SOCIAL CONFORMITY AND EQUILIBRIUM IN PURE STRATEGIES IN GAMES WITH MANY PLAYERS

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Abstract: We introduce a framework of noncooperative pregames, in which players are characterized by their attributes, and demonstrate that for all games with sufficiently many players, there exist approximate (ε) Nash equilibria in pure strategies. In fact, every mixed strategy equilibrium can be used to construct an ε-equilibrium in pure strategies, an 'ε-purification' result. Our main result is a social conformity theorem. Interpret a set of players, all with attributes in some convex subset of attribute space and all playing the same strategy, as a society. Observe that the number of societies may be as large as the number of players. Our social conformity result dictates that, given $\varepsilon > 0$, there is an integer L, depending on ε but not on the number of players, such that any sufficiently large game has an ε-equilibrium in pure strategies that induces a partition of the player set into fewer than L societies.

1 Social learning

A society is a group of individuals who have commonalities of language, social and behavioral norms, and customs. Social learning consists, at least in part, in learning the norms and behavior patterns of the society into which one is born and in those other societies which one may join – professional associations, faculty clubs, and communities, for example. Individuals may learn by observing and imitating individuals in the same society. A fundamental question is whether the outcome of such imitation can be consistent with self-interested behavior. This consistency requires the existence of a Nash equilibrium where individuals within the same society play the same or similar strategies and where societies are nontrivial in size. The existence of such an equilibrium is fundamental to the social sciences.

To address the question of whether imitation can be consistent with Nash equilibrium, we must first have an appropriate model. One of the main contributions of the current paper is the introduction of a non-cooperative counterpart to the pregame framework of cooperative game theory.[1] In cooperative game theory this framework has led to a number of results, especially results showing that large games with small effective groups[2] resemble, or in fact *are*, competitive economies. It appears that our framework of non-

[1] See, for example, Wooders (1983,1994).

[2] Small groups are effective if all or almost all gains to collective activities can be realized by cooperation only within groups of players that are small relative to the total population. This is an apparently mild condition; when there are sufficiently many players of each type that appears in the games, small group effectiveness is equivalent to boundedness of per capita payoffs (Wooders 1994).

cooperative games may be equally useful. We return to a comparison of noncooperative pregames and cooperative pregames in a concluding section.

The main component of a noncooperative pregame is a set Ω of attributes of players. A component of attribute space is a complete description of the possible characteristics of a player, such as gender, height, IQ, personality, as well as a player's payoff function. The two other components are a set S of pure strategies, constrained to be finite in the current paper, and a payoff function h summarizing the payoff functions of all players in all possible games derived from the pregame. A population consists of a finite player set and an attribute function, ascribing an attribute $\omega \in \Omega$ to each member of the population. The pregame induces a game on each population. In each game, a player's payoff is a function of his own strategy and of the joint distribution over pure strategies and attributes implied by the actions of the complementary player set. Thus, a player is indifferent when two players of the same attribute exchange strategies. A player is, however, not indifferent if two players with different attributes exchange strategies. This latter characteristic distinguishes the model of this paper with those used in much of the prior literature. The relationship of this paper to the literature is discussed further in Section 7.

We require two main assumptions for games with large player sets. The first is a *continuity* assumption on the payoff function with respect to changes in attributes. It dictates that a player is nearly indifferent to a small change in their own attribute. Further, it requires that a player be nearly indifferent to a slight perturbation in the attribute's of others - on the assumption that the strategies they play are unchanged. The second assumption, *global interaction*, states that each player's payoff is primarily a function of his own strategy and of the fraction of players with each attribute playing each strategy. Global interaction is essentially a statement of the type of game we model in this paper; games for which the actions of any one individual have relatively little effect on the payoffs of other players. A pregame is said to satisfy the large game property if continuity and global interaction holds for large games.

We first provide an existence and purification result; there exists a Nash equilibrium in pure strategies and 'near' to any mixed strategy vector is a strategy vector in pure strategies. More precisely:

Theorem 1: Existence. If a pregame satisfies the large game property then given any $\varepsilon > 0$ there exists an integer $\eta(\varepsilon)$ with the property that every game with at least $\eta(\varepsilon)$ players has an ε-equilibrium in pure strategies. Moreover, every strategy vector can be ε-purified.

Part of the interest of this result, as its antecedents in the literature, is that it overcomes the need to motivate mixed strategies. In a social conformity context, where players are imitating each other, the use of mixed strategies may be particularly difficult to explain.

For our next theorem we introduce the notion of a *society*; for any game induced by the pregame and for any convex subset of attribute space Ω_A, if the set of players A with attributes in Ω_A is such that every player in A plays the same pure strategy, then we interpret the set of players A as a society. Note that a strategy profile does not necessarily induce a partition of the set of players into societies (players with the same attribute play different strategies.) Moreover, if each player has a different attribute, the number of societies may be as large as the player set.

Theorem 2: Social conformity. Suppose a pregame satisfies the large game property, the space of attributes Ω satisfies an apparently mild property of 'convex separation' and that there is a bound B on the number of players with any one attribute; then, given any $\varepsilon > 0$, there is an integer $L(\varepsilon, B)$ such that for any game with at least $\eta(\varepsilon, B)$ players there exists an ε-equilibrium in pure strategies that induces a partition of the set of players into at most $L(\varepsilon, B)$ societies.

The integer L is a measure of social conformity; the smaller L the greater the possible dissimilarity of players who conform, in the ε-equilibrium, in their choice of strategy. Thus, it is an important feature of Theorem 2 that, given ε, the integer L is fixed, independent of the numbers of players.[3] More motivation for our notion of social conformity is provided in the following subsection and later in the paper.

Another important feature of Theorem 2 is the convexity aspect of societies. For some attribute spaces, for example, $\Omega = [0, 1]$, Theorem 2 implies more – the existence of an approximate equilibria in pure strategies with the property that most players are playing the same strategy as their closest neighbors in attribute space. Other examples of such instances could possibly be derived from exchange economies where, in equilibrium, similar players choose similar commodity bundles or from economies inducing consecutive games, as in Greenberg and Weber (1986). It may be, however, despite our Theorem 2, that there are games for which all ε-equilibria have the property that most players do not play the same strategy as their closest neighbor. Suppose, for example, that $\Omega = [0, 1] \times [0, 1]$. It may be that the

[3] We are grateful to Roland Benabou for suggesting we emphasize this aspect of Theorem 2.

partition required by Theorem 2 splits Ω into two convex subsets with the property that every player is in a different society than his closest neighbor; such an example is considered in Section 3.2. In some situations, however, it is natural for closest neighbors to play different strategies and additional conditions on our model to rule this out would be restrictive.

It is easy to see why Theorem 2 requires that the number of players with any particular attribute is bounded. Consider, for example, an n-person matching pennies game where all players have identical attributes. In this case, since all players are identical it is impossible to construct a partition of the total player set into two subsets of players where all the players in each subset play the same strategy and the attributes of the players in the two subsets are distinct. We return to this issue in a concluding section.

Our existence result, differs from a number of results in the literature in that the prior papers almost all have a continuum of players (for example, Schmeidler 1973, Mas-Colell 1984, Khan 1989, Pascoa 1993,1998, Khan et al. 1997, Khan and Sun 1999, and Araujo and Pascoa 2000). Of course for a number of these results, one could consider a sequence of large finite games with the player distribution converging to the distribution of player types in the given continuum, and from the results for the continuum, establish existence of ε-equilibrium in pure strategies for all sufficiently large games in the sequence.[4] Our results differ in that we are not restricted to one limiting distribution of player types; our results hold for all sufficiently large finite games derived from a non-cooperative pregame. The relationship between our framework and results and the existing literature are discussed in Section 7.

To the best of our knowledge, our social conformity result has no analogue in the extant literature. A related literature concerns dynamic models of social learning (for example, Blume 1993, Ellison 1993, Kandori, Mailath and Rob 1993, Ellison and Fudenberg 1993, 1995, Kirman 1993, Bjornerstedt and Weibull 1995, Young 2001). A primary motivation behind this literature is to see whether players can learn, using simple rules of thumb such as imitation, to play Nash equilibrium strategies. This motivation is analogous to ours in questioning the effectiveness of social learning. We note that through choice of model, however, in this literature the existence of the type of strategy vector we are concerned with in Theorem 2 is not an issue. In particular, there often trivially exists a strategy vector that is both a Nash equilibrium and an absorbing state of the social dynamic; the question

[4] Indeed, some of these papers discuss the implications of their results for games with many, but a finite number, of players; see, for example, Khan and Sun (1999).

is whether players can learn to play such a strategy vector. In contrast, our question is whether, more generally, such a strategy vector exists.

1.1 Further discussion of social conformity

There is no definitive definition of social conformity but typically it is used to describe situations in which a person's decision is influenced by a group of people with whom he interacts (Gross 1996) or by 'similar' players. In a widely accepted account of group influence, Deutsch and Gerard (1955) distinguish two fundamental reasons for group influence - informational influence and normative influence (see also Asch 1952). Group influence leads to conformity in the sense that people choose the same actions as those with whom they interact.

Informational influence reflects the fact that a person may be able to make a more informed decision by observing the actions of other individuals. This could be because a person is boundedly rational or has imperfect information and so imitates a person he believes is better informed (Gale and Rosenthal 1999). For example, in financial markets the actions of traders may signal private information (Schleifer 2000). Alternatively, in a game with multiple equilibria a player may be able to make a better strategy choice by observing the actions of others (Ellison and Fudenberg 1995). For example, there may be a choice between two types of computer software and it is in a player's interest to choose the same software as those other players with whom he interacts (Young 2001). In a financial market context it may be a best response to choose the same strategy as other traders on the basis 'there is less to explain' if that strategy ultimately does not succeed (Scharfstein and Stein 1990). Normative influence, in contrast to informational influence, reflects the fact that a person may be motivated by desires for prestige, popularity or acceptance etc. (Bernheim 1994). Thus, a player may choose a strategy purely so as to 'fit in' with a social norm.

It is clear that, in the presence of both informational and normative influences, a person will be influenced differently by different people. In particular, a person may not be influenced by the behaviour of certain individuals; he may not, for example, perceive them as better informed than himself or as people whose acceptance he desires. Reflecting this, the psychology literature also highlights that social influence may result from a process whereby a person perceives themselves as a part of a group whose members have similar characteristics (Gross 1996). A person is then influenced primarily by members of the group with which he identifies – similar individuals. In particular, a person may only be influenced by those people with whom he, a-priori, expects to choose the same action.

The above discussion motivates our notion of social conformity as a partition of the population into a relatively small number of societies, as defined above. It is apparent that different types of group influence appear, intuitively, more likely to be consistent with a small number of societies. For example, if players are influenced by normative concerns it appears more likely that social conformity can be consistent with rational behaviour. We do not, however, build into our model any explicit assumption that induces similar players to choose the same strategy. Normative influence for instance, may be present, but equally it may not. This means that our results can be applied to any type of conformity. For example, the results can be interpreted with respect to whether imitation as a method of learning by boundedly rational players can be consistent with rational behaviour. It may, however, be the case that the existence of normative influence, or other such factors, imply that a greater level of social conformity is consistent with rational behaviour. We explore this possibility further in two examples later in the paper.

1.2 Outline of the paper

In the following, we first introduce the framework of noncooperative pregames, games derived from pregames and the crucial 'large game' property (continuity and global interaction). A simple example illustrating the framework is presented at the end of the section. The next section presents two lemmas used in our purification results. The fourth section states and proves Theorem 1 (existence of approximate equilibrium in pure strategies) and the fifth section states and proves Theorem 2 (social conformity). In addition, in the fifth section, convex separation is defined and two additional lemmas, used in the proof of Theorem 2, are provided. A subsection continues the prior example to illustrate an equilibrium satisfying social conformity. A second subsection returns to the discussion of social conformity and provides two further examples. The sixth section provides and example of a compact metric space of attributes satisfying convex separation. The final two sections contain a discussion of relationships to the literature and then conclusions. An Appendix demonstrates that finite dimensional Euclidean space satisfies convex separation.

2 Noncooperative Pregames

Let Ω denote a set of *player attributes*, let S be a set of *pure strategies* and let $\Delta(S)$ denote the set of *mixed strategies*. A function w from $\Omega \times S$ into $\mathbb{R}_+$ is said to be a *weight function* if it satisfies $\sum_{s_k \in S} w(\omega, s_k) \in \mathbb{Z}_+$ for all $\omega \in \Omega$.[5] Let W denote the set of all weight functions.

A *pregame* is given by a triple

$$\mathcal{G} = (\Omega, S, h),$$

consisting of a metric space Ω of attributes, a set S of pure strategies and a function $h : \Omega \times \Delta(S) \times W \longrightarrow \mathbb{R}_+$. Throughout the following, we will assume that Ω is a compact metric space and S is a finite set, $S = \{s_1, ..., s_K\}$.

While a pregame is defined independently of any particular game, some interpretation here of the components of a pregame may be helpful to the reader. In the following section, we develop these ideas formally. First, given a finite player set N and an attribute function, assigning attributes to each player in N, the pregame induces a game on N. In such a game, each player will have the strategy choice set $\Delta(S)$. A strategy choice for each player in the game determines a weight function. The function h determines the payoff for each player in the game. The payoff to a player depends on the attributes of that player, his strategy choice, and the weight function induced by the strategy choices of the other players.

2.1 Populations and games

Let N be a finite set and let α be a mapping from N to Ω, called an *attribute function*. The pair (N, α) is a *population*. The *profile* of the population (N, α) is a function *profile*$(N, \alpha) : \Omega \to \mathbb{Z}_+$ given by

$$\mathit{profile}(N, \alpha)(\omega) = \left|\alpha^{-1}(\omega)\right|$$

Thus, the profile of a population tells us the number of players with each attribute in the population.

We say that a weight function w_α *corresponds* to population (N, α) when it satisfies

$$\sum_{s_k \in S} w_\alpha(\omega, s_k) = \mathit{profile}(N, \alpha)(\omega)$$

[5] Where $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote, respectively, the non-negative real numbers and non-negative integers.

for all $\omega \in \Omega$. In other words, given an attribute function α, a corresponding weight function is an assignment of a non-negative real number to each attribute-pure strategy pair (ω, s_k) so that the sum, over pure strategies, of the weights assigned to the pairs (ω, s_k) equals the number of players with that attribute. It follows that

$$\sum_{s_k \in S} \sum_{\omega \in \alpha(N)} w_\alpha(\omega, s_k) = |N| \, .$$

We let W_α denote *the set of weight functions corresponding to the population* (N, α).

A *strategy vector* for population (N, α) is given by

$$\sigma = (\sigma_1, ..., \sigma_{|N|}) \in \times_{i \in N} \Delta(S)$$

where σ_i denotes the *strategy of player* i. Let σ_{ik} denote the probability that player i plays pure strategy s_k. A strategy vector m is called *degenerate* if for each i, for some k, $m_{ik} = 1$; that is, each player's strategy assigns probability 1 to some pure strategy.

Given a population (N, α) and a strategy vector σ for the population (N, α) we say that weight function $w_{\alpha,\sigma}$ is *relative to strategy vector* σ *and attribute function* α if,

$$w_{\alpha,\sigma}(\omega, s_k) = \sum_{i \in N : \, \alpha(i) = \omega} \sigma_{ik}$$

for all $s_k \in S$ and all $\omega \in \Omega$. It is immediate that $w_{\alpha,\sigma}$ so defined is a weight function corresponding to population (N, α). We interpret $w_{\alpha,\sigma}(\omega, s_k)$, as stating, for each $\omega \in \Omega$, the total weight given to pure strategy s_k in the strategy vector σ by players assigned attribute ω by α. Thus, given the population (N, α) and the strategy vector σ,

$$\frac{w_{\alpha,\sigma}(\omega, s_k)}{\textit{profile}(N, \alpha)(\omega)}$$

is the expected proportion of times pure strategy s_k will be played by a player of attribute ω.

Every weight function $w \in W$ is relative to an essentially unique attribute function and a not-necessarily-unique strategy vector. Formally, if a weight function w corresponds to both population (N, α) and to population $(N, \overline{\alpha})$ then, in view of the identity $\sum_{s_k \in S} w(\omega, s_k) = \textit{profile}(N, \alpha)(\omega) = \textit{profile}(N, \overline{\alpha})(\omega)$ for all $\omega \in \Omega$, the two populations can differ only by a permutation of the attributes of the players in N; thus, up to a permutation of attributes of players, an attribute function is uniquely determined by the weight function w. If *profile*$(N,\alpha)(\omega) > 1$ for some $\omega \in \Omega$ then a weight function w corresponding to population (N,α) may be relative to many different strategy vectors. For example, if $\alpha(i) = \alpha(j) = \omega$ and $w(\omega, s_1) = w(\omega, s_2) = 1$, it may be that player i plays s_1 with probability 1 and player j plays s_2 with probability 1 or each of the two players may play each of the two strategies with probability 1/2.

