EQUILIBRIUM AGENDA FORMATION

Bhaskar Dutta,

Matthew O. Jackson

And

Michel Le Breton

No 628

WARWICK ECONOMIC RESEARCH PAPERS

THE UNIVERSITY OF WARWICK

DEPARTMENT OF ECONOMICS

Equilibrium Agenda Formation

Bhaskar Dutta, Matthew O. Jackson, Michel Le Breton*

September 2001
Draft: December 11, 2001

Abstract

We develop a definition of equilibrium for agenda formation in general voting settings. The definition is independent of any protocol. We show that the set of equilibrium outcomes for any Pareto efficient voting rule is uniquely determined. We also show that for such voting rules, if preferences are strict then the set of equilibrium outcomes coincides with that of the outcomes generated by considering all full agendas for voting by successive elimination and show that the set of equilibrium outcomes corresponds with the Banks set. We also examine the implications in several other settings.

Keywords: Agenda, Equilibrium, Voting

JEL Classification Numbers: D71, D72

---

*Dutta is at the Indian Statistical Institute, New Delhi 110016, India, and the University of Warwick, Coventry CV4 7AL, England (B.Dutta@warwick.ac.uk). Jackson is at the division of Humanities and Social Sciences, 228-77, California Institute of Technology, Pasadena, California 91125, USA, (jacksonm@hss.caltech.edu). Le Breton is at Université d'Aix-Marseille, 14 avenue Jules Ferry, Aix en Provence, 13621, France (lebreton@univ-aix.fr). Financial support under NSF grant SES-9986190 and an RTDF grant from the University of Warwick are also gratefully acknowledged.

1 Introduction

The importance of agenda formation in a wide variety of settings, ranging from committees to popular elections, is self-evident. In fact, in some legislative settings where the voting on speci¯c bills is highly predictable, one might argue that the most interesting strategic interaction takes place in the formation of the agenda.

Indeed, the wide literature that analyzes various aspects of voting recognizes the importance of the agenda, and has shown how important it can be (e.g., McKelvey (1976, 1979)). Nevertheless, we still lack tractable models of agenda formation, and a detailed understanding of how the formation of the agenda ultimately a®ects the outcome of voting. To quote Ordeshook (1993):

> More problematic is the issue of endogenous agendas, the process whereby agendas are formed via the sequential introduction and labeling of alternatives to be voted on. ... The particular problem is that to apply game theory we must provide a game form that speci¯es precisely the identity of decision makers, the sequence with which they make decisions, and the information at their disposal when they act. And although agenda voting, like simple descriptions of elections, lends itself readily to the construction of such form, the processes whereby agendas are formed is far less structured and, thereby, less amenable to unambiguous game-theoretical analysis.

Ordeshook's statement points out the di±culty of modeling agenda formation stemming from the lack of a clearly de¯ned game form.

In this paper we provide a model of agenda formation, and in particular one that does not rest on a speci¯c game form or protocol. The way in which we do this is to examine the continuation equilibria that might extend from any given agenda. We do this inductively, de¯ning sets of possible continuations from any given agenda up to some point, and imposing sequential rationality.

The main way in which we are able to make progress in de¯ning equilibrium agenda formation without reference to a speci¯c protocol is through a simple observation that ends up having powerful implications. That simple observation concerns when it is possible to stop at some agenda under an equilibrium: It is an equilibrium to stop at some agenda only if no agent prefers any continuation equilibrium. We show that the sequential rationality and stopping conditions alone provide strong conclusions for what the set of equilibrium agendas can include.

In particular, we show that if a voting rule always selects an outcome that is Pareto e±cient relative to the agenda, then sequential rationality and stopping conditions imply that equilibrium agendas will result in voting outcomes that are Pareto e±cient overall. Moreover, one of our main results states that for Pareto e±cient voting rules the equilibrium outcomes will always be a subset of what might arise from considering the set of complete agendas (including all outcomes). This result turns out to allow us to make fairly sharp predictions concerning equilibrium agendas in many settings. For example, if the voting rule does not depend on the speci¯c order of the agenda, then equilibrium agendas result in a unique outcome which is that when all alternatives are included in the voting. This also has important implications for voting rules where the order of the agenda does matter, such as the well-studied example of voting by successive elimination. There we show that equilibrium agendas always result in outcomes that lie in the Banks' set. Similarly, for voting rules that always pick outcomes that lie in the top cycle of the alternatives on the agenda, we show that the equilibrium agendas must result in outcomes that lie in the top-cycle of all alternatives. So, if for instance, a Condorcet winner exists and the voting rule is Condorcet consistent, then all equilibrium agendas include (and thus result in) the Condorcet winner.

