ARBITRAGE, EQUILIBRIUM AND NONSATIATION

Nizar Allouch

Cuong Le Van

And

Frank H. Page Jr

No 637

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF
WARWICK

Arbitrage, Equilibrium, and Nonsatiation

Nizar Allouch*
Department of Economics
University of Warwick
Coventry CV4 7AL
UK
ecsfn@csv.warwick.ac.uk

Cuong Le Van†
CNRS, CERMSEM
University of Paris 1
75013 Paris
France
Cuong.Le-Van@univ-paris1.fr

Frank H. Page, Jr.‡
Department of Finance
University of Alabama
Tuscaloosa, AL 35487
USA
fpage@cba.ua.edu

July, 2001
Current Version: January, 2002

Abstract

In his seminal paper on arbitrage and competitive equilibrium in unbounded exchange economies, Werner (*Econometrica, 1987)* proved the existence of a competitive equilibrium, under a price no-arbitrage condition, without assuming either local or global nonsatiation. Werner's existence result contrasts sharply with classical

*A substantial part of the paper was written while Allouch was at CERMSEM, University of Paris 1, and at the Department of Finance at the University of Alabama. Allouch thanks CERMSEM, Paris 1, and the Department of Finance at the University of Alabama for their generous support.

†This paper was begun while Le Van was visiting the Department of Finance at the University of Alabama. Le Van gratefully acknowledges the support and hospitality of the Department of Finance at the University of Alabama.

‡The current version of the paper was completed while Page was visiting CERMSEM at Paris 1 and the Department of Economics at Warwick. Page gratefully acknowledges the support and hospitality of CERMSEM, Paris 1, and the Department of Economics at Warwick. Page also acknowledges generous support from the Department of Finance at the University of Alabama.

existence results for bounded exchange economies which require, at minimum, global nonsatiation at rational allocations. Why do unbounded exchange economies admit existence without local or global nonsatiation? This question is the focus of our paper. We make two main contributions to the theory of arbitrage and competitive equilibrium. *First*, we show that, in general, in unbounded exchange economies (for example, asset exchange economies allowing short sales), even if some agents' preferences are satiated, the absence of arbitrage is sufficient for the existence of competitive equilibria, as long as each agent who is satiated has a nonempty set of useful net trades - that is, as long as agents' preferences satisfy weak nonsatiation. *Second*, we provide a new approach to proving existence in unbounded exchange economies. The key step in our new approach is to transform the original economy to an economy satisfying global nonsatiation such that all equilibria of the transformed economy are equilibria of the original economy. What our approach makes clear is that it is precisely the condition of weak nonsatiation - a condition considerably weaker than local or global nonsatiation - that makes possible this transformation. Moreover, as we show via examples, without weak nonsatiation, existence fails.

Keywords: Arbitrage, Asset Market Equilibrium, Nonsatiation, Recession Cones.

JEL Classification Numbers: C 62, D 50.

1 Introduction

Since the pioneering contributions of Grandmont ((1970), (1972), (1977)), Green (1973), and Hart (1974), the relationship between arbitrage and equilibrium in asset exchange economies allowing short sales has been the subject of much investigation[1]. When unlimited short sales are allowed, agents' choice sets are unbounded from below. As a consequence, asset prices at which agents can exhaust all gains from trade via mutually compatible net trades bounded in size may fail to exist. By assuming that markets admit "no arbitrage", the economy can be *bounded endogenously* - but this is not enough for existence. In addition to no-arbitrage conditions, two other conditions are frequently required: (i) *uniformity* of arbitrage opportunities,[2] and (ii) *nonsatiation*. Werner, in his seminal 1987 paper on arbitrage and competitive equilibrium, assumes uniformity of arbitrage opportunities and establishes the existence of a competitive equilibrium using a no-arbitrage condition on prices. An especially intriguing aspect of Werner's existence result is that it does not require local or global nonsatiation (see Werner (1987), Theorems 1).[3] This contrasts sharply with classical existence results for bounded exchange economies which require, at minimum, that agents' preferences be globally nonsatiated at rational allocations (e.g., see Debreu (1959), Gale and Mas-Colell (1975), and Bergstrom (1976)).[4] Why do unbounded exchange economies admit existence without local or global nonsatiation? This question is the focus of our paper.

Our starting point is Werner's notion of useful net trades. Stated informally, a useful net trade is a net trade that, for some endowments, represents a potential arbitrage.[5] Our main contribution is to show that, in general, in unbounded exchange economies (for example, asset exchange economies al-

[1]See also, for example, Milne (1976, 1980), Hammond (1983), Page (1987), Nielsen (1989), Page and Wooders (1996), Kim (1998), Dana, Le Van, Magnien (1999), Page, Wooders, and Monteiro (2000), and Allouch (2002).

[2]A vector of net trades y is said to be an arbitrage opportunity for agent i at x if starting at any x' weakly preferred to x, $x'+\lambda y$ is also weakly preferred to x for all $\lambda \geq 0$. If for each agent i an arbitrage opportunity y at x is also an arbitrage opportunity at *any other* x'', then uniformity holds (i.e., there is uniformity of arbitrage opportunities).

[3]Werner proves two existence results. In Theorem 1, each agent's choice set (or consumption set) is a closed, convex (not necessarily bounded) subset of $\mathbb{R}^l$. In Theorem 2, each agent's choice set is a closed, convex, bounded-from-below subset of $\mathbb{R}^l$.

[4]A rational allocation is an allocation such that each agent weakly prefers his piece of the allocation to his endowment.

[5]In order to formally define the notion of useful net trades, we must first define the notion of useless net trades. A vector of net trades y is said to be *useless* to agent i at x if agent i starting at x is indifferent to trading in the y or $-y$ directions on any scale. Thus, a vector of net trades y is useless to agent i at x if the agent is indifferent along the line $x+\lambda y$, $\lambda \in (-\infty, +\infty)$. A vector of net trades y is said to be *useful* to agent i at x if y is an arbitrage opportunity for agent i at x and if y is not useless to agent i at x.

lowing short sales), even if some agents' preferences are satiated, the absence of market arbitrage is sufficient for the existence of competitive equilibria, *as long as each agent who is satiated has a nonempty set of useful net trades* - that is, as long as agents' preferences satisfy *weak nonsatiation.*

Our second contribution is to provide a new approach to proving existence in unbounded exchange economies. In addition to being a technical innovation, our new approach makes clear the critical role played by unboundedness and weak nonsatiation in establishing existence in unbounded exchange economies where neither local nor global nonsatiation is satisfied. The key step in our new approach is a transformation of the original economy to a new economy satisfying global nonsatiation and having the property that all equilibria of the transformed economy are equilibria of the original economy. Existence for the transformed economy is then deduced using classical methods. It is precisely the condition of weak nonsatiation - a condition considerably weaker than local or global nonsatiation - that makes possible the transformation of the original economy to an equivalent economy satisfying global nonsatiation - even if the original economy fails to satisfy either local or global nonsatiation. Moreover, as we show via examples, without weak nonsatiation, existence fails.

In their classic paper on abstract exchange economies, Gale and Mas-Colell (1975) establish existence by transforming an exchange economy satisfying global nonsatiation to an exchange economy satisfying local nonsatiation. However, if global nonsatiation fails, then the Gale/Mas-Colell transformation cannot be applied. Here, we establish existence by transforming an exchange economy satisfying weak nonsatiation (in which global nonsatiation may fail) to an exchange economy satisfying global nonsatiation. Thus, while our transformation is similar in motivation to the Gale/Mas-Colell transformation, it goes beyond the Gale/Mas-Colell transformation by addressing the problem of global satiation.

As a prerequisite to proving existence in an exchange economy satisfying weak nonsatiation only, we must extend Werner's price no-arbitrage condition to allow for weak nonsatiation - and in particular, to allow for the possibility that some agents have empty sets of useful net trades at some rational allocations.[6] A third contribution of our paper is to show that this extended price no-arbitrage condition is equivalent to Hart's (1974) weak no-market-arbitrage condition.

In addition to extending Werner's price no-arbitrage condition and showing its equivalence to Hart's condition, we also extend Werner's model of an unbounded exchange economy in two ways. First, we weaken Werner's

[6]Werner's price no-arbitrage condition requires that each agent have a nonempty set of useful net trades. However, under weak nonsatiation, an agent is allowed to have an empty set of useful net trades at some rational allocations - provided the agent's preferences are globally nonsatiated at such rational allocations.

uniformity of arbitrage condition by assuming only uniformity of useless net trades (see Werner (1987), Assumption A3). We refer to our uniformity condition as weak uniformity.[7] Second, in our model we require only that agents' utility functions be upper semicontinuous, rather than continuous as in Werner (1987).