Before defining a game, it will be convenient to introduce one further definition. Given population (N,α) and player $i \in N$, define α_{-i} as the restriction of α to $N\backslash\{i\}$. Let $w_{\alpha_{-i},\sigma}$ be a weight function defined by its components as follows: for all $\omega \in \Omega$ and for all $\sigma_k \in S$

$$w_{\alpha_{-i},\sigma}(\omega, s_k) \stackrel{\text{def}}{=} \begin{cases} w_{\alpha,\sigma}(\omega, s_k) - \sigma_{ik} \text{ if } \alpha(i) = \omega \\ w_{\alpha,\sigma}(\omega, s_k) \text{ otherwise.} \end{cases}$$

Weight functions modified by the property that one player of some particular attribute is not included will play a role in the definition of games. We will use $W_{\alpha-\omega}$ to denote the set of weight functions corresponding to $(N\backslash\{i\}, \alpha_{-i})$ where $\omega = \alpha(i)$. Clearly, the set $W_{\alpha-\omega}$ is well-defined because, prior to the choice of strategies, players of the same attribute are anonymous.

2.1.1 Induced games

Given a population (N,α), a *game*

$$\Gamma(N,\alpha) = ((N,\alpha), S, \{h_\omega : \Delta(S) \times W_{\alpha-\omega} \longrightarrow \mathbb{R}_+ | \omega \in \alpha(N)\})$$

is induced from the pregame (Ω, S, h) by defining, for each $\omega \in \alpha(N)$,

$$h_\omega(t, w) \stackrel{\text{def}}{=} h(\omega, t, w)$$

for all $t \in \Delta(S)$ and all $w \in W_{\alpha-\omega}$. In interpretation, $h_{\alpha(i)}(t, w)$ is the payoff received by a player $i \in N$ of attribute $\alpha(i)$ from playing the strategy t when the strategies of other players are summarized by w. Note that players of the same attribute have the same payoff function, inherited from the pregame. Also note that the preferences of a player are partly determined by the population in which he lives. In particular, different attribute functions, say α and $\overline{\alpha}$, allow different supports for the weight functions $w \in W_{\alpha-\omega}$ and $w \in W_{\overline{\alpha}-\omega}$.

An anonymity assumption is implicit in defining the payoff $h_\omega(t, w)$ to depend on a weight function w rather than on the strategy choices of all members of the complementary player set. For example, consider two players $i, j \in N$, where $a(i) = \alpha(j)$, and two alternative scenarios. In the first scenario player i chooses pure strategy s_1 and player j chooses pure strategy s_2. In the second scenario, roles are reversed so that player i chooses s_2 and player j chooses s_1. Then, assuming everything else remains the same, the payoff to a third player $i' \in N$ is unaffected to this switch between i and j.

We make two standard assumptions. First, for all $i \in N$, player i's payoff to a mixed strategy p is a linear function of player i's mixing probability, that is,[6]

$$h_{\alpha(i)}(p, w_{\alpha_{-i},\sigma}) = \sum_{k=1}^{K} p_k h_{\alpha(i)}(s_k, w_{\alpha_{-i},\sigma}). \tag{1}$$

Second, we assume that, for all $i \in N$, player i's payoff to a mixed strategy vector is a linear function of the mixing probability of the other $N \backslash \{i\}$ players strategies; that is, for any three strategy vectors $\sigma, \sigma', \sigma''$, if

$$w_{\alpha_{-i},\sigma} = \lambda w_{\alpha_{-i},\sigma'} + (1-\lambda) w_{\alpha_{-i},\sigma''}$$

for some $\lambda \in [0,1]$ then,[7]

$$\begin{aligned} &h_{\alpha(i)}(p, w_{\alpha_{-i},\sigma}) \\ &= \lambda h_{\alpha(i)}(p, w_{\alpha_{-i},\sigma'}) + (1-\lambda) h_{\alpha(i)}(p, w_{\alpha_{-i},\sigma''}) \end{aligned} \tag{2}$$

for all $p \in \Delta(S)$.

Neither (1) or (2) are necessary. One motivation for using these assumptions is that they guarantee the existence of a Nash equilibrium strategy vector, possibly mixed. More general conditions are sufficient to guarantee such existence; see, for example, Reny (1999). The analysis of the current paper does not, however, even require the existence of a Nash equilibrium, but merely the existence of a mixed ε-Nash equilibrium for arbitrarily small, put positive, ε. We later confine our attention to games induced from a pregame satisfying a 'large game' property. It would appear that for any $\varepsilon > 0$ there exists a real number $\eta(\varepsilon)$ such that any game $\Gamma(N, \alpha)$ induced from a pregame satisfying the large game property and where $|N| > \eta(\varepsilon)$ has an ε-Nash equilibrium. The proof of this follows by showing that any game

[6] Here, for all k, s_k denotes the mixed strategy assigning unit weight to pure strategy s_k and p_k denotes the probability player i plays pure strategy s_k.

[7] Given the definition of a weight function, we note that if $\lambda_1 \in [0,1]$ then the sum $\lambda_1 w^1_{\alpha_{-i}}(\cdot,\cdot) + (1-\lambda_1) w^2_{\alpha_{-i}}(\cdot,\cdot)$, defined pointwise, gives a weight function $w_{\alpha_{-i}}(\cdot,\cdot) \in W_{\alpha_{-i}}$.

$\Gamma(N, \alpha)$ can be approximated by a game $\Gamma^A(N, \alpha)$ that is known to have a Nash equilibrium (a technique analogous to that used in Reny 1999).[8]

A second motivation for (1) is to enable purification. In particular, we need to rule out the possibility that a player's payoff to a mixed strategy is significantly higher than the payoff to the corresponding pure strategies in the support of that mixed strategy. We could, however, relax the condition to one stating that for any $\varepsilon > 0$ there exists some $\eta(\varepsilon)$ such that for all games $\Gamma(N, \alpha)$ where $|N| > \eta(\varepsilon)$ and for any player $i \in N$ player i's payoff to any mixed strategy $p \in \Delta(S)$ and any weight function $w \in W_\alpha$ is such that

$$h_{\alpha(i)}(p, w_{\alpha_{-i},\sigma}) \leq \sum_{k=1}^{K} p_k h_{\alpha(i)}(s_k, w_{\alpha_{-i},\sigma}) + \varepsilon.$$

We also note that the restriction to a finite number of strategies appears unnecessary. With these observations noted, for simplicity, we continue to use the standard assumptions as outlined above. Future research aims to look further at these issues.

2.1.2 Nash equilibrium

The standard definition of a Nash equilibrium applies. Let $\Gamma(N, \alpha)$ be a game. A strategy vector σ is a *Nash equilibrium* for $\Gamma(N, \alpha)$ only if, for each $i \in N$ it holds that

$$h_{\alpha(i)}(\sigma_i, w_{\alpha_{-i},\sigma}) \geq h_{\alpha(i)}(t, w_{\alpha_{-i},\sigma}) \text{ for all } t \in \Delta(S). \tag{3}$$

Given $\varepsilon \geq 0$, a strategy vector m is a *Nash ε-equilibrium in pure strategies* or, informally, *an approximate Nash equilibrium in pure strategies,* only if, for each $i \in N$, m_i is degenerate and

$$h_{\alpha(i)}(m_i, w_{\alpha_{-i},m}) \geq h_{\alpha(i)}(t, w_{\alpha_{-i},m}) - \varepsilon \text{ for all } t \in \Delta(S). \tag{4}$$

The notion of ε-purification was first introduced by Aumann et. al. (1983) in a different context to that of this paper (see Section 7). They define the notion of *ε-purification* as follows; given a game $\Gamma(N, \alpha)$, two strategies p and t are *ε-equivalent* for player i if for any weight function

[8] Such techniques have also been used in the study of cooperative pregames. For example, Wooders (1983) exploits the idea that cooperative games can be approximated by games satisfying "strong comprehensiveness", the property that boundaries of payoff sets do not contain segments parallel to the boundaries of $\mathbb{R}_+^S$ for any coalition S.

$w \in W_{\alpha-\alpha(i)}$, $\left|h_{\alpha(i)}(p, w) - h_{\alpha(i)}(t, w)\right| < \varepsilon$. An *$\varepsilon$-purification* of a strategy p is a pure strategy that is ε-equivalent to p. This definition proves too restrictive to be useful in the results of this paper. As such, we use a more relaxed definition of purification in which ε-equivalence is defined relative to strategy vectors rather than strategies. Formally, given a game $\Gamma(N, \alpha)$ we say that two strategy vectors σ and m are *ε-equivalent* if, for all $i \in N$,

$$\left|h_{\alpha(i)}(m_i, w_{\alpha_{-i},m}) - h_{\alpha(i)}(\sigma_i, w_{\alpha_{-i},\sigma})\right| < \varepsilon.$$

We say that a strategy vector σ can be *ε-purified* if there exists a strategy vector m which is degenerate and ε-equivalent to σ.

2.2 Large games

The following two concepts of continuity of payoff functions and global interaction allow us to introduce the *large game property.* This property places restrictions on the payoff function h of a pregame $\mathcal{G} = (\Omega, S, h)$. As a preliminary step, let $\gamma(\mathcal{G}, n)$ denote the set of games induced by the pregame $\mathcal{G}$ by populations of size n. That is, game $\Gamma(N, \alpha) \in \gamma(\mathcal{G}, n)$ if and only if $|N| = n$. We proceed with the two definitions and statement of the large game property before providing more discussion.

Continuity of payoff functions: Given positive real numbers $\varepsilon > 0$ and $\delta > 0$, the set of games $\gamma(\mathcal{G}, n)$ is said to satisfy *δ, ε-continuity of payoff functions* when for any two games $\Gamma(N, \alpha)$ and $\Gamma(N, \overline{\alpha})$ in $\gamma(\mathcal{G}, n)$, if, for all $i \in N$,

$$dist(\alpha(i), \overline{\alpha}(i)) < \delta$$

then, for any $j \in N$ with $\alpha(j) = \overline{\alpha}(j)$ and for any strategy vector σ,

$$\left|h_{\alpha(j)}(t, w_{\alpha_{-j},\sigma}) - h_{\alpha(j)}(t, w_{\overline{\alpha}_{-j},\sigma})\right| < \varepsilon$$

for all $t \in \Delta(S)$, where $w_{\alpha,\sigma}$ and $w_{\overline{\alpha},\sigma}$ are the weight functions relative to strategy vector σ and, respectively, attribute functions α and $\overline{\alpha}$.

Global interaction: Given positive real numbers $\varepsilon > 0$ and $\delta > 0$ the game $\Gamma(N, \alpha)$ is said to satisfy *δ, ε-global interaction* when for any two weight functions w_α and g_α, both relative to attribute function α, if,

$$\frac{1}{|N|} \sum_{s_k \in S} \sum_{\omega \in \alpha(N)} |w_\alpha(\omega, s_k) - g_\alpha(\omega, s_k)| < \delta$$

then,

$$\left|h_{\alpha(i)}(t, w_{\alpha_{-i}}) - h_{\alpha(i)}(t, g_{\alpha_{-i}})\right| < \varepsilon \tag{5}$$

for all $i \in N$ and all $t \in \Delta(S)$.

We can now introduce our main assumption,

Large game property: The pregame $\mathcal{G} = (\Omega, S, h)$ satisfies the *large game property* (with continuity of payoff functions) if for any $\varepsilon > 0$ there exists real numbers $\eta(\varepsilon)$, $\delta_c(\varepsilon) > 0$ and $\delta_g(\varepsilon) > 0$ such that for any $n > \eta(\varepsilon)$ the set of games $\gamma(\mathcal{G}, n)$ satisfies $\delta_c(\varepsilon), \varepsilon$-*continuity of payoff functions* and any game $\Gamma(N, \alpha) \in \gamma(\mathcal{G}, n)$ satisfies $\delta_g(\varepsilon), \varepsilon$-*global interaction*.

Thus, the pregame $\mathcal{G}$ satisfies the large game property if both global interaction and continuity of payoff functions are satisfied by large games. As mentioned above, the assumption of a large game property is an assumption on a pregame $\mathcal{G} = (\Omega, S, h)$ and in particular, on the function h. Specifically, the large game property implies a form of continuity of $h(\omega, t, w)$ with respect to changes in the weight function w while the attribute ω and strategy t remain constant. To appreciate the importance of this continuity we have to look at the games induced from the pregame (Ω, S, h).

Let us consider in more detail the assumption of continuity in payoff functions for large games. We can begin our explanation by imagining a population (N, α) and a change in the attribute function from α to $\overline{\alpha}$. As the attribute function changes so does the population, from (N, α) to $(N, \overline{\alpha})$, and the game from $\Gamma(N, \alpha)$ to $\Gamma(N, \overline{\alpha})$. Both games belong to the set $\gamma(\mathcal{G}, n)$ where $|N| = n$. Focus now on a player i who maintains the same attribute in both societies (N, α) and $(N, \overline{\alpha})$, that is $\alpha(i) = \overline{\alpha}(i)$. The payoff function of player i remains unchanged as h_ω. The set of weight functions that player i may face has, however, changed from $W_{\alpha_{-i}}$ to $W_{\overline{\alpha}_{-i}}$. It seems reasonable, though, given that the attribute function changes only slightly, that player i would feel as though the game has changed only slightly. This is the intuition formalized in the assumption of continuity of payoff functions. In particular, it dictates that, for any given strategy vector, if we change the attributes of some players only slightly, then for any player whose attribute is unchanged, the change in payoff is small. A key point to appreciate is how the existence of a pregame $\mathcal{G} = (\Omega, S, h)$ allows us to compare two different games $\Gamma(N, \alpha)$ and $\Gamma(N, \overline{\alpha})$ both induced from that pregame $\mathcal{G}$. We also highlight that, in the statement of the assumption, the strategy vector σ

remains unchanged. That is, while the attributes of the players may have changed (for player j they change from $\alpha(j)$ to $\overline{\alpha}(j)$) the *strategies* of the players remain unchanged. This is possible since a strategy vector lists a strategy for each player $i \in N$ and the set N remains unchanged.

We should perhaps explain further why continuity of payoff functions seems a relatively mild assumption. If one finds it reasonable that the payoff functions of players are affected only by the *actions* of others and not by their attributes, then the assumption of continuity in the attributes of others is very mild indeed. In reality, a player's payoff may depend on the attributes of others. To appreciate this we note, that an attribute can be seen to contain two pieces of information - a player's payoff function and a player's personal characteristics in terms of skills, physical appearance, etc.[9] (We discuss the nature of attributes in more detail in Section 6.) There are, therefore, two broad reasons why a player's payoff may be affected by the attributes of others. First, a player's payoff may depend on the payoff functions, or preferences, of other players.[10] Second, a player's payoff may depend on the personal characteristics of other players. For example, if someone offers to make you lunch it might matter a great deal how well he can cook. Both of these possibilities can be expected to be present in a broad range of situations. We only need consider, however, small changes in the attributes of other players. It seems unlikely that such small changes in attributes could lead to significant effects on a player's payoff when we, once more, recall that the strategies of players do not change.

Let us now turn to the second assumption of global interaction in large games. This assumption does not consider a change in the game $\Gamma(N, \alpha)$ but merely a change in the strategies of players within the game. Informally, global interaction dictates that a player is nearly indifferent to small changes in the *proportion*, relative to the total population of players, of players of each attribute playing each strategy. The term global interaction is used in the evolutionary game theory literature to describe a scenario in which

[9] A similar distinction is made in the literature of clubs and economies with local public goods, where these personal characteristics are called *crowding types* (cf. Conley and Wooders 1996,2001).

[10] There are situations where individuals claim to be affected by the feelings, loyalties or thoughts of others, independent of their actions. In Arthur Miller's celebrated book, *The Crucible*, Rachel has been a pious woman, known for her good deeds and kind works, all through her long life. But the witch hunters of Salem interpreted Rachel's apparent goodness as just a clever disguise to hide her love of the devil. Rachel was put to death as a witch; for witch hunters, the private feelings of others and their thoughts are significant.

every player plays against every other player.[11] If a player interacts with all members of the population equally then we would expect that his payoff should be dependent only on his own strategy and on the 'aggregate strategy of the population'. Furthermore, we would expect that a player is nearly indifferent to small changes in such an 'aggregate strategy of the population'. This is of the type of game considered in this paper.

2.3 A simple example

The following example illustrates the pregame framework and the large game property. Consider an economy in which there are two goods -'A' and 'B'. Each player has a unit of time which they devote to producing goods. There are two pure strategies - 'to spend all ones time producing A' or 'to spend ones time producing B'. We subsequently refer to these pure strategies as s^A and s^B.

The attribute space is given by $\Omega = [0,1] \times [0,1]$. If a player i has an attribute $\omega = (\omega^A, \omega^B)$ then the value ω^A is interpreted as the amount of good A player i produces if he spends all his time producing good A and similarly ω^B is interpreted as the amount of good B player i produces if he spends all his time producing good B.