While sequential rationality and stopping conditions already have a substantial impact on identifying agendas, we can impose further conditions to produce more speci¯c equilibrium sets and predictions. We also examine a consistency condition which requires that if one continuation is an equilibrium, and some agent prefers another continuation (which would be an equilibrium if the agenda is extended by the addition of one alternative), then this second continuation must also be an equilibrium. The converse is also imposed: unless there is a unique equilibrium agenda, all equilibrium agendas must be rationalizable in that at least some agent must weakly prefer them to some other equilibrium continuation. In the context of Pareto e±cient voting rules, we show that the consistency condition ties down the set of equilibrium agendas uniquely and provides a simple algorithm for identifying them.

Some Related Literature

Part of the motivation behind our analysis comes from the literature on \chaos" theorems. For instance, McKelvey (1976, 1979) has shown that in the context of majority rule and Euclidean settings, the top cycle of alternatives is either a singleton (a Condorcet winner) or the whole space. And, as Plott (1973) has shown, the second case is the generic one.[1] This implies that in most cases, starting from one alternative one can ¯nd a sequence of alternatives leading to any other, where each one in the sequence beats the previous one. While the conclusion that one should draw from such a result and whether or not \chaos" is an appropriate nickname has been debated, it is clear that such a result makes it critical to have an understanding of equilibrium agendas; as otherwise one is left without any prediction. This is essentially the primary motivation for our analysis. As such, we come back below to examine the predictions our equilibrium notion makes in the context of voting by successive elimination, and discuss the relation to chaos theorems.

An alternative approach to modeling agenda formation is to assume a speci¯c protocol and analyze its implications. For instance random recognition rules were studied in the context of multilateral bargaining (divide-the-dollar games) by Baron and Ferejohn (1989) (and the literature that followed). That approach provides for strong analytical conclusions. However, that approach is not so tractable outside of the distributive setting in which it is posed. Moreover, there are many applications where the protocol is not clear, as the above quote of Ordeshook points out. The advantages to the approach taken in this paper are that it can be applied to a general class of voting problems, where for instance, Euclidean preferences may not be appropriate; and it makes protocol-free predictions.

With regards to making protocol-free predictions, we remark that the sets of equilibria uncovered here should be viewed as a set of potential equilibria. Adding more knowledge of the speci¯c protocol may induce selections from the set we identify, and result in more speci¯c predictions. Nevertheless, as we shall show, fairly minimal requirements on the equilibrium set already allow for some tight predictions in the context of a variety of voting rules. Thus there are important aspects of equilibrium agendas that can be characterized

[1] See Austen-Smith and Banks (1999) for a nice discussion of this literature and extensions of McKelvey's theorem.

without detailed knowledge of the protocol.

Work on equilibrium agenda formation has also been done in other contexts. For example, Banks and Gasmi (1987) examined equilibrium agenda formation in three person committees. Their analysis is of a Euclidean setting and one where the three committee members can make only one proposal each, and so agendas are truncated. Specifying the problem to this level leads to sharp predictions. More recently, Penn (2001), in the context of three person divide-the-dollar games has extended the analysis to allow for arbitrary agenda lengths by a clever adaptation to in¯nite agendas, and shows that sharp predictions again result (but di®er from those of Banks and Gasmi). The above results are very encouraging in the face of \chaos" theorems, and may be thought of as answering those theorems by saying that if we do model agenda formation, then we can make speci¯c predictions. Nevertheless, the above analyses come in very speci¯c settings and are dependent upon the geometry of Euclidean preferences, and in some cases having three proposers and having a strong symmetry among them. Our analysis attempts to provide an equilibrium de¯nition that can be applied to a more general set of problems. Our main motivation is to develop a concept that does not require such speci¯c geometry, and at the same time does not demand detailed speci¯cation of the proposal protocol. [2] As such, the predictions our analysis makes are not always as crisp; but nevertheless are fairly speci¯c in many settings.

Equilibrium agenda formation has also been analyzed in another setting. That is the setting of strategic-candidacy. For instance, in Osborne and Slivinski (1996) and Besley and Coate's (1997) models of citizen- candidates, the decision to enter an election and take a position is studied under equilibrium. In other work (Dutta, Jackson, and Le Breton (1998, 2001)) we have examined the properties of equilibrium sets of candidates for a variety of voting rules and for voting by successive elimination. While the issue of strategic candidacy is an important example of endogenous agenda formation, modeling agenda formation more generally requires a di®erent approach. Most

[2] Another distinction is that our approach is based on one of inductively de¯ning equilibrium continuations, and so equilibria are de¯ned in a manner that can be thought of as analogous to subgame perfect equilibrium (but without a game form). Instead the Banks and Gasmi (1987) and Penn (2001) formulations use a \maximin-Stackelberg" based equilibrium notion.

importantly, the candidacy decision ultimately rests with the candidate.[3] This means that the proposal abilities of agents are limited. This provides for di®erent strategic considerations than, for instance, in a legislative setting where proposers are not restricted in the alternatives that they may propose. Below, we compare the outcomes of strategic agenda formation in the context of strategic candidacy and in the more general setting where proposers are not limited; and see that there are important distinctions.