We shall proceed as follows: In Section 2, we present the basic ingredients of our model, including the notions of arbitrage, useful and useless net trades, weak uniformity, and weak nonsatiation. In Section 3, we discuss the weak no-market-arbitrage condition of Hart (1974) and the price no-arbitrage condition of Werner (1987), and we extend Werner's price no-arbitrage condition to allow for weak nonsatiation. We then present our first Theorem which states that the extended price no-arbitrage condition is equivalent to Hart's weak no-market-arbitrage condition. In Section 4, we present our second Theorem which states that in an unbounded exchange economy (for example, in an asset exchange economy allowing short sales), if weak uniformity and weak nonsatiation hold, then the extended price no-arbitrage condition is sufficient to guarantee the existence of a quasi-equilibrium - and therefore is sufficient to guarantee the existence of a competitive equilibrium under the usual relative interiority conditions on endowments. In Section 5, we present two examples which show that our weak nonsatiation assumption is the weakest possible - without weak nonsatiation, existence fails. Finally, in Section 6, the Appendix, we present the proofs of Theorems 1 and 2. We preface our proof of Theorem 1 with a detailed discussion of the geometry of Hart's weak no-market-arbitrage condition. In the proof of Theorem 2, we present our new approach.

[7]Thus, weak uniformity holds if for each agent i, an arbitrage opportunity y at x' that is useless at x', for x' weakly preferred to the agent's endowment, is also useless at *any other* x'' weakly preferred to the agent's endowment.

2 The Model

We consider an economy $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ with m agents and l goods. Agent i has consumption set $X_i \subset \mathbb{R}^l$, utility function $u_i(\cdot)$, and endowment e_i. Agent $i's$ preferred set at $x_i \in X_i$ is

$$P_i(x_i) = \{x \in X_i \mid \ u_i(x) > u_i(x_i)\},$$

while the weakly preferred set at x_i is

$$\widehat{P}_i(x_i) = \{x \in X_i \mid \ u_i(x) \geq u_i(x_i)\}.$$

The set of *individually rational allocations* is given by

$$\mathcal{A} = \{(x_i) \in \prod_{i=1}^m X_i \mid \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \text{ and } \ x_i \in \widehat{P}_i(e_i), \forall i\}.$$

We shall denote by $\mathcal{A}_i$ the projection of $\mathcal{A}$ onto X_i.

Definition 1 *(a) A rational allocation $x^* \in \mathcal{A}$ together with a* nonzero *vector of prices $p^* \in \mathbb{R}^l$ is an equilibrium for the economy $\mathcal{E}$*

(i) if for each agent i and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x > p^* \cdot e_i$, and

(ii) if for each agent i, $p^* \cdot x_i^* = p^* \cdot e_i$.

(b) A rational allocation $x^* \in \mathcal{A}$ and a nonzero *price vector $p^* \in \mathbb{R}^l$ is a quasi-equilibrium*

(i) if for each agent i and $x \in X_i$, $u_i(x) > u_i(x_i^*)$ implies $p^* \cdot x \geq p^* \cdot e_i$, and

(ii) if for each agent i, $p^* \cdot x_i^* = p^* \cdot e_i$.

Given (x^*, p^*) a quasi-equilibrium, it is well-known that if for each agent i, (a) $p^* \cdot x < p^* \cdot e_i$ for some $x \in X_i$ and (b) $P_i(x_i^*)$ is relatively open in X_i, then (x^*, p^*) is an equilibrium. Conditions (a) and (b) will be satisfied if, for example, for each agent i, $e_i \in \text{int}X_i$, and u_i is continuous on X_i. Using irreducibility assumptions, one can also show that a quasi-equilibrium is an equilibrium.

We now introduce our first two assumptions: for agents $i = 1, 2, \ldots, m$,

[A.1] X_i is closed and convex with $e_i \in X_i$,

[A.2] u_i is upper semicontinuous and quasi-concave.

Under these two assumptions, the weak preferred set $\widehat{P}_i(x_i)$ is convex and closed for $x_i \in X_i$.

2.1 Arbitrage, Uniformity, and Nonsatiation

2.1.1 Arbitrage

We define the *i^{th} agent's arbitrage cone at $x_i \in X_i$* as the closed convex cone containing the origin given by

$$O^+\widehat{P}_i(x_i) = \{y_i \in \mathbb{R}^l \mid \forall x_i' \in \widehat{P}_i(x_i) \text{ and } \lambda \geq 0,\ x_i' + \lambda y_i \in \widehat{P}_i(x_i)\}.$$

Thus, if $y_i \in O^+\widehat{P}_i(x_i)$, then for all $\lambda \geq 0$ and all $x_i' \in \widehat{P}_i(x_i)$, $x_i' + \lambda y_i \in X_i$ and $u_i(x_i' + \lambda y_i) \geq u_i(x_i)$. The agent's arbitrage cone at x_i, then, is the recession cone corresponding to the weakly preferred set $\widehat{P}_i(x_i)$ (see Rockafellar (1970), Section 8).[8] If the agent's utility function, $u_i(\cdot)$, is concave, then for any $x_i \in X_i$ and $y_i \in O^+\widehat{P}_i(x_i)$, $u_i(x_i + \lambda y_i)$ is nondecreasing in $\lambda \geq 0$. Thus, starting at x_i, trading in the y_i direction on any scale is utility nondecreasing.

2.1.2 Uniformity

A set closely related to the i^{th} agent's arbitrage cone is the *lineality space*, $L_i(x_i)$, of $\widehat{P}_i(x_i)$ given by

$$L_i(x_i) = \{y_i \in \mathbb{R}^l \mid\ \forall x_i' \in \widehat{P}_i(x_i) \text{ and } \forall \lambda \in \mathbb{R},\ x_i' + \lambda y_i \in \widehat{P}_i(x_i)\}.$$

The set $L_i(x_i)$ consists of the zero vector and all the nonzero vectors y_i such that for each x_i' weakly preferred to x_i (i.e., $x_i' \in \widehat{P}_i(x_i)$), any vector z_i on the line through x_i' in the direction y_i, $z_i = x_i' + \lambda y_i$, is also weakly preferred to x_i (i.e., $z_i = x_i' + \lambda y_i \in \widehat{P}_i(x_i)$). The set $L_i(x_i)$ is a closed subspace of $\mathbb{R}^l$, and is the largest subspace contained in the arbitrage cone $O^+\widehat{P}_i(x_i)$ (see Rockafellar (1970)).

If for all agents, the lineality space $L_i(x_i)$ is the same for all $x_i \in \widehat{P}_i(e_i)$, then we say that the economy satisfies *weak uniformity*. We formalize this notion of uniformity in the following assumption:

[A.3][Weak Uniformity] for all agents i
$L_i(x_i) = L_i(e_i)$ for all $x_i \in \widehat{P}_i(e_i)$.

Under weak uniformity, we have for all $x_i \in \widehat{P}_i(e_i)$ and all $y_i \in L_i(e_i)$,

$$u_i(x_i + y_i) \leq u_i(x_i + y_i - y_i) \leq u_i(x_i + y_i).$$

[8] Equivalently, $y_i \in O^+\widehat{P}_i(x_i)$ if and only if y_i is a cluster point of some sequence $\{\lambda^k x_i^k\}_k$ where the sequence of positive numbers $\{\lambda^k\}_k$ is such that $\lambda^k \downarrow 0$, and where for all k, $x_i^k \in \widehat{P}_i(x_i)$; (see Rockafellar (1970), Theorem 8.2).

Thus, for all $x_i \in \widehat{P}_i(e_i)$ and all $y_i \in L_i(e_i)$,

$$u_i(x_i + y_i) = u_i(x_i).$$

Following the terminology of Werner (1987), we refer to arbitrage opportunities $y_i \in O^+\widehat{P}_i(x_i)$ such that

$$u_i(x_i + \lambda y_i) = u_i(x_i) \text{ for all } \lambda \in (-\infty, \infty)$$

as *useless* at x_i. Thus, under weak uniformity, the i^{th} agent's lineality space at his endowment, $L_i(e_i)$, is equal to the set of all net trades that are useless. Moreover, under weak uniformity the set of *useful* net trades at x_i is given by

$$O^+\widehat{P}_i(x_i)\backslash L_i(x_i) = O^+\widehat{P}_i(x_i)\backslash L_i(e_i).$$

Werner (1987) makes a uniformity assumption stronger than our assumption of uniformity of useless net trades (i.e., stronger than our assumption of weak uniformity, [A.3]). In particular, Werner assumes that *all* arbitrage opportunities are uniform. Stated formally,

$$\begin{gathered}\text{[Uniformity] for all agents } i \\ O^+\widehat{P}_i(x_i) = O^+\widehat{P}_i(e_i) \text{ for all } x_i \in \widehat{P}_i(e_i).\end{gathered}$$

If agents have concave utility functions, then Werner's uniformity assumption, and therefore weak uniformity, is satisfied automatically.