Given any population (N, α) and any weight function w_α corresponding to α the expected amount of good A that will be produced is given by,

$$x^A(w_\alpha) \stackrel{\text{def}}{=} \sum_{\omega=(\omega^A,\omega^B)\in\alpha(N)} w_\alpha(\omega, s^A) \times \omega^A$$

and the expected amount of good B that will be produced is given by,

$$x^B(w_\alpha) \stackrel{\text{def}}{=} \sum_{\omega=(\omega^A,\omega^B)\in\alpha(N)} w_\alpha(\omega, s^B) \times \omega^B.$$

Payoff functions are assumed to be the same for all $\omega \in \Omega$ and for all finite populations. Let (N, α) be any population and let player i be a member of N. Then the payoff function of player i is formally defined by

$$h_{\alpha(i)}(p, w_\alpha) = \frac{x^A(w_{\alpha_{-i}}) + x^B(w_{\alpha_{-i}}) + p^A\omega^A + p^B\omega^B}{|N|}$$

[11] The term is also used for the essentially identical situation in which the matching is random. That is, a player may only face one opponent but this opponent is randomly drawn from the remaining population.

where $\alpha(i) = (\omega^A, \omega^B)$ and where p^A and p^B denote, respectively, the probability that player i plays pure strategies s^A and s^B. Thus, a player's payoff is given by the per capita expected combined production of goods A and B. Our final task is to put a metric to the space Ω. We use the metric ρ_0 defined by:

$$\rho_0(\omega, \overline{\omega}) = \max\{\left|\omega^A - \overline{\omega}^A\right|, \left|\omega^B - \overline{\omega}^B\right|\}$$

for any $\omega = (\omega^A, \omega^B), \overline{\omega} = (\overline{\omega}^A, \overline{\omega}^B) \in \Omega$. This completes the definition of a pregame $\mathcal{G}$. It remains to show that this pregame satisfies the large game property.

For any two games $\Gamma(N, \alpha)$ and $\Gamma(N, \overline{\alpha})$ in $\gamma(\mathcal{G}, n)$, if, for all $i \in N$ it holds that

$$\rho_0(\alpha(i), \overline{\alpha}(i)) < \varepsilon$$

then, for any $j \in N$ and for any strategy vector σ,

$$\left|h_{\alpha(j)}(t, w_{\alpha_{-j},\sigma}) - h_{\overline{\alpha}(j)}(t, w_{\overline{\alpha}_{-j},\sigma})\right| < \varepsilon$$

for all $t \in \Delta(S)$. For any finite population (N, α) and for any two weight functions w_α and g_α, both corresponding to population (N, α), if

$$\frac{1}{|N|} \sum_{s_k \in S} \sum_{\omega \in \alpha(N)} |w_\alpha(\omega, s_k) - g_\alpha(\omega, s_k)| < \varepsilon$$

then

$$\left|h_{\alpha(i)}(t, w_{\alpha_{-i}}) - h_{\alpha(i)}(t, g_{\alpha_{-i}})\right| < \varepsilon \qquad (6)$$

for all $i \in N$ and all $t \in \Delta(S)$. Thus, by setting $\eta(\varepsilon) = 1$ and $\delta_c(\varepsilon) = \delta_g(\varepsilon) = \varepsilon$ for any $\varepsilon > 0$, it is apparent that this pregame satisfies the large game property.

3 Two Lemmas

This section states two lemmas. With these two lemmas in hand, in the following section, we prove our purification result and then, in the subsequent section, we prove our social conformity result. It is worth noting that our purification result could be proved using the Shapley-Folkman Theory (see, for example, Green and Heller 1991) and without need for the following the two lemmas. The method for such a proof becomes apparent from an understanding of the proofs to the theorems in this paper and those of Rashid (1983). The reasons for introducing the following two lemmas, and in particular Lemma 1, are two-fold. First, these results seem interesting in their own right. Indeed, Lemma 1 may be an extension of the Shapley-Folkman Theorem.[12] Second, the two lemmas provide a much more 'direct' way of proving Theorem 1 and thus ultimately simplify the proof.

Our two lemmas show that given a population (N, α) and any mixed strategy vector σ, there exists a degenerate strategy vector m such that (i) each player $i \in N$ plays a pure strategy s_k in the support of σ_i, and (ii) the aggregate weighting to strategies remains nearly unchanged; specifically,

$$\frac{|g_{\alpha,m}(\omega, s_k) - w_{\alpha,\sigma}(\omega, s_k)|}{|N|} < \frac{K}{|N|}$$

for all $\omega \in \Omega$ and $s_k \in S$, where $w_{\alpha,\sigma}$ is the weight function relative to strategy vector σ and attribute function α and $g_{\alpha,m}$ is the weight function relative to strategy vector m and attribute function α.

We need some additional notation. Let $a = (a_1, ..., a_n),\ b = (b_1, ..., b_n) \in \mathbb{R}^n$. We write $a \geq b$ if and only if $a_i \geq b_i$ for all $i = 1, ..., n$. Let $\mathbb{Z}_+^K$ denote the set of K dimensional vectors for which each component is a non-negative integer. When we apply Lemma 1 to prove our Theorems, we treat the set N as a subset of players all with the same attribute $\omega \in \Omega$.

Lemma 1: Let $N = \{1, ..., n\}$ be a finite set. For any strategy vector $\sigma = (\sigma_1, ..., \sigma_n) \in \Delta^{Kn}$ and for any vector $\overline{g} \in \mathbb{Z}_+^K$ such that $\sum_i \sigma_i \geq \overline{g}$, there exists a degenerate strategy vector $m = (m_1, ..., m_n)$ such that,

$$support(m_i) \subseteq support(\sigma_i)$$

for all $i \in N$ and

$$\sum_{i \in N} m_i \geq \overline{g}.$$

The proofs of Lemmas 1 and 2 are presented in the Appendix.

Weight functions corresponding to degenerate strategies play an important role in our proofs. A weight function $w_\alpha \in W_\alpha$ is called *integer-valued*

[12] This issue is taken further in Cartwright and Wooders (2001).

if $w_\alpha(\omega, s_k) \in \mathbb{Z}_+$ for each $\omega \in \Omega$ and each $s_k \in S$. We typically denote an integer-valued weight function by g_α. If g_α is an integer-valued weight function there exists a strategy vector m such that m_i is degenerate for all players $i \in N$ and $g_\alpha(\omega, s_k) = g_{\alpha,m}(\omega, s_k)$ for all $s_k \in S$ and all $\omega \in \Omega$. Moreover, every degenerate strategy vector m generates an integer-valued weight function. Given a degenerate strategy vector m, the interpretation is that, for each attribute ω and strategy $s_k \in S$, $g_{\alpha,m}(\omega, s_k)$ denotes the number of players i in N with attribute ω whose strategy m_i places weight 1 on pure strategy s_k.

Lemma 2: Let σ denote a strategy vector for population (N, α) and let $w_{\alpha,\sigma}$ denote the weight function relative to σ and α. Then there exists a degenerate strategy vector m and weight function $g_{\alpha,m}$ relative to m and α, with the properties that:

$$support(m_i) \subseteq support(\sigma_i)$$

for all $i \in N$ and

$$|g_{\alpha,m}(\omega, s_k) - w_{\alpha,\sigma}(\omega, s_k)| < K$$

for all $\omega \in \Omega$ and all $s_k \in S$.

4 Existence of ε-equilibrium in pure strategies

In the following Theorem, we demonstrate that, given $\varepsilon > 0$ there is an integer η_1 sufficiently large so that every game $\Gamma(N, \alpha)$ induced from a pregame with the large game property and such that $|N| > \eta_1$ has a Nash ε-equilibrium in pure strategies. To obtain this result, at a point in the proof we arbitrarily select a Nash equilibrium for each game in a sequence and show that if there are sufficiently many players, this Nash equilibrium can be used to construct a Nash ε-equilibrium in pure strategies. Since the selection of the Nash equilibrium was arbitrary, our result can be viewed as a purification theorem on the set of Nash equilibrium. In fact, it is clear from the proof that any strategy vector can be purified.

Theorem 1: If a pregame $\mathcal{G}$ satisfies the large game property then for any real number $\varepsilon > 0$ there exists an integer $\eta_1(\varepsilon)$ such that for any population (N, α) where $|N| > \eta_1(\varepsilon)$ the induced game $\Gamma(N, \alpha)$ has a Nash ε-equilibrium in pure strategies. Moreover, any strategy vector of the game $\Gamma(N,\alpha)$ can be ε-purified.

Proof: Suppose that the statement of the Theorem is false. Then there is some $\varepsilon > 0$ such that, for each integer ν there is a population (N^ν, α^ν) and induced game $\Gamma(N^\nu, \alpha^\nu)$ with $|N^\nu| > \nu$ for which there does not exist an ε-equilibrium in pure strategies. That is, for each induced game $\Gamma(N^\nu, \alpha^\nu)$ there does not exist a degenerate strategy vector m^ν, with corresponding integer-valued weight function $g^\nu_{\alpha^\nu,m^\nu}$ such that:

$$h_{\alpha(i)}(m_i^\nu, g^\nu_{\alpha^\nu_{-i},m^\nu}) \geq h_{\alpha(i)}(t, g^\nu_{\alpha^\nu_{-i},m^\nu}) - \varepsilon \tag{7}$$

for all $t \in \Delta(S)$ and for all $i \in N^\nu$.

From Nash's well known theorem, for each ν, the game $\Gamma(N^\nu, \alpha^\nu)$ has a mixed strategy Nash equilibrium. Denote a Nash equilibrium of the game $\Gamma(N^\nu, \alpha^\nu)$ by σ^ν and the relative weight function by $w^\nu_{\alpha^\nu,\sigma^\nu}$. Since σ^ν is a Nash equilibrium, for each ν and for each $i \in N^\nu$ we have:

$$h_{\alpha^\nu(i)}(\sigma_i^\nu, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) \geq h_{\alpha^\nu(i)}(t, w^\nu_{\alpha^\nu_{-i},\sigma^\nu})$$

for all $t \in \Delta(S)$ and

$$h_{\alpha^\nu(i)}(p, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) \geq h_{\alpha^\nu(i)}(t, w^\nu_{\alpha^\nu_{-i},\sigma^\nu})$$

for all $t \in \Delta(S)$ and for all p where *support*$(p) \subset$ *support*(σ_i^ν).

Since the pregame $\mathcal{G}$ satisfies the large game property, we may choose non-negative real numbers $\delta_c\left(\frac{\varepsilon}{6}\right), \delta_g\left(\frac{\varepsilon}{6}\right)$ and $\eta\left(\frac{\varepsilon}{6}\right)$ such that for any $n > \eta(\frac{\varepsilon}{6})$ the set of games $\gamma(\mathcal{G}, n)$ satisfies $\delta_c(\frac{\varepsilon}{6}), (\frac{\varepsilon}{6})$-continuity of payoff functions and any game $\Gamma(N,\alpha) \in \gamma(\mathcal{G}, n)$ satisfies $\delta_g(\frac{\varepsilon}{6}), (\frac{\varepsilon}{6})$-global interaction.

Set $\delta = \delta_c\left(\frac{\varepsilon}{6}\right)$. Use compactness of Ω to write Ω as the disjoint union of a finite number of non-empty subsets $\Omega_1, ..., \Omega_A$, each of diameter less than δ.[13] For each a, choose and fix a point $\omega_a \in \Omega_a$.

For all ν and for all $i \in N^\nu$ we define the attribute function $\overline{\alpha}^\nu$ as follows:

$$\overline{\alpha}^\nu(i) = \omega_a \text{ if and only if } \alpha(i) \in \Omega_a.$$

For all ν, let $w^\nu_{\overline{\alpha}^\nu,\sigma^\nu}$ denote the weight function relative to attribute function $\overline{\alpha}^\nu$ and strategy vector σ^ν.

[13] A subset A of (the metric space) Ω has diameter δ, if, for any $\omega, \omega' \in A$, *distance*$(\omega, \omega') \leq \delta$.

For each $a = 1, ..., A$ and for each $k = 1, .., K$ define θ^{ν}_{ak} as follows:

$$\theta^{\nu}_{ak} = \frac{w^{\nu}_{\overline{\alpha}^{\nu},\sigma^{\nu}}(w_a, s_k)}{|N^{\nu}|}.$$

By passing to a subsequence if necessary, assume that the $\lim_{\nu\to\infty} \theta^{\nu}_{ak} = \theta_{ak}$ exists for all $a = 1, .., A$ and all $k = 1, ..., K$.

By Lemma 2 there exists a sequence $\{m^{\nu}\}$ of degenerate strategy vectors and a sequence $\{g^{\nu}_{\overline{\alpha}^{\nu},m^{\nu}}\}$ of integer-valued weight functions where $g^{\nu}_{\overline{\alpha}^{\nu},m^{\nu}}$ is relative to attribute function $\overline{\alpha}^{\nu}$ and strategy vector m^{ν}, such that:

1. for all ν and for all $s_k \in S$ and all $\omega_a \in \Omega$,

$$\lim_{\nu\to\infty} \frac{g^{\nu}_{\overline{\alpha}^{\nu},m^{\nu}}(w_a, s_k)}{|N^{\nu}|} = \lim_{\nu\to\infty} \frac{g^{\nu}_{\overline{\alpha}^{\nu},m^{\nu}}(w_a, s_k) - 1}{|N^{\nu}|} = \theta_{ak} \qquad (8)$$

2. for all ν and for all $i \in N^{\nu}$, $support(m^{\nu}_i) \subset support(\sigma^{\nu}_i)$.

For all ν, given the weight function $g^{\nu}_{\overline{\alpha}^{\nu},m^{\nu}}$ relative to attribute function $\overline{\alpha}^{\nu}$ and strategy vector m^{ν}, let $g^{\nu}_{\alpha^{\nu},m^{\nu}}$ denote the integer valued weight function relative to attribute function α^{ν} and strategy vector m^{ν}.

Restrict attention to those populations (N^{ν}, α^{ν}) where $|N^{\nu}| > \eta\left(\frac{\varepsilon}{6}\right)$. Consider the change in payoff to a player $i \in N^{\nu}$ from the change in strategy vector from σ^{ν} to m^{ν}. By the assumption of continuity of payoff functions and the choice of δ it holds that:

$$\left| h^{\nu}_{\alpha^{\nu}(i)}(t, w^{\nu}_{\overline{\alpha}^{\nu}_{-i},\sigma^{\nu}}) - h^{\nu}_{\alpha^{\nu}(i)}(t, w^{\nu}_{\alpha^{\nu}_{-i},\sigma^{\nu}}) \right| < \frac{\varepsilon}{6}$$

for all $t \in \Delta(S)$. Similarly,

$$\left| h^{\nu}_{\alpha^{\nu}(i)}(t, g^{\nu}_{\overline{\alpha}^{\nu}_{-i},m^{\nu}}) - h^{\nu}_{\alpha^{\nu}(i)}(t, g^{\nu}_{\alpha^{\nu}_{-i},m^{\nu}}) \right| < \frac{\varepsilon}{6}$$

for all $t \in \Delta(S)$.

In view of (8) and the assumption of global interaction there exists a ν_1 such that for all $\nu > \nu_1$,

$$\left| h^{\nu}_{\alpha^{\nu}(i)}(t, w^{\nu}_{\overline{\alpha}^{\nu}_{-i},\sigma^{\nu}}) - h^{\nu}_{\alpha^{\nu}(i)}(t, g^{\nu}_{\overline{\alpha}^{\nu}_{-i},m^{\nu}}) \right| < \frac{\varepsilon}{6}$$

for any $t \in \Delta(S)$.