Another branch of the literature that has touched on equilibrium agenda formation is that which has looked at sophisticated voting by successive elimination. In particular, a de¯nition of equilibrium agendas appears in work by Miller, Grofman, and Feld (1990). In their analysis an agenda is an equilibrium if nobody would gain by adding some alternative to the current agenda. The important di®erences between such a de¯nition and the ones presented here are in the beliefs of the proposers. The de¯nition just described does not account for the fact that in many cases the agenda will not end, but instead will be subject to further modi¯cations. Thus, proposers are acting myopically.[4] If proposers can make any predictions about continuations, rather than myopically assuming the agenda will end, then the outcome could be quite di®erent. This emphasizes an important aspect of our de¯nitions. Incorporating such sequential rationality and anticipating equilibrium continuations is the foundation on which we build our de¯nitions. We come back to examine the impact of this feature below, when we apply our de¯nitions to voting by successive elimination.

Finally, we mention a distantly related literature in terms of applications and speci¯cs; but more closely related in terms of ¯nding equilibrium de¯nitions that are not tied down to protocol speci¯cation. In particular, the literature on coalition formation (and on coalitional bargaining) faces a similar di±culty to that expressed in the quote of Ordeshook above. Writing down speci¯c bargaining protocols allows for sharp predictions, but ones that

[3] Even if one allows candidates to be nominated, they usually have the option to decline to run.

[4] Austen-Smith (1987) and Groseclose and Krehbiel (1993) also examine equilibrium agenda formation under voting by successive elimination. Their approach does not have the myopic problem of Miller, Grofman, and Feld; but they avoid this by assuming a ¯xed ordering over individuals who can each make a single proposal. However, their focus is on analyzing sophisticated sincerity, and not on characterizing equilibrium agendas more broadly.

may not be robust and are not so easily adapted to settings where the protocol is not obvious. Chwe (1994) provides a definition of consistent sets of alternatives that might come out of coalitional bargaining settings, that is not dependent on any specific protocol and yet still makes intuitively appealing predictions in many applications. Our approach here is intended to do the same thing for agenda formation problems. While there is a parallel in spirit, the actual equilibrium definition that we provide and the issues we face bear little resemblance to that in Chwe's work.[5]

2 Definitions

Alternatives

There is a set of alternatives X. Generic elements are denoted x, y, and z.

We begin the analysis with the case where X is finite and with $\#X = m$, as this brings out the intuitions most clearly. We then return to show how our analysis extends to the infinite case in Section 5.7.

Society will select one of these alternatives. These may be potential bills that a legislature might enact, a set of candidates that a society might elect, or a list of potential decisions that a committee might reach.

Voters or Decision Makers

The set $N = \{1, \ldots, n\}$ is a finite set of voters.

These are the individuals who are involved in determining the agenda and the outcome from that agenda. In Section 6 we discuss the possibility of having special roles for some individuals.

Preferences

Individuals have preferences over the set of alternatives represented by a complete and transitive binary relation, R_i. The strict preference relation associated with R_i is denoted P_i, and is defined by xP_iy if and only if not yR_ix. As usual, knowing P_i similarly defines R_i, and so we keep track of the strict relationship with the weak one being inferred.

[5] As a note, our use of the word consistency has no relationship to that of Chwe's consistent sets.

Let $\mathcal{P}$ denote the set of admissible profiles of preference relations. The notation $P \in \mathcal{P}$ denotes a generic profile $P = (P_1, \ldots, P_n)$.

In some applications $\mathcal{P}$ will be a restricted domain. A number of different examples appear in what follows.

Agendas

An agenda of length $k \in \{1, \ldots, m\}$ is a finite vector of alternatives $(x_1, \ldots, x_k) \in X^k$, with the restriction that $x_i \neq x_j$ for each $i \neq j$.

Let $\mathcal{A}^k$ denote the set of agendas of length k, and let $\mathcal{A} = \cup_{k=1}^{m} \mathcal{A}^k$ be the set of all agendas.

The restriction that the same alternative not appear more than once in an agenda is common to many legislative and committee settings. Given that the set of alternatives X could be quite large and dense, this does not prevent an alternative and a close approximation of it from appearing in an agenda.

Depending on how the voting procedure works, the sequence of the agenda may or may not matter. For instance if the agenda is simply a list of nominated candidates and some neutral voting procedure is used, then the agendas (x, y, z) and (z, y, x) would be equivalent. However, if the voting procedure is non-neutral, then the sequence can be important. For instance, under voting by successive elimination where proposed alternatives are voted upon in reverse order of their proposal the agendas (x, y, z) and (z, y, x) are not equivalent and could lead to different outcomes.

Extensions of an Agenda

In many situations of interest, some part of an agenda will already be on the table. For example, if there is a status quo, then it may take the first place in any agenda that follows. More generally, in building a definition of equilibrium we need to be able to make predictions starting from various existing agendas and so it is useful to consider the concept of the extensions of a given agenda.