For notational simplicity, we will denote each agent's arbitrage cone and lineality space at endowments in a special way. In particular, we will let

$$R_i := O^+\widehat{P}_i(e_i), \text{ and } L_i := L(e_i).$$

2.1.3 Nonsatiation

We begin by recalling the classical notions of global and local nonsatiation:

$$\begin{gathered}\text{[Global Nonsatiation] for all agents } i, \\ P_i(x_i) \neq \emptyset \text{ for all } x_i \in \mathcal{A}_i;\end{gathered}$$

$$\begin{gathered}\text{[Local Nonsatiation] for all agents } i, \\ P_i(x_i) \neq \emptyset \text{ and } \mathrm{cl}P_i(x_i) = \widehat{P}_i(x_i) \text{ for all } x_i \in \mathcal{A}_i.\end{gathered}$$

Here, cl denotes closure. Werner assumes uniformity and then, rather than assume global or local nonsatiation, assumes that

$$\begin{gathered}\text{[Werner Nonsatiation] for all agents } i \\ R_i\backslash L_i \neq \varnothing.\end{gathered}$$

This assumption is weaker than the classical assumptions. We will weaken Werner's nonsatiation assumption as follows:

$$\text{[A.4][Weak Nonsatiation] for all agents } i$$
$$\forall x_i \in \mathcal{A}_i, \text{ if } P_i(x_i) = \emptyset, \text{ then } O^+\widehat{P}_i(x_i) \setminus L_i(x_i) \neq \emptyset.$$

Note that weak nonsatiation holds if global nonsatiation, local nonsatiation, or Werner nonsatiation holds. Also, note that under weak nonsatiation if $x_i \in \mathcal{A}_i$ is a satiation point for agent i, then, as in Werner, there is a *useful* net trade vector y_i such that $u_i(x_i+\lambda y_i) = u_i(x_i)$ for all $\lambda \geq 0$. Thus, if there are satiation points, then the set of satiation points must be unbounded.

3 The No-Arbitrage Conditions of Hart and Werner

Hart's (1974) no-arbitrage condition is a condition on net trades. In particular, Hart's condition requires that all mutually compatible arbitrage opportunities be *useless*.[9] We shall refer to Hart's condition as the *weak no-market-arbitrage* condition (WNMA). We have the following definition:

Definition 2 *The economy $\mathcal{E}$ satisfies the* WNMA *condition if*

$$\sum_{i=1}^{m} y_i = 0 \text{ and } y_i \in R_i \text{ for all } i, \text{ then}$$
$$y_i \in L_i \text{ for all } i.$$

Werner's (1987) no-arbitrage condition is a condition on prices. In particular, Werner's condition requires that there be a nonempty set of prices such that each price in this set assigns a strictly positive value to any vector of *useful* net trades belonging to any agent.[10] We shall refer to Werner's condition as the *price no-arbitrage* condition (PNA). We have the following definition:

[9]Hart's condition is stated within the context of an asset exchange economy model where uncertainty concerning asset returns is specified via a joint probability distribution function. Page (1987) shows that in an asset exchange economy, if there are no perfectly correlated assets, then Hart's condition and Page's (1987) no-unbounded-arbitrage condition are equivalent.

[10]Translating Werner's condition to an asset exchange economy, it is easy to show that if there are no perfectly correlated assets and if agents are sufficiently risk averse, then Werner's condition is equivalent to Hammond's overlapping expectation condition. Page (1987) shows that in an asset exchange economy if there are no perfectly correlated assets and if agents are sufficiently risk averse, then Hammond's overlapping expectations condition and Page's no-unbounded-arbitrage condition are equivalent. Thus, in an asset exchange economy with no perfectly correlated assets populated by sufficiently risk averse agents, the conditions of Hart (1974), Werner (1987), Hammond (1983), and Page (1987) are all equivalent.

Definition 3 *In an economy $\mathcal{E}$ satisfying* [Werner Nonsatiation]*, Werner's* PNA *condition is satisfied if*

$$\bigcap_{i=1}^{m} S_i^W \neq \emptyset,$$

where

$$S_i^W = \{p \in \mathbb{R}^\ell \mid \ p \cdot y > 0, \forall y \in R_i \setminus L_i\}$$

is Werner's cone of no-arbitrage prices.

Here we extend Werner's condition to allow for the possibility that for some agent the set of useful net trades is empty - that is, to allow for the possibility that for some agent, $R_i \setminus L_i = \emptyset$. More importantly, we shall prove, under very mild conditions, that our extended version of Werner's condition is equivalent to Hart's condition. This result extends an earlier result by Page, Wooders, and Monteiro (2000) on the equivalence of the Hart and Werner conditions.

We begin by extending the definition of Werner's cone of *no-arbitrage prices*:

Definition 4 *For each agent i, define*

$$S_i = \begin{cases} S_i^W & \text{if } R_i \setminus L_i \neq \emptyset, \\ L_i^\perp & \text{if } R_i \setminus L_i = \emptyset. \end{cases}$$

Given this expanded definition of the no-arbitrage-price cone, the *extended price no-arbitrage* condition (EPNA) is defined as follows:

Definition 5 *The economy $\mathcal{E}$ satisfies the* EPNA *condition if*

$$\bigcap_{i=1}^{m} S_i \neq \emptyset.$$

Remark Note that if the economy $\mathcal{E}$ satisfies Werner's nonsatiation condition, i.e., $R_i \setminus L_i \neq \emptyset$, $\forall i$, then the EPNA condition given in Definition 5 above reduces to Werner's original condition PNA given in Definition 3.

Page, Wooders and Monteiro (2000) show that under assumptions [A.1]-[A.2], [Uniformity] and [Werner Nonsatiation], WNMA holds if and only if $\bigcap_{i=1}^m S_i^W \neq \emptyset$ (i.e., Hart's condition holds if and only if Werner's condition holds). Here, we extend this result by proving, under [A.1]-[A.2] only, that WNMA holds if and only if $\bigcap_{i=1}^m S_i \neq \emptyset$.

Theorem 1 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]*. The following statements are equivalent:*

1. *$\mathcal{E}$ satisfies* WNMA.

2. *$\mathcal{E}$ satisfies* EPNA.

Proof. See Appendix.

4 The Existence of Equilibrium

Our next result extends Werner's (1987) main result on arbitrage and the existence of equilibrium in two ways:

(1) Werner assumes uniformity of arbitrage opportunities. Here, we assume only weak uniformity of agents' lineality spaces [A.3].

(3) Werner assumes that for each agent i, $O^+\widehat{P}_i(x_i) \setminus L_i(x_i) \neq \emptyset$, $\forall x_i \in X_i$. Here, we weaken Werner's nonsatiation assumption to allow $O^+\widehat{P}_i(x_i) = L_i(x_i)$ for some agents i and some $x_i \in \mathcal{A}_i$. But in this case we require that $P_i(x_i) \neq \emptyset$. In particular, we require only weak nonsatiation [A.4].

Theorem 2 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2], *weak uniformity* [A.3], *and weak nonsatiation* [A.4]. *If $\mathcal{E}$ satisfies Hart's condition,* WNMA, *or equivalently, if $\mathcal{E}$ satisfies the extended Werner condition,* EPNA, *then $\mathcal{E}$ has a quasi-equilibrium.*

Moreover, if $(x_1^*, \ldots, x_m^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}$ such that for each agent i,

1. *$\inf_{x \in X_i} \langle x, p \rangle < \langle \omega_i, p \rangle$, and*

2. *$P_i(x_i^*)$ is relatively open in X_i,*

then $(x_1^*, \ldots, x_m^*, p^*)$ is an equilibrium.

Proof. See Appendix.

In addition to extending Werner (1987), we also introduce a new method for proving existence in exchange economies with short selling. In particular, we prove existence by first transforming the economy $\mathcal{E}$ to an economy $\mathcal{E}'$ satisfying global nonsatiation and having the property that any equilibrium of $\mathcal{E}'$ is an equilibrium of $\mathcal{E}$. We accomplish via a modification of agents' utility functions. Our assumption of weak nonsatiation is crucial - it allows us to modify agents' utility functions in precisely the right way. We then prove existence for the modified economy $\mathcal{E}'$ using the excess demand approach via the Gale-Nikaido-Debreu Lemma.

