COST MONOTONICITY, CONSISTENCY AND MINIMUM COST SPANNING TREE GAMES

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No 629

WARWICK ECONOMIC RESEARCH PAPERS

THE UNIVERSITY OF WARWICK

DEPARTMENT OF ECONOMICS

Cost Monotonicity, Consistency and Minimum Cost Spanning Tree Games

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July 2001

Revision : January 2002

Abstract

We propose a new cost allocation rule for minimum cost spanning tree games. The new rule is a core selection and also satisfies cost monotonicity. We also give characterization theorems for the new rule as well as the much-studied Bird allocation. We show that the principal difference between these two rules is in terms of their consistency properties. JEL Classification Numbers: D7

Keywords: spanning tree, cost allocation, core selection, cost monotonicity, consistency.

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1 Introduction

There is a wide range of economic contexts in which "aggregate costs" have to be allocated amongst individual agents or components who derive the benefits from a common project. A firm has to allocate overhead costs amongst its different divisions. Regulatory authorities have to set taxes or fees on individual users for a variety of services. Partners in a joint venture must share costs (and benefits) of the joint venture. In many of these examples, there is no external force such as the market, which determines the allocation of costs. Thus, the final allocation of costs is decided either by mutual agreement or by an "arbitrator" on the basis of some notion of fairness.

A central problem of cooperative game theory is how to divide the benefits of cooperation amongst individual players or agents. Since there is an obvious analogy between the division of costs and that of benefits, the tools of cooperative game theory have proved very useful in the analysis of cost allocation problems.[1] Much of this literature has focused on "general" cost allocation problems, so that the ensuing cost game is identical to that of a typical game in characteristic function form. This has facilitated the search for "appropriate" cost allocation rules considerably given the corresponding results in cooperative game theory.

The purpose of this paper is the analysis of allocation rules in a special class of cost allocation problems. The common feature of these problems is that a group of users have to be connected to a single supplier of some service. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network. There is a non-negative cost of connecting each pair of users (towns) as well as a cost of connecting each user (town) to the common supplier (power plant). A cost game arises because cooperation reduces aggregate costs - it may be cheaper for town A to construct a link to town B which is "nearer" to the power plant, rather than build a separate link to the plant. Clearly, an efficient network must be a tree, which connects all users to the common supplier. That is

[1] Moulin[10] and Young [15] are excellent surveys of this literature.

why these games have been labelled minimum cost spanning tree games.

Notice that in the example mentioned above, it makes sense for town B to demand some compensation from A in order to let A use its own link to the power plant. But, how much should A agree to pay? This is where both strategic issues as well as considerations of fairness come into play. Of course, these issues are present in any surplus-sharing or cost allocation problem. What is special in our context is that the structure of the problem implies that the domain of the allocation rule will be smaller than that in a more general cost problem. This smaller domain raises the possibility of constructing allocation rules satisfying "nice" properties which cannot always be done in general problems. For instance, it is known that the core of a minimum cost spanning tree game is always non-empty.[2]

Much of the literature on minimum cost spanning tree games has focused on algorithmic issues.[3] In contrast, the derivation of attractive cost allocation rules or the analysis of axiomatic properties of di¤erent rules has received correspondingly little attention.[4] This provides the main motivation for this paper. We show that the allocation rule proposed by Bird [1], which always selects an allocation in the core of the game, does not satisfy cost monotonicity. Cost monotonicity is an extremely attractive property, and requires that the cost allocated to agent i does not increase if the cost of a link involving i goes down, nothing else changing. Notice that if a rule does not satisfy cost monotonicity, then it may not provide agents with the appropriate incentives to reduce the costs of constructing links.

The cost allocation rule, which coincides with the Shapley value of the cost game, satis…es cost monotonicity. However, the Shapley value is unlikely to be used in these contexts because it is not in the core. This implies that some group of agents may well …nd it bene…cial to construct their own network if the Shapley value is used to allocate costs. We show that cost monotonicity and the core are not mutually exclusive[5] by constructing a new rule, which satis…es cost monotonicity and also

[2] See, for instance, Bird[1], Granot and Huberman [7].

[3] See for instance Granot and Granot [5], Granot and Huberman [6], Graham and Hell [4].

[4] Exceptions are Feltkampf [3], Kar [9]. See Sharkey [13] for a survey of this literature.

[5] This is where the small domain comes in useful. Young [15] shows that in the context of

selects an allocation in the core of the game.

We then go on to provide axiomatic characterizations of the Bird rule as well as the new rule constructed by us. An important type of axiom used by us is closely linked to the reduced game properties which have been extensively used in the axiomatic characterization of solutions in cooperative game theory.[6] These are consistency conditions, which place restrictions on how solutions of different but related games defined on different player sets behave. We show that Bird rule and the new allocation rule satisfy different consistency conditions.

The plan of this paper is the following. In section 2, we define the basic structure of minimum cost spanning tree games. The main purpose of Section 3 is the construction of the new rule as well as the proof that it satisfies cost monotonicity and also selects an allocation in the core of the game. Section 4 contains the characterization results.

2 The Framework

Let $\mathcal{N} = \{1, 2, \dots\}$ be the set of all possible agents. We are interested in graphs or networks where the nodes are elements of a set $N \cup \{0\}$, where $N \subset \mathcal{N}$, and 0 is a distinguished node which we will refer to as the source or root.

Henceforth, for any set $N \subset \mathcal{N}$, we will use N^+ to denote the set $N \cup \{0\}$.

A typical graph over N^+ will be represented by $g_N = \{(ij) | i, j \in N^+\}$. Two nodes i and $j \in N^+$ are said to be connected in g_N if $\exists (i_1 i_2), (i_2 i_3), \dots, (i_{n-1} i_n)$ such that $(i_k i_{k+1}) \in g$, $1 \leq k \leq n-1$, and $i_1 = i, i_n = j$. A graph g_N is called connected over N^+ if i, j are connected in g_N for all $i, j \in N^+$. The set of connected graphs over N^+ is denoted by Γ_N.

Consider any $N \subset \mathcal{N}$, where $\#N = n$. A cost matrix $C = (c_{ij})$ represents the cost of direct connection beween any pair of nodes. That is, c_{ij} is the cost of directly

transferable utility games, there is no solution concept which picks an allocation in the core of the game when the latter is nonempty and also satisfies a property which is analogous to cost monotonicity.

[6] See Peleg[11], Thomson [14].

connecting any pair $i, j \in N^+$. We assume that each $c_{ij} > 0$ whenever $i \neq j$. We also adopt the convention that for each $i \in N^+$, $c_{ii} = 0$. So, each cost matrix is nonnegative, symmetric and of order $n + 1$. The set of all cost matrices for N is denoted by $\mathcal{C}_N$. However, we will typically drop the subscript N whenever there is no cause for confusion about the set of nodes.

Consider any $C \in \mathcal{C}_N$. A minimum cost spanning tree (m.c.s.t.) over N^+ satisfies $g_N = \operatorname{argmin}_{g \in \mathcal{G}_N} \sum_{(ij) \in g} c_{ij}$. Note that an m.c.s.t. need not be unique. Clearly a minimum cost spanning network must be a tree. Otherwise, we can delete an extra edge and still obtain a connected graph at a lower cost.

An m.c.s.t. corresponding to $C \in \mathcal{C}_N$ will typically be denoted by $g_N(C)$.

Example 1: Consider a set of three rural communities $\{A, B, C\}$, which have to decide whether to build a system of irrigation channels to an existing dam, which is the source or root. Each community has to be connected to the dam in order to draw water from the dam. However, some connection(s) could be indirect. For instance, community A could be connected directly to the dam, while B and C are connected to A, and hence indirectly to the source.

There is a cost of building a channel connecting each pair of communities, as well as a channel connecting each community directly to the dam. Suppose, these costs are represented by a cost matrix C.

$$C = \begin{pmatrix} 0 & 2 & 4 & 1 \\ 2 & 0 & 1 & 3 \\ 4 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix}$$

The minimum cost of building the system of irrigation channels will be 4 units. Our object of interest in this paper is to see how the total cost of 4 units is to be distributed amongst A, B and C.

This provides the motivation for the next definition.

Definition 1: A cost allocation rule is a family of functions $\{\psi^N\}_{N \subseteq \mathcal{N}}$ with $N \subseteq \mathcal{N}$, $\psi^N : \mathcal{C}_N \to \Re^N_+$ satisfying $\sum_{i \in N} \psi^N_i(C) \geq \sum_{(ij) \in g_N(C)} c_{ij}$ for all $C \in \mathcal{C}_N$.

We will drop the superscript N for the rest of the paper.

So, given any set of nodes N and any cost matrix C of order $(|N|+1)$, a cost allocation rule specifies the costs attributed to agents in N. Note that the source 0 is not an active player, and hence does not bear any part of the cost.

A cost allocation rule can be generated by any single-valued game-theoretic solution of a transferable utility game. Thus, consider the transferable utility game generated by considering the aggregate cost of a minimum cost spanning tree for each coalition $S \subseteq N$. Given C and $S \subseteq N$, let C_S be the cost matrix restricted to S. Then, consider a m.c.s.t. $g_S(C_S)$ over S^+, and the corresponding minimum cost of connecting S to the source. Let this cost be denoted c_S. For each $N \subset \mathcal{N}$, this defines a cost game $(N; c)$ where for each $S \subseteq N$, $c(S) = c_S$. That is, c is the cost function, and is analogous to a TU game. Then, if Ψ is a single-valued solution, $\Psi(N; c)$ can be viewed as the cost allocation rule corresponding to the cost matrix which generates the cost function c.[7]

One particularly important game-theoretic property, which will be used subsequently is that of the core. If a cost allocation rule does not always pick an element in the core of the game, then some subset of N will find it profitable to break up N and construct its own minimum cost tree. This motivates the following definition.

Definition 2: A cost allocation rule ϕ is a *core selection* if for all $N \subseteq \mathcal{N}$ and for all $C \in \mathcal{C}_N$, $\sum_{i \in S} \phi_i(C) \leq c(S)$, where $c(S)$ is the cost of the m.c.s.t. for S, $\forall S \subseteq N$.

However, cost allocation rules can also be defined without appealing to the underlying cost game. For instance, this was the procedure followed by Bird [1]. In order to describe his procedure, we need some more notations.

The (unique) path from i to j in tree g, is a set $U(i; j; g) = \{i_1, i_2, \ldots, i_K\}$, where each pair $(i_{k-1} i_k) \in g$, and $i_1, i_2, \ldots, i_K$ are all distinct agents with $i_1 = i, i_K = j$. The predecessor set of an agent i in g is defined as $P(i; g) = \{k \mid k \neq i, k \in U(0; i; g)\}$. The immediate predecessor of agent i, denoted by $\alpha(i)$, is the agent who comes immediately before i, that is, $\alpha(i) \in P(i; g)$ and $k \in P(i; g)$ implies either

[7] See Kar [9] for an axiomatic characterization of the Shapley value in m.c.s.t. games.

$k = \alpha(i)$ or $k \in P(\alpha(i); g)$.[8] The followers of agent i, are those agents who come immediately after i; $F(i) = \{j | \alpha(j) = i\}$.