With this in mind, for any k and $a \in \mathcal{A}^k$ let $\mathcal{A}(a)$ to be the set of all agendas that agree with a in the first k spots. That is,

$$\mathcal{A}(a) = \{a' \in \mathcal{A} \mid a'_h = a_h \ \forall h \in \{1, \ldots, k\}\}.$$

Voting Procedures

A voting procedure is a function $V : \mathcal{A} \times \mathcal{P} \to X$ such that $V(a; P) \in a$ for all $a \in \mathcal{A}$ and $P \in \mathcal{P}$.

A voting procedure thus summarizes the choice the society would make from a given agenda at a given preference profile. This formulation is very flexible and allows for many applications. For instance, it could be that V is determined by strategic voting or instead by sincere voting. Also, V might depend on the ordering of the agenda or it might not; and V might be anonymous, or it might treat some voters specially.

The details of how V is determined will not be important in developing our definition of equilibrium agenda formation. Later, in providing some results about the properties of equilibria, we will specify some properties of potential voting rules V and examine some specific voting rules.

3 Equilibrium Agendas

Before presenting the formal definitions of equilibrium, we begin with a simple example to motivate and illustrate the definitions.

Example 1

$X = \{x; y; z\}$ and x is the status quo.

The voters' preferences form a classic cycle:

- xP_1yP_1z
- yP_2zP_2x
- zP_3xP_3y

Here x beats y, y beats z, and z beats x under majority rule.

The voting rule is sincere voting by successive elimination. For instance, if the agenda is $(x; z; y)$, then first a vote is held between y and z, and then the winner is matched against x. Under sincere voting, the outcome of this agenda would be x, as y would defeat z and then x would beat y.[6] Here,

[6] A situation which approximately fits this one is that of the Powell amendment discussed by Denzau, Riker, and Shepsle (1985) and others. The alternative x would be the status quo of no U.S. federal funding of local public schools. The bill z under consideration

the only possible outcomes are x from agendas $(x;y;z)$, $(x;z;y)$, $(x;y)$ and x; and z from agenda $(x;z)$.

Let us discuss equilibrium conditions based on this example. Once an agenda of three alternatives has been reached, there are no alternatives left to propose, and so an equilibrium continuation is simply the agenda in question. Next let us step back and consider an agenda of length 2 that starts with the status quo x. There are only two such agendas to consider. One is the agenda $(x;z)$. If this agenda is reached, then agent 1 by adding the alternative y would change the outcome from z to x. This would make agent 1 better off, and so the agenda $(x;z)$ would not be stable to amendment.[7] This suggests one of the conditions in our equilibrium definition: that stopping at a given agenda is an equilibrium if and only if there is no agent who can benefit from advancing the agenda to some further continuation equilibrium. So, the only continuation equilibrium following $(x;z)$ is the agenda $(x;z;y)$. Next, let us back things up. Given the agenda x in place, if some agent proposes z next, then if she should anticipate that the result will be the full agenda $(x;z;y)$ with outcome x. This embodies another part of the equilibrium definition: agents should anticipate equilibrium continuations from extensions of an agenda. In this case, no matter what happens after x, any continuation equilibrium must lead to the outcome of x. This actually means that stopping at x can be an equilibrium. Whether or not the other agendas that lead to x are also included as equilibrium continuations from x, is something that is not mandated by our basic definitions of equilibrium.

in the House of Representatives was one that would introduce some federal funding of local public schools. The amendment to the bill y introduced by Powell was to deny federal funding to public schools that practiced segregation (this was in the 1950's). As Denzau, Riker and Shepsle argue, sincere voting could be explained by the difficulty in explaining voting against the Powell amendment to one's constituency. In fact, the situation had some mixture of sincere and sophisticated voting, as some representatives who opposed funding (and supported segregation) may have voted for the Powell amendment in the first round and then against it in the second round. So there may have been some conservative representatives who had the preferences of voter 1 except with z and y reversed, but who when voting strategically would vote the same as voter 1 would vote when voting sincerely.

[7] Interestingly, in this example if we require a second agent to support a proposal in order for it to become part of the agenda, neither of the remaining agents would second the proposal. This turns out to be an artifact of the sincere voting and also the fact that there is only one agent with any given preference profile. We discuss how this is not a problem for sophisticated voting below.

However, a further consistency condition that we add would imply that the other agendas leading to x would also be equilibria in this example.

With some of the basic ideas from this simple example in hand, let us now consider the full definition of equilibrium agendas.

First, notice as in the above example, defining behavior at one agenda requires having some notion of what will happen following various extensions of the given agenda. Thus, the definition involves sets of continuation equilibria to be defined from each starting point. This is necessarily a set of sets, where a set of continuation equilibria is specified starting from each possible agenda.

We deliberately impose only weak requirements in defining equilibrium sets. Although taking such an approach allows for various collections to satisfy the definition, these weak requirements already have substantial implications for which outcomes might be reached.

A collection of sets of continuation equilibria for a profile of preferences $P \in \mathcal{P}$ is a collection $\{CE_V(a;P)\}_{a \in \mathcal{A}}$, where $CE_V(a;P) \subset \mathcal{A}(a)$ for each $a \in \mathcal{A}$, that satisfies the following properties.