5 Examples

Weak nonsatiation [A.4] plays a critical role in our proof of existence. In this section, we present two examples which show that our weak nonsatiation assumption is the weakest possible. In example 1, the economy fails to satisfy global nonsatiation and also fails to satisfy Werner nonsatiation. However, the economy does satisfy weak nonsatiation, as well as all the assumptions of our Theorem2 - and there exists a quasi-equilibrium. In example 2, all the assumptions of Theorem2 are satisfied except weak nonsatiation [A.4] and existence fails. In both examples, as in Werner (1987), there is uniformity of arbitrage opportunities.

Example 1

Consider an economy with 2 agents and 2 goods. Agent 1 has consumption set $X_1 = [0,1] \times \mathbb{R}$ and endowment $e_1 = (\frac{1}{4}, 0)$. Agent 1's utility function is given by

$$u_1(x_{11}, x_{21}) \quad \begin{cases} x_{11}, & \text{if } x_{11} \in [0, \frac{1}{4}], \\ \frac{1}{4}, & \text{if } x_{11} \in [\frac{1}{4}, \frac{1}{2}], \\ x_{11} - \frac{1}{4}, & \text{if } x_{11} \in [\frac{1}{2}, 1]. \end{cases}$$

For agent 1, Werner nonsatiation fails because $R_1 = L_1 = \{0\} \times \mathbb{R}$. Moreover, for agent 1

$$\mathcal{A}_1 = \{(x_{11}, x_{21}) \mid \frac{1}{4} \leq x_{11} \leq \frac{7}{16}, x_{21} \in \mathbb{R}\}.$$

Thus, global nonsatiation is satisfied - and thus for agent 1 weak nonsatiation is satisfied.

Agent 2 has consumption set $X_2 = \mathbb{R}_+ \times \mathbb{R}$ and endowment $e_2 = (\frac{1}{4}, 0)$. Agent 2's utility function is given by

$$u_2(x_{12}, x_{22}) \quad \begin{cases} \sqrt{x_{12}} & \text{if } x_{12} \in [0, \frac{1}{16}], \\ \frac{1}{4} & \text{if } x_{12} \geq \frac{1}{16}. \end{cases}$$

For agent 2, global nonsatiation fails because

$$\mathcal{A}_2 = \{(x_{12}, x_{22}) \mid \frac{1}{16} \leq x_{12} \leq \frac{1}{4}, x_{22} \in \mathbb{R}\}.$$

Moreover, for agent 2 the arbitrage cone is $R_2 = \mathbb{R}_+ \times \mathbb{R}$, while the space of useless net trades (i.e., the lineality space) is given by $L_2 = \{0\} \times \mathbb{R}$. Thus, for agent 2 Werner nonsatiation is satisfied - and thus for agent 2 weak nonsatiation is satisfied.

It is easy to see that Hart's condition (WNMA) is satisfied, and it is easy to check that

$$(x_1^*, x_2^*, p^*) = ((x_{11}^*, x_{21}^*), (x_{12}^*, x_{22}^*), (p_1^*, p_2^*)) = ((\frac{1}{4}, 0), (\frac{1}{4}, 0), (1, 0))$$

is a quasi-equilibrium.

Example 2

In this example, again there are two agents and two goods, but agent 1's preferences *do not* satisfy assumption [A.4], weak nonsatiation.

Agent 1 has consumption set $X_1 = [0,1]\times\mathbb{R}$ and endowment $e_1 = (\frac{1}{4}, 0)$. But now agent 1's utility function is given by $u_1(x_{11}, x_{21}) = -x_{11}$. As in our first example, Werner nonsatiation fails for agent 1. In particular, agent 1's arbitrage cone is $R_1 = L_1 = \{0\} \times\mathbb{R}$. Thus, for agent 1, the arbitrage cone is equal to the space of useless net trades (i.e., the lineality space).

Agent 2 has consumption set $X_2 = \mathbb{R}_+\times\mathbb{R}$ and endowment $e_2 = (\frac{1}{4}, 0)$. Agent 2's utility function is given by $u_2(x_{12}, x_{22}) = x_{12}$. For agent 2, the arbitrage cone is $R_2 = \mathbb{R}_+\times\mathbb{R}$, while the space of useless net trades (i.e., the lineality space) is given by $L_2 = \{0\} \times\mathbb{R}$.

It is easy to see that Hart's condition (WNMA) is satisfied. It is also easy to check that for agent 1

$$\mathcal{A}_1 = \{(x_{11}, x_{21}) \mid 0 \leq x_{11} \leq \frac{1}{4}, x_{21} \in \mathbb{R}\}.$$

But note that for agent 1, global nonsatiation fails at $(0, x_{21}) \in \mathcal{A}_1$, for all $x_{21} \in \mathbb{R}$. Thus, since for agent 1, $R_1 = L_1 = \{0\} \times\mathbb{R}$, weak nonsatiation [A.4] fails for agent 1, and thus in this example weak nonsatiation does not hold. Does there exist an equilibrium?

In this economy, for each agent i, $e_i \in \text{int}X_i$ and utility functions are continuous. Hence any quasi-equilibrium is an equilibrium. Moreover, if an equilibrium exists, it must be the case that $p^* = (1, 0)$. Given p^*, agent 1's choice problem is given by

$$\max\{u_1(x_{11}, x_{21}) \mid x_{11} \in [0, \frac{1}{4}], x_{21} \in \mathbb{R}\}.$$

All solutions to agent 1's choice problem are of the form: $x_1^* = (x_{11}^*, x_{21}^*) = (0, x_{21}^*)$ for $x_{21}^* \in \mathbb{R}$. Given p^*, agent 2's choice problem is given by

$$\max\{u_2(x_{12}, x_{22}) \mid x_{12} \in [0, \frac{1}{4}], x_{22} \in \mathbb{R}\}.$$

All solutions to agent 2's choice problem are of the form: $x_2^* = (x_{12}^*, x_{22}^*) = (\frac{1}{4}, x_{22}^*)$ for $x_{22}^* \in \mathbb{R}$. But $x_1^* + x_2^* \neq e_1 + e_2 = (\frac{1}{2}, 0)$. Thus, in this example weak nonsatiation fails and there does not exist a quasi-equilibrium.

6 Appendix

6.1 The Geometry of Hart's Condition

In order to better understand the weak-no-market-arbitrage condition, let us consider the basic geometry underlying the condition. To begin, let

$L_i^\perp := L_i^\perp(e_i)$ denote the space orthogonal to agent $i's$ lineality space $L_i := L_i(e_i)$. Recall that under weak uniformity, L_i is the i^{th} agent's set of useless net trades. The vector space $\mathbb{R}^l$ can be decomposed into the direct sum of the lineality space L_i and its orthogonal complement, $L_i^\perp$. Thus, we have

$$\mathbb{R}^l = L_i^\perp \oplus L_i,$$

and thus, each vector $x \in \mathbb{R}^l$ has a *unique* representation as the sum of two vectors, one from L_i and one from $L_i^\perp$. In particular, for each $x \in \mathbb{R}^l$, there exists uniquely two vectors, $y \in L_i^\perp$ and $z \in L_i$, such that $x = y + z$. Now let

$$\mathcal{A}^\perp \text{be the projection of } \mathcal{A} \text{ onto} \prod_{i=1}^m L_i^\perp.$$

For each rational allocation $x = (x_1, \ldots, x_m)$ there exists uniquely two m-tuples, $y = (y_1, \ldots, y_m) \in \mathcal{A}^\perp$ and $z = (z_1, \ldots, z_m) \in \prod_{i=1}^m L_i$, such that $x = y + z$. Thus, we can think of each rational allocation as being uniquely decomposable into a potentially useful component and a potentially useless component. Our first result, a lemma, tells us that Hart's condition holds if and only if the set $\mathcal{A}^\perp$ of all useful components of the set of rational allocation is compact. We will use this lemma in our proof of existence.