Thus, for any $\nu > \nu_1$ and any $i \in N^\nu$ we have that:

$$\begin{aligned}
& \left| h^\nu_{\alpha^\nu(i)}(t, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) - h^\nu_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i},m^\nu}) \right| \\
\leq & \left| h^\nu_{\alpha^\nu(i)}(t, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) - h^\nu_{\alpha^\nu(i)}(t, w^\nu_{\overline{\alpha}^\nu_{-i},\sigma^\nu}) \right| \\
& + \left| h^\nu_{\alpha^\nu(i)}(t, w^\nu_{\overline{\alpha}^\nu_{-i},\sigma^\nu}) - h^\nu_{\alpha^\nu(i)}(t, g^\nu_{\overline{\alpha}^\nu_{-i},m^\nu}) \right| \\
& + \left| h^\nu_{\alpha^\nu(i)}(t, g^\nu_{\overline{\alpha}^\nu_{-i},m^\nu}) - h^\nu_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i},m^\nu}) \right| \\
< & \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}
\end{aligned}$$

for any $t \in \Delta(S)$. We recall, however, that

$$h^\nu_{\alpha^\nu(i)}(p, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) - h^\nu_{\alpha^\nu(i)}(t, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) \geq 0$$

for all p such that *support*(p) $\subset$*support*(σ_i^ν), for all $i \in N$, for all $t \in \Delta(S)$ and for all ν. Given that *support*(m_i^ν) $\subset$*support*(σ_i^ν) for all ν, this implies that for $\nu > \nu_1$ and for all $i \in N^\nu$,

$$\begin{aligned}
& h^\nu_{\alpha^\nu(i)}(m_i^\nu, g^\nu_{\alpha^\nu_{-i},m^\nu}) - h^\nu_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i},m^\nu}) \\
& \geq - \left| h^\nu_{\alpha^\nu(i)}(t, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) - h^\nu_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i},m^\nu}) \right| \\
& - \left| h^\nu_{\alpha^\nu(i)}(m_i^\nu, g^\nu_{\alpha^\nu_{-i},m^\nu}) - h^\nu_{\alpha^\nu(i)}(m_i^\nu, w^\nu_{\alpha^\nu_{-i},\sigma^\nu}) \right| \\
& > -\tfrac{\varepsilon}{2} - \tfrac{\varepsilon}{2} = -\varepsilon
\end{aligned}$$

for all $t \in \Delta(S)$ which contradicts the supposition that (7) does not hold and completes the proof.

We note that the equilibrium mixed strategy vector with which we started our proof was arbitrary. Thus, any Nash equilibrium can be ε-purified.■

Related literature is discussed in Section 7.

5 Social conformity

Besides permitting results such as Theorem 1 and various extensions, our framework has the advantage, as exemplified by the following social conformity theorem, that it allows us to address different questions than currently in the game-theoretic literature. The interpretation of our social conformity result also requires us to look more deeply at the attribute space. We proceed with the statement and proof of our conformity result and postpone discussion of indexing attributes to the next Section.

One additional assumption is required. Technically, this is a stronger version of continuity of payoff functions. The role of this assumption in our proofs is to allow us to construct a ε-Nash equilibrium by interchanging strategies of similar players. The statement of the assumption is:

Continuity in attributes: Given positive real numbers $\varepsilon > 0$ and $\delta > 0$, the set of games $\gamma(\mathcal{G}, n)$ is said to satisfy δ, ε*-continuity in attributes* when for any two games $\Gamma(N, \alpha)$ and $\Gamma(N, \overline{\alpha})$ in $\gamma(\mathcal{G}, n)$, if, for all $i \in N$,

$$dist(\alpha(i), \overline{\alpha}(i)) < \delta$$

then, for any $j \in N$ and for any strategy vector σ,

$$\left| h_{\alpha(j)}(t, w_{\alpha_{-j},\sigma}) - h_{\overline{\alpha}(j)}(t, w_{\overline{\alpha}_{-j},\sigma}) \right| < \varepsilon$$

for all $t \in \Delta(S)$, where $w_{\alpha,\sigma}$ and $w_{\overline{\alpha},\sigma}$ are the weight functions relative to strategy vector σ and, respectively, attribute functions α and $\overline{\alpha}$.

We extend the definition of the large game property in the obvious way, distinguishing between the large game property with continuity in attributes and the large game property with continuity of payoff functions. Note that continuity in attributes implies continuity of payoff functions.

The assumption of continuity in attributes for large games goes further than continuity of payoff functions in dictating that a player i is relatively unaffected by, not only a change in the weight function, but also a change in his own attribute, that is, a change in attribute from $\alpha(i)$ to $\overline{\alpha}(i)$. Thus, while the large game property (with continuity in attributes) still remains an assumption on the function h of a pregame $\mathcal{G} = (\Omega, S, h)$, it now implies a form of continuity of $h(\omega, t, w)$ with respect to changes in the attribute ω as well as changes in the weight function w. If the assumption of continuity of payoff functions is accepted, then this is a relatively mild extension that says a small change in attribute implies only a small change in the payoff function. Again the assumption of continuity in attributes demonstrates how the use of a pregame allows us to move between different games induced by that pregame.

For the purposes of our social conformity result we assume that Ω is a compact subset of a normed, real linear space. An example of such an attribute space is provided in the following section. The choice of attribute space allows us to treat convex subsets of Ω and to define a society. Before defining a society the issue of convexity requires further consideration. Throughout this paper we will treat convexity as a property relative to an attribute space Ω rather than the linear space of which Ω is a subset. Formally, given an attribute space Ω we say that $A \subset \Omega$ is convex when for any two points $a, b \in A$ and for any $\lambda \in [0, 1]$,

$$\begin{aligned} \text{if } \lambda a + (1-\lambda)b &\in \Omega \\ \text{then } \lambda a + (1-\lambda)b &\in A. \end{aligned}$$

To appreciate why this may be necessary suppose that $\Omega = \mathbb{Q} \cap [0, 1]$, i.e. the set of rationals on the unit interval. The above definition of convexity allows us to partition Ω into a finite number of convex subsets. This would not be possible under a standard definition of convexity .

Given a population (N, α) and strategy vector σ we interpret a set of players D as a society (relative to α and σ) if (i) there exists some strategy $t \in \Delta(S)$ such that $\sigma_i = t$ for all $i \in D$, and (ii) for any player $i \in N$ if $i \in$ *convex hull*$(\alpha(D))$ then $i \in D$. Thus, any two players belonging to a society D must play the same strategy. Furthermore, to any society D we can associate a convex subset Ω_D of attribute space Ω with the properties that any player i belonging to D has attributes in Ω_D while there exists no other player $j \in N \backslash D$ who has attributes in Ω_D.

We say that a strategy vector σ *induces a partition of the population* (N, α) into a set of societies $\mathcal{S} = \{N_1, ..., N_C\}$ if each player $i \in N$ belongs to a unique society $N_c \in \mathcal{S}$ and if each society $N_c \in \mathcal{S}$ is relative to α and σ. We also say that σ induces a partition of the population (N, α) into C societies. Thus, if σ induces a partition of the population (N, α) into C societies there exists a partition of Ω into C convex subsets $\{\Omega_c\}_{c=1}^{C}$ such that for any two players $i, j \in N$ and any c if $\alpha(i), \alpha(j) \in \Omega_c$ then $\sigma_i = \sigma_j$.

To illustrate the concept of a society we make two observations. First, for some populations (N, α) there may exist a strategy vector σ which does not induce a partition of the population (N, α) into societies. In particular, if there exists two players $i, j \in N$ such that $\alpha(i) = \alpha(j)$ but $\sigma_i \neq \sigma_j$ then the population (N, α) cannot be partitioned into societies. Conversely, if *profile*$(N, \alpha)(\omega) \in \{0, 1\}$ for all $\omega \in \Omega$ then any strategy vector σ partitions the population (N, α) into $|N|$ societies. That is, each individual constitutes a society.

An important aspect of the following result is that the population (N, α) can be partitioned into a bounded number of societies $A(\varepsilon, B)K$ and this bound is independent of the size of the total player set. The smaller the bound $A(\varepsilon, B)K$, the stronger social conformity, since the smaller $A(\varepsilon, B)K$, the more dissimilar the players in the same society.

Before stating Theorem 2 we require the following definition.

convex separation. We say that a linear space Ω satisfies convex separation if there exists a binary relation $<$ that is a strict linear order on Ω and such that, for any two finite sets of points $\Omega_J = \{\omega_1, ..., \omega_J\}$ and $\Omega_Q = \{\omega'_1, ..., \omega'_Q\}$, where $\omega_1, ..., \omega_J, \omega'_1, ..., \omega'_Q \in \Omega$, if,

$$\max_{\omega j \in \Omega_J} \{\omega_j\} < \min_{\omega'_q \in \Omega_Q} \{\omega'_q\}$$

then the convex hulls of the sets Ω_J and Ω_Q are distinct.[14]

It appears that convex separation, a new property, is satisfied by an interesting class of metric spaces. In the appendix we demonstrate that both finite and infinite dimensional Euclidean space satisfy convex separation (see Lemma 5).

We can now state and prove our second theorem:

Theorem 2: Given any real numbers $\varepsilon > 0$ and $B \geq 1$ there exists a real number $\eta_2(\varepsilon, B)$ and an integer $A(\varepsilon, B)$ such that, if a pregame $\mathcal{G}$ satisfies the large game property (with continuity in attributes) then for any population (N, α) where $|N| > \eta_2(\varepsilon, B)$ and *profile*$(N, \alpha)(\omega) \leq B$ for all $\omega \in \Omega$, the induced game $\Gamma(N, \alpha)$ has a Nash ε-equilibrium in pure strategies which partitions the population (N, α) into $C \leq A(\varepsilon, B)K$ societies.

We given some intuition behind the proof of Theorem 2 before stating two lemmas, both proved in the appendix, and providing the formal proof. Theorem 1 allows to take as given, for a large game $\Gamma(N, \alpha)$, the existence of a Nash $\frac{\varepsilon}{9}$-equilibrium in pure strategies m. The proof of Theorem 2 proceeds by constructing from this a Nash ε-equilibrium in pure strategies $\overline{\overline{m}}$ which, provided $|N|$ is sufficiently large, partitions the population (N, α) into $C \leq A(\varepsilon, B)K$ societies.

We begin by partitioning the attribute set Ω into a finite number of convex subsets $\Omega_1, ..., \Omega_A$, each of relatively small diameter. Interpret two

[14] A binary relation $<$ is a strict linear order on Ω if (1) for any $\omega \in \Omega$ it is not true that $\omega < \omega$, (2) for any $\omega, \overline{\omega} \in \Omega$ either $\omega < \overline{\omega}$ or $\overline{\omega} < \omega$, (3) for any $\omega, \overline{\omega}, \widetilde{\omega} \in \Omega$ if $\omega < \overline{\omega}$ and $\overline{\omega} < \widetilde{\omega}$ then $\omega < \widetilde{\omega}$.

players with attributes in the same subset Ω_a as players of the same class.[15] Select a binary relation $<$ on Ω satisfying the requirements of convex separation. We can use $<$ to order players of the same class in a line. To form the smallest number of societies we would like it to be the case that the set of players with class a who are playing pure strategy s_k form a connected subset of this line. If this were the case for all a and all s_k, given the definitions of convex separation, and of a society, the number of societies would be at most KA. It should be noted that we are making no use of the attribute index in placing players along a line. In particular, two players with very different attributes could potentially end up being 'next to each other' in terms of the ordering of attributes. The implications of this become clear in Section 5.1.

In the first stage of the proof we consider an arbitrary exchange of strategies between players of the same class. That is, an initial degenerate strategy vector m is compared to a degenerate strategy vector $\overline{m}$ where the number of players with class a playing pure strategy s_k is the same for $\overline{m}$ as for m, for any class and pure strategy. Intuitively, we would expect that if m is an approximate Nash equilibrium then $\overline{m}$ is an approximate Nash equilibria for a slightly more relaxed bound. This is the intuition formalized in Lemma 3.

Lemma 3: Given a pregame $\mathcal{G}$ suppose that the set of games $\gamma(\mathcal{G}, n)$ satisfies $\delta, \frac{\varepsilon}{9}$-continuity in attributes. Then for any game $\Gamma(N, \alpha) \in \gamma(\mathcal{G}, n)$, any partition of Ω into a finite number of subsets $\Omega_1, ..., \Omega_A$, each of diameter less that δ, and any two degenerate strategy profiles m and $\overline{m}$, where for all a and all $s_k \in S$,

$$\sum_{i:\ \alpha(i) \in \Omega_a} m_{ik} = \sum_{i:\ \alpha(i) \in \Omega_a} \overline{m}_{ik},$$

if m is a Nash $\frac{\varepsilon}{9}$-equilibrium, then $\overline{m}$ is a Nash $\frac{\varepsilon}{3}$-equilibrium of the game $\Gamma(N, \alpha)$.

Given the initial Nash $\frac{\varepsilon}{9}$-equilibrium m consider a degenerate strategy vector $\overline{m}$ such that, (1) $\sum_{i:\ \alpha(i)\in\Omega_a} m_{ik} = \sum_{i:\ \alpha(i)\in\Omega_a} \overline{m}_{ik}$, and (2) when $\alpha(i), \alpha(j) \in \Omega_a$ for some a and $m_{ik} = 1$ and $m_{i\overline{k}} = 1$ where $k < \overline{k}$ then $\alpha^{\nu}(i) \leqq \alpha^{\nu}(j)$. That is, $\overline{m}$ is derived from m by exchanging the strategies of players with the same class so that those players with the same class and

[15] We can think of players of the same class as being approximate substitutes for each other or as being approximately the same 'type.' We avoid using the word 'type', however, as this may create confusion with the notion of 'type' due to Harsanyi.

playing the same strategy are 'next to' each other in terms of the ordering of attributes. By Lemma 3 we know that such an $\overline{m}$ is a Nash $\frac{\varepsilon}{3}$-equilibrium.

This does not complete the proof of Theorem 2. The remaining possibility is that given strategy vector $\overline{m}$ there may be players with the same attribute who are playing different strategies. This prevents us from forming societies. The answer is to construct a strategy profile $\overline{\overline{m}}$ in which for any two players i, j with the same attributes who are playing different strategies we simply reallocate one player i the strategy of player j. Provided the number of players whose strategy could change is potentially small we would expect such a change to have a relatively minor effect on payoffs. This is formalized by Lemma 4.

Lemma 4: Suppose the pregame $\mathcal{G}$ satisfies convex separation and the large game property. Then for any $\varepsilon > 0$, any $B \geq 1$, and any partition of Ω into a finite number of subsets $\Omega_1, ..., \Omega_A$ there exists a real number $\eta_4(\varepsilon, B, A)$ such that for any game $\Gamma(N, \alpha)$ induced from $\mathcal{G}$ where $|N| > \eta_4(\varepsilon, B, A)$ and *profile*$(N, \alpha)(\omega) \leq B$ for all $\omega \in \Omega$, if there exists a Nash $\frac{\varepsilon}{3}$-equilibrium in pure strategies $\overline{m}$ with the property that,

$$\begin{aligned} \text{when } \alpha(i), \alpha(j) &\in \Omega_a \text{ for some } a \text{ and } \overline{m}_{ik} = 1 \text{ and } \overline{m}_{i\bar{k}} = 1 \\ \text{where } k &< \bar{k}, \text{ then } \alpha^\nu(i) \leq \alpha^\nu(j), \end{aligned}$$

then there exists a Nash ε-equilibrium in pure strategies $\overline{\overline{m}}$ with the property that,

$$\begin{aligned} \text{when } \alpha(i), \alpha(j) &\in \Omega_a \text{ for some } a \text{ and } \overline{\overline{m}}_{ik} = 1 \text{ and } \overline{\overline{m}}_{i\bar{k}} = 1 \\ \text{where } k &< \bar{k} \text{ then } \alpha^\nu(i) < \alpha^\nu(j). \end{aligned}$$

The formal statement of the proof now follows.

Proof of Theorem 2: Suppose not. Then, there is some $\varepsilon > 0$ and some $B \geq 1$ such that for each integer ν there is a population (N^ν, α^ν) where $|N^\nu| > \nu$, and induced game $\Gamma(N^\nu, \alpha^\nu)$, which satisfy the conditions of the statement of the Theorem but for which there exists no Nash ε-equilibrium with the required properties.

We note that, by Theorem 1, for any ε there exists a real number $\eta_1\left(\frac{\varepsilon}{9}\right)$ such that if $\nu \geq \eta_1\left(\frac{\varepsilon}{9}\right)$ the game $\Gamma(N^\nu, \alpha^\nu)$ has a Nash $\frac{\varepsilon}{9}$ equilibrium in pure strategies. For each ν, denote a Nash $\frac{\varepsilon}{9}$ equilibrium in pure strategies by m^ν and let $g^\nu_{\alpha^\nu, m^\nu}$ be the corresponding weight function.

Since the pregame $\mathcal{G}$ satisfies the large game property, we may choose non-negative real numbers $\delta_c\left(\frac{\varepsilon}{9}\right)$ and $\eta\left(\frac{\varepsilon}{9}\right)$ such that for any $n > \eta(\frac{\varepsilon}{9})$ the set of games $\gamma(\mathcal{G}, n)$ satisfies $\delta_c(\frac{\varepsilon}{9}), (\frac{\varepsilon}{9})$-continuity of payoff functions. Use compactness of Ω to write Ω as the disjoint union of a finite number A of *convex* non-empty subsets $\Omega_1, ..., \Omega_A$, each of diameter less than $\delta = \delta_c\left(\frac{\varepsilon}{9}\right)$.