Given $\{CE_V(a;P)\}_{a \in \mathcal{A}}$, let

$$C_V^+(a;P) = \cup_{x \notin a} CE_V((a;x);P).$$

So $C_V^+(a;P)$ is the set of all continuation equilibria that could result if some alternative is added to an existing agenda a.[8]

A continuation equilibrium set satisfies the following for each $a \in \mathcal{A}$:

(CE1) (Equilibrium Continuations) $CE_V(a;P)$ is a nonempty subset of $a \cup C_V^+(a;P)$ and

(CE2) (Stopping Requirements) $a \in CE_V(a;P)$ if and only if $V(a;P) R_i V(a';P)$ for all $a' \in C_V^+(a;P)$ and for all $i \in N$.

Part (CE1) is a sequential rationality condition that simply says that the possibilities from any agenda a are either to stop at a, or to add a new alternative to the agenda and then follow some continuation equilibrium from

[8] We remark that if $a \in \mathcal{A}^m$, then $C_V^+(a;P) = \emptyset$. Under (CE1) and (CE2) below, this implies that $CE_V(a;P) = \{a\}$ if $a \in \mathcal{A}^m$.

the resulting agenda. This is a condition that essentially just requires that the sets of equilibria for different agendas have some minimal relationship to each other: if agents anticipate that $a' = (a, x, \ldots)$ is a continuation equilibrium starting at a, then they must also expect it to still be a continuation equilibrium when they have reached (a, x).

Part (CE2) describes conditions under which it can be an equilibrium for agents to 'stop' at a. If every agent finds that $V(a, P)$ is at least as good as the outcome corresponding to any other possible continuation equilibrium, then no agent has an incentive to extend a. Conversely, if some agent i finds the voting outcome corresponding to some continuation equilibrium strictly preferred to $V(a, P)$, then this i will rather make a proposal and follow the preferred continuation equilibrium, and the agenda will not stop at a.

One of our main themes developed below is that these minimal conditions already have some very strong implications and imply a great deal about sets of equilibria.

While imposing some restrictions on collections of sets of continuation equilibria, conditions (CE1) and (CE2) can still allow for a multiplicity of collections of equilibrium continuations that satisfy the definition. Essentially, (CE1) and (CE2) give us some weak limitations on what can be in the set of equilibria, but they do not tell us much about which agendas must be included in the set. Consistency (CE3), below, addresses this issue.

We say that an agenda $a' = (a, x, \ldots) \in C_V^+(a, P)$ is rationalizable if there exists $i \in N$ and $a'' \in CE_V(a, P)$ with either $a'' = (a, y, \ldots)$ with $y \neq x$ or $a'' = a$ such that $V(a', P) R_i V(a'', P)$.

The idea of rationalizability is that i proposes adding x to the agenda a under the belief that it will result in the agenda a', and that if i does not propose adding x then instead the continuation would be a''. As a'' is a continuation equilibrium, this belief can be justified.

We say that a collection of sets of continuation equilibria is consistent if it satisfies

(CE3) (Consistency) If $a' \in C_V^+(a, P)$ is rationalizable, then $a' \in CE_V(a, P)$. Conversely, if $a' = (a, x, \ldots) \in CE_V(a, P)$ and either $a \in CE_V(a, P)$ or $a'' = (a, y, \ldots) \in CE_V(a, P)$ for some $y \neq x$, then a' is rationalizable.

Part (CE3) is a consistency condition on the collections of sets of continuation equilibria. It says the equilibrium continuations are those which are rationalizable, subject to two exceptions. One is that stopping is handled under (CE2), and so the rationalization of a itself is already addressed. The second is that an equilibrium continuation agenda does not need to be rationalizable if it is a "unique" equilibrium continuation. Note that in this second case, the first part of the condition implies that all agents unanimously find the outcomes under $(a; x; \ldots)$ preferred to stopping or adding any other alternative to a.

Later, we come back to discuss other notions of rationalizability and consistency.

We point out some important aspects of the above definitions.

First, the definitions necessarily involve a whole collection of $\{CE_V(a; P)\}$, one set for each $a \in A$. This reflects the forward-looking aspect of the definition. In order to know what is an equilibrium starting at one agenda, one has to be able to anticipate what will happen starting at extensions of that agenda.[9]

Second, there always exists at least one collection $\{CE_V(a; P)\}$ satisfying (CE1)-(CE3), which is easily seen via a backwards induction argument, starting with agendas of full length, and then working back to smaller agendas.

Third, the set $CE_V(a; P)$ is not always uniquely determined. That is, there may be several different sets which satisfy conditions (CE1) and (CE2); even when consistency (CE3) is imposed. This stems from the fact that the conditions are designed to be weak, to specify conditions that an equilibrium set should satisfy, but not so strong as to always uniquely determine that set. Again, this traces back to our deliberate avoidance of any reliance on an ad hoc formulation of the proposal process. To see an easy example of the potential multiplicity of equilibrium continuations, consider a somewhat degenerate voting rule as follows.