Lemma 3 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]. *The following statements are equivalent:*

1. *The set $\mathcal{A}^\perp$ is compact.*
2. *$\mathcal{E}$ satisfies Hart's condition, weak-no-market-arbitrage.*

Proof. First, we will show that $\mathcal{A}^\perp$ is closed. For any $x_i \in \widehat{P}_i(e_i)$, write $x_i = x_i^\perp + \widehat{x}_i$ for $x_i^\perp \in \widehat{P}_i(e_i) \cap L_i^\perp$ and $\widehat{x}_i \in L_i$. Let $\{(x_i^{\perp n})\}_n$ be a sequence in $\mathcal{A}^\perp$ such that $\lim_{n \to +\infty}(x_i^{\perp n}) = (x_i^\perp)$. For each n, there exists $(\widehat{x}_i^n) \in \prod_{i=1}^m L_i$, such that

$$\sum_{i=1}^m x_i^{\perp n} + \sum_{i=1}^m \widehat{x}_i^n = \sum_{i=1}^m e_i.$$

Hence,

$$\lim_{n \to +\infty} \sum_{i=1}^m \widehat{x}_i^n = \zeta \in \sum_{i=1}^m L_i$$

since $\sum_{i=1}^m L_i$ is a finite dimensional subspace and hence closed. Now write $\zeta = \sum_{i=1}^m \zeta_i$, where for each i, $\zeta_i \in L_i$. One can check that for each i,

$$x_i^\perp \in \widehat{P}_i(e_i) \cap L_i^\perp \text{ and } (x_i^\perp + \zeta_i) \in \mathcal{A}.$$

Hence $(x_i^\perp) \in \mathcal{A}^\perp$.

(1) $\Rightarrow$ (2) : Let $y = (y_1, \ldots, y_m)$ be such that $y_i \in R_i$ for all i and $\sum_{i=1}^m y_i = 0$. For each i, write

$$y_i = \widehat{y}_i + y_i^\perp \text{ for } \widehat{y}_i \in L_i \text{ and } y_i^\perp \in L_i^\perp$$
$$\text{and}$$
$$e_i = \widehat{e}_i + e_i^\perp \text{ for } \widehat{e}_i \in L_i \text{ and } e_i^\perp \in L_i^\perp.$$

We have

$$(e_1^\perp + \lambda y_1^\perp, \ldots, e_m^\perp + \lambda y_m^\perp) \in \mathcal{A}^\perp \text{ for all } \lambda \geq 0.$$

If $\mathcal{A}^\perp$ is bounded, we must have $y_i^\perp = 0$ for all i. Thus $y_i \in L_i$ for all i.

(2) $\Rightarrow$ (1) : In order to show that Hart's condition implies that $\mathcal{A}^\perp$ is compact, it suffices to show that Hart's condition implies that $\mathcal{A}^\perp$ is bounded. Suppose not. Let $\{(x_i^{\perp n})\}_n$ be a sequence in $\mathcal{A}^\perp$ such that such that

$$\sum_{i=1}^m \left\| x_i^{\perp n} \right\| \to \infty.$$

Now let $\{(\widehat{x}_i^n)\}_n$ be a sequence in $\prod_{i=1}^m L_i$ such that

$$\sum_{i=1}^m x_i^{\perp n} + \sum_{i=1}^m \widehat{x}_i^n = \sum_{i=1}^m e_i := \overline{e}.$$

Without loss of generality, we can assume that for all i,

$$\frac{x_i^{\perp n}}{\sum_{i=1}^m \left\| x_i^{\perp n} \right\| + \| \sum_{i=1}^m \widehat{x}_i^n \|} \to x_i^*, \text{ and } \frac{\sum_{i=1}^m \widehat{x}_i^n}{\sum_{i=1}^m \left\| x_i^{\perp n} \right\| + \| \sum_{i=1}^m \widehat{x}_i^n \|} \to \zeta.$$

Note that since $\sum_{i=1}^m L_i$ is a finite-dimensional subspace, it is closed. Thus,

$$\frac{\sum_{i=1}^m \widehat{x}_i^n}{\sum_{i=1}^m \left\| x_i^{\perp n} \right\| + \left\| \sum_{i=1}^m \widehat{x}_i^n \right\|} \in \sum_{i=1}^m L_i \text{ for all } n,$$
$$\text{implies that}$$
$$\zeta \in \sum_{i=1}^m L_i.$$

Write $\zeta = \sum_{i=1}^m \zeta_i$ where $\zeta_i \in L_i$ for each i. We have

$$\sum_{i=1}^m x_i^* + \sum_{i=1}^m \zeta_i = 0.$$

Since for all i, $x_i^* + \zeta_i \in R_i$, by Hart's condition, we have $x_i^* + \zeta_i \in L_i$ for all i. Since $\zeta_i \in L_i$, $x_i^* + \zeta_i \in L_i$ implies that $x_i^* \in L_i$. But $x_i^* \in L_i^\perp$. Thus, for all i, $x_i^* = 0$, so that $\sum_{i=1}^m \zeta_i = 0$. Observe that for all n,

$$\frac{\sum_{i=1}^m \left\| x_i^{\perp n} \right\|}{\sum_{i=1}^m \left\| x_i^{\perp n} \right\| + \| \sum_{i=1}^m \widehat{x}_i^n \|} + \frac{\| \sum_{i=1}^m \widehat{x}_i^n \|}{\sum_{i=1}^m \left\| x_i^{\perp n} \right\| + \| \sum_{i=1}^m \widehat{x}_i^n \|} = 1,$$

and hence

$$\sum_{i=1}^m \| x_i^* \| + \left\| \sum_{i=1}^m \zeta_i \right\| = 1.$$

Thus, we have a contradiction. ■

6.2 The Equivalence of the Hart's Condition and the Generalized Werner Condition

In order to prove the equivalence of Hart (WNMA) and the extended Werner condition (EPNA), we need two additional results.

Lemma 4 *Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying* [A.1]-[A.2]. *The following statements are true:*

1. *For any i, such that $R_i \setminus L_i \neq \emptyset$, we have:*

$$S_i = \{p \in L_i^\perp \mid \ p.y > 0, \forall y \in (R_i \cap L_i^\perp) \setminus \{0\}\}.$$

2. *$\forall i = 1, \ldots, m, S_i = -\mathrm{ri}(R_i^0)$ where (R_i^0) is the polar cone of R_i, and* ri *denotes relative interior (i.e., the interior relative to the affine hull,* $\mathrm{aff}(R_i^0)$*).*

Proof. (1) See Dana, Le Van and Magnien (1999, p.182).

(2) It is clear that if $R_i = L_i$ then $R_i^0 = L_i^\perp = S_i$. Thus, $S_i = \mathrm{ri}(-{R_i}^0)$. Now let us suppose that $R_i \setminus L_i \neq \emptyset$. First, we show that $\mathrm{aff}(R_i^0) = L_i^\perp$. Indeed, since $L_i \subset R_i$ we have $R_i^0 \subset L_i^\perp$ and then $\mathrm{aff}(R_i^0) \subset L_i^\perp$. Furthermore, if $\mathrm{aff}(R_i^0)$ is a proper vector subspace of $L_i^\perp$, then L_i is a proper vector subspace of $(\mathrm{aff}(R_i^0))^\perp$. But $(\mathrm{aff}(R_i^0))^\perp \subset R_i$, which contradicts the fact that the lineality space L_i is the maximal vector subspace contained in R_i.

It is easy to check that $R_i = (R_i \cap L_i^\perp) + L_i$ (also see Allouch, Le Van, and Page (2001)). By Corollary 16.4.2 in Rockafellar (1970), we have

$$\begin{aligned} R_i^0 &= (R_i \cap L_i^\perp)^0 \cap L_i^\perp \qquad (1) \\ &= \{p \in L_i^\perp \mid \ p.y \leq 0, \forall y \in (R_i \cap L_i^\perp)\}. \qquad (2) \end{aligned}$$

We notice that the positive dual of $R_i \cap L_i^\perp$ in $L_i^\perp$ is also R_i^0, and that $R_i \cap L_i^\perp$ is pointed cone, that is:

$$(R_i \cap L_i^\perp) \bigcap -(R_i \cap L_i^\perp) = 0.$$

Then, it follows from (2)

$$\mathrm{ri}R_i^0 = \mathrm{int}_{L_i^\perp} R_i^0 = \{p \in L_i^\perp \mid \ p \cdot y < 0, \forall y \in (R_i \cap L_i^\perp) \setminus \{0\}\}.$$

From (1) of the present lemma, we get $S_i = -\mathrm{ri}(R_i^0)$. ■

In addition to Lemma 4 above, we need the following lemma, a restatement of Corollary 16.2.2 in Rockafellar (1970).

