$$E(r, \varepsilon) = \{\alpha \in R^N \mid \Pi_{i=1}^r \alpha \in V^\varepsilon(N_r)\}.$$

We require some minimal assumption on the economy to ensure that equal-treatment utilities derived from the economy do not become infinite. To this purpose we introduce the following assumption:

Desirability of wealth: Assume that there is a bundle of private goods, x^* and a replication number r^*, such that for any other bundle of private goods x, for some network of the r^*th economy, it holds that, for any r, for any t,

$$u^t(x^t + x^*, \mathcal{N}_{r^*}[t]) \geq u^t(x^t, \mathcal{N}_r[t])$$

for any network $\mathcal{N}_r$ of the rth economy.

Informally, this assumption ensures that wealth, in terms of private goods, can substitute for large clique memberships, no matter how large the economy. Because of the possibility of ever-increasing returns to clique size, due to public goods for example, in our model agents may derive more and more utility from larger and larger cliques. Desirability of wealth ensures that if an individual were sufficiently wealthy, however, he could provide clique goods for himself and his friends. Note that x^* is independent of the type of the player; this is for simplicity of statement.

Theorem 1. Assume desirability of wealth. Then there is a positive real number K such that for any replication number r and for any utility equal treatment feasible state of the rth economy,

$$\sup u^t(x, \mathcal{N}_r[t]) < K.$$

Proof. To show per-capita boundedness of $(N_r, V_r^\varepsilon)_{r=1}^\infty$ we construct a sequence of $*$-economies and consider the sequence of games, denoted by $(N_r, V_r^*)_{r=1}^\infty$, derived from the sequences of $*$-economies. We construct the sequence of $*$-economies so that $V_r^\varepsilon(N_r) \subset V_r^*(N_r)$ and show that $(N_r, V_r^*)_{r=1}^\infty$ satisfies per-capita boundedness to obtain the conclusion of the theorem.

For each r^{th} $*$-economy, let the utility function of agent tq be defined by

$$u^{*t}(x^{tq}) = \max\ u^t(x^{tq} + x^*, \mathcal{N}_{r^*}[t]),$$

where r^* satisfies desirability of wealth.

The utility functions u^{*t} are well defined and are quasi-concave. Also. it is clear that given any $(x^{tq}, \mathcal{N}_r[tq])$ we have

$$u^{*t}(x^{tq}) \geq u^{tq}(x^{tq}, \mathcal{N}_r[tq]).$$

For each r, the allocation (x^{N_r}), is *$*$-feasible* if

$$\sum_{tq \in N_r} (x^{tq} - w^{tq}) \leq 0.$$

The set of all *-feasible allocations is denoted by A_r^*. Let K be a real number such that

$$K > \sup_{x_t \in A_1^*} u^{*t}(x^t).$$

From the closeness of A_1^* and quasi concavity there is a such real number. Obviously, K is a per-capita bound .□

Desirability of wealth thus implies that the sequence of economies satisfies per capita boundedness and, given any $\varepsilon \in [0, \varepsilon_0]$, the c($\varepsilon$)-core of the game derived from the economy, is non-empty for a subsequence of economies, that is, given any r there is an integer m_0 such that for all integers ℓ, the $\ell m_0 rth$ economy has a nonempty c(ε)-core. However, from desirability of endowments, monotonicity, and divisibility of private commodities, we also have the following result.

Theorem 2: Assume desirability of wealth. Then, given $\varepsilon > 0$ there is a replication number $r(\varepsilon)$ such that for all $r > r(\varepsilon)$, the c(ε)-core of the game derived from the r^{th} economy is nonempty.

Proof. This follows from Wooders (1983) and Wooders (1988) Theorem 2.

Theorem 3: if $(x^{N_r}, \mathcal{N}_r)$ is in the c(ε)-core of the economy for all replications of the economy, then $(x^{N_r}, \mathcal{N}_r)$ is an ε-equilibrium state of the economy.

We postpone the proof of Theorem 3 to an Appendix.