Example 2 Multiple Collections of Sets of Continuation Equilibria:

Under V the outcome is always the second alternative proposed in the agenda (or the first if the agenda is a singleton), regardless of the preference profile. So $V(a; P) = a_2$ if $a \in A^k$ with $k \geq 2$ and $V(a; P) = a_1$ if $a \in A^1$.

[9] Of course, this is similar to a definition such as subgame perfect equilibrium where continuation strategies must be specified for each possible subgame.

This is a peculiar voting rule, but one that allows for a simple illustration of the multiplicity of equilibria. Note that in this case, $CE_V(a; P) = A(a)$ is uniquely determined for any $a \in A^k$ for $k \geq 2$. This follows since once the second alternative has been proposed the outcome is already determined and the rest of the agenda is completely irrelevant and so under (CE2) and (CE3) all continuations are then equilibria. Now consider the outcome that is proposed in the second place in the agenda. In particular, let $X = \{w; x; y; z\}$ and consider a preference profile where some agents have preferences z, y, x, w, and others have preferences z, x, y, w; where the ordering specifies the strict preferences where w is the worst alternative. Consider starting at the agenda $a = \{w\}$. So, w is the status quo. Conditions (CE1) and (CE2) have only very weak implications here: it cannot be an equilibrium to stop at $\{w\}$. Beyond that, they allow for a variety of continuation equilibrium sets. Once consistency is added, however, things are tied down to a greater degree. In particular, there are two sets which satisfy (CE1), (CE2) and (CE3). The first such set consists of all extensions of a with z in second place (i.e., $CE_V(a; P) = A((w; z))$); and the second such set consists of all extensions of a with any of x, y, or z in second place (i.e., $CE_V(a; P) = A((w; x)) \cup A((w; y)) \cup A((w; z))$).

In this example, consistency (CE3) still does not uniquely tie things down. One might argue that extensions of $(w; z)$ are really the only sensible equilibrium continuations in the above example, as they are unanimously preferred to proposals x and y. One may wish to impose such additional conditions on the notion of equilibrium (and we discuss this more fully in Section 6). However, as we shall see, if we restrict attention to more sensible voting rules, such as those which satisfy a Pareto efficiency condition, consistency will already tie things down uniquely without the imposition of any additional conditions.

Given the potential multiplicity of collections of equilibria, we now show that in many cases of interest the set of continuation equilibria is in fact uniquely determined under consistency. This allows us to develop an equivalent definition that is not self-referential.

4 Equilibrium Agendas for Pareto Efficient Voting Rules

An alternative $x \in B \subset X$ is Pareto efficient relative to P and B if there does not exist $y \in B$ such that yR_ix for all $i \in N$ and yP_jx for some $j \in N$.

V is Pareto efficient if $V(a;P)$ is Pareto efficient relative to P and the alternatives in a, for each $a \in A$ and $P \in \mathcal{P}$.

Given a collection $\{CE_V(a;P)\}_{a \in A}$ and any $a \in A$, let $PE_V(a;P)$ denote the set of agendas in $C_V^+(a;P) \cup a$ that result in Pareto efficient alternatives (considering all of X).

Theorem 1 For any Pareto efficient voting rule V and preference profile $P \in \mathcal{P}$ and collection of sets of continuation equilibria $\{CE_V(a;P)\}_{a \in A}$, $V(a';P)$ is Pareto efficient (considering all alternatives) for all a and $a' \in CE_V(a;P)$. [10] Moreover, if consistency is satisfied, then $\{CE_V(a;P)\}_{a \in A}$ is uniquely defined and described by

$$CE_V(a;P) = \begin{cases} PE_V(a;P) & \text{if } V(a;P)R_iV(a';P)\ \forall i \text{ and } a' \in C_V^+(a;P) \\ PE_V(a;P) \setminus a & \text{otherwise.} \end{cases}$$

The first result in Theorem 1 is that equilibrium agendas of Pareto efficient voting rules must result in outcomes that are Pareto efficient overall. This conclusion is not quite as obvious as it seems. For instance, it could be that x is Pareto dominated by y, but that $V(a';P) \neq y$ for all $a' \in A(a)$. This means that since y is never in the range of V, it does not threaten x. The proof uses the fact that if y is added to an agenda containing x, then the outcome cannot be x and must instead be some other outcome that some voter prefers to x. Building on this reasoning we rule out equilibrium agendas leading to x. The details are provided in the proof in the appendix.

The second result in Theorem 1 is that under consistency the continuation equilibria of Pareto efficient voting rules are uniquely determined and described by a simple algorithm.

[10] Theorem 1 also holds if one replaces Pareto efficiency everywhere by weak Pareto efficiency, where an alternative $x \in B \subset X$ is weakly Pareto efficient relative to P and B if there does not exist $y \in B$ such that yP_ix for all $i \in N$. This weakens the assumptions of the theorem, but then also the conclusions.