Bird's method is defined with respect to a specific tree. Let g_N be some m.c.s.t. corresponding to the cost matrix C. Then,

$$B_i(C) = c_{i\alpha(i)} \; \forall i \in N.$$

So, in the Bird allocation, each node pays the cost of connecting to its immediate predecessor in the appropriate m.c.s.t.

Notice that this does not define an allocation rule if C gives rise to more than one m.c.s.t. However, when C does not induce a unique m.c.s.t., one can still use Bird's method on each m.c.s.t. derived from C and then take some convex combination of the allocations corresponding to each m.c.s.t. as the cost allocation rule. In general, the properties of the resulting cost allocation rule will not be identical to those of the cost allocation rule given by Bird's method on cost matrices, which induce unique m.c.s.t. s.

In section 4, we will use two domain restrictions on the set of permissible cost matrices. These are defined below.

Definition 3: $\mathcal{C}^1 = \{C \in \mathcal{C} | C$ induces a unique m.c.s.t. $\forall N \subseteq \mathcal{N}\}$.

Definition 4: $\mathcal{C}^2 = \{C \in \mathcal{C}^1 |$ no two edges of the unique m.c.s.t. have the same cost$\}$.

Notice if C is not in $\mathcal{C}^2$, then even a "small" perturbation of C produces a cost matrix which is in $\mathcal{C}^2$. So, even the stronger domain restriction is relatively mild, and the permissible sets of cost matrices are large.

3 Cost Monotonicity

The Bird allocation is an attractive rule because it is a core selection. In addition, it is easy to compute. However, it fails to satisfy cost monotonicity.

Definition 5: Fix $N \subseteq \mathcal{N}$. Let $i, j \in N^+$, and $C, C' \in \mathcal{C}_N$ be such that $c_{kl} = c'_{kl}$ for all $(kl) \neq (ij)$ and $c_{ij} > c'_{ij}$. Then, the allocation rule ψ satisfies Cost Monotonicity

[8] Note that since g is a tree, the immediate predecessor must be unique.

if for all $m \in N \setminus \{i,j\}$, $\psi_m(C) \geq \psi_m(C')$.

Cost monotonicity is an extremely appealing property. The property applies to two cost matrices which differ only in the cost of connecting the pair (ij), c'_{ij} being lower than c_{ij}. Then, cost monotonicity requires that no agent in the pair $\{i,j\}$ be charged more when the cost matrix changes from C to C'.

Despite its intuitive appeal, cost monotonicity has a lot of bite.[9] The following example shows that the Bird allocation rule does not satisfy cost monotonicity.

Example 2: Let $N = \{1,2\}$. The two cost matrices are specified below.

(i) $c_{01} = 4; c_{02} = 4.5; c_{12} = 3.$

(ii) $c'_{01} = 4; c'_{02} = 3.5; c'_{12} = 3.$

Then, $B_1(C) = 4; B_2(C) = 3$, while $B_1(C') = 3; B_2(C') = 3.5$. So, 2 is charged more when the cost matrix is C' although $c'_{02} < c_{02}$ and the costs of edges involving 1 remain the same.

The cost allocation rule corresponding to the Shapley value of the cost game does satisfy cost monotonicity. However, it does not always select an outcome which is in the core of the cost game. Our main purpose in this section is to define a new allocation rule which will be a core selection and satisfy cost monotonicity. We are able to do this despite the impossibility result due to Young because of the special structure of minimum cost spanning tree games - these are a strict subset of the class of balanced games. Hence, monotonicity in the context of m.c.s.t. games is a weaker restriction.

We describe an algorithm whose outcome will be the cost allocation prescribed by the new rule. Our rule is defined for all cost matrices in $\mathcal{C}$. However, in order to economise on notation, we describe the algorithm for a cost matrix in $\mathcal{C}^2$. We then indicate how to construct the rule for all cost matrices.

[9] In fact, Young [15] shows that an analogous property in the context of TU games cannot be satisfied by any solution which selects a core outcome in balanced games.

Fix some $N \subset \mathcal{N}$, and choose some matrix $C \in \mathcal{C}^2_N$. Also, for any $A \subset N$, define A_c as the complement of A in N^+. That is $A_c = N^+ \setminus A$.

The algorithm proceeds as follows.

Let $A^0 = \{0\}$, $g^0 = \emptyset$, $t^0 = 0$.

Step 1: Choose the ordered pair (a^1b^1) such that $(a^1b^1) = \text{argmin}_{(i,j)\in A^0 \times A^0_c} c_{ij}$.

Define $t^1 = \max(t^0, c_{a^1b^1})$, $A^1 = A^0 \cup \{b^1\}$, $g^1 = g^0 \cup \{(a^1b^1)\}$.

Step k: Define the ordered pair $(a^kb^k) = \text{argmin}_{(i,j)\in A^{k-1} \times A^{k-1}_c} c_{ij}$, $A^k = A^{k-1} \cup \{b^k\}$, $g^k = g^{k-1} \cup \{(a^kb^k)\}$, $t^k = \max(t^{k-1}, c_{a^kb^k})$. Also,

$$\psi^*_{b^{k-1}}(C) = \min(t^{k-1}, c_{a^kb^k}). \tag{1}$$

The algorithm terminates at step $\#N = n$. Then,

$$\psi^*_{b^n}(C) = t^n \tag{2}$$

The new allocation rule ψ^* is described by equations (1), (2).

At any step k, A^{k-1} is the set of nodes which have already been connected to the source 0. Then, a new edge is constructed at this step by choosing the lowest-cost edge between a node in A^{k-1} and nodes in A^{k-1}_c. The cost allocation of b^{k-1} is decided at step k. Equation (1) shows that b^{k-1} pays the minimum of t^{k-1}, which is the maximum cost amongst all edges which have been constructed in previous steps, and $c_{a^kb^k}$, the edge being constructed in step k. Finally, equation (2) shows that b^n, the last node to be connected, pays the maximum cost.[10]

Remark 1: The algorithm has been described for cost matrices in $\mathcal{C}^2$. Suppose that $C \notin \mathcal{C}^2$. Then, the algorithm is not well-defined because at some step k, two distinct edges (a^kb^k) and $(\hat{a}^k\hat{b}^k)$ may minimise the cost of connecting nodes in A^{k-1} and A^{k-1}_c. But, there is an easy way to extend the algorithm to deal with matrices not in $\mathcal{C}^2$. Let σ be a strict ordering over N. Then, σ can be used as a tie-breaking rule - for instance, choose (a^kb^k) if b^k is ranked over $\hat{b}^k$ according to σ. Any such tie-breaking rule makes the algorithm well-defined. Now, let Σ be the set of all strict

[10] From Prim[12], it follows that g^n is also the m.c.s.t. corresponding to C.

orderings over N. Then, the eventual cost allocation is obtained by taking the simple average of the "component" cost allocations obtained for each ordering $\sigma \in \Sigma$. That is, for any $\sigma \in \Sigma$, let $\psi^*_\sigma(C)$ denote the cost allocation obtained from the algorithm when σ is used as the tie-breaking rule. Then,

$$\psi^*(C) = \frac{1}{\#\Sigma} \sum_{\sigma \in \Sigma} \psi^*_\sigma(C). \tag{3}$$

We illustrate this procedure in Example 5 below.

Remark 2: Notice that ψ^* only depends on the m.c.s.t.s corresponding to any cost matrix. This property of Tree Invariance adds to the computational simplicity of the rule, and distinguishes it from rules such as the Shapley Value and nucleolus.

We now construct a few examples to illustrate the algorithm.

Example 3: Suppose C^1 is such that the m.c.s.t. is unique and is a line. That is, each node has at most one follower. Then the nodes can be labelled $a_0, a_1, a_2, \ldots, a_n$, where $a_0 = 0$, $\#N = n$, with the predecessor set of a_k, $P(a_k, g) = \{0, a_1, \ldots, a_{k-1}\}$. Then,

$$\forall k < n, \psi^*_{a_k}(C^1) = \min(\max_{0 \le t < k} c_{a_t a_{t+1}}, c_{a_k a_{k+1}}) \tag{4}$$

and

$$\psi^*_{a_n}(C^1) = \max_{0 \le t < n} c_{a_t a_{t+1}} \tag{5}$$

Example 4: Let $N = \{1, 2, 3, 4\}$, and

$$C^2 = \begin{pmatrix} 0 & 4 & 5 & 5 & 5 \\ 4 & 0 & 2 & 1 & 5 \\ 5 & 2 & 0 & 5 & 5 \\ 5 & 1 & 5 & 0 & 3 \\ 5 & 5 & 5 & 3 & 0 \end{pmatrix}$$

There is only one m.c.s.t. of C^2.

Step 1: We have $(a^1 b^1) = (01)$, $t^1 = c_{01} = 4$; $A^1 = \{0, 1\}$.

Step 2: Next, $(a^2 b^2) = (13)$, $\psi^*_1(C^2) = \min(t^1, c_{13}) = 1$, $t^2 = \max(t^1, c_{13}) = 4$, $A^2 = \{0, 1, 3\}$.

Step 3: We now have $(a^3 b^3) = (12)$, $\psi_3^*(C^2) = \min(t^2; c_{12}) = 2$, $t^3 = \max(t^2; c_{12}) = 4$, $A^3 = \{0; 1; 2; 3\}$:

Step 4: Next, $(a^4 b^4) = (34)$, $\psi_2^*(C^2) = \min(t^3; c_{34}) = 3$, $t^4 = \max(t^3; c_{34}) = 4$, $A^4 = \{0; 1; 2; 3; 4\}$.

Since $A^4 = N^+$, $\psi_4^*(C^2) = t^4 = 4$, and the algorithm is terminated.

So, $\psi^*(C^2) = (1; 3; 2; 4)$. This example shows that it is not necessary for a node to be assigned the cost of its preceding or following edge. Here 2 pays the cost of the edge (34), while 3 pays the cost of the edge (12).[11]

The next example involves a cost matrix which has more than one m.c.s.t., with one of the trees having edges which cost the same.

Example 5: Let $N = \{1; 2; 3\}$, and

$$C^3 = \begin{pmatrix} 0 & 4 & 4 & 5 \\ 4 & 0 & 2 & 2 \\ 4 & 2 & 0 & 5 \\ 5 & 2 & 5 & 0 \end{pmatrix}$$

C^3 has two m.c.s.t.s - $g_N = \{(01); (12); (13)\}$ and $g_N^1 = \{(02); (12); (13)\}$. Also, in g_N, the edges (12) and (13) have the same cost.[12]

Suppose the algorithm is first applied to g_N. Then, we have $b^1 = 1$. In step 2, $a^2 = 1$, but b^2 can be either 2 or 3. Taking each in turn, we get the vectors $x^1 = (2; 2; 4)$ and $x^2 = (2; 4; 2)$.

Now, consider g_N^1, which is a line. So, as we have described in Example 3, the resulting cost allocation is $\hat{x} = (2; 2; 4)$.

The algorithm will "generate" g_N^1 instead of g_N for all $\sigma \in \Sigma$ which ranks 2 over 1. Hence, the "weight" attached to g_N^1 is half. Similarly, the weight attached to x^1 and x^2 must be one-sixth and one-third.

Hence, $\psi^*(C^3) = (2; \frac{8}{3}; \frac{10}{3})$.