5 Relationships to the literature

In his seminal paper, Tiebout (1956) suggests that, if public goods are subject to congestion, the benefits of sharing costs over a large number of agents will eventually be offset by the negative effects of crowding. When it is optimal to have many jurisdictions providing such public goods, Tiebout conjectures that the movement of consumers to their preferred jurisdictions will lead to a 'market-type,' near-optimal outcome and the free rider problem of economies with pure public goods will not arise, the 'Tiebout Hypothesis.' Early contributions to this literature include Pauly (1972), treating an economy consisting of identical agents and Wooders (1978,1980), who allows several types of agents and nondifferentiated (or anonymous) crowding.[9] A short survey of the vast literature is provided in Wooders (1999). In Conley and Wooders (1996,1997), and Cole and Prescott (1997) the model is further developed in that the crowding types of agents (external characteristics) are separated from taste types and

[9] The agents appear to differ but, as shown in Wooders (1976), the fact that both types of agents make the identical marginal contributions to coalitions implies that only the size and not the composition of coalitions is relevant.

it is shown that first best prices can be defined to depend only on the crowding types of agents; no private information is required.[10]

In the economic models of those papers listed above agents are allowed to belong to only one jurisdiction. Allowing multiple memberships in clubs, Shubik and Wooders (1982) demonstrated nonemptiness of approximate cores of economies with many agents but price-taking equilibrium was not studied. Kovalenkov and Wooders (1997) demonstrated conditions under which large finite games and economies with clubs and permitting multiple memberships have nonempty approximate cores. Subsequently, Ellickson et al (2001) introduced a model of an economy with multiple memberships and obtained approximate versions of existence of equilibrium and equivalence of the core and the set of equilibrium outcomes. Their model is more restrictive than the prior model of Kovalenkov and Wooders, since there are only a finite number of distinct sorts of clubs, so clubs become negligible as the economy grows large. As Conley and Wooders (1997) and Cole and Prescott (1997), Ellickson et al. make a distinction between the crowding types (in their language, 'external characteristics') of agents and their taste (and endowment) types.

While Ellickson et al. extend the prior equivalence results for economies with local public goods/cliques in several aspects, there is one important aspect where there model is more restrictive than the prior literature. In particular, they model the economy as one with a finite number of sorts of clubs, which enables them to treat the economy similarly to an economy with a finite number of indivisible commodities. The major difference of the economy from one with private goods appears to be in the feasibility condition on club memberships. Unfortunately, this rules out situations where there are ever-increasing returns to club size, which are permitted in Wooders (1989,1993) and in Shubik and Wooders (1982), and of course the Kovalenkov-Wooders papers.

In view of the prior literature on large games and large economies one might hope for approximate equivalence in large finite economies even with multiple memberships in cliques and with potentially ever-increasing returns to clique size. The crucial restriction appears to be that all or *almost all gains* to collective activities are realized by groups bounded in size; that is, that small groups are effective. In the case of one-private-good, the restrictions of Ellickson et al. transform the economy into one satisfying *strict* small group effectiveness, that is, *all* gains to collective activities can be realized by groups of players bounded in absolute size.

In this paper, we have taken a first step towards obtaining equivalence of the core and price-taking equilibrium outcomes when club sizes may be unbounded. We start with a replica economy situation. More specifically, we show that if a

[10] When crowding is anonymous – that is, individuals care only about the numbers of agents in the same jurisdiction and not their characteristics – then prices are anonymous (Wooders 1978). With differentiated (also known as nonanonymous) crowding, until the works of Conley and Wooders (1996,1997) and Cole and Prescott (1997), prices were also nonanonymous, depending on tastes. Conley and Wooders (1996,1997) and Cole and Prescott (1997) separated taste types from crowding types and showed that first best prices can be defined to depend only on crowding types or external characteristics of agents.

state of the economy is in the core for all replications of that economy, then it is an equilibrium state of the economy. We also show that an equilibrium state is in the core.

The results in the current version of this paper are preliminary and conclude with the replication model. Approximate cores are supported by approximate equilibria. To relax the replication restriction, it is only necessary to ensure that small group effectiveness is satisfied. Here we can exploit the results of Kovalenkov and Wooders (1997, 2000, 2001). In brief, once it is known that approximate cores are nonempty, based on the techniques in this preliminary version of our research, it can be shown that approximate equilibria exist and are in approximate cores. In addition, we show that approximate cores converge to equilibrium outcomes.

For the reader familiar with the previous literature on large games and economies with clubs/local public goods, it is apparent that one can build substantially on our results for replica games/economies. Thus, we spend some time describing our approach, which itself builds on the research of Debreu and Scarf (1963), Foley (1970) and Wooders (1989). Recall that, given a state of the economy that is in the core for all replications of the total agent set, Debreu and Scarf (1963) define the set of preferred net trades of each agent in the economy and show that the convex hull of union of these sets can be separated from the origin. For an economy with pure public goods, Foley (1970) extends the commodity space to make the public good a separate good for each consumer. Wooders (1989) further extends the commodity space to make local public goods for each consumer in each possible jurisdiction separate commodities. In this paper, we build on these three approaches. Precisely, we extend the public good space so that each clique and its membership is a different commodity for each agent in the clique. Thus, in this respect, the techniques of Debreu and Scarf (1963) apply. We also introduce a virtual production set. Even though we have no production in the current paper, our virtual production set plays a similar role to the extended production sets in Foley (1970) and Wooders (1989). In particular, the feasibility requirements ensuring the clique choices are consistent are imposed on the virtual production set.