Lemma 5 *Let $f_1, \ldots f_m$ be a proper convex functions on $\mathbb{R}^m$. In order that there do not exist vectors $x_1^*, \ldots, x_m^*$ such that*

$$x_1^* + \ldots + x_m^* = 0, \tag{3}$$

$$f_1^* O^+(x_1^*) + \ldots + f_m^* O^+(x_m^*) \leq 0, \tag{4}$$

$$f_1^* O^+(-x_1^*) + \ldots + f_m^* O^+(-x_m^*) > 0, \tag{5}$$

it is necessary and sufficient that

$$\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom} f_i) \neq \emptyset.$$

We recall that for a convex function $\mathrm{dom} f_i = \{x \in \mathbb{R}^m \mid \ f_i(x) < +\infty\}$ and $f_i^* O^+$ is the support function of $\mathrm{dom} f_i$, that is,

$$f_i^* O^+(x_i^*) = \sup\{x_i^* \cdot x \mid \ x \in \mathrm{dom} f_i\}.$$

Proof of Theorem 1 (The Equivalence of Hart and Werner)

For every $i = 1, \ldots, m$, let

$$f_i(x) = \begin{cases} 0 & \text{if } x \in R_i^0, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence

$$f_i^* O^+(x_i^*) = \sup\{x_i^* \cdot x \mid \ x \in R_i^0\}. \tag{6}$$

Since $0 \in R_i^0$, it follows that $f_i^* O^+(x_i^*) \geq 0$ for all i. Then (4) is satisfied if and only if $f_i^* O^+(x_i^*) = 0$ for all i and therefore from (6) if and only if $x_i^* \in R_i$. Quite similarly, (5) is not satisfied if and only if $-x_i^* \in R_i$. Since $L_i = R_i \cap -R_i$, it follows that the first assertion of Lemma 5 is satisfied if and only if the WNMA condition is satisfied. Furthermore, from Lemma 4 one gets

$$\bigcap_{i=1}^{m} S_i = \bigcap_{i=1}^{m} \mathrm{ri}(-{R_i}^0) = -\bigcap_{i=1}^{m} \mathrm{ri}(\mathrm{dom} f_i).$$

Hence, the equivalence follows from Lemma 5. ■

6.3 Existence

6.3.1 Modifying the economy

Our method of proving existence is new. Our starting point is an exchange economy $\mathcal{E}$ satisfying assumptions [A.1]-[A.2] and weak nonsatiation [A.4]. To deal with the problem of satiation, we construct a new economy $\mathcal{E}'$ in which agents' utility functions have been modified. In the new economy $\mathcal{E}'$ agents' preferences are such that no agent is satiated at a rational allocation. Below, we establish that if the economy $\mathcal{E}$ satisfies assumptions [A.1]-[A.2] and weak nonsatiation [A.4], then the modified economy $\mathcal{E}'$ satisfies assumptions [A.1]-[A.2], and global nonsatiation. Moreover, we show that if $\mathcal{E}$ satisfies Hart's condition, then the modified economy $\mathcal{E}'$ also satisfies Hart's condition. Finally, we show that a quasi-equilibrium for the modified economy $\mathcal{E}'$ is also a quasi-equilibrium for the original economy $\mathcal{E}$.

Let $\mathcal{E} = (X_i, u_i, e_i)_{i=1}^m$ be an economy satisfying [A.1]-[A.2], and weak nonsatiation [A.4]. We begin by modifying agents' utility functions. Suppose that for some agent i there exists a satiation point $x_i^* \in \mathcal{A}_i$, that is,

$$u_i(x_i^*) = \sup_{x_i \in X_i} u_i(x_i).$$

It follows from weak nonsatiation [A.4] that there exists

$$r_i \in O^+ \widehat{P}_i(x_i^*) \setminus L_i(x_i^*).$$

Using r_i we define the function

$$\rho_i(\cdot) : \widehat{P}_i(x_i^*) \to \mathbb{R}_+$$

as follows:

$$\rho_i(x_i) = \sup\{\beta \in \mathbb{R}_+ \mid \ (x_i - \beta r_i) \in \widehat{P}_i(x_i^*)\}.$$

Now using the function $\rho_i(\cdot)$, we can define a new utility function, $v_i(\cdot)$, for agent i:

$$v_i(x_i) = \begin{cases} u_i(x_i) + \rho_i(x_i), & \text{if } x_i \text{ is a satiation point,} \\ u_i(x_i), & \text{otherwise.} \end{cases}$$

Claim 6.1 *The function ρ_i is well-defined. Moreover, for all $x_i \in \widehat{P}_i(x_i^*)$ we have $(x_i - \rho_i(x_i) r_i) \in \widehat{P}_i(x_i^*)$.*

Proof of Claim 6.1. Let

$$W = \{\beta \in \mathbb{R}_+ \mid \ (x_i - \beta r_i) \in \widehat{P}_i(x_i^*)\}.$$

We first notice that $0 \in W$. Thus, $\emptyset \neq W \subset \mathbb{R}_+$. We claim that W is bounded. Suppose the contrary. Then $-r_i \in O^+ \widehat{P}_i(x_i^*)$ and therefore

$r_i \in L_i(x_i^*)$, which contradicts $r_i \in O^+\widehat{P}_i(x_i^*) \setminus L_i(x_i^*)$. Finally, we have $(x_i - \rho_i(x_i)r_i) \in \widehat{P}_i(x_i^*)$ since $\widehat{P}_i(x_i^*)$ is closed.□

Claim 6.2 *Let $\lambda \geq 0$. Then*

$$\{x \in \widehat{P}_i(x_i^*) \mid \ \rho(x) \geq \lambda\} = \{\lambda r_i\} + \widehat{P}_i(x_i^*).$$

Proof of Claim 6.2. First it is obvious that

$$\{\lambda r_i\} + (\widehat{P}_i(x_i^*)) \subset \{x \in \widehat{P}_i(x_i^*) \mid \ \rho_i(x) \geq \lambda\}.$$

Furthermore, let $x_i \in \{x \in \widehat{P}_i(x_i^*) \mid \ \rho_i(x) \geq \lambda\}$. Then, $(x_i - \rho(x_i)r_i) \in \widehat{P}_i(x_i^*)$ and therefore $x_i \in \{\lambda r_i\} + \widehat{P}_i(x_i^*)$, since $\widehat{P}_i(x_i^*)$ is convex.□

Claim 6.3 *We have* $\sup_{x_i \in \widehat{P}_i(x_i^*)} \rho_i(x_i) = +\infty$.

Proof of Claim 6.3. It is obvious that $(x_i + \lambda r_i) \in \widehat{P}_i(x_i^*)$, for all $\lambda \geq 0$, since $r_i \in O^+\widehat{P}_i(x_i^*)$. Moreover, $\rho_i(x_i + \lambda r_i) \geq \lambda$. Then, $\sup_{x_i \in \widehat{P}_i(x_i^*)} \rho_i(x_i) = +\infty$.□

Consider the level set $E_\lambda = \{x \in X_i \mid \ v_i(x) \geq \lambda\}$, for every $\lambda \in \mathbb{R}$.

Claim 6.4 *The function v_i is upper semicontinuous and quasi-concave. Moreover, for all $x_i \in \widehat{P}_i(e_i)$*

$$O^+E_{v_i(x_i)} = \begin{cases} O^+\widehat{P}_i(x_i^*), & \textit{if } x_i \textit{ is a satiation point,} \\ O^+\widehat{P}_i(x_i), & \textit{otherwise.} \end{cases}$$

Proof of Claim 6.4. The function v_i is upper semicontinuous and quasi-concave if and only if E_λ is closed and convex for all $\lambda \in \mathbb{R}$.

first case. Suppose $\lambda \leq u_i(x_i^*)$. Then, $E_\lambda = \{x \in X_i \mid \ u_i(x) \geq \lambda\}$. Thus, E_λ is closed and convex, since u_i is upper semicontinuous and quasi-concave.

second case. Suppose $\lambda > u_i(x_i^*)$. Then

$$\begin{aligned} E_\lambda &= \{x \in X_i \mid \ v_i(x) \geq \lambda\} \\ &= \{x \in \widehat{P}_i(x_i^*) \mid \ \rho_i(x) \geq (\lambda - u_i(x_i^*))\} \\ &= \{(\lambda - u_i(x_i^*))r_i\} + \widehat{P}_i(x_i^*). \end{aligned}$$

Thus, E_λ is convex and closed.□

Now, we consider the modified economy $\mathcal{E}' = (X_i, v_i, e_i)_{i=1....,m}$. Let

$$\mathcal{A}' = \{(x_i) \in \prod_{i=1}^{m} X_i \mid \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i \text{ and } \ v_i(x_i) \geq v_i(e_i), \forall i\},$$

be the set of rational allocations of $\mathcal{E}'$.