The implications of Theorem 1 are even stronger when preferences satisfy a mild restriction.

Let P^* be the set of all profiles satisfying the restriction:

$$\forall x, y \in X, \exists i \in N \text{ such that } xP_i y \text{ or } yP_i x.$$

So, P^* is the set of profiles such that it is never the case that all individuals are indifferent between some pair of alternatives x, y. Of course, this condition is satisfied when individual preferences are strict, but also holds more generally including where some transfers or distribution of resources are possible. In this case, we obtain a characterization of continuation equilibrium outcomes that does not even require an inductive definition.[11]

Theorem 2 Consider a Pareto efficient voting rule V and profile of preferences $P \in P^*$. If $\{CE_V(a; P)\}_{a \in A}$ is a collection of sets of continuation equilibria, then the outcomes corresponding to continuation equilibria following some agenda a are a subset of those that can be found by considering only full length agendas that are extensions of a. That is,

$$\cup_{a' \in CE_V(a;P)} V(a'; P) \subseteq \cup_{a' \in A(a) \cap A^m} V(a'; P).$$

If in addition consistency is satisfied, then these sets are equal:

$$\cup_{a' \in CE_V(a;P)} V(a'; P) = \cup_{a' \in A(a) \cap A^m} V(a'; P).$$

Theorem 2 shows how powerful the implications of the simple stopping condition are. It states that the equilibrium outcomes correspond to those where complete agendas are considered. The idea behind this follows an inductive proof. Suppose this is true once an agenda is of length k or more. Now suppose that some agenda of length $k - 1$ is an equilibrium agenda and results in an outcome that differs from all full length agendas, and thus all continuation equilibria if any outcome is added. Given Pareto efficiency, some agent must prefer some outcome of a longer agenda that is a continuation

[11] To see an example of why this condition is needed in the theorem, consider a situation where all voters are indifferent between all alternatives, and when there is a tie in voting the last item in the agenda wins. It can be an equilibrium to stop at any agenda (including the status quo) given full indifference, and yet the status-quo can never be reached by a full length agenda.

equilibrium if some alternative is added to the current agenda to that of stopping. Then (CE2) implies that stopping cannot be an equilibrium.

The proof of Theorem 2 is in the Appendix. The second half of the proof actually follows from a stronger claim which does not invoke Pareto e±ciency of the voting procedure. Since this is of independent interest, we state it here.[12]

Claim 1 For any voting procedure V, preference pro¯le P 2 P and a 2 A, if f$CE_V(a;P)$g$_{a2A}$ is a collection of sets of continuation equilibria satisfying consistency, then any Pareto e±cient alternative that can be reached via some full length continuation of a is an equilibrium continuation outcome following a at P.[13]

5 Applications to Speci¯c Voting Rules and Settings

In order to demonstrate the implications and usefulness of Theorems 1 and 2, we apply them to a number of settings including some prominent ones.

5.1 Order Independent Voting Rules

A voting rule V is order independent if $V(a;P) = V(a^0;P)$ whenever fx 2 ag = fx 2 a^0g.

Order independent voting rules are those for which the ordering of the agenda does not matter. Neutral voting rules are order independent, but there are also important order independent voting rules that are non-neutral. Consider the following example: candidates are people who are seeded according to their age (or experience, rank, etc.). Regardless of the order in which they are proposed or nominated, the two youngest candidates are voted

[12] In fact we prove stronger statements in the appendix, showing that even for ine±cient voting rules there is a minimal consistent set of equilibria (in terms of set inclusion), which corresponds to the de¯nition under the algorithm above. It is under Pareto e±ciency that this must coincide with all consistent sets of equilibria.

[13] Since we show in the appendix that there is a minimal consistent set of equilibria (in terms of set inclusion), this must hold for the minimal consistent set of equilibria.

upon, then the winner of that vote is pitted against the next youngest, etc.. This rule is independent of the order in which the candidates are proposed, and yet it is still a sequential rule and is clearly not neutral. Therefore, we emphasize that “order independence” refers only to the order of the agenda and does not mean that the voting rule itself is not based on some implicit ordering of alternatives.

Note that for any order independent voting rule, $V(a;P) = V(a';P)$ for any a and a' in A^m. With an abuse of notation, we write this outcome as $V(X;P)$.

The following is a direct corollary of Theorem 2.

Corollary 1 For any Pareto efficient and order independent voting rule V, preference profile of preferences $P \in \mathcal{P}^*$, collection of sets of continuation equilibria $\{CE_V(a;P)\}_{a \in A}$ (i.e., satisfying (CE1) and (CE2)), and agenda $a \in A$, there is a unique continuation equilibrium outcome

$$\cup_{a' \in CE_V(a;P)} V(a';P) = V(X;P):$$

An important remark about Corollary 1 is that it does not require consistency, but follows from (CE1) and (CE2) in the definition.

The following example shows how Borda's rule is covered under Corollary 1.