The proof now proceeds in two stages. In the first stage consider a change of degenerate strategy vector from m^ν to $\overline{m}^\nu$, for all ν, where $\overline{m}^\nu$ satisfies:

1.
$$\sum_{i:\, \alpha(i)\in\Omega_j} m^\nu_{ik} = \sum_{i:\, \alpha(i)\in\Omega_j} \overline{m}^\nu_{ik}$$
for all $a = 1, ..., A$ and for all $s_k \in S$ and,

2.
$$\begin{aligned} \text{when } \alpha^\nu(i), \alpha^\nu(j) &\in \Omega_a \text{ for some } a \text{ and } m^\nu_{ik} = 1 \text{ and } m^\nu_{i\overline{k}} = 1 \\ \text{where } k &< \overline{k} \text{ then } \alpha^\nu(i) \leqq \alpha^\nu(j), \end{aligned}$$
for all $i, j \in N^\nu$.

Given that the finite set of points $\alpha(N^\nu)$ is well ordered for all ν, it is always possible to construct such a degenerate strategy vector $\overline{m}^\nu$ by a simple 'reallocation' of strategies. Given the choice of δ and that m^ν is a Nash $\frac{\varepsilon}{9}$-equilibrium for all ν it is immediate from Lemma 3 that $\overline{m}^\nu$ is a Nash $\frac{\varepsilon}{3}$-equilibrium of the game $\Gamma(N^\nu, \alpha^\nu)$.

In the second stage of the proof we apply Lemma 4 to show the existence of a degenerate strategy profile $\overline{\overline{m}}^\nu$ for all $\nu > \eta_4(\varepsilon, B, A)$ where for any $i, j, \in N^\nu$,

$$\begin{aligned} \text{when } \alpha(i), \alpha(j) &\in \Omega_a \text{ for some } a \text{ and } \overline{\overline{m}}_{ik} = 1 \text{ and } \overline{\overline{m}}_{i\overline{k}} = 1 \\ \text{where } k &< \overline{k} \text{ then } \alpha^\nu(i) < \alpha^\nu(j), \end{aligned}$$

and where $\overline{\overline{m}}^\nu$ is a Nash ε-equilibrium. From the definition of convex separation and the fact that each Ω_a is convex it is clear that $\overline{\overline{m}}^\nu$ is a Nash ε-equilibrium in pure strategies that partitions the population into at most AK societies, giving the desired contradiction.■

Some form of condition that *profile*$(N, \alpha)(\omega) \leq B$ for all $\omega \in \Omega$ in the statement of the Theorem is necessary because of our insistence on pure strategy equilibria. If all players have the same attribute, for example, and a Nash equilibrium exists with the property that a positive fraction of the players choose one strategy and a positive fraction choose another strategy, then a result such as Theorem 2 cannot be obtained. The condition could, however, possibly be relaxed to one requiring only that the percentage of players with any one attribute is small, say less than $\frac{\varepsilon}{B}$.

5.1 The example revisited

It is worth briefly returning to the example in Section 2.3 of a pregame satisfying the large game property given earlier. As highlighted in the introduction, for some attribute spaces Theorem 2 implies the existence of an approximate equilibria in pure strategies with the property that most players are playing the same strategy as their closest neighbors in attribute space. In particular, this is the case when $\Omega = [0, 1]$. It may be, however, despite our Theorem 2, that there are games for which all ε-equilibria have the property that most players do not play the same strategy as their closest neighbor. Our earlier example allows us to illustrate why this may be reasonable.

Consider a sequence of populations $\{(N^\nu, \alpha^\nu)\}_{\nu>1}$ such that for all ν and any player $i \in N^\nu$ the attribute of player i is given by,

$$\left(\frac{i + \frac{1}{n^2}}{n+2}, \frac{i}{n+2}\right) \text{ if } i \text{ is even, and}$$
$$\left(\frac{i+1}{n+2}, \frac{i+1+\frac{1}{n^2}}{n+2}\right) \text{ if } i \text{ is odd,}$$

where $n = |N^\nu| > \nu$. Figure 1 illustrates the attribute space (not to scale) for a population with 10 players. This demonstrates how, within this sequence of populations, players are in pairs of the form $(i, i+1)$ where i is an odd number. Players in the same pair are basically identical in terms of attribute but one is slightly more efficient at producing good A and the other at producing good B. Two players within the same pair are clearly their closest neighbors in terms of attributes. It seems intuitively reasonable, however, for two players within the same pair to belong in different societies.

5.2 Examples of social conformity

In a number of game theoretic models of social situations, there is some feature built into the model that ensures social conformity. For example, Bernheim (1994) obtains such a result for a model in which individuals care about status and behaviour (strategy) serves as a signal of status. Conformity within subgroups (clubs or jurisdictions) of the population has also been an issue in models of cooperative and price-taking behaviour; see, for example, Greenberg and Weber (1986) or Conley and Wooders (1996) where equilibrium jurisdictions consist of similar individuals. The following two examples illustrate that when there is some feature of the model motivating individuals to conform, it is 'easier' to have social conformity as defined in this paper. Indeed, we demonstrate that the existence of normative conformity can imply the existence of exact equilibria satisfying social conformity with a fewer number of societies.

Example 2: Normative influence can reduce the number of societies.

There are 2 pure strategies - to dress down (D) or to dress smart (U). The attribute space is $\Omega = [0,1]$. An attribute is a measure of a person's preference for dressing smart. Specifically, for any finite population (N,α) we can think of a player's payoff function, assuming an absence of normative influence, as given by,

$$h^0_\omega(p,w) = (2-\omega)p_D + (1+\omega)p_U$$

for all $w \in W_{\alpha - w}$, where p_D is the probability of dressing down and p_U the probability of dressing smart. The larger the attribute of an individual the more he enjoys dressing smart. Players may, however, exhibit normative influence. Given a population (N,α) and any weight function w define $d(w)$ and $u(w)$ as follows,

$$\begin{aligned} d(w) &= \frac{\sum_\omega \sum_{i\in N} w(D,\omega)}{|N|}, \\ u(w) &= \frac{\sum_\omega \sum_{i\in N} w(U,w)}{|N|}. \end{aligned}$$

Thus, $d(w)$ is the proportion of the population who dress down. Let β be a non-negative real number referred to as the degree of normative influence. A player's payoff function for any finite game $\Gamma(N,\alpha)$ is then given by

$$h_\omega(p,w) = (2-\omega)p_D(1+\beta d(w)) + (1+\omega)p_U(1+\beta u(w))$$

for all $w \in W_{\alpha - w}$. If $\beta = 0$ payoffs do not exhibit normative influence.

It is easily checked that this pregame satisfies the large game property (for any suitable metric on Ω). By applying Theorem 2 we can show

> Given real numbers $\varepsilon > 0$ and $B \geq 1$ for any population (N, α) where $profile(N, \alpha)(\omega) \leq B$ for all $\omega \in \Omega$, the induced game $\Gamma(N, \alpha)$ has a Nash ε-equilibrium in pure strategies that partitions the population (N, α) into $C \leq 2$ societies.'

Consider a population (N, α) where $|N| = 100$ and where $profile(N, \alpha)(0)$ $= profile(N, \alpha)(1) = 50$. If β is small (e.g. 0) then there will be a unique Nash 0-equilibrium in which there are two societies - one in which people dress smart and one in which people dress down. If β is large (e.g. 2) then there are two Nash 0-equilibrium - one in which there exists a unique society of people who dress smart and one in which there exists a unique society of people who dress down. That is, there exists a unique society.

Example 3: Normative influence can imply the existence of exact Nash equilibria.

This example is based around the 2 player matching pennies game whose payoff matrix is given below:

row \ column	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

The attribute space Ω is given by $[0, 1] \times \{R, C\}$. The first element of a player's attribute can interpreted as his geographical position. The second element is interpreted as whether he is a row (R) or column player (C).

Induced games are such that given any population (N, α) every column player ($i \in N$ such that $\alpha(i) = (\cdot, C)$) *meets* every row player ($j \in N$ such that $\alpha(j) = (\cdot, R)$). When two players i and j meet they play the matching pennies game. There are two pure strategies, to 'choose heads' (H) and to 'choose tails' (T). If a player chooses strategy H then he will play heads against any player he meets.

Given a population (N, α) and weight function w let $CH(w)$ be defined by,

$$CH(w) = \sum_{\omega \in \alpha(N) : \omega = (\cdot, C)} w(\omega, H).$$

That is, $CH(w)$ denotes the weight being put on H by column (C) players. We provide similar interpretations to $CT(w)$, $RH(w)$ and $RT(w)$. If there is no normative conformity then player i's payoff for any game $\Gamma(N,\alpha)$, any weight function w_α and any strategy $p = (p_H, p_T)$ is given by

$$h^0_{\alpha(i)}(p, w_{\alpha_{-i}}) = \frac{1}{|N|}\left[p_H(RT(w_{\alpha_{-i}}) - RH(w_{\alpha_{-i}})) + p_T(RH(w_{\alpha_{-i}}) - RT(w_{\alpha_{-i}}))\right]$$

if $\alpha(i) = (\cdot, C)$ and by

$$h^0_{\alpha(i)}(p, w_{\alpha_{-i}}) = \frac{1}{|N|}\left[p_H(CH(w_{\alpha_{-i}}) - CT(w_{\alpha_{-i}})) + p_T(CT(w_{\alpha_{-i}}) - CH(w_{\alpha_{-i}}))\right]$$

if $\alpha(i) = (\cdot, R)$. That is, their payoff is the total amount they earn from playing the matching pennies game, divided the number of players in the population.

Let β be a non-negative real number referred to as the degree of normative influence. Given a value for β player i's payoff for any game $\Gamma(N,\alpha)$, any weight function w_α and any strategy $p = (p_T, p_H)$ is given by

$$h^\beta_{\alpha(i)}(p, w_{\alpha_{-i}}) = h^0_{\alpha(i)}(p, w_{\alpha_{-i}}) + \beta\frac{1}{|N|}\left[p_H(RH(w_{\alpha_{-i}}) + CH(w_{\alpha_{-i}})) + p_T(RT(w_{\alpha_{-i}}) + CT(w_{\alpha_{-i}}))\right].$$

Thus, depending on the level of β a player's payoff is higher if they are playing the same strategy as other players in the population.

The definition of a pregame is completed by providing a metric ρ for Ω. For any $\omega_1, \omega_2 \in \Omega$, let

$$\begin{aligned} \rho((\omega_{11}, \omega_{21}), (\omega_{12}, \omega_{22})) &= |\omega_{11} - \omega_{12}| \text{ if } \omega_{21} = \omega_{22} \\ \text{and } \rho(\omega_1, \omega_2) &= 2 \qquad \text{otherwise.} \end{aligned}$$

This pregame satisfies the large game property as can be verified by setting $\delta_c(\varepsilon) = 1$, $\delta_g(\varepsilon) = \frac{\varepsilon}{2}$ and $\eta(\varepsilon) = 1$ for all $\varepsilon > 0$. On reflection of the proof of Theorem 2, for this example, we have that

'Given real numbers $\varepsilon > 0$ and $B \geq 1$ for any population (N,α) where $profile(N,\alpha)(\omega) \leq B$ for all $\omega \in \Omega$, the induced game $\Gamma(N,\alpha)$ has a Nash ε-equilibrium in pure strategies that partitions the population (N,α) into $C \leq 4$ societies.'

For example, there will be a society of column players living in a convex subset of $[0,1]$ who all play H.

Given the existence of normative influence can we obtain a stronger result? To simplify matters we restrict attention to games where the number of column players is equal to the number of row players. Begin by assuming that $\beta < 1$. For any induced game $\Gamma(N, \alpha)$, a Nash equilibrium σ clearly satisfies $CH(w_{\alpha,\sigma}) = CT(w_{\alpha,\sigma})$ and $RH(w_{\alpha,\sigma}) = RT(w_{\alpha,\sigma})$. It is immediate from this, that for arbitrarily large populations, if there is an odd number of row and column players, there need not exist an Nash 0-equilibrium in pure strategies. That is, there can only exist an approximate equilibrium in pure strategies. Further, it is also clear that any such approximate Nash equilibrium will induce a partition of the population into four societies. Assume now that $\beta > 1$. In this case there will exist two Nash 0-equilibria in pure strategies - 'everybody play H' and 'everybody play T'. Thus, because of the high degree of normative influence there exists an exact Nash equilibrium in pure strategies. Further, the number of societies falls to two.[16]

6 An example of an attribute space.

Theorem 2 assumes that the space of attributes is both a compact metric space and a space that satisfies convex separation. Clearly this imposes restrictions on the type of attributes permissible. Our purpose in this section is to provide an illustrative example of a space that has the desired properties. This example is by no means the most general possible but will hopefully demonstrate that the restrictions imposed on the attribute space are relatively mild.

Let Ω be the space of sequences of bounded rational numbers, let $[a, b]$ be closed interval contained in the real line $\mathbb{R}$, and let $\mathbb{Q}$ denotes the set of rational numbers. An element of Ω is of the form $\omega = (\omega_1, \omega_2, ...)$ where $\omega_\ell \in [a, b] \cap \mathbb{Q}$ for each ℓ .[17] As demonstrated in the appendix, Ω satisfies convex separation. Further, given a suitable metric, the space Ω is a compact

[16] It is well know that the symmetric mixed equilibrium may be a good predictor of the aggregate distribution of play, but a poor predictor of individual play from round to round in experimental games; see, for example, O'Neil.(1987) and, for a recent discussion, Walker and Wooders (2001). Glen Ellison has suggested to us that one explanation may be that play is converging to an Nash ε-equilibria in pure strategies that look like the symmetric mixed Nash equilibrium. While this example illustrates such a possibility, investigation of Ellison's suggestion is beyond the scope of the current paper, but one we hope to consider in future research.

[17] When indexing an attribute $\omega_j \in \Omega$ we use the notation that $\omega_j = (\omega_{j1}, \omega_{j2}, ...)$.

metric space. We will discuss below such a suitable metric, but for now, let us proceed on the basis that Ω is indeed a compact metric space satisfying convex separation. We thus need to demonstrate how an element ω of Ω can be thought of as describing a player.

Generally, we can think of a player $i \in N$ as being described by two pieces of information: first, the player's payoff function and second his personal characteristics such as gender, height, intelligence, metabolic rate and endowment etc.

Let u_ω denote the payoff function of a player with attribute ω. We note that equations (1) and (2) imply that a payoff function need merely detail a player's payoff at those strategy-weight function combinations (t, w) for which t is a *pure* strategy choice and w is a *degenerate* weight function. Let G denote the set of degenerate weight functions and assume that u_ω maps $S \times G$ into $[a, b] \cap \mathbb{Q}$. We note that given the function u_ω for all $\omega \in \Omega$ it is clearly possible to derive a function $h : \Omega \times \Delta(S) \times W \longrightarrow \mathbb{R}_+$ which summarises the payoff characteristics of players and enters as a component of the pregame.

We can now introduce a function p_ω to represent the personal characteristics of a player i. Given the attribute space Ω let $\mathcal{A}$ denote the set of functions mapping $N \to \Omega$ for any finite set $N = \{1, ..., |N|\}$. That is, $\mathcal{A}$ is the set of attribute functions possible given attribute space Ω. Assume that $p_\omega : \mathcal{A} \to [a, b]^P \cap \mathbb{Q}^P$ for some finite P. In other words, the personal characteristics of a player with attribute ω are given by a finite list of bounded rational numbers and are determined by the attribute function. Note, that a player's personal characteristics may be dependent in part on the attributes of other players in the population.

Take a population (N, α) as given. We note that because there are a finite number of strategies K and finite number of players $|N|$, there are only a finite number of possible degenerate weight functions $g_\alpha \in G_\alpha$. Furthermore, this implies that there are only a finite number of plausible pure strategy-integer valued weight function pairs $(s_k, g_{\alpha_{-i}})$ with respect to a player $i \in N$ for the population (N, α).[18] We know also that the personal characteristics of player $i \in N$ are given by the finite list $p_{\alpha(i)}(\alpha)$. Thus, for any population (N, α) and for any player $i \in N$, player i can be described, in that population, by a finite sequence of bounded rational numbers.

An attribute must, however, specify the personal characteristics and pay-off's of a player with attribute ω for any possible population (N, α) induced by the pregame $\mathcal{G}$. We note, however, that Ω has a countable number of elements. This further implies that there are only a countable number of possible societies (N, α) for finite $|N|$.[19] Thus, given that a finite sequence of bounded rational numbers describes a player's payoff and personal characteristics for any given population, a complete description of an attribute can be given by a countable sequence of bounded rational numbers. That is, we can think of a player's attribute as a list of the payoff and personal characteristics that the player will have for all of the possible societies that they might live in.

This demonstrates how an element ω of Ω can represent the attribute of a player. It remains to put a metric to the space Ω. We begin by introducing some notation: Given a sequence of real numbers $\omega = (\omega_1, \omega_2, ...)$ and a finite set of positive integers $\mathcal{P} = (1, ..., P)$, a sequence $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, ...)$ is a *permutation of* ω *with respect to* $\mathcal{P}$ if:

1. $\overline{\omega}_i = \omega_i$ for all $i \notin \mathcal{P}$ and,
2. there exists a one-to-one mapping p from $\mathcal{P}$ to $\mathcal{P}$ such that $\overline{\omega}_i = \omega_{p(i)}$ for all $i \in \mathcal{P}$.