Given C, let $g_N(C)$ be the (unique) m.c.s.t. of C. Suppose $g_N(C) = g_{N_1} \cup g_{N_2} \ldots \cup g_{N_K}$, where each g_{N_k} is the m.c.s.t. on N_k for the cost matrix C restricted

[11] For m.c.s.t. of C^2 see Fig. 1.

[12] For m.c.s.t. of C^3 see Fig. 2.

to N_k^+, with $\cup_{k=1}^K N_k = N$ and $N_i \cap N_j = \emptyset\ \forall i \neq j$. We will call such a partition the m.c.s.t. partition of $g_N(C)$.

We now show that ψ^* is a core selection and also satisfies Cost Monotonicity.

Theorem 1: The cost allocation rule ψ^* satisfies Cost Monotonicity and is a core selection.

Proof: We first show that ψ^* satisfies Cost Monotonicity.

Fix any $N \subset \mathcal{N}$. We give our proof for cost matrices in $\mathcal{C}^2$, and then indicate how the proof can be extended to cover all cost matrices. Let $C, \hat{C} \in \mathcal{C}^2$ be such that for some $i, j \in N^+$, $c_{ij} > \hat{c}_{ij}$, and $c_{kl} = \hat{c}_{kl}$ for all other pairs (kl). We need to show that $\psi_k^*(C) \geq \psi_k^*(\hat{C})$ for $k \in N \setminus \{i, j\}$.

In describing the algorithm which is used in constructing ψ^*, we fixed a specific cost matrix, and so did not have to specify the dependence of A^k, t^k, a^k, b^k etc. on the cost matrix. But, now we need to distinguish between these entities for the two cost matrices C and $\hat{C}$. We adopt the following notation in the rest of the proof of the thorem. Let A^k, t^k, a^k, b^k, g_N etc. refer to the cost matrix C, while $\hat{A}^k, \hat{t}^k, \hat{a}^k, \hat{b}^k, \hat{g}_N$ etc. will denote the entities corresponding to $\hat{C}$.

Case 1: $(ij) \notin \hat{g}_N$.

Then, $\hat{g}_N = g_N$. Since the cost of all edges in g_N remain the same, $\psi_k^*(\hat{C}) = \psi_k^*(C)$ for all $k \in N$.

Case 2: $(ij) \in \hat{g}_N$.

Without loss of generality, let i be a predecessor of j in $\hat{g}_N$. Since the source never pays anything, we only consider the case where i is not the source.

Suppose $i = \hat{b}^{k-1}$. As the cost of all other edges remain the same, $A^{k-1} = \hat{A}^{k-1}$ and $t^{k-1} = \hat{t}^{k-1}$. Now, $\psi_i^*(\hat{C}) = \min(\hat{t}^{k-1}, \hat{c}_{\hat{a}^k \hat{b}^k})$ and $\psi_i^*(C) = \min(t^{k-1}, c_{a^k b^k})$. Since $\hat{c}_{\hat{a}^k \hat{b}^k} \leq c_{a^k b^k}$, $\psi_i^*(\hat{C}) \leq \psi_i^*(C)$.

We now show that $\psi_j^*(\hat{C}) \leq \psi_j^*(C)$. Let $j = b^l$ and $j = \hat{b}^m$. Note that $l \geq m$, and that $\hat{A}^m \subseteq A^l$, and $t^l \geq \hat{t}^m$.

Now, $\psi_j^*(\hat{C}) = \min(\hat{t}^m, \hat{c}_{\hat{a}^{m+1} \hat{b}^{m+1}})$, while $\psi_j^*(C) = \min(t^l, c_{a^{l+1} b^{l+1}})$. Since $t^l \geq \hat{t}^m$, we only need to show that $\hat{c}_{\hat{a}^{m+1} \hat{b}^{m+1}} \leq c_{a^{l+1} b^{l+1}}$.

Case 2(a): Suppose $a^{l+1} \in \bar{A}^m$. Since $b^{l+1} \in N^+ \setminus \bar{A}^m$, $\bar{c}_{\bar{a}^{m+1}\bar{b}^{m+1}} \leq \bar{c}_{a^{l+1}b^{l+1}} \leq c_{a^{l+1}b^{l+1}}$.

Case 2(b): Suppose $a^{l+1} \notin \bar{A}^m$. Then, $a^{l+1} \neq j$. Also, $a^{l+1} \in A^l$, and so

$$c_{a^{l+1}b^{l+1}} \geq c_{a^l b^l} \tag{6}$$

We need to consider two sub-cases.

Case 2(bi): $a^l \in A^{l-1} \setminus \bar{A}^{m-1}$.

Then, since $A^l = A^{l-1} \cup \{j\}$ and $\bar{A}^m = \bar{A}^{m-1} \cup \{j\}$, $a^l \in A^l \setminus \bar{A}^m$.

Now since $j \in \bar{A}^m$ and $a^l \notin \bar{A}^m$, $\bar{c}_{\bar{a}^{m+1}\bar{b}^{m+1}} \leq \bar{c}_{ja^l} \leq c_{ja^l} = c_{a^l b^l}$. Using equation 6, $c_{a^{l+1}b^{l+1}} \geq c_{a^l b^l} \geq \bar{c}_{\bar{a}^{m+1}\bar{b}^{m+1}}$.

Case 2(bii): $a^l \in \bar{A}^{m-1} = A^{m-1}$.

Then, $c_{a^l b^l} \geq c_{a^m b^m}$ since $m \leq l$.

Also, $\bar{A}^m \subseteq A^l$ and $a^{l+1} \in A^l \setminus \bar{A}^m$ imply that $\#\bar{A}^m < \#A^l$. That is, $l > m$. So, $b^m \neq j = b^l$. This implies $b^m \notin (\bar{A}^{m-1} \cup \{j\}) = \bar{A}^m$.

Now, $a^m \in A^{m-1} = \bar{A}^{m-1}$. So, $a^m \in \bar{A}^m$. But $a^m \in \bar{A}^m$ and $b^m \notin \bar{A}^m$ together imply that $\bar{c}_{\bar{a}^{m+1}\bar{b}^{m+1}} \leq \bar{c}_{a^m b^m} \leq c_{a^m b^m}$.

So, using equation 6, $\bar{c}_{\bar{a}^{m+1}\bar{b}^{m+1}} \leq c_{a^m b^m} \leq c_{a^l b^l} \leq c_{a^{l+1}b^{l+1}}$.

Hence, ψ^* satisfies cost monotonicity.[13]

We now show that for all $C \in \mathcal{C}$, $\psi^*(C)$ is an element in the core of the cost game corresponding to C.

Again, we present the proof for any $C \in \mathcal{C}^2$ in order to avoid notational complications.[14] We want to show that for all $S \subseteq N$, $\sum_{i \in S} \psi^*_i(C) \leq c(S)$.

Without loss of generality, assume that for all $i \in N; b^i = i$ and denote $c_{a^k b^k} = c^k$.

Claim 1: If $S = \{1, 2, \ldots, K\}$ where $K \leq \#N$, then $\sum_{i \in S} \psi^*_i(C) \leq c(S)$.

[13] Suppose $C \notin \mathcal{C}^2$. What we have shown above is that the outcome of the algorithm for each tie-breaking rule satisfies cost monotonicity. Hence, the average must also satisfy cost monotonicity.

[14] Suppose instead that $C \notin \mathcal{C}^2$. Then, our subsequent proof shows that the outcome of the algorithm is in the core for each $\sigma \in \Sigma$. Since the core is a convex set, the average (that is, ψ^*) must be in the core if each ψ^*_σ is in the core.

Proof of Claim: Clearly, $g = \cup_{k=1}^{K}\{a^k k\}$ is a connected graph over $S \cup \{0\}$. Also, g is in fact the m.c.s.t. over S.

So, $c(S) = \sum_{k=1}^{K} c^k$. Also, $\sum_{i \in S} \psi_i^*(C) = \sum_{k=1}^{K+1} c^k - \max_{1 \leq k \leq K+1} c^k \leq \sum_{k=1}^{K} c^k = c(S)$. ■

Hence, a blocking coalition cannot consist of an initial set of integers, given our assumption that $b^k = k$ for all $k \in N$.

Now, let S be a largest blocking coalition. That is,

(i) $\sum_{i \in S} \psi_i^*(C) > c(S)$.

(ii) If $S \subset T$, then $\sum_{i \in T} \psi_i^*(C) \leq c(T)$.

There are two possible cases.

Case 1: $1 \notin S$.

Let $K = \min_{j \in S} j$. Consider $T = \{1, \ldots, K-1\}$. We will show that $S \cup T$ is also a blocking coalition, contradicting the description of S.

Now,

$$\sum_{i \in T \cup S} \psi_i^*(C) = \sum_{i \in S} \psi_i^*(C) + \sum_{i \in T} \psi_i^*(C) > c(S) + \sum_{k=1}^{K} c^k - \max_{1 \leq k \leq K} c^k \geq c(S) + \sum_{k=1}^{K} c^k - c_{0s},$$

where $(0s) \in g_S$, the m.c.s.t. of S. Note that the last inequality follows from the fact that $c^k \leq c_{0s}$ for all $k \in \{1, \ldots K\}$.

Since $g = (\cup_{k=1}^{K} a^k b^k) \cup (g_S \setminus \{(0s)\})$ is a connected graph over $(T \cup S \cup \{0\})$, $c(S) + \sum_{k=1}^{K} c^k - c_{0s} \geq c(S \cup T)$. Hence, $\sum_{i \in S \cup T} \psi_i^*(C) > c(S \cup T)$, establishing the contradiction that $S \cup T$ is a blocking coalition.

Case 2: $1 \in S$.

From the claim, S is not an initial segment of the integers. So, we can partition S into $\{S_1, \ldots, S_K\}$, where each S_k consists of consecutive integers, and $i \in S_k, j \in S_{k+1}$ implies that $i < j$. Assume $m = \max_{j \in S_1} j$ and $n = \min_{j \in S_2} j$. Note that $n > m + 1$. Define $T = \{m+1, \ldots, n-1\}$. We will show that $S \cup T$ is a blocking coalition, contradicting the assumption that S is a largest blocking coalition.

Now,

$$\begin{aligned}
\sum_{i \in S \cup T} \psi_i^*(C) &= \sum_{i \in S} \psi_i^*(C) + \sum_{i \in T} \psi_i^*(C) \\
&> c(S) + \sum_{i \in S_1 \cup T} \psi_i^*(C) - \sum_{i \in S_1} \psi_i^*(C) \\
&= c(S) + \Big(\sum_{i=1}^{n} c^i - \max_{1 \le i \le n} c^i\Big) - \Big(\sum_{i=1}^{m+1} c^i - \max_{1 \le i \le m+1} c^i\Big) \\
&= c(S) + \Big(\sum_{i=m+2}^{n} c^i - \big(\max_{1 \le i \le n} c^i - \max_{1 \le i \le m+1} c^i\big)\Big)
\end{aligned}$$

Of course, $\max_{1 \le i \le n} c^i \ge \max_{1 \le i \le m+1} c^i$. If $\max_{1 \le i \le n} c^i = \max_{1 \le i \le m+1} c^i$, then $\sum_{i \in S \cup T} \psi_i^*(C) > c(S) + \sum_{i=m+2}^{n} c^i \ge c(S \cup T)$, where the latter inequality follows from the fact that $[\cup_{k=m+2}^{n} \{(a^k b^k)\} \cup g_S]$ is a connected graph over $S \cup T \cup \{0\}$.