From the point of view of the literature on large economies with private goods only, a paper that is relevant to our longer range objectives is Kirman, Oddou and Weber (1986). The purpose of the (stochastic) networks in that paper is to facilitate trade in private commodities. We conjecture that similar results of convergence of cores may be obtainable in models of socially networked economies, with large networks, as long as some condition of boundedness of per capita payoffs or effectiveness of small groups (that is, all or almost all gains to collective activities can be realized by groups bounded in size of membership).

6 Conclusions

With the fundamental results of this paper it is possible to extend the results to hold in all respects in the same or more generality as the extant literature.

The major economic importance of our research is that equilibrium cliques and networks may be unbounded – they do not necessarily become infinitesimal as the economy grows large. This aspect of our modelling is especially relevant for questions of political economy, for example, and to issues of regulation of large firms, such as multinationals. We hope to study these issues, as well as other issues relating to labor markets in economies with large firms/jurisdictions in future research.

7 Appendix: Proof of Theorem 3

The proof of the Theorem is an adaptation of proofs of convergence of the core to equilibrium states due to Debreu and Scarf (1963) and existence proof of Foley (1970) and Wooders (1989). Without any loss of generality we can assume $N = N_r$. Let $\{S_1, ..., S_k, ..., S_K\}$ denote the set of all cliques in N. Let $(\tilde{x}^N, \mathcal{N})$ be a state of the economy, with $\mathcal{N} = \{J_1, ..., J_g, ..., J_G\}$. Assume that $(\tilde{x}^N, \mathcal{N})$ is in the $c(\varepsilon)$-core for all replications of the economy.

Preliminaries: We first consider the following space $A = R^{TK}$ where T is the number of agents and K is the number of cliques. Let $a = (a^1, ..., a^t, ..., a^T)$ be a vector where, for each t, $a_t = (a_1^t, ..., a_k^t, ..., a_K^t)$ and for each k, $a_k^t \in \mathbf{R}$. Let A_t be the set of elements in $\mathbf{R}^K$ defined by

$$A_t = \{a \in R^{TK} : a_k^{t'} = 0 \text{ if } t \neq t' \text{or if } \ t \notin S_k\}.$$

For a given $\mathcal{S}[t] \in \mathcal{I}[t]$, we represent $\mathcal{S}[t]$ in A_t by a, such that a_k^t equal one if S_k belongs to $\mathcal{S}[t]$ and equal zero otherwise.

We next define a 'virtual' production set in the extended commodity space. For each k define $b[k] \in R^{TK}$ as a vector having the properties that:

(i) $b[k]_{k'}^t = 0$ if $k \neq k'$ or if $t \notin S_k$

(ii) for (any) t in S_k, $b[k]_k^t = 1$

Define the virtual production set Y as the convex cone generated by the $\{(z_{S_k}, b[k]) : k = 1, \ldots, K\}$, where z_{S_k} is the input required to form the clique z_{S_k}. The set Y is precisely the set of all positive linear combinations of $\{(z_{S_k}, b[k]) : k = 1, \ldots, K\}$.
Step 1: **The sets of preferred allocations** $_t$. Let $_t$ denote the set of members of $(x^t - w^t - \varepsilon \bar{z}, a^t)$ in $X^t \times A_t$ such that, for every network $\mathcal{S}$ with the property that $\mathcal{S}[t] = \{S_k \mid a_k^t = 1\}$, we have $u^t(x^t, \mathcal{S}[t]) > u^t(\tilde{x}^t, \mathcal{N}[t])$.

The set $_t \subset \mathbf{R}^{L+TK}$ describes the set of net trades of private goods and clique memberships for agent t strictly preferred to his allocation in the given state of the economy $(\tilde{x}^N, \mathcal{N})$. It is clear that $_t$ is not convex.
Step 2: The preferred set . Let denote the convex hull of the union of the sets $_t$, $t = 1, ..., T$. We now show, in the remainder of Step 2, that

$$\cap Y = \emptyset.$$

Suppose, on the contrary, that $(x, a) \in \quad \cap Y$. Then, by the definition of there exist an integer J and $\lambda \in R^J$ such that $(x, a) = \sum_{j=1}^{J} \lambda_j (x^j, a^j)$ with $\lambda_j > 0$, $\sum \lambda_j = 1$.