For any $\omega \in \Omega$ let $\pi_{\mathcal{P}}(\omega)$ denote the set of permutations of x with respect

[18] A simple example may be clarifying. Consider a society of three people indexed $1, 2$ and 3 and two strategies indexed A and B. Suppose further that $\alpha(1) = \alpha(2) = \omega_1$ and $\alpha(3) = \omega_3 \neq \omega_1$. With regard to player 1 there are 8 possible strategy/integer valued weight function pairs $(s_k, g_{\alpha_{-i}})$ of which four are:-

$$(A; (1 \times A; 1 \times A)) \; ; \; (A; (1 \times A; 1 \times B)) \; ;$$
$$(A; (1 \times B; 1 \times A)) \; ;(A; (1 \times B; 1 \times B)$$

where $(1 \times A; 1 \times B)$ is interpreted as the integer valued weight function in which one player with attribute ω_1 plays strategy A and one player of attribute ω_3 plays strategy B.

[19] A standard result from real analysis is that a countable union of countable sets is itself countable. Repeated application of this result shows, for any finite N, that there are a countable number of possible societies of size N. Given this, application of the 'standard result' one more time demonstrates that there are a countable number of possible societies of finite size.

to $\mathcal{P}$. We define a metric, given $\mathcal{P}$, by:

$$\rho_{\mathcal{P}}(\omega, \overline{\omega}) = \sup_{z \in \pi_{\mathcal{P}}(\omega - \overline{\omega})} \sum_{i=1}^{\infty} \frac{|z_i|}{1 + |z_i|} 2^{-i}$$

This is indeed a metric for any finite set $\mathcal{P}$.

To explain the metric and then its motivation, we begin by ignoring the possibility of permutations. The difference between two sequences $\omega = (\omega_1, \omega_2, ...)$ and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, ...)$ is then given by a standard metric $\rho_s(\omega, \overline{\omega}) = \sum_{i=1}^{\infty} \frac{|\omega_i - \overline{\omega}_i|}{1+|\omega_i - \overline{\omega}_i|} 2^{-i}$. (Indeed, such a metric makes Ω a metric space furnished with the *product topology*.) The difference between ω and $\overline{\omega}$ is, in this case, given by the 'weighted sum' of the element-wise differences in the two sequences, with differences in elements 'early in the sequence' given more weight. In the more general metric $\rho_{\mathcal{P}}$ we essentially pick a finite set of 'representative points' along the sequence. In calculating the difference between two sequences ω and $\overline{\omega}$ all permutations of ω and $\overline{\omega}$ about these representative points are considered. The difference is then taken as the maximum difference in the corresponding metric $\rho_s(\omega, \overline{\omega})$. The metric space $(\Omega, \rho_{\mathcal{P}})$ is indeed compact for any set $\mathcal{P}$.

We conclude by explaining the motivation behind the metric $\rho_{\mathcal{P}}$. We begin by highlighting that, while an infinite sequence of numbers is needed to fully describe a player's attribute, to describe the payoff function and personal characteristics of a player with that attribute in any given population requires only a finite set of components of that sequence. Thus, informally, in choosing the finite set of points $\mathcal{P}$, we can select any finite number of games of finite size and measure the distance between the attributes of two players in terms of the payoffs and characteristics that players with these attributes have in such games.[20] As such, the notion of a finite set of 'representative points' can be superseded by the notion of a finite set of 'representative games'. The assumptions of continuity, compactness and global interaction suggest that such a set of 'representative games' should exist.

7 Some relationships to the literature

The literature on the existence of an approximate non-cooperative equilibrium in pure strategies was initiated by Schmeidler (1973) for non-atomic games, i.e. games with a continuum of players. (The introduction to Khan,

[20] Clearly, we are assuming that the sequence of numbers in the attribute are ordered (or constructed) in such a way that element wise comparison is appropriate.

Rath and Sun 1997 summarizes much of this literature.) The relationship between this literature and the results of the current paper are worth exploring. It is useful to begin by discussing some related results, for games with a finite player set, developed in Kalai (2000).

Kalai introduces the appealing concept of information proof Bayesian equilibrium, a type of equilibrium immune to changes in the prior probability of types, in the probabilities of mixed pure actions, and in the order of play, as well as to information leakage and the possibility of revision. Remarkably, he demonstrates that information proofness is a property of Bayesian equilibrium for a broad class of large games. Throughout, it is assumed that there is a finite strategy set. As an application of the law of large numbers, for games with complete information Kalai's results imply the existence of ε-Nash equilibria in pure strategies for all sufficiently large games satisfying the anonymity properties of his model. A similar result could also be derived from Rashid (1983).

The anonymity, or global interaction, property in Kalai (2000) differs significantly from that of this paper. In particular, given any population (N, α) and strategy profile σ let

$$w_{\alpha,\sigma}(s_k) = \sum_{i \in N} \sigma_{ik}.$$

Given σ, the *empirical distribution* of play defined by Kalai is given by $\frac{w_{\alpha,\sigma}(\cdot)}{|N|}$. We note that this empirical distribution is itself a strategy. A player's payoff function is assumed to map $S \times \Delta(S)$ into a closed subset of the real line. The interpretation is that a player's payoff depends on his own strategy and on the empirical distribution of opponent's strategies.

There are two ways in which the anonymity assumption of Kalai differs from the global interaction assumption of this paper. These two differences are also reflected in the literature on non-atomic games. A first difference is that within our framework payoffs depend on the *joint* distribution over strategies and attributes (or player characteristics) and not just on the distribution over strategies. That is, recalling the definition of a weight function,

$$w_{\alpha,\sigma}(\omega, s_k) = \sum_{i \in N:\, \alpha(i) = \omega} \sigma_{ik},$$

within our framework payoffs depend on who plays which strategy. It is typical within the literature on nonatomic games to formulate the model such that payoffs depend on the distribution over strategies and not the joint distribution over strategies and attributes (for example, Schmeidler 1973, Mas-Colell 1984, Khan 1989). Exceptions include Pascoa (1993a, 1993b) where this issue is discussed in more detail.[21] Within the framework of this paper it is relatively trivial to prove both Theorems 1 and 2 in the case where payoffs depend on the distribution over strategies and not on the joint distribution over strategies and attributes. Indeed, such results are easily derived from Lemma 1.

A second difference between the anonymity property in Kalai (2000) and the global interaction assumption of this paper is that in Kalai (2000) payoffs are a function of the average, $\frac{w_{\alpha,\sigma}(\cdot)}{|N|}$, while in this paper payoffs are a function of the distribution $w_{\alpha,\sigma}(\cdot,\cdot)$. In fact, our framework is such that payoffs depend on the distribution $w_{\alpha,\sigma}(\cdot,\cdot)$ but are nearly invariant to changes in the average $\frac{w_{\alpha,\sigma}(\cdot,\cdot).}{|N|}$. An important implication of this is that the size of the population can have an influence on payoffs.[22] It is typical within the literature to assume payoffs depend on averages (e.g. Mas-Colell, Pascoa 1993a.) An exception is Schmeidler (1973) and Khan, Rath and Sun (1997) where this issue is discussed further. Again, it is trivial that both Theorems 1 and 2 could be derived within our framework if payoffs were to depend on averages and not on the distribution.

With payoffs being a function of the distribution and not the average, and furthermore, being a function of the joint distribution over pure strategies and attributes and not just the distribution over strategies, it becomes clear that the framework of this paper is a relatively general one. The only real restriction would appear to be the assumption of a finite strategy set. With this restriction noted, Theorem 1 represents a finite analogue to most of the pure strategy existence results for non-atomic games. It is worth highlighting that the motivation behind assuming a continuum player set is usually to approximate a large but finite player set. Furthermore, games with finite player sets are significantly different from games with a continuum of players, and thus, it is not trivial true that results based on a nonatomic game can be applied to large but finite games (Green 1984). We also note that the assumption of a finite strategy space could be relaxed. In particular, there seems no obvious reason why Theorem 1 and, possibly Theorem 2, could not be extended to the case of a countable strategy space. It is also clear within our framework how we could derive results if there is a compact metric space

[21] Mas-Colell (1984) also remarks that assuming payoffs depend only on the distribution over strategies need not be restrictive when there exists an infinite strategy set. In this case strategies can be 'indexed' so as to indicate a player's attribute.

[22] Note, with a finite strategy set and continuum player set, the distinction between averages and distributions is of no real significance (Khan, Rath and Sun 1997).

of strategies. Such a space of strategies could be treated in an analogous way to that in which the compact metric space of attributes is treated in this paper – we approximate by finite sets of attributes.

The literature on non-atomic games has also addressed the question of when there exists a symmetric non-competitive equilibrium in pure strategies (Mas-Colell 1984, Pascoa 1993a, Rath, Sun, Yamshige 1995). A symmetric equilibrium has the property that players with the same attribute (usually this means same payoff function) play the same strategy. It is immediate that our Theorem 2 implies the existence of such an equilibrium in large finite games. It does, of course, much more - it shows how different players may be who still conform in their choice of strategy.

A related literature concerns purification of Nash equilibria in finite games with imperfect information. Assume that before the game is played a player receives a private signal on which he can condition his action. Provided there is sufficient uncertainty over the signals players will receive, any mixed strategy can be purified. That is, any mixed strategy can be replaced by a pure strategy that yields all players similar payoffs. One line of the literature concerns exact purification (e.g. Radner and Rosenthal 1982) and a second line considers approximate purification (e.g. Aumann et. al. 1983). In this paper we use a notion of ε-purification related to that used by Aumann et. al. (1983) by demonstrating that any strategy vector can be ε-purified if there are sufficiently many players in the game. We note that the literature on nonatomic games does not demonstrate the possibility of such purification. This, however, would merely appear to reflect the method of proving the existence of a pure strategy non-cooperative equilibrium.

8 Conclusions

Assume that players learn and choose strategies by observing and conforming to the strategy choices of similar players in the population. Is it possible that through such a learning dynamic players can learn to behave 'as if' fully rational? That is, can players learn to play Nash equilibria? Such conformity effects appear to be a significant element in human decision making and thus the answer to this question is one of fundamental importance. (For further discussion see Durlauf and Young 2001.)

A first step in addressing the question of whether players can learn to play Nash equilibria is to consider the possible existence of an approximate equilibrium with conformity, that is, to demonstrate the existence of an equilibrium that is stable in terms of the learning dynamic and also an approximate Nash equilibrium. In this paper we demonstrate existence of such an equilibrium in large games. The result is made more interesting by observing that large games are games for which it seems intuitively most plausible that players will base decisions on processes such as conformity and imitation.

Two issues still remaining are: (i) We only demonstrate the existence of a Nash equilibrium with conformity; we do not address directly, the question of whether players actually learn to play that equilibrium. (ii). Social conformity in mixed strategy equilibrium is not treated. We take up each of these issues in turn.

With regard to the dynamics of the learning process and convergence to a Nash equilibrium there are two potential problems. First, we may get convergence to a cycle and, second, we may get convergence to a non-Nash equilibrium stable state. (See Fudenberg and Levine 1998 for a more detailed discussion of these issues.) Informally, convergence to a cycle is usually the result of aggregate play 'circling around' a mixed strategy Nash equilibria. The purification result of this paper suggests a possible solution to such problems. The intuition for this claim is that 'nearby' any Nash equilibrium in mixed strategies is an approximate equilibrium in pure strategies. Thus, it seems reasonable to conjecture that evolution cannot get caught in a permanent cycle around a mixed strategy equilibrium but is ultimately drawn to the corresponding pure strategy equilibrium.

A more fundamental problem related to convergence may concern 'mistakes' in measuring similarities leading to convergence on non Nash equilibrium stable states. In particular, players may mimic individuals they *perceive* as similar. We demonstrate the existence of an equilibrium in which individuals that are *in reality* similar play the same strategies. Thus, the metric by which individuals measure similarity is crucial. A bright and highly capable boy, living in a low income area in Liverpool, for example, may aspire to be a football hooligan rather than the rocket scientist he is capable of becoming. It may be that if the similarity metrics that people use are biased to place too much weight on similarities of gender, race, color, or religion rather than on similarities of ability, interests, and so on, there may be (non-Nash) 'stable equilibrium' outcomes that are quite different than Nash outcomes. Part of the motivation for developing the current model is to explore such issues.

The second issue is that there may be situations in which the requirement that individuals from the same society play the same pure strategies is too restrictive; it may be more natural to require only that similar individuals play similar mixed strategies. In Cartwright and Wooders (2001) we formulate social conformity in terms of mixed strategies and demonstrate conditions under which social conformity obtains.[23] This formulation of social conformity permits an arbitrary compact set of strategies.[24]

Let us now turn to other motivations for our research, from the literature of general equilibrium and cooperative game theory. One possible issue concerns the so called "Equivalence Principle" of cooperative outcomes of large ("competitive") exchange economies. In exchange economies with many players, the set of equilibrium outcomes, represented by the induced utilities of members of the economy, coincides with the core of the game generated by the economy and the value outcomes; see Debreu and Scarf (1963), Aumann (1964, 1985). This sort of result holds for diverse models of economies with many players (cf. Conley and Wooders 1997, 2001 for economies with clubs and with endogenous choice of crowding types – skills and other external characteristics). For abstract models of large economies as cooperative games, under mild assumptions quite similar in spirit to those of this paper, it is known that approximate cores are nonempty and treat similar players similarly (cf. Kovalenkov and Wooders 1999a, 2001). We conjecture that when noncooperative games derived from pregames are required to satisfy the conditions of this paper (satisfied, in spirit, for exchange economies for which the Equivalence Principle holds) and, in addition, the condition of self-sufficiency – that what a coalition of players can achieve is at least asymptotically independent of the population in which it is embedded – then analogues of the Equivalence Principle can be obtained for large noncooperative games. More precisely, we conjecture that under self sufficiency, (approximate) strong equilibrium outcomes are close to Pareto optimal and also treat similar individuals similarly in terms of their expected equilibrium payoffs.

To compare our noncooperative pregame framework to the cooperative pregame framework, it is important to note a major and significant difference. In the cooperative framework, the payoff to a coalition is fixed and independent of the population in which that coalition is embedded. Although this is possible within the current noncooperative framework, it is not built into the model and thus may or may not hold. Noncooperative games derived from a (noncooperative) pregame are parameterized by the numbers of players of each type in the player set and may vary considerably

[23] We are grateful to Francis Bloch, Alan Kirman and Michel Le Breton for stressing the importance of this issue.

[24] A third issue that may be related, but which we do not explore here is under what conditions evolutionary stable strategies must be pure strategies; see Selten (1980) and Binmore and Samuelson (2001) for discussions of this issue.

depending on the attributes of the players actually represented in the population – even in a large population the presence or absence of one player with a certain attribute may significantly change the game. For example, there may be little relationship between derived games where all players have the characteristic, male for example, and games where even one player has a different characteristic, female, for example. Moreover, even in the case where all players are identical, there is no necessary relationship between a game with n players and another with $n + 1$ players.

One interesting similarity between the two frameworks is that, in the cooperative pregame framework, the condition of *small group effectiveness* plays an important role, cf., Wooders (1994). This condition dictates that all or almost all gains to collective activities can be realized by groups of players bounded in size. An equivalent condition is that *small groups are negligible*: in large cooperative games derived from pregames, small groups are effective if and only if small groups cannot have significant effects on aggregate per capita payoff (Wooders 1993). The main substantive condition of the current paper, global interaction, can be interpreted as the negligibility of small groups of players; that is, the effects of the actions of any small set of players on the complementary set of players become negligible in games with many players. For cooperative pregames with side payments, the condition of small group negligibility implies that large games are market games, as defined by Shapley and Shubik (1969). The full implications of the condition of global interaction for noncooperative pregames have not been fully explored, but we expect there to be many.

The noncooperative framework that allowed us to derive our main result promises to be fruitful and the techniques developed in this paper may be useful in other applications other than those mentioned above. One potential application, currently in draft form, is to games with incomplete information (Cartwright and Wooders 2001). In particular, note that Lemma 1 applies to *any* game and an extension of our model to incomplete information can be obtained using that Lemma similarly to how it is used in this paper.

9 Appendix: Euclidean space satisfies convex separation.

To provide a simple example of convex separation, we show that a closed subset of finite dimensional Euclidean space satisfies convex separation:

Lemma 5: For any finite integer M the linear space $\times_{m=1}^{M}[0,1]$ satisfies convex separation.

Proof. We define the binary relations $<$ and $=$ as follows: Take any two points $x = (x_1, ..., x_M), x' = (x'_1, ..., x'_M) \in \times_{m=1}^{M}[0,1]$. We say that $x = x'$ if $x_m = x'_m$ for all $m = 1, ..., M$. We say that $x < x'$ if either:

1. $\sum_m x_m < \sum_m x'_m$ or,
2. $\sum_m x_m = \sum_m x'_m$ and for some $m^* \in \{1, ..., M-1\}$ $x_{m^*} < x'_{m^*}$ and $x_m = x'_m$ for all $m < m^*$.