If $\max_{1 \le i \le n} c^i > \max_{1 \le i \le m+1} c^i$, then $\max_{1 \le i \le n} c^i = \max_{m+2 \le i \le n} c^i$. Then,

$$\begin{aligned}
\sum_{i \in S \cup T} \psi_i^*(C) &> c(S) + \max_{1 \le i \le m+1} c^i + \sum_{i=m+2}^{n} c^i - \max_{m+2 \le i \le n} c^i \\
&\ge c(S) + \sum_{i=m+1}^{n} c^i - \max_{m+2 \le i \le n} c^i \\
&\ge c(S) + \sum_{i=m+1}^{n} c^i - c_{s_1 s_2}
\end{aligned}$$

where $(s_1 s_2) \in g_S$ with $s_1 \in S_1 \cup \{0\}; s_2 \in N \setminus S_1$.[15] Since the edge $(s_1 s_2)$ could have been connected (but was not) in steps $m+2, \ldots, n$ of the algorithm for N, we must have $c_{s_1 s_2} > c^k$ for $k \in \{m+2, \ldots, n\}$. Hence, the last inequality follows.

But, note that $[g_S \setminus \{(s_1 s_2)\}] \cup_{k=m+1}^{n} \{(a^k b^k)\}$ is a connected graph over $S \cup T \cup \{0\}$. So,

$$\begin{aligned}
\sum_{i \in S \cup T} \psi_i^*(C) &> c(S) + \sum_{i=m+1}^{n} c^i - c_{s_1 s_2} \\
&\ge c(S \cup T).
\end{aligned}$$

So, $S \cup T$ is a blocking coalition, establishing the desired contradiction.

This concludes the proof of the theorem. ∎

[15] Such $(s_1 s_2)$ must exist in g_S since g_S is the m.c.s.t. over S.

4 Characterization Theorems

In this section, we present characterizations of the allocation rules ψ^* and B.[16] We first describe the axioms used in the characterization. These characterization theorems will be proved for the restricted domains $\mathcal{C}^1$ for B and $\mathcal{C}^2$ for ψ^*.

Efficiency (EF): $\sum_{i \in N} \psi_i(C) = \sum_{(ij) \in g_N(C)} c_{ij}$.

This axiom ensures that the agents together pay exactly the cost of the efficient network.

Before moving on to our next axiom, we introduce the concept of an extreme point. Let $C \in \mathcal{C}_N$ be such that the m.c.s.t. $g_N(C)$ is unique. Then, $i \in N$ is called an extreme point of $g_N(C)$ (or equivalently of C), if i has no follower in $g_N(C)$.

Extreme Point Monotonicity (EPM): Let $C \in \mathcal{C}_N$, and i be an extreme point of C. Let C^{-i} be the restriction of C over the set $N^+ \setminus \{i\}$. An allocation rule satisfies Extreme Point Monotonicity if $\psi_k(C^{-i}) \geq \psi_k(C)$ $\forall k \in N^+ \setminus \{i\}$.

Suppose i is an extreme point of $g_N(C)$. Note that i is of no use to the rest of the network since no node is connected to the source through i. Extreme Point Monotonicity essentially states that no "existing" node k will agree to pay a higher cost in order to include i in the network.

The next two axioms are consistency properties, analogous to reduced game properties introduced by Davis and Maschler [2] and Hart and Mas-Collel [8].[17]

We need some further notation before we can formally describe the consistency axioms. Consider any C with a unique m.c.s.t. $g_N(C)$, and suppose that $(i0) \in g_N(C)$. Let x_i be the cost allocation 'assigned' to i. Suppose i 'leaves' the scene (or stops bargaining for a different cost allocation), but other nodes are allowed to connect through it. Then, the effective reduced cost matrix changes for the remaining nodes. We can think of two alternative ways in which the others can use node i.

(i) The others can use node i only to connect to the source.

[16] See Feltkamp [3] for an alternative characterization of B.

[17] Thomson[14] contains an excellent discussion of consistency properties in various contexts.

(ii) Node i can be used more widely. That is, node j can connect to node k through i.

In case (i), the connection costs on $N^+ \setminus \{i\}$ are described by the following equations:

$$\text{For all } j \neq i, \hat{c}_{j0} = \min(c_{j0}, c_{ji} + c_{i0} - x_i) \tag{7}$$

$$\text{If } \{j,k\} \cap \{i,0\} = \emptyset, \text{ then } \hat{c}_{jk} = c_{jk} \tag{8}$$

Equation 7 captures the notion that node j's cost of connecting to the source is the cheaper of two options - the first option being the original one of connecting directly to the source, while the second is the indirect one of connecting through node i. In the latter case, the cost borne by j is adjusted for the fact that i pays x_i. Equation 8 captures the notion that node i can only be used to connect to the source.

Let $C^{sr}_{x_i}$ represent the reduced cost matrix derived through equations 7, 8.

Consider now case (ii).

$$\text{For all } j,k \in N^+ \setminus \{i\}, \hat{c}_{jk} = \min(c_{jk}, c_{ji} + c_{ki} - x_i). \tag{9}$$

Equation 9 captures the notion that j can use i to connect to any other node k, where k is not necessarily the source.

Let $C^{tr}_{x_i}$ represent the reduced cost matrix derived through equation 9.

We can now define the two consistency conditions.

Source Consistency (SR): Let $C \in \mathcal{C}^1_N$, and $(0i) \in g_N(C)$. Then, the allocation rule ψ satisfies Source Consistency if $\psi_k(C^{sr}_{\psi_i(C)}) = \psi_k(C)$ for all $k \in N \setminus \{i\}$ whenever $C^{sr}_{\psi_i(C)} \in \mathcal{C}^1_{N \setminus i}$.

Tree Consistency (TR): Let $C \in \mathcal{C}^2_N$, and $(0i) \in g_N(C)$. Then, the allocation rule ψ satisfies Tree Consistency if $\psi_k(C^{tr}_{\psi_i(C)}) = \psi_k(C)$ for all $k \in N \setminus \{i\}$ whenever $C^{tr}_{\psi_i(C)} \in \mathcal{C}^2_{N \setminus i}$.

The two consistency conditions require that the cost allocated to any agent be the same on the original and reduced cost matrix. This ensures that once an agent connected to the source agrees to a particular cost allocation and then subsequently allows other agents to use its location for possible connections, the remaining agents do not have any incentive to reopen the debate about what is an appropriate allocation of costs.

The following lemmas will be used in the proofs of Theorems 2 and 3.

Lemma 1 : Let $C \in \mathcal{C}^1_N$, and $i \in N$. If $c_{ik} = \min_{l \in N^+ \setminus \{i\}} c_{il}$, then $(ik) \in g_N(C)$.

Proof : Suppose $(ik) \notin g_N(C)$. As $g_N(C)$ is a connected graph over N^+, $\exists\, j \in N^+ \setminus \{i, k\}$ such that $(ij) \in g_N(C)$ and j is on the path between i and k. But, $\{g_N \cup (ik)\} \setminus \{(ij)\}$ is still a connected graph which costs no more than $g_N(C)$, as $c_{ik} \leq c_{ij}$. This is not possible as $g_N(C)$ is the only m.c.s.t. of C. ∎

Lemma 2 : Let $C \in \mathcal{C}^2_N$, and $(01) \in g_N(C)$. Let $\psi_1(C) = \min_{k \in N^+ \setminus \{1\}} c_{1k}$. Then, $C^{tr}_{\psi_1(C)} \in \mathcal{C}^2_{N \setminus \{1\}}$.

Proof : We will denote $C^{tr}_{\psi_1(C)}$ by $\hat{C}$ for the rest of this proof.

Let $\psi_1(C) = \min_{k \in N^+ \setminus \{1\}} c_{1k} = c_{1k^*}$ (say).

Suppose there exists $(ij) \in g_N(C)$ such that $i, j \neq 1$. Since $(ij) \in g_N(C)$, either i precedes j or vice versa. Without loss of generality assume i precedes j in $g_N(C)$. Since $(01), (ij) \in g_N(C)$, $(1j) \notin g_N(C)$. Then, $c_{1j} > c_{ij}$. As $\psi_1(C) \leq c_{i1}$, $c_{i1} + c_{1j} - \psi_1(C) \geq c_{1j} > c_{ij}$. Hence $\hat{c}_{ij} = c_{ij} \forall (ij) \in g_N(C)$; such that $i, j \neq 1$.

Now, suppose there is $j \in N$ such that $j \neq k^*$ and $(1j) \in g_N(C)$. Since $(1j), (1k^*) \in g_N(C)$, $(jk^*) \notin g_N(C)$. Hence, $c_{1j} < c_{k^*j}$. Thus,

$\hat{c}_{k^*j} = \min\{(c_{1j} + c_{1k^*} - \psi_1(C)), c_{k^*j}\} = \min(c_{1j}, c_{k^*j}) = c_{1j}$.

Next, let $\hat{g}_{N \setminus \{1\}}$, be a connected graph over $N^+ \setminus \{1\}$, defined as follows.

$$\hat{g}_{N \setminus \{1\}} = \{(ij) \mid \text{either } (ij) \in g_N(C) \text{ s.t. } i, j \neq 1 \text{ or } (ij) = (k^*l) \text{ where } (1l) \in g_N(C)\}.$$

Note that no two edges have equal cost in $\hat{g}_{N \setminus \{1\}}$.

Also,

$$\sum_{(ij) \in \hat{g}_{N \setminus \{1\}}} \hat{c}_{ij} = \sum_{(ij) \in g_N(C)} c_{ij} - c_{1k^*}. \tag{10}$$

We prove that $\hat{C}$ belongs to $\mathcal{C}^2_{N \setminus \{1\}}$ by showing that $\hat{g}_{N \setminus \{1\}}$ is the only m.c.s.t. of $\hat{C}$.

Suppose this is not true, so that $g^*_{N\setminus\{1\}}$ is an m.c.s.t. corresponding to $\hat{C}$. Then, using 10,

$$\sum_{(ij)\in g^*_{N\setminus\{1\}}} \hat{c}_{ij} \leq \sum_{(ij)\in g_N(C)} c_{ij} - c_{1k^*} \tag{11}$$

Let $g^*_{N\setminus\{1\}} = g^1 \cup g^2$, where

$$\begin{aligned} g^1 &= \{(ij) \mid (ij) \in g^*_{N\setminus\{1\}}; c_{ij} = \hat{c}_{ij}\} \\ g^2 &= g^*_{N\setminus\{1\}} \setminus g^1 \end{aligned}$$

If $(ij) \in g^2$, then

$$\begin{aligned} \hat{c}_{ij} &= \min(c_{ij}; c_{1i} + c_{1j} - \psi_1(C)) \\ &= c_{1i} + c_{1j} - \psi_1(C) \\ &\geq \max(c_{1i}; c_{1j}) \end{aligned}$$

where the last inequality follows from the assumption that $\psi_1(C) = \min_{k\in N^+\setminus\{1\}} c_{1k}$.