From the definition of Y there exist a $K' \in \{1, \ldots, K\}$ and $\mu \in R_{++}^{K'}$ such that

$$(x, a) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k]).$$

Let us consider $J[t] = \{j \mid (x^j, a^j) \in \quad_t\}$. Then, it follows from

$$\sum_{j=1}^{J} \lambda_j (x^j, a^j) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k])$$

that for each $k \in K'$ and each $t \in S_k$ we have

$$\sum_{j \in J[t]} \lambda_j a_k^{j,t} = \mu_k$$

For a given (x^j, a^j) in $\quad_t$ and a given sequence $\{(\beta^n)\}_n$ of real numbers. Suppose that $\beta^n \geq 1$ for each n and that $(\beta^n x^j, a^j)$ converges to one as n goes to infinity. Then, because of the continuity of preferences, for all n sufficiently large, we have $(\beta^n x^j, a^j)$ is in $\quad_t$.

We now show that, since we have supposed that $\quad \cap Y \neq \emptyset$, we can form a blocking coalition for some sufficiently large replication. We will use the following lemma.

Lemma. There exists a sequence of rational numbers $(\lambda_1^n, \ldots, \lambda_j^n, \ldots, \lambda_J^n)$ converging to $(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_J)$ having the properties that:

(i) $\lambda_j^n \leq \lambda_j$

(ii) for (any) k, and for any $t, t' \in S_k$ we have:

$$\sum_{j \in J[t]} \lambda_j^n a_k^{j,t} = \sum_{j \in J[t']} \lambda_j^n a_k^{j,t'}.$$

Proof. Let us consider the closed line segment $[0_{R^J}, \lambda]$ in $\mathbb{R}^J$. From convexity it follows that, for any $\alpha \in [0_{R^J}, \lambda]$, for (any) k and for any $t, t' \in S_k$ we have

$$\sum_{j \in J[t]} \alpha_j a_k^{j,t} = \sum_{j \in J[t']} \alpha_j a_k^{j,t'}.$$

But we know that Q^J, where Q is the set of rational number, is dense in R^J. Hence, $Q^J \cap [0_{R^J}, \lambda]$ is dense in $[0_{R^J}, \lambda]$ and therefore we can choose a sequence satisfying (i) and (ii).□

Let us consider the sequence $(\lambda_1^n, \ldots, \lambda_j^n, \ldots, \lambda_J^n)$ defined above, and let us select a positive integer n, which will eventually tend to infinity. For each j define $x^{jn} = \frac{\lambda_j}{\lambda_j^n} x^j$. From the concluding paragraph of the last Step, for all n sufficiently large $(x^{jn}, a^j) \in {}_t$. Let n satisfy the property that $(x^{jn}, a^j) \in {}_t$ for each t. Recall that λ_j^n is a rational number.

Now, let us define $\mu_k^n = \sum_{j \in J[t]} \lambda_j^n a_n^{j,t'}$. Since $\sum_{j=1}^J \lambda_j^n x^{jn} = \sum_{k \in K'} \mu_k z_k$, $\mu_k^n \leq \mu_k$ and $z_k \in -R_+^L$, it follows that

$$\sum_{j=1}^{J} \lambda_j^n x^{jn} \leq \sum_{k \in K'} \mu_k^n z_k$$

Let r' be a replication number such that $r'\lambda_j^n$ is an integer for all j. Let $\delta_j = r'\lambda_j^n$ and $\gamma_k = \sum_{j \in J[t]} \delta_j$. It holds that

$$\sum_{j=1}^{J} \delta_j x^{jn} \geq \sum_{k \in K'} \gamma_k z_k.$$

Recall that $\mathbf{1}$ is the profile of N so $r\mathbf{1}$ is the profile of N_r. Let r^* be sufficiently large so that $\sum_k \gamma_k s_k \leq r^*\mathbf{1}$. From our choice of r^* it holds that N_{r*} contains a subset, say S with profile s, such that $s = \sum_{k \in K'} \gamma_k s_k$. This implies that there is a state of the economy for the group S that can c(ε)-improve upon the initially given state of the economy. The state of the economy for S described by the consumption plans (x^{jn}, a^j), for δ_j consumers, for each j is c(ε)-feasible and preferred by all members of the replication of the initially given state of the economy. Consequently, S can c(ε)- improve upon the $r*^{th}$ replication of $(\tilde{x}^N, \mathcal{N})$, which is a contradiction. Therefore $\cap Y = \emptyset$.