Claim 6.5 *If in addition to satisfying assumptions* [A.1]-[A.2], *and weak nonsatiation* [A.4], $\mathcal{E}$ *also satisfies weak uniformity* [A.3], *then the following statement is true:*

If the original economy $\mathcal{E}$ *satisfies Hart's condition* (WNMA), *then the modified economy* $\mathcal{E}'$ *also satisfies Hart's condition.*

Proof of Claim 6.5. It follows from Claim 6.4 that for all $x_i \in E_{v_i(e_i)}$ we have

$$L_i \subset O^+ E_{v_i(x_i)} \subset O^+ E_{v_i(e_i)} \subset R_i.$$

Since, L_i is the maximal subspace in R_i, one gets v_i has uniform lineality space equal to L_i. Furthermore, $\sum_{i=1}^m y_i = 0$ with $\forall i$, $y_i \in O^+ E_{v_i(e_i)}$ implies that $\sum_{i=1}^m y_i = 0$ with $\forall i$, $y_i \in R_i$. Since $\mathcal{E}$ satisfies the WNMA condition, $y_i \in L_i$, $\forall i$. Therefore, $\mathcal{E}'$ also satisfies the WNMA condition .□

Claim 6.6 *We have:*

(i) The modified economy $\mathcal{E}'$ *satisfies Global Nonsatiation.*

(ii) If (x^*, p^*) *is a quasi-equilibrium of* $\mathcal{E}'$, *then* (x^*, p^*) *is a quasi-equilibrium of* $\mathcal{E}$.

Proof of Claim 6.6. (*i*) It follows from Claim 6.3.

(*ii*) It is clear that $x^* \in \mathcal{A}' \subset \mathcal{A}$. Moreover, let $x_i \in X_i$ be such that $u_i(x_i) > u_i(x_i^*)$. Then, x_i^* is not a satiation point and therefore $v_i(x_i^*) = u_i(x_i^*)$. Since $v_i(x_i) \geq u_i(x_i)$, it follows that $v_i(x_i) > v_i(x_i^*)$. Since (x^*, p^*) is a quasi-equilibrium of $\mathcal{E}'$, we can conclude that $p^* \cdot x_i \geq p^* \cdot e_i$. Thus, (x^*, p^*) is a quasi-equilibrium of $\mathcal{E}$.□

6.3.2 Proof of Theorem 2 (Existence Result)

First, it follows from Claim 6.5 that $\mathcal{E}'$ also satisfies the WNMA. From Claim 6.6 it is sufficient to show that $\mathcal{E}'$ has a quasi-equilibrium.

We consider a sequence of truncated economies with consumption sets

$$X_i^n = \widehat{P}_i(e_i) \cap L_i^\perp \cap \mathrm{cl}B(0, n),$$

where $B(0,n)$ is the open ball of radius n centered at 0. We choose n large enough so that $e_i \in B(0,n)$ for each i.
Let $D = \cap L_i^{\perp}$ and Π is the unit sphere of $\mathbb{R}^{l+1}$.

For $(p,q) \in (D \times \mathbb{R}_+) \cap \Pi$, we consider

$$\varphi_i^n(p,q) = \{x_i \in X_i^n \mid \ p \cdot x_i \leq p \cdot e_i + q\},$$

and

$$\zeta_i^n(p,q) = \{x_i \in \varphi_i^n(p,q) \mid \ y \in \widetilde{P}_i^n(x_i) \Rightarrow p \cdot y \geq p \cdot e_i + q\},$$

where

$$\widetilde{P}_i^n(x_i) = \{(1-\lambda)x_i + \lambda z_i \mid \ 0 < \lambda \leq 1, v_i(x_i) < v_i(z_i) \text{ and } z_i \in X_i^n\}.$$

We have the following result:

Lemma 6.1 *For n large enough, ζ_i^n is upper semicontinuous nonempty, compact and convex valued, for every i.*

Proof. First we show that $\zeta_i^n(p,q)$ is nonempty for n large enough.

For n large enough, $e_i \in \varphi_i^n(p,q)$. Let $\widehat{x}_i$ be a maximizer of v_i on $\varphi_i^n(p,q)$. If $\widetilde{P}_i^n(\widehat{x}_i) = \emptyset$, we end the proof, since $\widehat{x}_i \in \zeta_i^n(p,q)$. If not, let $z_i \in X_i^n$, such that $v_i(z_i) > v_i(\widehat{x}_i)$. By the very definition of $\widehat{x}_i$, we have $p \cdot z_i > p \cdot e_i + q$. Let t_i, contained in the segment $[\widehat{x}_i, z_i]$, be such that

$$p \cdot t_i = p \cdot e_i + q.$$

By quasi-concavity of the utility function, $v_i(t_i) \geq v_i(\widehat{x}_i)$. By the definition of $\widehat{x}_i, v_i(t_i) \leq v_i(\widehat{x}_i)$. Hence t_i is another maximizer of v_i on $\varphi_i^n(p,q)$. We claim that $t_i \in \zeta_i^n(p,q)$. Indeed, let $z' \in X_i^n$ such that $v_i(z_i') > v_i(t_i)$. We have $p \cdot z' > p \cdot e_i + q$. Thus,

$$\forall \lambda \in]0,1], p \cdot ((1-\lambda)t_i + \lambda z') > p \cdot e_i + q.$$

Second we show that $\zeta_i^n(p,q)$ is convex valued.

Let x and x' be contained in $\zeta_i^n(p,q)$ and let $y \in \widetilde{P}_i^n(\lambda x + (1-\lambda)x')$ for $\lambda \in]0,1[$.

(a) First assume $p \cdot x < p \cdot e_i + q$ and $p \cdot x' \leq p \cdot e_i + q$. If $v_i(x) > v_i(x')$ then $p \cdot x \geq p \cdot e_i + q$, which is a contradiction. Hence $v_i(x) \leq v_i(x')$. If $v_i(x') > v_i(x)$, then $p \cdot x' = p \cdot e_i + q$. Because $v_i(x') > v_i(x)$, we have $\lambda x + (1-\lambda)x' \in \widetilde{P}_i^n(x)$ which implies that

$$p \cdot (\lambda x + (1-\lambda)x') \geq p \cdot e_i + q,$$

Thus, we have a contradiction because

$$p \cdot (\lambda x + (1-\lambda)x') < p \cdot e_i + q.$$

Therefore $v_i(x') = v_i(x)$. But now by quasi-concavity, we have

$$v_i(\lambda x + (1-\lambda)x') \geq v_i(x) = v_i(x').$$

If $v_i(\lambda x + (1-\lambda)x') > v_i(x)$, then

$$p \cdot (\lambda x + (1-\lambda)x') \geq p \cdot e_i + q,$$

a contradiction as before. Hence,

$$v_i(\lambda x + (1-\lambda)x') = v_i(x) = v_i(x').$$

Let $y \in \widetilde{P}_i^n(\lambda x + (1-\lambda)x')$, i.e., $y = \alpha(\lambda x + (1-\lambda)x') + (1-\alpha)z$ for some $\alpha \in [0,1[$, and some $z \in X_i^n$ such that $v_i(z) > v_i(\lambda x + (1-\lambda)x')$. We have the identity

$$\alpha(\lambda x + (1-\lambda)x') + (1-\alpha)z = \lambda(\alpha x + (1-\alpha)z) + (1-\lambda)(\alpha x' + (1-\alpha)z).$$

But we have, $p \cdot (\alpha x + (1-\alpha)z) \geq p \cdot e_i + q$, and $p \cdot (\alpha x + (1-\alpha)z') \geq p \cdot e_i + q$. Therefore, $p \cdot y \geq p \cdot e_i + q$.

(b) Assume now $p \cdot x = p \cdot e_i + q$ and $p \cdot x' = p \cdot e_i + q$. In this case $p \cdot (\lambda x + (1-\lambda)x') \geq p \cdot e_i + q$. Let

$$y = \alpha(\lambda x + (1-\lambda)x') + (1-\alpha)z$$

for some $\alpha \in [0,1[$ and some $z \in X_i^n$ such that $v_i(z) > v_i(\lambda x + (1-\lambda)x')$. We have

$$v_i(z) > v_i(\lambda x + (1-\lambda)x') \geq \min\{v_i(x), v_i(x')\}.$$

Hence $p \cdot z \geq p \cdot e_i + q$, and $p \cdot y \geq p \cdot e_i + q$.

Finally, we show that $\zeta_i^n(\cdot,\cdot)$ has a closed graph. Let

$$x_i^\nu \in \zeta_i^n(p^\nu, q^\nu),\ x_i^\nu \to x,\ (p^\nu, q^\nu) \to (p,q),$$

and let

$$z = (1-\lambda)x_i + \lambda y,$$

for $\lambda \in]0,1]$ and $y \in X_i^n$ such that $v_i(y) > v_i(x)$. By the u.s.c. of v_i, for ν large enough, $v_i(y) > v_i(x_i^\nu)$. Let

$$z^\nu = (1-\lambda)x_i^\nu + \lambda y.$$

Clearly, $z^\nu \in \widetilde{P}_i^n(x_i^\nu)$, so that

$$p^\nu \cdot z^\nu \geq p^\nu \cdot e_i + q^\nu.$$

Since $\lim_{\nu\to+\infty} z^\nu = z$,

$$p \cdot z \geq p \cdot e_i + q.$$

Thus, $x \in \zeta_i^n(p,q)$.□

Now, define

$$Z^n(p,q) := [\sum_{i=1}^m (\zeta_i^n(p,q) - e_i)] \times \{-m\}.$$

It is clear that,

$$\forall (p,q) \in (D \times \mathbb{R}_+) \cap \Pi, \forall x \in Z^n(p,q), (p,q).x \leq 0.$$

We can now apply the Debreu fixed point lemma (see Florenzano and Le Van (1986)).