So,

$$\begin{aligned} \hat{c}_{ij} &= c_{ij} \text{ if } (ij) \in g^1 \\ &\geq \max(c_{1i}; c_{1j}) \text{ if } (ij) \in g^2. \end{aligned} \tag{12}$$

Now, extend $g^*_{N\setminus\{1\}}$ to a connected graph g'_N over N^+ as follows. Letting $g = \{(1i) \mid (ij) \in g^2; j \in U(i; k^*; g^*_{N\setminus\{1\}})\}$, define

$$g'_N = g^1 \cup (1k^*) \cup g.$$

Claim: g'_N is a connected graph over N^+.

Proof of Claim: It is sufficient to show that every $i \in N^+ \setminus \{1\}$ is connected to 1 in g'_N. Clearly, this is true for $i = k^*$. Take any $i \in N^+ \setminus \{1; k^*\}$. Let $U(i; k^*; g^*_{N\setminus\{1\}}) = \{(m_0m_1); \ldots; (m_pm_{p+1})\}$ where $m_0 = i$ and $m_{p+1} = k^*$. If all these edges are in g^1, then they are also in g'_N, and there is nothing to prove.

So, suppose there is $(m_tm_{t+1}) \in g^2$ while all edges in $\{(m_0m_1); \ldots; (m_{t-1}m_t)\}$ belong to g^1. In this case, (m_t1) as well as all edges in $\{(m_0m_1); \ldots; (m_{t-1}m_t)\}$ belong to g'_N. Hence, i is connected to 1. ∎

To complete the proof of the lemma, note that

$$\sum_{(ij)\in g_N^0} c_{ij} = \sum_{(ij)\in g^1} c_{ij} + c_{1k^*} + \sum_{(ij)\in g} c_{1i}.$$

Using (12),

$$\sum_{(ij)\in g_N^0} c_{ij} \le \sum_{(ij)\in g^1} \hat{c}_{ij} + c_{1k^*} + \sum_{(ij)\in g^2} \hat{c}_{ij} = \sum_{(ij)\in g^*_{N\setminus\{1\}}} \hat{c}_{ij} + c_{1k^*}$$

Finally, using (11),

$$\sum_{(ij)\in g_N^0} c_{ij} \le \sum_{(ij)\in g_N(C)} c_{ij}.$$

But, this contradicts the assumption that $g_N(C)$ is the unique m.c.s.t. for C. ∎

Lemma 3 : Let $C \in \mathcal{C}_N^1$, $(10) \in g_N(C)$. Suppose $\psi_1(C) = c_{01}$. Then $C^{sr}_{\psi_1(C)} \in \mathcal{C}^1_{N\setminus\{1\}}$.

Proof : Throughout the proof of this lemma, we denote $C^{sr}_{\psi_1(C)}$ by $\hat{C}$.

We know $\psi_1(C) = c_{01}$. Suppose $(ij) \in g_N(C)$ such that $\{i,j\} \cap \{0,1\} = \emptyset$. Then $\hat{c}_{ij} = c_{ij}$.

On the other hand if $(i0) \in g_N(C)$, and $i \neq 1$, then $\hat{c}_{0i} = \min\{(c_{i1} + c_{10} - \psi_1(C)), c_{0i}\} = \min(c_{i1}, c_{i0}) = c_{i0}$. Note that the last equality follows from the fact that $(i0) \in g_N(C)$ but $(i1) \notin g_N(C)$ implies that $c_{i1} > c_{i0}$.

If $(i1) \in g_N(C)$, then $\hat{c}_{i0} = \min\{(c_{i1} + c_{10} - \psi_1(C)), c_{i0}\} = \min(c_{i1}, c_{i0}) = c_{i1}$, as $(i1) \in g_N(C)$ but $(i0) \notin g_N(C)$.

Now we construct $\hat{g}_{N\setminus\{1\}}$, a connected graph over $N^+ \setminus \{1\}$ as follows.

$$\hat{g}_{N\setminus\{1\}} = \{(ij) \mid \text{either } (ij) \in g_N(C) \text{ s.t. } i,j \neq 1 \text{ or } (ij) = (l0) \text{ where } (l1) \in g_N(C)\}$$

Then, $\hat{g}_{N\setminus\{1\}}$ must be the only m.c.s.t. of $\hat{C}$. For if there is another $g^*_{N\setminus\{1\}}$ which is also an m.c.s.t. of $\hat{C}$, then one can show that $g_N(C)$ cannot be the only m.c.s.t. coresponding to C.[18] ∎

Lemma 4: Suppose ψ satisfies TR, EPM and EF. Let $C \in \mathcal{C}_N^2$. If $(i0) \in g_N(C)$, then $\psi_i(C) \ge \min_{k\in N^+\setminus\{i\}} c_{ik}$.

[18] The proof of this assertion is analogous to that of the corresponding assertion in Lemma 2, and is hence omitted.

Proof: Consider any $C \in \mathcal{C}^2_N$, $(i0) \in g_N(C)$, and ψ satisfying TR, EPM, EF. Let $\psi(C) = x$, and $c_{im} = \min_{k \in N^+ \setminus \{i\}} c_{ik}$. We want to show that $x_i \geq c_{im}$.

Choose $j \notin N$, and define $\hat{N} = N \cup \{j\}$. Let $\hat{C} \in \mathcal{C}^2_{\hat{N}}$ be such that

(i) $\hat{C}$ coincides with C on N^+.

(ii) For all $k \in N^+ \setminus \{i\}$, $\hat{c}_{jk} > \hat{c}_{ij} > \sum_{(pq) \in g_N(C)} c_{pq}$.[19]

Hence, $g_{\hat{N}}(\hat{C}) = g_N(C) \cup \{(ij)\}$.

Notice that j is an extreme point of $\hat{C}$. Denoting $\psi(\hat{C}) = \hat{x}$, EPM implies that

$$x_i \geq \hat{x}_i \tag{13}$$

We prove the lemma by showing that $\hat{x}_i \geq \hat{c}_{im} = c_{im}$.

Let $\hat{C}^{tr}_{\hat{x}_i} = C'$, and $N' = \hat{N} \setminus \{i\}$; $\psi(C') = x'$. Assume $\hat{x}_i < \hat{c}_{im}$.

Case 1: $C' \in \mathcal{C}^2_{N'}$.

Suppose there is some $k \in N'$ such that $(ik) \notin g_{\hat{N}}(\hat{C})$. Let l be the predecessor of k in $g_{\hat{N}}(\hat{C})$. Since $(kl) \in g_{\hat{N}}(\hat{C})$ and $(ik) \notin g_{\hat{N}}(\hat{C})$, $\hat{c}_{kl} < \hat{c}_{ki}$. Also, $\hat{c}_{il} \geq \hat{c}_{im} > \hat{x}_i$. Hence,

$$c'_{kl} = \min(\hat{c}_{kl}, \hat{c}_{ki} + \hat{c}_{li} - \hat{x}_i) = \hat{c}_{kl} \tag{14}$$

Now, consider $k \in (N' \cup \{0\}) \setminus \{m, j\}$ such that $(ik) \in g_{\hat{N}}(\hat{C})$. Note that $(km) \notin g_{\hat{N}}(\hat{C})$ since $(im) \in g_{\hat{N}}(\hat{C})$ from lemma 1. Hence, $\hat{c}_{km} > \hat{c}_{ik}$ since $(ik) \in g_{\hat{N}}(\hat{C})$ and $(km) \notin g_{\hat{N}}(\hat{C})$. So,

$$c'_{km} = \min(\hat{c}_{km}, \hat{c}_{ik} + \hat{c}_{im} - \hat{x}_i) > \hat{c}_{ik} \tag{15}$$

Take any $(kl) \notin g_{\hat{N}}(\hat{C})$. Suppose $(s_1 s_2) \in U(k, l, g_{\hat{N}}(\hat{C}))$. Then, $\hat{c}_{s_1 s_2} < \min(\hat{c}_{kl}, \hat{c}_{ik}, \hat{c}_{il})$. So, $\forall (s_1 s_2) \in U(k, l, g_{\hat{N}}(\hat{C}))$

$$c'_{kl} = \min(\hat{c}_{ki} + \hat{c}_{il} - \hat{x}_i, \hat{c}_{kl}) > \hat{c}_{s_1 s_2} \tag{16}$$

[19] Note that the exact lower bound on $\hat{c}_{ij}$ will play no role in the subsequent proof. All that we require is that $\hat{c}_{ij}$ is "high" so that the reduced cost matrix C', to be defined below, has appropriate properties.

Next, note that since $\hat{c}_{jm}$ can be chosen arbitrarily high,

$$c'_{jm} = \min(\hat{c}_{jm}, \hat{c}_{ij} + \hat{c}_{im} - \hat{x}_i) = \hat{c}_{ij} + \hat{c}_{im} - \hat{x}_i \tag{17}$$

Since for all $t \in (N' \cup \{0\}) \setminus \{m, j\}$, $c'_{jt} = \hat{c}_{ij} + \hat{c}_{it} - \hat{x}_i > c'_{jm}$, $(jm) \in g_{N'}(C')$.

From TR, we have $x'_k = \hat{x}_k$ for all $k \in N'$. Using EF, and equations 14, 15, 16, 17,

$$\sum_{k \in \hat{N} \setminus \{i\}} \hat{x}_k = \sum_{k \in N'} x'_k = c(g_{N'}(C')) > c(g_{\hat{N}}(\hat{C})) - \hat{x}_i \tag{18}$$

But, this violates EF since $\sum_{k \in \hat{N}} \hat{x}_k > c(g_{\hat{N}}(\hat{C}))$.

Case 2: $C' \notin \mathcal{C}^2_{N'}$.

This implies that there exist $(pn), (kl)$ such that $c'_{pn} = c'_{kl}$, and both $(pn), (kl)$ belong to some m.c.s.t. (not necessarily the same one) corresponding to C'.

Note that $i \notin \{p, n, k, l\}$. So, if $(pn) \in g_{\hat{N}}(\hat{C})$, then $\hat{c}_{pn} = c'_{pn}$. Similarly, if $(kl) \in g_{\hat{N}}(\hat{C})$, then $\hat{c}_{kl} = c'_{kl}$. So, both pairs cannot be in $g_{\hat{N}}(\hat{C})$ since $\hat{C} \in \mathcal{C}^2_{\hat{N}}$.

Without loss of generality, assume that $(pn) \notin g_{\hat{N}}(\hat{C})$. There exists a path from p to n denoted by $U(p, n; g_{\hat{N}}(\hat{C})) = \{s_1, s_2, \ldots, s_K\}$. We first want to show that

$$c'_{pn} > \hat{c}_{s_k s_{k+1}} \text{ for all } k = 1, \ldots K - 1. \tag{19}$$

Note that $c'_{pn} = \min(\hat{c}_{pi} + \hat{c}_{in} - \hat{x}_i, \hat{c}_{pn})$. If (19) is not true, then either (a)$\hat{c}_{pn} \leq \hat{c}_{s_k s_{k+1}}$ or (b)$\hat{c}_{pi} + \hat{c}_{ni} - \hat{x}_i \leq \hat{c}_{s_k s_{k+1}}$ for some k.

Suppose (a) holds. Then, $(g_{\hat{N}}(\hat{C}) \cup \{(pn)\}) \setminus \{(s_k s_{k+1})\}$ is a m.c.s.t. for $\hat{N}$ corresponding to $\hat{C}$, contradicting the fact that $\hat{C} \in \mathcal{C}^2_{\hat{N}}$.