Step 3: Prices. From the Minkowski Separating Hyperplane Theorem, there is a hyperplane with normal $(p, \pi) \neq 0$, where p is in the private goods price space, and $\pi \in R^{TK}$ such that, for some constant C,

$$p \cdot x + \pi \cdot a \geq C \text{ for all } (x, a) \in \quad \text{and}$$

$$p \cdot z + \pi \cdot b \leq C \text{ for all } (z, b) \in Y.$$

Since Y is a closed convex cone with vertex zero, it follows that we can choose $C = 0$. Then, in particular, it follows that for each $(x^t, a^t) \in {}_t$

$$p \cdot (x^t - w^t - \varepsilon \bar{z}) + \sum_{\{k | a_k^t = 1\}} \pi^t(S) \geq 0,$$

and for each clique $S \subset N$ we have

$$p \cdot z_S + \sum_{t \in S} \pi^t(S) \leq 0.$$

Recall that $(\tilde{x}^N, \mathcal{N})$ is a c(ε)-core state of the economy relative to the network $\mathcal{N} = \{J_1, ..., J_G\}$ of N. Observe that we can represent the total consumption of each agent t by $(\tilde{x}^t, \tilde{a}^t) \in R^{L+TK}$.

From monotonicity it follows that $p \geq 0$. Suppose that $p = 0$. Therefore, from the separating hyperplane it follows that for each S_k we have

$$\sum_{t \in S_k} \pi^t(S_k) \leq 0,$$

and for each $t \in S_k$ we have $\pi^t(S_k) \geq 0$. Thus $\pi^t(S_k) = 0$, for each S_k and each $t \in S_k$, which is a contradiction to the fact that $(p, \pi) \neq 0$.

Since, for each t, $(\tilde{x}^t - w^t - \varepsilon\bar{z}, \tilde{a}^t)$ is in the closure of $\ _t$, it holds that

$$p \cdot (\tilde{x}^t - w^t - \varepsilon\bar{z}) + \sum_{\{k|\tilde{a}^t_k=1\}} \pi^t(S) \geq 0.$$

Moreover, for each clique J_g we have

$$p \cdot z_{J_g} + \sum_{\{t \in J_g\}} \pi^t(J_g) \leq 0.$$

Summing the above inequalities over consumers one obtains

$$p \cdot \sum_{t \in N} (\tilde{x}^t - w^t - \varepsilon\bar{z}) \geq p \cdot \sum_g z_{J_g},$$

and summing over clubs one obtains

$$\sum_g p \cdot z_{J_g} + \sum_g \sum_{\{t \in J_g\}} \pi^t(J_g) \leq 0.$$

Since $p \in R^L_+ \setminus \{0\}$ and $\sum_{t \in N} (\tilde{x}^t - w^t) \leq \sum_g z_{J_g}$ it follows that
Then from the fact that $p \cdot z_{J_g} + \sum_{\{t \in J_g\}} \pi^t(J_g) \leq 0$ it follows that

$$\sum_{t \in N} p \cdot x^t + \sum_{t \in N} \sum_{S \in \mathcal{N}[t]} \pi^t(S) + \sum_{t \in N} \varepsilon p \cdot \bar{z} \leq \sum_{t \in N} p \cdot w^t,$$

and

$$\sum_g p \cdot z_{J_g} + \sum_g \sum_{\{t \in J_g\}} \pi^t(J_g) \geq \sum_{t \in N} p \cdot w^t.$$

Now we claim that $((\tilde{x}^N, \mathcal{N}), p, \Pi)$ is an ε-equilibrium. Checking the proof so far, it remains only to show that individual consumers are optimizing, i.e., that the prices p, Π and the state $(x^N, \mathcal{N})$ satisfy condition (iii) of the definition of an equilibrium.

Suppose that for some consumer t, and some consumption (x^t, a^t),

$$u^t(x^t, a^t) > u^t(\tilde{x}^t, \tilde{a}^t) \text{ and}$$

$$p \cdot (x^t - w^t - \varepsilon \bar{z}) + \sum_{t \in S} \pi^t(S) \leq 0.$$

From our desirablity of wealth assumption, there is a consumption $x^0 \in X^t$ such that

$$p \cdot (x^0 - w^t - \varepsilon \bar{z}) + \sum_{t \in S} \pi^t(S) < 0.$$

It follows that for some x'^t in the segment $[x^0, x^t]$

$$u^t(x'^t, a^t) > u^t(\tilde{x}^t, \tilde{a}^t)$$

and

$$p \cdot (x'^t - w^t + \varepsilon \bar{z}) + \sum_{t \in S} \pi^t(S) < 0,$$

which is a contradiction.

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