Lemma 6.2 *Let $P \subset \mathbb{R}^{\ell+1}$ be a convex cone which is not a linear subspace. Let P^0 and Π denote respectively the polar of P and the unit sphere of $\mathbb{R}^{\ell+1}$. Let Z be an upper semicontinuous (u.s.c.), nonempty, compact and convex valued correspondence from $P \cap \Pi$ into $\mathbb{R}^{\ell+1}$ such that*

$$\forall p \in P \cap \Pi, \ \exists z \in Z(p) \text{ such that } p \cdot z \leq 0.$$

Then there exists $\bar{p} \in P \cap \Pi$ such that $Z(\bar{p}) \cap P^0 \neq \emptyset$.

Thus, it follows from the above lemma that

$$\begin{gathered} \exists (p^n, q^n) \in (D \times \mathbb{R}_+) \cap \Pi, \\ \exists x_i^n \in \zeta_i(p^n, q^n), \forall i, \\ \text{and} \\ \exists z^n \in \textstyle\sum_{i=1}^m L_i \text{ such that } \sum_{i=1}^m (x_i^n - e_i) = z^n. \end{gathered}$$

One can write $z^n = \sum_{i=1}^m l_i^n$, where $l_i^n \in L_i, \forall i$. Then one has

$$\sum_{i=1}^m (x_i^n - l_i^n) = \sum_{i=1}^m e_i,$$

and therefore $(x_i^n) \in \mathcal{A}^\perp$. Passing to a subsequence if necessary, it follows from the compactness of $\mathcal{A}^\perp$ and $(D \times \mathbb{R}_+) \cap \Pi$ that

$$\lim_{n\to+\infty} (x_i^n) = x^* \in \mathcal{A}^\perp \text{ and } \lim_{n\to+\infty} (p^n, q^n) = (p^*, q^*) \in (D \times \mathbb{R}_+) \cap \Pi.$$

Since $x^* \in \mathcal{A}^\perp$ there exists $(l_i) \in \prod_{i=1}^m L_i$ such that

$$\sum_{i=1}^m (x_i^* - l_i) = \sum_{i=1}^m e_i,$$

and $x_i'^* = x_i^* - l_i$. By Global Nonsatiation for v_i there exists $z_i \in X_i$, such that

$$v_i(z_i) > v_i(x_i'^*) = v_i(x_i^*).$$

Then, by weak uniformity, [A.3], there exists $z_i^\perp \in X_i \cap L_i^\perp$, such that $v_i(z_i^\perp) > v_i(x_i^*)$. For n large enough, $z_i^\perp \in X_i^n$, and therefore $v_i(z_i^\perp) > v_i(x_i^n)$ (since v_i is u.s.c.). It follows from $x_i^n \in \zeta_i^n(p^n, q^n)$, that

$$p^n \cdot y_i^n \geq p^n \cdot e_i + q^n, \text{ for } y_i^n = (1-\lambda)x_i^n + \lambda z_i, \ \lambda \in]0,1].$$

Let $n \to \infty$. Then

$$p^* \cdot ((1-\lambda)x_i^* + \lambda z_i) \geq p^* \cdot e_i + q^*.$$

Let $\lambda \to 0$. Then

$$p^* \cdot x_i^* \geq p^* \cdot e_i + q^*.$$

But, $p^* \cdot x_i \leq p^* \cdot e_i + q^*$. Hence

$$p^* \cdot x_i^* = p^* \cdot e_i + q^*, \forall i,$$

and also

$$p^* \cdot x_i'^* = p^* \cdot e_i + q^*, \forall i,$$

since $l_i \in L_i$. Summing over i, one gets $q^* = 0$, and $p^* \cdot x_i'^* = p^* \cdot e_i, \forall i$.

We claim that $(x_i'^*, p^*)$ is a quasi-equilibrium of $\mathcal{E}'$. Thus, it remains to check that $v_i(x_i) > v_i(x_i'^*)$ implies $p^* \cdot x_i \geq p^* \cdot e_i$. For such an x_i, let $x_i^\perp$ be the projection of x_i on $L_i^\perp$. For n large enough, $x_i^\perp \in X_i^n$, and $v_i(x_i^\perp) > v_i(x_i^n)$. Since $x_i^n \in \zeta_i^n(p^n, q^n)$, we have

$$p^n \cdot x_i^\perp \geq p^n \cdot e_i + q^n,$$

which implies $p^* \cdot x_i^\perp \geq p^* \cdot e_i$, and therefore $p^* \cdot x_i \geq p^* \cdot e_i$.□

References

[1] Allouch N., An equilibrium existence result with short selling, Journal of Mathematical Economics, forthcoming, (2002).

[2] Allouch N., Le Van, C., Page, F.H.Jr., The geometry of arbitrage and the existence of competitive equilibrium, Warwick Economic Research Papers, No 598, Department of Economics, University of Warwick, (2001).

[3] Bergstrom, T. C., How to discard 'free disposability' - at no cost, Journal of Mathematical Economics **3**, 131-134 (1976).

[4] Dana, R.-A., Le Van, C., Magnien F., On the different notions of arbitrage and existence of equilibrium, Journal of Economic Theory **86**, 169-193 (1999).

[5] Debreu, G., Theory of Value, John Wiley and Sons, New-York, (1959).

[6] Florenzano, M., Le Van, C., A note on Gale-Nikaido-Debreu lemma and the existence of general equilibrium, Economics Letters **32**, 107-110, (1986).

[7] Gale, D., Mas-Colell, A., An equilibrium existence theorem for a general model without ordered preferences, Journal of Mathematical Economics **2**, 9-15 (1975).

[8] Geistdoerfer-Florenzano, M., The Gale-Nikaido-Debreu lemma and the existence of transitive equilibrium with or without the free-disposal assumption, Journal of Mathematical Economics **9**, 113-134 (1982).

[9] Grandmont, J.M., On the Temporary Competitive Equilibrium, Working Paper, No. 305, Center for Research in Management Science, University of California, Berkeley, August (1970).

[10] Grandmont, J.M., Continuity Properties of a von Neumann-Morgenstern Utility, Journal of Economic Theory **5**, 45-57, (1972).

[11] Grandmont, J.M., Temporary general equilibrium theory, Econometrica **45**, 535-572, (1977).

[12] Green, J.R., Temporary general equilibrium in a sequential trading model with spot and futures transactions, Econometrica **41**, 1103-1124, (1973).

[13] Hammond, P.J., Overlapping expectations and Hart's condition for equilibrium in a securities model, Journal of Economics Theory **31**, 170-175 (1983).

[14] Hart, O., On the existence of an equilibrium in a securities model, Journal of Economic Theory **9**, 293-311 (1974).

[15] Kim, C., Stochastic dominance, Pareto optimality and equilibrium asset pricing, Review of Economic Studies **65**, 341-356 (1998).

[16] Milne, F., Default risk in a general equilibrium asset economy with incomplete market, International Economic Review **17**, 613-625 (1976).

[17] Milne, F., Short selling, default risk and the existence of equilibrium in a securities model, International Economic Review **21**, 255-267 (1980).

[18] Nielsen, L.T., Asset market equilibrium with short selling, Review of Economic Studies **56**, 467-474 (1989)

[19] Page, F.H.Jr., On equilibrium in Hart's securities exchange model, Journal of Economic Theory **41**, 392-404 (1987).

[20] Page, F.H.Jr., Wooders, M.H., A necessary and sufficient condition for compactness of individually rational and feasible outcomes and the existence of an equilibrium, Economics Letters **52**, 153-162 (1996).

[21] Page, F.H.Jr., Wooders, M.H., Monteiro, P.K., Inconsequential arbitrage, Journal of Mathematical Economics **34**, 439-469 (2000).

[22] Rockafellar, R.T., Convex Analysis, Princeton University Press, (1970).

[23] Werner J., Arbitrage and the existence of competitive equilibrium, Econometrica **55**, 1403-1418 (1987).