Suppose (b) holds. Then, $\hat{c}_{pi} \leq \hat{c}_{s_k s_{k+1}}$ and $\hat{c}_{ni} \leq \hat{c}_{s_k s_{k+1}}$ since $\min(\hat{c}_{pi}, \hat{c}_{ni}) > \hat{x}_i$.

If (pi)(or (ni)) $\notin g_{\hat{N}}(\hat{C})$, then we can delete $(s_k s_{k+1})$, join (pi)(or (ni)), and contradict the fact that $g_{\hat{N}}(\hat{C})$ is the unique m.c.s.t.

So, the only remaining possibility is that the path $U(p, n; g_{\hat{N}}(\hat{C})) = \{(ni), (pi)\}$. But, in this case, we already know that $\hat{c}_{ni} + \hat{c}_{pi} - \hat{x}_i > \max(\hat{c}_{ni}, \hat{c}_{pi})$ since $\hat{x}_i < \min(\hat{c}_{ni}, \hat{c}_{pi})$.

So, (19) is true.

Now, choose $q \notin \bar{N}$, and define $\hat{N} = \bar{N} \cup \{q\}$. Consider a cost matrix $\hat{C} \in \mathcal{C}^2_{\hat{N}}$ such that

(i) $\hat{C}$ coincides with $\bar{C}$ on $\bar{N}^+$.

(ii) $\hat{c}_{qp} = \min_{k \in \bar{N}^+} \hat{c}_{qk}$.

(iii) $c'_{pn} > \hat{c}_{qn} > \max_{(st) \in U(p,n,g_{\bar{N}}(\bar{C}))} \bar{c}_{st}$.[20]

(iv) $\hat{c}_{qt}$ is "sufficiently" large for all $t \neq p, n$.

Then, we have $g_{\hat{N}}(\hat{C}) = g_{\bar{N}}(\bar{C}) \cup \{(qp)\}$, so that q is an extreme point of $\hat{C}$. Let $\psi(\hat{C}) = \hat{x}$. From EPM,

$$\bar{x}_i \geq \hat{x}_i \tag{20}$$

Now, consider the reduced cost matrix $\tilde{C} \equiv \hat{C}^{tr}_{\hat{x}_i}$. We assert that $\tilde{C} \in \mathcal{C}^2_{\hat{N} \setminus \{i\}}$.[21] This is because (pn) is now "irrelevant" since in the m.c.s.t. corresponding to $\tilde{C}$, p and n will be connected through the path (pq) and (qn). To see this, note the following.

First,

$$\begin{aligned} c'_{pn} &= \min(\bar{c}_{pi} + \bar{c}_{in} - \bar{x}_i, \bar{c}_{pn}) \\ &\leq \min(\hat{c}_{pi} + \hat{c}_{in} - \hat{x}_i, \hat{c}_{pn}) \\ &= \tilde{c}_{pn} \end{aligned}$$

since $\bar{c}_{pn} = \hat{c}_{pn}, \bar{c}_{pi} = \hat{c}_{pi}, \bar{c}_{in} = \hat{c}_{in}$ and $\bar{x}_i \geq \hat{x}_i$ from (20).

Second, $c'_{pn} > \hat{c}_{qn}$ by construction. Lastly, $\hat{c}_{qn} = \tilde{c}_{qn}$ since $\tilde{c}_{qn} = \min(\hat{c}_{qn}, \hat{c}_{qi} + \hat{c}_{in} - \hat{x}_i)$ and $\hat{c}_{qi}$ has been chosen sufficiently large.

So, $\tilde{c}_{pn} > \tilde{c}_{qn}$. Since $(qp) \in g_{\hat{N} \setminus \{i\}}(\tilde{C})$ from Lemma 1, this shows that $(pn) \notin g_{\hat{N} \setminus \{i\}}(\tilde{C})$.

Since $\tilde{C} \in \mathcal{C}^2_{\hat{N} \setminus \{i\}}$, we apply the conclusion of Case 1 of the lemma to conclude that $\hat{x}_i \geq \hat{c}_{im} = \bar{c}_{im}$. Equation (20) now establishes that $\bar{x}_i \geq \bar{c}_{im}$. ■

[20] Note that this specification of costs is valid because (19) is true.

[21] This assertion is contingent on $(pn), (kl)$ being the only pairs of nodes in some m.c.s.t. of C' having the same cost. However, the proof described here can be adapted to establish a similar conclusion if there are more such pairs.

We state without proof the corresponding lemma when TR is replaced with SR. The proof is almost identical to that of Lemma 4.

Lemma 5: Suppose ψ satisfies SR, EPM and EF. Let $C \in \mathcal{C}_N^1$. If $(i0) \in g_N(C)$, then $\psi_i(C) \geq \min_{k \in N^+ \setminus \{i\}} c_{ik}$.

We now present a characterization of ψ^* in terms of Tree Consistency, Efficiency and Extreme Point Monotonicity.

Theorem 2 : Over the domain $\mathcal{C}^2$, a cost allocation rule ψ satisfies TR, EF and EPM if and only if $\psi = \psi^*$.

Proof : First, we prove that ψ^* satisfies all the three axioms.

Let $C \in \mathcal{C}^2$.

Efficiency follows trivially from the algorithm which defines the allocation.

Next, we show that ψ^* satisfies TR.

Let $(10) = \operatorname{argmin}_{k \in N} c_{k0}$. Hence, the algorithm yields $b^1 = 1$, and $\psi_1^*(C) = \min(c_{10}; c_{a^2 b^2})$. There are two possible choice of a^2.

Case 1: $a^2 = 1$. Then, we get $c_{1b^2} = \min_{k \in N \setminus \{1\}} c_{1k}$. Therefore $\psi_1^*(C) = \min(c_{10}; c_{1b^2}) = \min_{k \in N^+ \setminus \{1\}} c_{1k}$.

Case 2: $a^2 = 0$. Then, $c_{b^2 0} \leq c_{1k}\ \forall k \in N \setminus \{1\}$. Since $c_{10} \leq c_{b^2 0}$, we conclude $\psi_1^*(C) = \min(c_{10}; c_{b^2 0}) = c_{10} = \min_{k \in N^+ \setminus \{1\}} c_{1k}$.

So, in either case, 1 pays its minimum cost.

Let $\psi_1^*(C) = x_1 = \min_{k \in N^+ \setminus \{1\}} c_{1k} = c_{1k^*}$. Denoting $C^{tr}_{x_1}$ by $\hat{C}$, we know from Lemma 2, that $\hat{C} \in \mathcal{C}^2$. Hence, the algorithm is well defined on $\hat{C}$.

Let $\hat{a}^k; \hat{b}^k; \hat{t}^k$, etc denote the relevant variables of the algorithm corresponding to $\hat{C}$.

Claim: $\forall i \in N \setminus \{1\}$, $\psi_i^*(C) = \psi_i^*(\hat{C})$. That is, ψ^* satisfies Tree Consistency.

Proof of Claim: From the proof of lemma 2,

(i) $\hat{c}_{ij} = c_{ij}\ \forall (ij) \in g_N(C)$ s.t. $i; j \neq 1$.

(ii) $\hat{c}_{k^* j} = c_{1j}$ for $j \in N^+ \setminus \{k^*\}$ s.t. $(1j) \in g_N(C)$.

Also,

$$g_{N\setminus\{1\}}(\hat{C}) = \{(ij) \mid (ij) \in g_N(C) \text{ if } i,j \neq 1 \text{ and } (ij) = (k^*l) \text{ if } (1l) \in g_N(C)\}.$$

Let $b^2 = i$. Either $k^* = 0$ or $k^* = i$. In either case, $\hat{c}_{0i} = c_{01} < \hat{c}_{0j}$ for $j \notin \{0,1,i\}$. Hence, $\hat{b}^1 = i$.

Now, $t^2 = \max(c_{a^1b^1}, c_{a^2b^2}) = \max(c_{10}, c_{a^2 i}) = \hat{c}_{0i} = \hat{t}^1$.

Also, $a^3 \in \{0,1,i\}$, while $b^3 \in \{0,1,i\}^c$. If $a^3 \in \{0,i\}$, then $\hat{a}^2 = a^3$. If $a^3 = 1$, then $\hat{a}^2 = k^*$. In all cases, $b^3 = \hat{b}^2$, and $c_{a^3b^3} = \hat{c}_{\hat{a}^2\hat{b}^2}$. So,

$$\tilde{A}^*_i(C) = \min(t^2, c_{a^3b^3}) = \min(\hat{t}^1, \hat{c}_{\hat{a}^2\hat{b}^2}) = \tilde{A}^*_i(\hat{C}). \qquad (21)$$

The claim is established for $\{b^3, \ldots, b^n\}$ by using the strucure of $g_{N\setminus\{1\}}(\hat{C})$, the definition of $\hat{C}$ given above, and the following induction hypothesis. The details are left to the reader.

For all $i = 2, \ldots, k-1$,

(i) $\hat{b}^{i-1} = b^i$.

(ii $\hat{t}^{i-1} = t^i$.

(iii) $\hat{a}^{i-1} = a^i$ if $a^i \neq 1$, and $\hat{a}^{i-1} = k^*$ if $a^i = 1$.

We now have to show that $\tilde{A}^*$ satisfies EPM.

Let $i \in N$ be an extreme point of $g_N(C)$, and $\bar{C}$ be the restriction of C over $N \setminus \{i\}$. Of course, $\bar{C} \in C^2$.

In order to differentiate between the algorithms on C and $\bar{C}$, we denote the outcomes corresponding to the latter by $\bar{a}^k, \bar{b}^k, \bar{t}^k$, etc.

Suppose $b^k = i$. Clearly, the algorithm will produce the same outcomes till step $(k-1)$, and so $\tilde{A}^*_j(C) = \tilde{A}^*_j(\bar{C})$ for all $j \in \{b^1, \ldots, b^{k-2}\}$, and $t^{k-1} = \bar{t}^{k-1}$.

Now, let us calculate $\tilde{A}^*_j(C)$ where $j = b^{k-1}$. As i is an extreme point of g_N, and $(a^k i) \in g_N$, $a^{k+1} \neq i$. Also, $A^k = A^{k-1} \cup \{i\}$. Hence, $a^{k+1} \in A^{k-1}$. This implies $c_{a^k i} \leq c_{a^{k+1}b^{k+1}}$. But $i \notin \bar{A}^{k-1}_c$. Hence $(\bar{a}^k\bar{b}^k) = (a^{k+1}b^{k+1})$. Thus,

$$\tilde{A}^*_j(C) = \min(t^{k-1}, c_{a^k i}) \leq \min(\bar{t}^{k-1}, c_{a^{k+1}b^{k+1}}) = \tilde{A}^*_j(\bar{C}) \qquad (22)$$

Also, $\hat t^k = \max(\hat t^{k-1}; c_{a^{k+1}b^{k+1}}) \geq \max(t^{k-1}; c_{a^k i}; c_{a^{k+1}b^{k+1}}) = t^{k+1}$. The algorithm on C determines the cost allocation for i in step $(k+1)$. Since i is an extreme point of g_N, $i \neq a^s$ for any s. Hence, the choice of a^j and b^j must be the same in C and $\hat C$ for $j \geq k+1$. So, for all $j \in \{k+1, \ldots, \#N\}$, $a^j = \hat a^{j-1}$; $b^j = \hat b^{j-1}$; $t^j \leq \hat t^{j-1}$. Hence,

$$\psi^*_{b^j}(C) = \min(t^j; c_{a^{j+1}b^{j+1}}) \leq \min(\hat t^{j-1}; \hat c_{\hat a^j \hat b^j}) = \psi^*_{\hat b^{j-1}}(\hat C) \tag{23}$$

Hence, we can conclude that ψ^* satisfies Extreme Point Monotonicity.