Suppose $\Omega_J = \{x_1, x_2, ..., x_J\}$ and $\Omega_Q = \{x'_1, x'_2, ..., x'_Q\}$ are two sets of points in $\times_{m=1}^{M}[0,1]$. Then, if $x_j < x'_q$ for all $x_j \in \Omega_J$ and all $x_q \in \Omega_Q$ we claim that: the convex hulls of Ω_J and Ω_Q are disjoint. Denote the convex hull of a set of points $S = \{S_1, ..., S_n\}$ by *con* S where

$$con\ S = \left\{ \begin{array}{c} x|x = \sum_{i=1}^{n} \beta_i s_i \text{ for some numbers } \beta_i, 0 \leq \beta_i \leq 1, \\ \sum_{i=1}^{n} \beta_i = 1, \text{ and for some set of points } s_i \in S \end{array} \right\}.$$

Thus, suppose the claim is false. Then there exists a point x such that $x \in con\ \Omega_J$ and $x \in con\ \Omega_Q$. Thus, for some numbers $\beta_1, ..., \beta_Q$ and $\gamma_1, ..., \gamma_J$ we have that:

$$x = \sum_{j=1}^{J} \gamma_j x_j \text{ and } x = \sum_{q=1}^{Q} \beta_q x'_q. \tag{9}$$

which implies that:

$$\sum_{j=1}^{J} \gamma_j \sum_{m=1}^{M} x_{jm} = \sum_{q=1}^{Q} \beta_q \sum_{m=1}^{M} x'_{qm}$$

Suppose, for some x_{j^*} and x'_{q^*} that $\sum_m x_{jm} < \sum_m x'_{qm}$. Then, given that $x_j < x'_q$ for all $x_j \in \Omega_J$ and all $x'_q \in \Omega_Q$, we must have that either $\gamma_{j^*} = 0$ or $\beta_{q^*} = 0$. Let Ω_J^+ denote the set of $x_j \in \Omega_J$ given positive weight $\gamma_j > 0$ and Ω_Q^+ the set of all $x'_q \in \Omega_Q$ given positive weight $\beta_q > 0$. Then, it is immediate from the above that $\sum_m x_{jm} = \sum_m x'_{qm}$ for all $x_j \in \Omega_J^+$ and $x'_q \in \Omega_Q^+$. For any pair of points $x_j \in \Omega_J^+$ and $x'_q \in \Omega_Q^+$ there must exist some $m^* \in \{1, ..., M-1\}$ for which $x_{jm^*} < x'_{qm^*}$ and $x_m = x'_m$ for all

$m < m^*$. Take the minimum of these m^* over all points $x_j \in \Omega_J^+$ and $x'_q \in \Omega_Q^+$. By 9 we have that:

$$\sum_{j=1}^{J} \gamma_j x_{jm^*} = \sum_{q=1}^{Q} \beta_q x'_{qm^*}$$

However, by choice of m^* we have that $x_{jm^*} \leq x_{qm^*}$ for all j, q and $x_{jm^*} < x_{qm^*}$ for some j, q which implies that:

$$\sum_{j=1}^{J} \gamma_j > \sum_{q=1}^{Q} \beta_q$$

giving the desired contradiction.■

The same line of reasoning exemplified by the example above can be extended to other spaces. We can use the following binary relation on the space of bounded sequences of real numbers: two sequences $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ are equal, written $x = y$, if and only if $x_i = y_i$ for all $i \in \mathcal{Z}^+$. We say that the sequence x is strictly less than the sequence y if:

$$\sum_{i=1}^{\infty} \frac{x_i}{1+|x_i|} 2^{-i} < \sum_{i=1}^{\infty} \frac{y_i}{1+|y_i|} 2^{-i}$$

The remaining case is one in which the above sum is equal but the two sequences are not equal. This implies that there is some smallest integer $i \in \mathcal{Z}^+$ at which they differ, say $x_i < y_i$, and in this case we say that $x < y$.

10 Appendix 2. Proofs of Lemmas

Lemma 1: Let $N = \{1, ..., n\}$ be a finite set. For any strategy vector $\sigma = (\sigma_1, ..., \sigma_n) \in \Delta^{Kn}$ and for any vector $\overline{g} \in \mathbb{Z}_+^K$ such that $\sum_i \sigma_i \geq \overline{g}$, there exists a degenerate strategy vector $m = (m_1, ..., m_n)$ such that,

$$support(m_i) \subseteq support(\sigma_i)$$

for all $i \in N$ and

$$\sum_i m_i \geq \overline{g}.$$

Proof: Given any strategy vector σ let $M(\sigma)$ denote the set of strategy vectors such that $m \in M(\sigma)$ if and only if (1) m is degenerate and (2) *support*(m_i) $\subseteq$*support*(σ_i) for all $i \in N$. It is trivial that $M(\sigma)$ is non-empty for any σ.

Suppose the statement of the lemma is false. Then, there exists a strategy vector $\sigma = (\sigma_1, ..., \sigma_n)$ and a vector $\overline{g} \in \mathbb{Z}_+^K$ where $\sum_{i \in N} \sigma_i \geq \overline{g}$ and such that, for any vector $m = (m_1, ..., m_n) \in M(\sigma)$ there must be at least one $\widehat{k}$ where $s_{\widehat{k}} \in S$ and for which $\sum_i m_{i\widehat{k}} < \overline{g}_{\widehat{k}}$. For each vector $m \in M(\sigma)$ let L be defined as follows:

$$L(m) = \sum_{s_k \in S : \sum_i m_{ik} < \overline{g}_k} \left(\overline{g}_k - \sum_i m_{ik} \right)$$

Select $m^0 \in M(\sigma)$ for which $L(m)$ attains its minimum value over all $m \in M(\sigma)$. Intuitively the vector m^0 is 'as close' as we can get to satisfying the lemma. Pick a strategy $s_{\widehat{k}}$ such that $\overline{g}_{\widehat{k}} - \sum_i m^0_{i\widehat{k}} > 0$.

For any subset I of N let the set $S(I) \subset S$ be such that:

$$S(I) = \left\{ s_{\widehat{k}} \right\} \cup \left\{ s_k \in S : m^0_{ik} = 1 \text{ for some } i \in I \right\}$$

We can now define sets $I^t\left(\widehat{k}\right)$ for $t = 0, 1, ...$ as follows:

$$I^0\left(\widehat{k}\right) = \left\{ i \in N : m^0_{i\widehat{k}} = 1 \right\},$$

for all $t \geq 1$,

$$I^t\left(\widehat{k}\right) = I^{t-1}\left(\widehat{k}\right) \cup \left\{ \begin{array}{c} j \in N : \sigma_{jk} > 0 \text{ and } m^0_{jk} = 0 \\ \text{for some } k \in S\left(I^{t-1}\left(\widehat{k}\right)\right) \end{array} \right\}$$

Consider a player $i_1 \in I^1\left(\widehat{k}\right) \backslash I^0\left(\widehat{k}\right)$, if one exists. Then, $m^0_{i_1\widehat{k}} = 0$, $\sigma_{i_1\widehat{k}} > 0$ and $m^0_{i_1k_1} = 1$ for some $s_{k_1} \in S$. Thus, there exists an $m^* \in M(\sigma)$ such that $m^*_i = m^0_i$ for all $i \neq i_1$ while $m^*_{i_1\widehat{k}} = 1$ and $m^*_{i_1k_1} = 0$. Suppose that $\sum_{i \in N} m^0_{ik_1} > \overline{g}_{k_1}$. This implies, given that $m^0_{ik_1}$ and $\overline{g}_{k_1}$ are integers, that $\sum_{i \in N} m^0_{ik_1} \geq \overline{g}_{k_1} + 1$. Then, it follows, by the definition of $L(m)$ that $L(m^*) = L(m^0) - 1$. This contradicts that we chose the vector $m^0 \in M(\sigma)$ with minimum L.

Consider a player $i_t \in I^t\left(\widehat{k}\right) \backslash I^{t-1}\left(\widehat{k}\right)$, if one exists, where $t \geq 2$ and $m^0_{i_tk_t} = 1$ for some pure strategy s_{k_t}. By the construction of $I^t\left(\widehat{k}\right)$ if

$i_t \in I^t\left(\widehat{k}\right) \backslash I^{t-1}\left(\widehat{k}\right)$ then there must exist some strategy

$$s_{k_{t-1}} \in S\left(I^{t-1}\left(\widehat{k}\right)\right) \backslash S\left(I^{t-2}\left(\widehat{k}\right)\right)$$

such that $\sigma_{i_t k_{t-1}} > 0$ and $m^0_{i_t k_{t-1}} = 0$. This implies, by the definition of $S(I)$, that there exists a player

$$i_{t-1} \in I^{t-1}\left(\widehat{k}\right) \backslash I^{t-2}\left(\widehat{k}\right)$$

such that $m^0_{i_{t-1} k_{t-1}} = 1$.

Continuing this chain as far as necessary, it follows that there exists a set of players $C = \{i_1, ..., i_t\} \subset I^t\left(\widehat{k}\right)$, where for all $i_t \in C$, we have, $i_t \in I^t\left(\widehat{k}\right)$ and $m^0_{i_t k_t} = 1$. Further, there exists a vector $m^* \in M(\sigma)$ such that:

for i_1, $m^*_{i_1 k_1} = 0$ and $m^*_{i_1 \widehat{k}} = 1$,
for all $i_t \in C \backslash \{i_1\}$, $m^*_{i_t k_t} = 0$ and $m^*_{i_t k_{t-1}} = 1$, and
for all $i \notin C$ and all $s_k \in S$, $m^*_{ik} = m^0_{ik}$.

Suppose that

$$\sum_{i \in N} m^0_{ik_t} > \overline{g}_{k_t}.$$

Then, again noting this implies that

$$\sum_{i \in N} m^0_{ik_t} \geq \overline{g}_{k_t} + 1,$$

we have that

$$L(m^*) = L(m^0) - 1.$$

To see this we note that

$$\sum_{i \in N} m^0_{ik_r} = \sum_{i \in N} m^*_{ik_r}$$

for all $1 \leq r < t$ while

$$\sum_{i \in N} m^0_{ik_t} - 1 = \sum_{i \in N} m^*_{ik_t}$$

and

$$\sum_{i \in N} m^0_{i\widehat{k}} + 1 = \sum_{i \in N} m^*_{i\widehat{k}}.$$

Then use the definition of $L(m)$. This contradicts the choice of $m^0 \in M(\sigma)$ to minimize $L(\cdot)$.

Ultimately, for some $t^* \geq 1$ we must have that $I^{t^*+1}\left(\widehat{k}\right) = I^{t^*}\left(\widehat{k}\right)$. This is an immediate consequence of the finiteness of the player set.

The above has shown that if there exists a strategy $s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)$ where:

$$\sum_{i \in N} m^0_{ik} > \overline{g}_k$$

then we have the desired contradiction. This implies, for all $s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)$ that:

$$\sum_{i \in N} m^0_{ik} \leq \overline{g}_k. \tag{10}$$

Using the definition of $I^t\left(\widehat{k}\right)$ and that $I^{t^*+1} = I^{t^*}$, there can exist no player $j \in N \backslash I^{t^*}\left(\widehat{k}\right)$ such that $\sigma_{jk} > 0$ for some $s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)$, unless $m^0_{jk} = 1$. This implies that:

$$\sum_{i \in N \backslash I^{t^*}\left(\widehat{k}\right)} m^0_{ik} \geq \sum_{i \in N \backslash I^{t^*}\left(\widehat{k}\right)} \sigma_{ik} \tag{11}$$

for all $s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)$.

Using the definition of $S\left(I^{t^*}\left(\widehat{k}\right)\right)$, we have that:

$$\sum_{s_k : s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)} \sum_{i \in I^{t^*}\left(\widehat{k}\right)} m^0_{ik} \geq \sum_{s_k : s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)} \sum_{i \in I^{t^*}\left(\widehat{k}\right)} \sigma_{ik}. \tag{12}$$

Combining 11 and 12 and using the statement of the lemma, we see that:

$$\sum_{s_k : s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)} \sum_{i \in N} m^0_{ik} \geq \sum_{s_k : s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)} \sum_{i \in N} \sigma_{ik} \geq \sum_{s_k : s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)} \overline{g}_k$$

However, by assumption:

$$\overline{g}_{\widehat{k}} > \sum_{i \in N} m^0_{i\widehat{k}}$$

and also by assumption, $s_{\widehat{k}} \in S\left(I^{t^*}\left(\widehat{k}\right)\right)$. Thus, there must exist at least one $s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)$ such that:

$$\overline{g}_k < \sum_{i \in N} m^0_{ik}.$$

This contradicts 10 and completes the proof.■

Here we provide an intuitive explanation of the sets $I^t\left(\widehat{k}\right)$ with reference to Figure 1 below.

The set $I^0\left(\widehat{k}\right)$ contains those players who are 'assigned' the pure strategy $s_{\widehat{k}}$ by vector m^0. That is, if $i_0 \in I^0(\widehat{k})$ then $m^0_{i_0\widehat{k}} = 1$. The set $I^1\left(\widehat{k}\right) \backslash I^0\left(\widehat{k}\right)$ consists of those players who could have been assigned the strategy $\widehat{k}$ according to the definition of $M(\sigma)$, but were not. That is, if $i_1 \in I^1\left(\widehat{k}\right) \backslash I^0\left(\widehat{k}\right)$, then $\sigma_{i_1\widehat{k}} > 0$ but $m^0_{i_1\widehat{k}} = 0$.

Suppose that $m^0_{i_1k_1} = 1$. That is, player 1 was assigned pure strategy s_{k_1}. In looking at the set $I^2\left(\widehat{k}\right) \backslash I^1\left(\widehat{k}\right)$ pure strategies $k \neq \widehat{k}$ play a role. In particular, if there exists a player i_2 such that $m^0_{i_2k_2} = 1$ and $\sigma_{i_2k_1} > 0$ then player $i_2 \in I^2\left(\widehat{k}\right) \backslash I^1\left(\widehat{k}\right)$. That is, player i_2 could have been assigned the strategy s_{k_1} but was actually assigned strategy s_{k_2}. Further, there exists a player $i_1 \in I^1\left(\widehat{k}\right)$ using pure strategy s_{k_1}.[25] In adding the next group of players, $I^3\left(\widehat{k}\right) \backslash I^2\left(\widehat{k}\right)$, we can now take into account the pure strategy s_{k_2}. Thus, we start looking for some player i_3 such that $m^0_{i_3k_3} = 1$ and $\sigma_{i_3k_2} > 0$. And so on.

[25] We also require that $\sigma_{i_2\widehat{k}} = 0$ otherwise player i_2 would be included in the set $I^1\left(\widehat{k}\right)$.

$i_0 \in I^0\left(\widehat{k}\right)$	$i_1 \in I^1\left(\widehat{k}\right) \setminus I^0\left(\widehat{k}\right)$	$i_2 \in I^2\left(\widehat{k}\right) \setminus I^1\left(\widehat{k}\right)$	$i_3 \in I^3\left(\widehat{k}\right) \setminus I^2\left(\widehat{k}\right)$	$\cdots$
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	
$\sigma_{i_0\widehat{k}} > 0$ $m^0_{i_0\widehat{k}} = 1.$	$\sigma_{i_1\widehat{k}} > 0$ $m^0_{i_1\widehat{k}} = 0$	$\sigma_{i_2\widehat{k}} = 0$ $m^0_{i_2\widehat{k}} = 0.$	$\sigma_{i_3\widehat{k}} = 0$ $m^0_{i_3\widehat{k}} = 0.$	
	and for some k_1, $\sigma_{i_1k'} > 0$ $m^0_{i_1k_1} = 1$.	For k_1, $\sigma_{i_2k'} > 0,$ $m^0_{i_2k_1} = 0$	For k_1, $\sigma_{i_3k_1} = 0,$ $m^0_{i_3k_1} = 0.$	
		and for some k_2 $\sigma_{i_2k_2} > 0$ $m^0_{i_2k_2} = 1$.	For k_2 $\sigma_{i_3k_2} > 0$ $m^0_{i_3k_2} = 0$	
			and for some k_3 $\sigma_{i_3k_3} > 0$ $m^0_{i_3k_3} = 1.$	

Figure 1

Any of the players in $I^1\left(\widehat{k}\right) \setminus I^0\left(\widehat{k}\right)$, such as i_1, could have been assigned the pure strategy $\widehat{k}$ (since $\sigma_{i_1\widehat{k}} > 0$) but instead i_1 was assigned the pure strategy s_{k_1}. Player i_2 could have been assigned pure strategy i_1 (since $\sigma_{i_2k_1} > 0$) but was assigned pure strategy s_{k_2}. Generalizing, any player $i_t \in I^t\left(\widehat{k}\right) \setminus I^{t-1}\left(\widehat{k}\right)$ could have been assigned pure strategy $s_{k_{t-1}}$ but was actually assigned pure strategy s_{k_t}.