Next, we will prove that only one allocation rule over $\mathcal{C}^2$ satisfies all three axioms. Let ψ be an allocation rule satisfying all the three axioms. We will show by induction on the cardinality of the set of nodes that ψ is unique.

Let us start by showing that the result is true for $|N| = 2$. There are several cases.

Case 1: $c_{12} > c_{10}, c_{20}$. From Lemma 4, $\psi_1(C) \geq c_{10}, \psi_2(C) \geq c_{20}$. By EF, $\psi_1(C) + \psi_2(C) = c_{10} + c_{20}$. Thus $\psi_1(C) = c_{10}$, and $\psi_2(C) = c_{20}$. So, the allocation is unique.

Case 2: $c_{20} > c_{12} > c_{10}$. Introduce a third agent 3 and costs $c_{20} < \bar c_{13} < \min(\bar c_{32}, \bar c_{30})$. Let the restriction of $\bar C$ on $\{1,2\}^+$ coincide with C. Hence, $g_{\{1,2,3\}} = \{(01),(12),(13)\}$. Let $\psi(\bar C) = \bar x$. From Lemma 4, $x_1 \geq c_{10}$.

Denote the reduced matrix $\bar C^{1\,tr}_{\bar x_1}$ as $\hat C$. Now, $\hat c_{02} = \min(\bar c_{01} + \bar c_{12} - \bar x_1; \bar c_{02}) = \bar c_{01} + \bar c_{12} - \bar x_1$. Similarly, $\hat c_{23} = \min(\bar c_{13} + \bar c_{12} - x_1; \bar c_{23})$. Noting that $x_1 \geq \bar c_{10}; \bar c_{23} > \bar c_{12}$ and $\bar c_{13} > \bar c_{10}$, we conclude that

$$\hat c_{02} < \hat c_{23}.$$

Analogously, $\hat c_{03} = \bar c_{01} + \bar c_{13} - \bar x_1 < \hat c_{23}$.

Hence, $g_{\{2,3\}}(\hat C) = \{(02),(03)\}$. So, $\hat C \in \mathcal{C}^2$. Using TR,

$$\psi_2(\hat C) = \psi_2(\bar C); \psi_3(\hat C) = \psi_3(\bar C) \tag{24}$$

From Case 1 above,

$$\psi_2(\hat C) = \bar c_{01} + \bar c_{12} - \bar x_1; \psi_3(\hat C) = \bar c_{01} + \bar c_{13} - \bar x_1 \tag{25}$$

From (24) and (25),

$$\begin{aligned} \psi_2(\hat{C}) + \psi_3(\hat{C}) &= \hat{c}_{01} + \hat{c}_{12} - \hat{x}_1 + \hat{c}_{01} + \hat{c}_{13} - \hat{x}_1 \\ \text{or } \hat{x}_1 + \psi_2(\hat{C}) + \psi_3(\hat{C}) &= \hat{c}_{01} + \hat{c}_{12} + \hat{c}_{13} + (\hat{c}_{01} - \hat{x}_1) \end{aligned}$$

But, from EF, $\hat{x}_1 + \psi_2(\hat{C}) + \psi_3(\hat{C}) = \hat{c}_{01} + \hat{c}_{12} + \hat{c}_{13}$. So, $\hat{x}_1 = \hat{c}_{01}$. So, $\psi_2(\bar{C}) = \psi_2(\hat{C}) = \hat{c}_{12} = c_{12}$.

By EPM, $\hat{x}_1 \leq \psi_1(C)$, and $\psi_2(\hat{C}) \leq \psi_2(C)$. Using EF, it follows that $\psi_1(C) = c_{01}$ and $\psi_2(C) = c_{12}$. Hence, ψ is unique.

The case $c_{10} > c_{12} > c_{20}$ is similar.

Case 3: $c_{20} > c_{10} > c_{12}$.

We again introduce a third agent (say 3). Consider the cost matrix $\hat{C}$, coinciding wih C on $\{1, 2\}^+$, and such that

(i) $\hat{c}_{32} > \hat{c}_{13} > \hat{c}_{20}$.

(ii) $\hat{c}_{30} > \hat{c}_{10} + \hat{c}_{13}$.

Then, $\hat{C} \in \mathcal{C}^2$ since it has the unique m.c.s.t. $g_N(\hat{C}) = \{(01), (12), (13)\}$, where no two edges have the same cost.

Note that 3 is an extreme point of the m.c.s.t. corresponding to $\hat{C}$. Using EPM, we get

$$\psi_1(C) \geq \psi_1(\hat{C}), \psi_2(C) \geq \psi_2(\hat{C}). \tag{26}$$

Now, $\hat{c}_{10} < \min(\hat{c}_{20}, \hat{c}_{30})$. Consider the reduced cost matrix $\hat{C}^{tr}_{\psi_1(\hat{C})}$ on $\{2, 3\}$. Denote $\hat{C}^{tr}_{x_1} = \bar{C}$ for ease of notation. Since $\psi_1(\hat{C}) \geq \hat{c}_{12}$ from Lemma 4, it follows that $\hat{c}_{12} + \hat{c}_{10} - \psi_1(\hat{C}) \leq \hat{c}_{10} < \hat{c}_{20}$, and $\hat{c}_{12} + \hat{c}_{13} - \psi_1(\hat{C}) \leq \hat{c}_{13} < \hat{c}_{23}$. Hence,

$$\bar{c}_{20} = \hat{c}_{12} + \hat{c}_{10} - \psi_1(\hat{C}); \bar{c}_{23} = \hat{c}_{12} + \hat{c}_{13} - \psi_1(\hat{C}); \bar{c}_{30} = \hat{c}_{13} + \hat{c}_{10} - \psi_1(\hat{C}) \tag{27}$$

Note that

$$\hat{c}_{21} + \hat{c}_{10} - \psi_1(\hat{C}) < \hat{c}_{21} + \hat{c}_{13} - \psi_1(\hat{C}) < \hat{c}_{10} + \hat{c}_{13} - \psi_1(\hat{C})$$

Hence, $g_{\{23\}}(\bar{C}) = \{(02), (23)\}$.

Applying case 2, $\phi_2(\hat{\hat{C}}) = \hat{\hat{c}}_{20} = \hat{c}_{12} + \hat{c}_{10} - \phi_1(\hat{C})$ and $\phi_3(\hat{\hat{C}}) = \hat{\hat{c}}_{23} = \hat{c}_{21} + \hat{c}_{13} - \phi_1(\hat{C})$. Using TR, $\phi_2(\hat{\hat{C}}) = \phi_2(\hat{C}); \phi_3(\hat{\hat{C}}) = \phi_3(\hat{C})$. EF on $\hat{C}$ gives,

$$\phi_1(\hat{C}) + \phi_2(\hat{C}) + \phi_3(\hat{C}) = \hat{c}_{10} + \hat{c}_{12} + \hat{c}_{13}$$

$$\text{or} \quad \phi_1(\hat{C}) + (\hat{c}_{12} + \hat{c}_{10} - \phi_1(\hat{C})) + (\hat{c}_{12} + \hat{c}_{13} - \phi_1(\hat{C})) = \hat{c}_{10} + \hat{c}_{12} + \hat{c}_{13}$$

$$\text{or} \quad \phi_1(\hat{C}) = \hat{c}_{12}$$

Hence $\phi_2(\hat{C}) = \hat{c}_{10}; \phi_3(\hat{C}) = \hat{c}_{13}$. From equation 26, $\phi_1(C) \geq \hat{c}_{12}; \phi_2(C) \geq \hat{c}_{10}$. Using EF on C we can conclude that, $\phi_1(C) = c_{12}$ and $\phi_2(C) = c_{10}$, i.e. the allocation is unique.

The case $c_{10} > c_{20} > c_{12}$ is similar.

This completes the proof of the case $|N| = 2$.[22]

Suppose the theorem is true for all $C \in \mathcal{C}^2_N$, where $|N| < m$. We will show that the result is true for all $C \in \mathcal{C}^2_N$ such that $|N| = m$.

Let $C \in \mathcal{C}^2_N$. Without loss of generality, assume $c_{10} = \min_{k\in N} c_{k0}$.[23] Thus $(10) \in g_N(C)$. There are two possible cases.

Case 1: $c_{10} = \min_{k\in N^+\setminus\{1\}} c_{1k}$.

Then choose $j \in N$ such that $(j0) \in g_N(C)$ or $(j1) \in g_N(C)$.

Case 2: $c_{1j} = \min_{k\in N^+\setminus\{1\}} c_{1k}$.

Then from Lemma 1, $(1j) \in g_N(C)$.

In either Case 1 or 2, let $\hat{C}$ denote the restriction of C on $\{1, j\}$. Then, from the case when $\# N = 2$, it follows that $\phi_1(\hat{C}) = \min_{k\in N^+\setminus\{1\}} c_{1k}$.

Now, by iterative elimination of extreme points and repeated application of EPM, it follows that $\phi_1(C) \leq \phi_1(\hat{C}) = \min_{k\in N^+\setminus\{1\}} c_{1k}$. But, $C \in \mathcal{C}^2_N$, and ϕ satisfies EF, TR and EPM. So, from lemma 4, it follows that $\phi_1(C) \geq \min_{k\in N^+\setminus\{1\}} c_{1k}$. Hence, $\phi_1(C) = \min_{k\in N^+\setminus\{1\}} c_{1k} = x_1$ (say).

We remove 1 to get reduced cost matrix $C^{tr}_{x_1}$. From lemma 2, $C^{tr}_{x_1} \in \mathcal{C}^2$. By TR, $\phi_k(C^{tr}_{x_1}) = \phi_k(C)\ \forall k \neq 1$. From the induction hypothesis, the allocation is unique on $C^{tr}_{x_1}$ and hence on C.

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[22] Note that these three cases cover all possibilities since equality between different costs will result in the cost matrix not being in $\mathcal{C}^2_N$.

[23] This is unique as $C \in \mathcal{C}^2_N$.

This completes the proof of the theorem. ∎

We now show that the three axioms used in the theorem are independent. The examples constructed below are all variants of ψ^*. So, a^k, b^k are derived from the algorithm used to construct ψ^*.

Example 6: We construct a rule ϕ which satisfies EPM and TR but violates EF.

Let $\phi_k(C) = \psi^*_k(C)\ \forall k \neq b^n;\ \phi_{b^n}(C) = \psi^*_{b^n}(C) + \epsilon;$ where $\epsilon > 0$.

Using the result that ψ^* satisfies EPM and TR it can be easily checked that ϕ also satisfies these axioms. Also note that $\sum_{k=1}^{n} \phi_k(C) = \sum_{k=1}^{n} \psi^*_k(C) + \epsilon > c(N)$, and hence ϕ violates EF.

Example 7: We now construct a rule μ which satisfies EF and TR, but does not satisfy EPM.

For $n = 1$, $\mu_1(C) = c_{10}$. For $n \geq 2$,

(i) $\mu_k(C) = \psi^*_k(C)\ \forall k \neq b^{n-1}; b^n$.

(ii) $\mu_{b^{n-1}}(C) = \mu_{b^n}(C) = \frac{(\psi^*_{b^{n-1}}(C) + \psi^*_{b^n}(C))}{2}$.

This allocation satisfies EF and TR but violates EPM. In order to see the latter, consider the following cost matrix C.

$$C = \begin{pmatrix} 0 & 3 & 5 & 7 \\ 3 & 0 & 1 & 2 \\ 5 & 1 & 0 & 4 \\ 7 & 2 & 4 & 0 \end{pmatrix}.$$

Then, $g_N(C) = \{(01); (12); (13)\}$. Clearly, 3 is an extreme point of C. Let $\hat{C}$ denote the restriction of C over {0,1,2}. Then, $\mu_2(C) = 2.5 > 2 = \mu_2(\hat{C})$ and hence EPM is violated.

We remark in the next theorem that the Bird rule B satisfies EF and EPM. Since $B \neq \psi^*$, it follows that B does not satisfy TR. Here is an explicit example to show that B violates TR.

$$C = \begin{pmatrix} 0 & 2 & 3.5 & 3 \\ 2 & 0 & 1.5 & 1 \\ 3.5 & 1.5 & 0 & 2.5 \\ 3 & 1 & 2.5 & 0 \end{pmatrix}$$

Then, $B_1(C) = 2; B_2(C) = 1.5$ and $B_3(C) = 1$. The reduced cost matrix is $C^{tr}_{x_1}$ is shown below.

$$C^{tr}_{x_1} = \begin{pmatrix} 0 & 1.5 & 1 \\ 1.5 & 0 & 0.5 \\ 1 & 0.5 & 0 \end{pmatrix}.$$

Then, $B_2(C^{tr}_{x_1}) = 0.5$ and $B_3(C^{tr}_{x_1}) = 1$. Therefore TR is violated.

However, B does satisfy Source Consistency on the domain $\mathcal{C}^1$. In fact, we now show that B is the only rule satisfying EF, EPM and SR.

Theorem 3 : Over the domain $\mathcal{C}^1$, an allocation rule ψ satisfies SR, EF and EPM iff $\psi = B$.

Proof : We first show that B satisfies all the three axioms. EF and EPM follows trivially from the definition. It is only necessary to show that B satisfies SR.

Let $(10) \in g_N$. Then, $B_1(C) = c_{01}$. Let us denote the reduced cost matrix $C^{sr}_{B_1}$ by $\hat{C}$. From Lemma 3, $\hat{C} \in \mathcal{C}^1$. Also, the m.c.s.t. over $N \setminus \{1\}$ corresponding to $\hat{C}$ is

$$g_{N\setminus\{1\}} = \{(ij) \mid \text{either } (ij) \in g_N \text{ with } i, j \neq 1 \text{ or } (ij) = (l0) \text{ where } (1l) \in g_N\}.$$

Also, for all $i, j \in N \setminus \{1\}$, $\hat{c}_{ij} = c_{ij}$ if $(ij) \in g_N$, and for $k \in N \setminus \{1\}; \hat{c}_{k0} = c_{1k}$ if $(1k) \in g_N$. Hence, for all $k \in N \setminus \{1\}; \hat{c}_{k\alpha(k)} = c_{k\alpha(k)}$. So, $B_k(\hat{C}) = B_k(C)$ for all $k \in N \setminus \{1\}$ and B satisfies Source Consistency.

Next, we show that B is the only allocation rule over $\mathcal{C}^1$, which satisfies all the three axioms. This proof is by induction on the cardinality of the set of agents.

We remark that the proof for the case $|N| = 2$ is virtually identical to that of Theorem 2, with SR replacing TR and Lemma 5 replacing Lemma 4.

Suppose B is the only cost allocation rule satisfying the three axioms, for all $C \in \mathcal{C}^1$, where $|N| < m$. We will show that the result is true for all $C \in \mathcal{C}^1$ such that $|N| = m$.

Let $C \in \mathcal{C}^1$. Without loss of generality, assume $(10) \in g_N(C)$. There are two possible cases.

Case 1 : There are at least two extreme points of C, say m_1 and m_2.

First, remove m_1 and consider the cost matrix C^{m_1}, which is the restriction of C over $(N^+ \setminus \{m_1\})$. By EPM, $\psi_i(C) \le \psi_i(C^{m_1})$ for all $i \ne m_1$. As C^{m_1} has $(m-1)$ agents, the induction hypothesis gives $\psi_i(C^{m_1}) = c_{i\alpha(i)}$. So, $\psi_i(C) \le c_{i\alpha(i)}$ $\forall i \ne m_1$. Similarly by eliminating m_2 and using EPM, we get $\psi_i(C) \le c_{i\alpha(i)}$ $\forall i \ne m_2$. Combining the two, we get $\psi_i(C) \le c_{i\alpha(i)}$ $\forall i \in N$.

But from EF, we know that $\sum_{i\in N} \psi_i(C) = c(N) = \sum_{i \in N} c_{i\alpha(i)}$. Therefore $\psi_i(C) = c_{i\alpha(i)}$ $\forall i \in N$, and hence the allocation is unique.

Case 2: If there is only one extreme point of C, then $g_N(C)$ must be a line, i.e. each agent has atmost one follower. Without loss of generality, assume 1 is connected to 2 and 0. Let $\hat{C}$ be the restriction of C over the set $\{0, 1, 2\}$. By iterative elimination of the extreme points and use of EPM we get $\psi_i(C) \le \psi_i(\hat{C})$. Using the induction hypothesis, we get $\psi_1(C) \le c_{10}$ and $\psi_2(C) \le c_{12}$.

Suppose $\psi_1(C) = x_1 = c_{10} - \epsilon$, where $\epsilon \ge 0$. Now consider the reduced cost matrix $C^{sr}_{x_1}$, which will be denoted by $\bar{C}$. It can be easily checked that $g_{N\setminus\{1\}}$ is also a line where 2 is connected to 0. Thus $\psi_2(\bar{C}) = \bar{c}_{20} = c_{21} + c_{10} - \psi_1(C) = c_{21} + \epsilon$. By SR, $\psi_2(C) = \psi_2(\bar{C}) = c_{21} + \epsilon$. But from EPM $\psi_2(C) \le \psi_2(\hat{C}) = c_{21}$. This is possible only if $\epsilon = 0$. Therefore, $\psi_1(C) = c_{10}$. Using SR and the induction hypothesis, we can conclude that $\psi = B$. ∎

We now show that the three axioms used in Theorem 3 are independent.

Example 8: The following allocation rule ϕ satisfies EPM and SR but violates EF.

Let $\phi_k(C) = B_k(C)$ $\forall k \ne i$, where i is an extreme point of C with $(i0) \notin g_N(C)$, and $\phi_i(C) = B_i(C) + \epsilon$; $\epsilon > 0$.

Using the result that B satisfies EPM and SR it can be easily checked that ϕ also satisfies these axioms. Also note that $\sum_{k=1}^{n} \phi_k(C) = \sum_{k=1}^{n} B_k(C) + \epsilon > c(N)$, and hence ϕ violates EF.

Example 9: The following example shows that EPM is independent of other axioms.

For $n \geq 2$, let μ coincide with B on all m.c.s.t. s which are not lines. Over a line g if k is the extreme point, and $(kl);(lm) \in g$, then $\mu_i(C) = B_i(C)\ \forall i \neq k;l$, $\mu_k(C) = \mu_l(C) = \frac{c_{kl}+c_{lm}}{2}$.

This rule satisfies EF and SR but violates EPM. Let the cost matrix C be

$$C = \begin{pmatrix} 0 & 3 & 5 & 7 \\ 3 & 0 & 1 & 2 \\ 5 & 1 & 0 & 4 \\ 7 & 2 & 4 & 0 \end{pmatrix}$$

Then, $g_N(C) = \{(01);(12);(13)\}$. Here, 3 is an extreme point of C. Let $\hat{C}$ be the restriction of C over {0,1,2}, and $g_{N\setminus\{3\}}(\hat{C}) = \{(01);(12)\}$. Then $\mu_1(C) = 3 > 2 = \mu_1(\hat{C})$ and hence EPM is violated.

Our new allocation rule ψ^* satisfies all the axioms but SR. The fact that ψ^* satisfies EF and EPM is proved in the previous theorem. Here is an example to show that our allocation rule may violates SR.

$$C = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 1.5 & 1 \\ 3 & 1.5 & 0 & 3.5 \\ 4 & 1 & 3.5 & 0 \end{pmatrix}$$

Then, $\psi^*_1(C) = 1; \psi^*_2(C) = 2$ and $\psi^*_3(C) = 1.5$. The reduced cost matrix is $\bar{C}$,

$$\bar{C} = \begin{pmatrix} 0 & 2.5 & 2 \\ 2.5 & 0 & 3.5 \\ 2 & 3.5 & 0 \end{pmatrix}$$

$\psi^*_2(\bar{C}) = 2.5$ and $\psi^*_3(\bar{C}) = 2$. Therefore SR is violated.

In Theorem 2, we have restricted attention to cost matrices in $\mathcal{C}^2$. This is because ψ^* does not satisfy TR outside $\mathcal{C}^2$. The next example illustrates.

Example 10: Consider

$$C = \begin{pmatrix} 0 & 3 & 4 & 3 \\ 3 & 0 & 2 & 5 \\ 4 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{pmatrix}$$

Then, $g_N^1(C) = \{(10), (12), (23)\}$ and $g_N^2(C) = \{(30), (32), (21)\}$ are the two m.c.s.t. s corresponding to C. Taking the average of the two cost allocations derived from the algorithm, we get $\psi^*(C) = (2.5, 1.5, 2)$. If we remove 1, which is connected to 0 in g_N^1, the reduced cost matrix $\hat{C}$ is:

$$\hat{C} = \begin{pmatrix} 0 & 2.5 & 3 \\ 2.5 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Then, $\psi_2^*(\hat{C}) = 1$ and $\psi_3^*(C) = 2.5$. So, TR is violated.

Remark 3: Note that in the previous example C lies outside $\mathcal{C}^1$. If we take a cost matrix in $\mathcal{C}^1 \cap \mathcal{C}^2$, then Lemma 2 will no longer be valid - the reduced cost matrix may lie outside $\mathcal{C}^1$ even when a node connected to the source pays the minimum cost amongst all its links. Thus, ψ^* will satisfy TR vacuously. But there may exist allocation rules other than ψ^* which satisfies EF, TR and EPM over $\mathcal{C}^1$.

Similarly, B does not satisfy SR outside $\mathcal{C}^1$.

Example 11: Consider the same cost matrix as in Example 10. Recall that $B(C) = (2.5, 1.5, 2)$.

If we remove 1, which is connected to 0 in g_N^1, the reduced cost matrix $\hat{C}$ is:

$$\hat{C} = \begin{pmatrix} 0 & 2.5 & 3 \\ 2.5 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Then, $B_2(\hat{C}) = 2.5$ and $B_3(\hat{C}) = 1$. Therefore SR is violated.

Remark 4: An interesting open question is the characterization of ψ^* using cost monotonicity and other axioms.

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