Now, suppose that there exists a strategy $s_k \in S\left(I^{t^*}\left(\widehat{k}\right)\right)$ such that:

$$\sum_{i \in N} m^0_{ik} \geq \overline{g}_k + 1$$

We can do the following reallocation: put $m^0_{i_tk_t}$ equal to zero and set $m^0_{i_rk_{r-1}} = 1$ for all $r \geq 2$ and set $m_{i_1k_1} = 0$ and $m_{i_1\widehat{k}} = 1$. This leaves the numbers allocated to strategy s_{k_r} for all $1 \leq r \leq n-1$ unchanged. The number playing s_{k_t} reduces by one and the number playing $s_{\widehat{k}}$ increases by one. Essentially, the player i_t has been allocated to a strategy s_{k_t} where 'it is not needed'. Thus, we can take player i_t away from strategy s_{k_t} and allocate it to strategy $s_{k_{t-1}}$. Repeating this chain we finish by putting $m_{i_1\widehat{k}} = 1$. If there was a shortfall in the number of players using strategy $s_{\widehat{k}}$ we thus reduce this shortfall to at the worst one less than we began with. At best, we can, of course, overcome the shortfall completely and repeatedly applying the above procedure will eventually do so.

Lemma 2: Let σ denote a strategy vector for population (N, α) and let $w_{\alpha,\sigma}$ denote the weight function relative to σ and α, then there exists a degenerate strategy vector m and weight function $g_{\alpha,m}$ relative to m and α, with the properties that:

$$\textit{support}(m_i) \subseteq \textit{support}(\sigma_i)$$

for all $i \in N$ and

$$|g_{\alpha,m}(\omega, s_k) - w_{\alpha,\sigma}(\omega, s_k)| < K$$

for all $\omega \in \Omega$ and all $s_k \in S$.

Proof: We consider in turn each $\omega \in \alpha(N)$. Thus, pick some $\overline{\omega} \in \alpha(N)$. Choose a vector $\overline{g} \in \mathbb{Z}_+^K$ such that for each $k = 1, ..., K$,

$$\sum_{i \in N : \alpha(i) = \overline{\omega}} \sigma_{ik} - 1 < \overline{g}_k \leq \sum_{i \in N : \alpha(i) = \overline{\omega}} \sigma_{ik}.$$

By Lemma 1 there exists a degenerate strategy vector $m = (m_1, ..., m_{|N|})$ such that

$$\sum_{i \in N : \alpha(i) = \overline{\omega}} m_i \geq \overline{g}.$$

and the $\textit{support}(m_i) \subset \textit{support}(\sigma_i)$ for all $i \in N$ with $\alpha(i) = \overline{\omega}$. Thus, for any $k = 1, .., K$,

$$\sum_{i \in N : \alpha(i) = \overline{\omega}} \sigma_{ik} - \sum_{i \in N : \alpha(i) = \overline{\omega}} m_{ik} < 1. \tag{13}$$

Note, that $\sum_{s_k \in S} \sum_{i \in N:\alpha(i)=\overline{\omega}} \sigma_{ik} = \sum_{s_k \in S} \sum_{i \in N:\alpha(i)=\overline{\omega}} m_{ik} = \mathit{profile}(N,\alpha)(\overline{\omega})$. Thus, for any $s_k \in S$,

$$\begin{aligned}\sum_{i \in N:\alpha(i)=\overline{\omega}} m_{ik} &= \mathit{profile}(N,\alpha)(\overline{\omega}) - \sum_{s_{k'} \in S: s_{k'} \neq s_k} \sum_{i \in N:\alpha(i)=\overline{\omega}} m_{ik} \\ &= \sum_{s_{k'} \in S: s_{k'} \neq s_k} \sum_{i \in N:\alpha(i)=\overline{\omega}} (\sigma_{ik} - m_{ik}) + \sum_{i \in N:\alpha(i)=\overline{\omega}} \sigma_{ik} \\ &< K + \sum_{i \in N:\alpha(i)=\overline{\omega}} \sigma_{ik}.\end{aligned}$$

Thus, by (13)

$$-1 < \sum_{i \in N:\alpha(i)=\overline{\omega}} m_{ik} - \sum_{i \in N:\alpha(i)=\overline{\omega}} \sigma_{ik} < K.$$

Repeating this argument for each $\omega \in \alpha(N)$ completes the proof.■

Lemma 3: Given a pregame $\mathcal{G}$ suppose that the set of games $\gamma(\mathcal{G}, n)$ satisfies $\delta, \frac{\varepsilon}{9}$-continuity in attributes. Then for any game $\Gamma(N,\alpha) \in \gamma(\mathcal{G}, n)$, any partition of Ω into a finite number of subsets $\Omega_1, ..., \Omega_A$, each of diameter less that δ, and any two degenerate strategy profiles m and $\overline{m}$, where for all a and all $s_k \in S$,

$$\sum_{i:\ \alpha(i) \in \Omega_a} m_{ik} = \sum_{i:\ \alpha(i) \in \Omega_a} \overline{m}_{ik}, \tag{14}$$

if m is a Nash $\frac{\varepsilon}{9}$-equilibrium, then $\overline{m}$ is a Nash $\frac{\varepsilon}{3}$-equilibrium of the game $\Gamma(N,\alpha)$.

Proof: Given (14) and the fact that both m and $\overline{m}$ are degenerate there must exist a one-to-one mapping $R(i) : N \to N$ such that,

$$\overline{m}_i = m_{R(i)} \text{ for all } i \in N \tag{15}$$

and,

$$\mathit{dist}(\alpha(i), \alpha(R(i))) < \delta. \tag{16}$$

That is, in the strategy profile $\overline{m}$ player i plays the strategy that player $R(i)$ plays in strategy profile m. Further, the attributes of players $R(i)$ and i must be within δ of each other. The proof of the lemma, therefore, requires us to show the existence of such a mapping R, and the fact that m is a Nash $\frac{\varepsilon}{9}$-equilibrium, implies that,

$$h_{\alpha(i)}(\overline{m}_i, g_{\alpha_{-i},\overline{m}}) > h_{\alpha(i)}(t, g_{\alpha_{-i},\overline{m}}) - \frac{\varepsilon}{3} \tag{17}$$

for all $i \in N$ where $g_{\alpha,\overline{m}}$ denotes the weight function relative to attribute function α and strategy vector $\overline{m}$.

Consider a change in population from (N, α) to $(N, \overline{\alpha})$ where,

$$\overline{\alpha}(R(i)) = \alpha(i) \text{ for all } i \in N. \tag{18}$$

That is, instead of player i taking the strategy of player $R(i)$, we now think of player $R(i)$ as taking the attribute of player i. Denote by $g_{\overline{\alpha},m}$ the weight function relative to attribute function $\overline{\alpha}$ and strategy vector m. By the assumption of continuity in attributes and (16) we have that,

$$\left|h_{\alpha(i)}(t, g_{\alpha_{-i},m}) - h_{\overline{\alpha}(i)}(t, g_{\overline{\alpha}_{-i},m})\right| < \frac{\varepsilon}{9}$$

for all $t \in \Delta(S)$ and all $i \in N$. Given that m is a Nash ε-equilibrium,

$$h_{\alpha(i)}(m_i, g_{\alpha_{-i},m}) > h_{\alpha(i)}(t, g_{\alpha_{-i},m}) - \frac{\varepsilon}{9}$$

for all $t \in \Delta(S)$ and all $i \in N$. Therefore,

$$\begin{aligned} h_{\overline{\alpha}(i)}(m_i, g_{\overline{\alpha}_{-i},m}) \;\; > \;\; & h_{\overline{\alpha}(i)}(t, g_{\overline{\alpha}_{-i},m}) - \frac{\varepsilon}{9} && (19) \\ & - \left|h_{\alpha(i)}(t, g_{\alpha_{-i},m}) - h_{\overline{\alpha}(i)}(t, g_{\overline{\alpha}_{-i},m})\right| \\ & - \left|h_{\alpha(i)}(m_i, g_{\alpha_{-i},m}) - h_{\overline{\alpha}(i)}(m_i, g_{\overline{\alpha}_{-i},m})\right| && (20) \end{aligned}$$

and thus,

$$h_{\overline{\alpha}(i)}(m_i, g_{\overline{\alpha}_{-i},m}) > h_{\overline{\alpha}(i)}(t, g_{\overline{\alpha}_{-i},m}) - \frac{\varepsilon}{3} \tag{21}$$

for all $i \in N$ and for all $t \in \Delta(S)$. In other words, the strategy vector m, which we knew was a Nash $\frac{\varepsilon}{9}$-equilibrium for game $\Gamma(N, \alpha)$, is a Nash $\frac{\varepsilon}{3}$-equilibrium in game $\Gamma(N, \overline{\alpha})$.

We note, by (15) and (18) that, for any $i \in N$

$$\left(\overline{\alpha}(R(i)), m_{R(i)}\right) = \left(\alpha(i), m_{R(i)}\right) = \left(\alpha(i), \overline{m}_i\right). \tag{22}$$

It follows from (22) that

$$g_{\overline{\alpha},m} = g_{\alpha,\overline{m}}$$

and that

$$g_{\overline{\alpha}_{-R(i)},m} = g_{\alpha_{-i},\overline{m}}$$

for all $i \in N$. Furthermore,

$$h_{\overline{\alpha}(R(i))}(t, g_{\overline{\alpha}_{-R(i)},m}) = h_{\alpha(i)}(t, g_{\alpha_{-i},\overline{m}})$$

for all $i \in N$ and all $t \in \Delta(S)$. Thus, as a consequence of (21),

$$h_{\alpha(i)}(\overline{m}_i, g_{\alpha_{-i},\overline{m}}) > h_{\alpha(i)}(t, g_{\alpha_{-i},\overline{m}}) - \frac{\varepsilon}{3}$$

for all $i \in N$ and for all $t \in \Delta(S)$. (To see this last equation it may be helpful to recall that $R(i)$ is just some player $i' \in N$.) This demonstrates (17).■

Lemma 4: Suppose the pregame $\mathcal{G}$ satisfies convex separation and the large game property. Then for any $\varepsilon > 0$, any $B \geq 1$, and any partition of Ω into a finite number of subsets $\Omega_1, ..., \Omega_A$ there exists a real number $\eta_4(\varepsilon, B, A)$ such that for any game $\Gamma(N, \alpha)$ induced from $\mathcal{G}$ where $|N| > \eta_4(\varepsilon, B, A)$ and *profile*$(N,\alpha)(\omega) \leq B$ for all $\omega \in \Omega$, if there exists a Nash $\frac{\varepsilon}{3}$-equilibrium in pure strategies m with the property that,

$$\text{when } \alpha(i), \alpha(j) \in \Omega_a \text{ for some } a \text{ and } m_{ik} = 1 \text{ and } m_{i\overline{k}} = 1 \quad (23)$$
$$\text{where } k < \overline{k} \text{ then } \alpha^\nu(i) \leqq \alpha^\nu(j), \quad (24)$$

then there exists a Nash ε-equilibrium in pure strategies $\overline{m}$ with the property that,

$$\text{when } \alpha(i), \alpha(j) \in \Omega_a \text{ for some } a \text{ and } \overline{m}_{ik} = 1 \text{ and } \overline{m}_{i\overline{k}} = 1 \quad (25)$$
$$\text{where } k < \overline{k} \text{ then } \alpha^\nu(i) < \alpha^\nu(j). \quad (26)$$

Proof of Lemma 4: Suppose not. Then, there is some $\varepsilon > 0$ and some $B \geq 1$ such that for each integer ν there is a population (N^ν, α^ν), where $|N^\nu| > \nu$, and induced game $\Gamma(N^\nu, \alpha^\nu)$, which satisfy the conditions of the statement of the Theorem but for which there exists no Nash ε-equilibrium with the required properties. For each ν denote by m^ν a Nash $\frac{\varepsilon}{3}$-equilibrium in pure strategies m with the property (23).

Consider, for each ν, changing the strategy vector m^ν to a degenerate strategy vector $\overline{m}^\nu$ which satisfies the following conditions,

(i) for any $i, j \in N^\nu$, if $\alpha^\nu(i) = \alpha^\nu(j)$ then $\overline{m}_i^\nu = \overline{m}_j^\nu$,

(ii) for all $\omega \in \alpha^\nu(N^\nu)$ there exists some player $i \in N^\nu$ such that $\alpha^\nu(i) = \omega$ and $\overline{m}_i^\nu = m_i^\nu$.

That is, we consider changing the strategy vector m^ν to a strategy vector $\overline{m}^\nu$ such that any two players with the same attribute play the same pure strategy. We require also that for any attribute $\omega \in \alpha^\nu(N^\nu)$ there is at least one player who is playing the same pure strategy under $\overline{m}^\nu$ as they did under m^ν. Such a strategy $\overline{m}^\nu$ satisfies property (25) but it remains to show it is a Nash ε-equilibrium.

For each ν let $\Omega^{\nu *}$ be the set of attributes such that $\omega \in \Omega^{\nu *}$ if and only there exists players $i, j \in N^\nu$ such that $\alpha^\nu(i) = \alpha^\nu(j)$ and $m_i^\nu \neq m_j^\nu$. It is clear that in changing from m^ν to $\overline{m}^\nu$ only those players with attributes $\alpha(i) \in \Omega^{\nu *}$ could have changed strategy. The ordering, as implied by (23), implies that there are at most $K-1$ attributes ω belonging to $\Omega^{\nu *} \cap \Omega_a$ for each a. We can therefore, for each ν, put an upper bound on the number of players $i \in N^\nu$ for whom $m_i^\nu \neq \overline{m}_i^\nu$ of $(K-1)A(B-1)$.

Let $g^\nu_{\alpha^\nu, m^\nu}$ and $g^\nu_{\alpha^\nu, \overline{m}^\nu}$ denote respectively the weight functions relative to attribute function α^ν and strategy vectors m^ν and $\overline{m}^\nu$. Since the pregame $\mathcal{G}$ satisfies the large game property, we may choose non-negative real numbers $\delta_g\left(\frac{\varepsilon}{3}\right)$ and $\eta\left(\frac{\varepsilon}{3}\right)$ such that for any $n > \eta(\frac{\varepsilon}{3})$, any game $\Gamma(N, \alpha) \in \gamma(\mathcal{G}, n)$ satisfies $\delta_g(\frac{\varepsilon}{3}), (\frac{\varepsilon}{3})$-global interaction. Given $(K-1)A(B-1)$ is a finite number, independent of the size of the player set there must exists some ν_1 such that for all $\nu > \nu_1$,

$$\left| h_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i}, \overline{m}^\nu}) - h_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i}, m^\nu}) \right| < \frac{\varepsilon}{3}$$

for all $i \in N^\nu$ and all $t \in \Delta(S)$. Given that m is a Nash $\frac{\varepsilon}{3}$-equilibrium,

$$h_{\alpha^\nu(i)}(m_i^\nu, g^\nu_{\alpha^\nu_{-i}, m^\nu}) > h_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i}, m^\nu}) - \frac{\varepsilon}{3}$$

for all $t \in \Delta(S)$ and all $i \in N$. Thus, for all $\nu > \nu_1$ and all those players $i \in N^\nu$ for whom $m_i^\nu = \overline{m}_i^\nu$,

$$\begin{aligned} h_{\alpha^\nu(i)}(\overline{m}_i^\nu, g^\nu_{\alpha^\nu_{-i}, \overline{m}^\nu}) \quad > \quad & h_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i}, \overline{m}^\nu}) - \frac{\varepsilon}{3} \\ & - \left| h_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i}, \overline{m}^\nu}) - h_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i}, m^\nu}) \right| \\ & - \left| h_{\alpha^\nu(i)}(\overline{m}_i^\nu, g^\nu_{\alpha^\nu_{-i}, \overline{m}^\nu}) - h_{\alpha^\nu(i)}(\overline{m}_i^\nu, g^\nu_{\alpha^\nu_{-i}, m^\nu}) \right| \end{aligned}$$

and therefore,

$$h_{\alpha^\nu(i)}(\overline{m}_i^\nu, g^\nu_{\alpha^\nu_{-i},\overline{m}^\nu}) > h_{\alpha^\nu(i)}(t, g^\nu_{\alpha^\nu_{-i},\overline{m}^\nu}) - \varepsilon$$

for all $t \in \Delta(S)$, for all $i \in N$ and for all $\nu > \nu_1$. Furthermore, condition (ii) above allows us to extend this to all $i \in N^\nu$ including those for whom $m_i^\nu \neq \overline{m}_i^\nu$. This gives the desired contradiction.■

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