Example 3

Voters' preferences are :

- $xP_1wP_1yP_1z$
- $xP_2wP_2yP_2z$
- $zP_3wP_3yP_3x$

Voting is (sincere) voting according to Borda's rule. An alternative receives three points for a first place ranking in a voter's preferences, two points for a second place ranking, one point for a third place ranking, and no points for a fourth place ranking; and the alternative with the highest score is the outcome (with ties broken according to any deterministic rule). These are adjusted for the restricted ranking if some subset of alternatives is considered.

This is a Pareto efficient and order independent voting rule.

It is easily checked that w wins whenever it is on the agenda. Also, x wins if it is present but w is not. If just y and z are present, then y wins. z only wins if it is the only proposed alternative.

Corollary 1 implies that the outcome of any equilibrium agenda must be w in this example. Indeed, it is easily seen that no agenda leading to y or z can be an equilibrium, as adding w to the agenda will lead to a continuation equilibrium outcome of w which would be preferred over y or z by some agent. Similarly, if an agenda leads to x, then adding w will lead to a continuation equilibrium of w, which is better for voter 3 than x.

5.2 Tournaments and Top Cycle Consistent Rules

The following definitions are useful in some of the remaining applications.

Tournaments

In many contexts, the preferences of the voters can be summarized (even for strategic purposes) by the majority voting relation that is induced over pairs of alternatives. A tournament is a binary relation that summarizes the important aspects of voters' preferences in some contexts.[14] More formally, the majority voting tournament is defined as follows.

Given $P \in \mathcal{P}$, denote by $T(P)$ the binary relation defined by

$$xT(P)y \quad \Leftrightarrow \quad \#\{i \in N : xP_iy\} > \#\{i \in N : yP_ix\}$$

$T(P)$ is always asymmetric and if n is odd and individual preferences are strict then $T(P)$ is complete. If we break ties in some deterministic manner, then even in cases with an even number of voters or indifferences $T(P)$ is also complete, and therefore a tournament (an asymmetric and complete binary relation). In what follows, unless specified otherwise, we will assume that ties are broken so that $T(P)$ is complete. $T(P)$ is referred to as the majority tournament induced by P.

The Top Cycle

[14] See Laslier [11] for an illuminating account of the principal results in the vast literature on tournaments.

As the majority tournament is not necessarily transitive, it can have cycles. A prominent cycle that we refer to in the sequel is the top cycle associated with a tournament.

The top cycle of $T(P)$, denoted by $TC(X;P)$ is the set $\{x \in X : \forall y \in X; \exists x_1; \ldots; x_k$ in X such that $x_1 = x; x_k = y$ and $x_i T(P) x_{i+1} \forall i = 1; \ldots; k - 1\}$ i.e. the set of alternatives that can reach any other alternative in X via a $T(P)$-chain of arbitrary length. For subsets of alternatives, $B \subset X$, there is a corresponding definition and we denote that set $TC(B;P)$. When there is no $B \subset X$ indicated, then we are referring to the top cycle relative to X, and we use the notation $TC(a;P)$ to denote the top cycle relative to the set of alternatives in the agenda a under the tournament $T(P)$.

A voting rule is top cycle consistent at a P such that $T(P)$ is a tournament if $V(a;P) \in TC(a;P)$ for any $a \in A$.

Condorcet Winners and Consistency

An alternative $\{x\}$ is a Condorcet winner relative to $B \subset X$ if $TC(B;P) = \{x\}$. That is, a Condorcet winner is an alternative that beats every other alternative in B under $T(P)$.

A voting rule V is Condorcet consistent if $V(a;P)$ selects a Condorcet winner whenever one exists relative to $T(P)$ and the alternatives in a.

5.3 Equilibrium Agendas for Top Cycle and Condorcet Consistent Voting Rules

If the voting procedure V arises from strategic voting on a binary tree, then it follows from McKelvey and Niemi (1978) that V is top cycle consistent. Thus, the following proposition covers a wide variety of applications.

Proposition 1 Consider a P such that $T(P)$ is a well-defined tournament and a collection of sets of continuation equilibria $\{CE_V(a;P)\}_{a \in A}$ (i.e., satisfying (CE1) and (CE2)). If V is top cycle consistent, then all equilibrium outcomes following any agenda are in the (overall) top cycle. Moreover, if V is Condorcet consistent and there exists a Condorcet winner x at P, then all of the equilibrium continuations from any agenda lead to x.

Again, remark that Proposition 1 does not require consistency (CE3).

The second statement does not quite follow from the first, since Condorcet consistency does not imply top cycle consistency. The proof of Proposition

1 is straightforward for the case where the preference profiles in P^* are strict and the voting rule is Pareto efficient. Then, from Theorem 2 we know that the equilibrium outcomes coincide with those that are full agendas and extensions of the starting agenda. These must select from the top cycle. The proof when the preference profiles are not necessarily in P^* or the voting rule is inefficient is slightly more complicated, as then Theorem 2 cannot be applied. The proof is still relatively short and appears in the appendix.

A direct corollary of Proposition 1 is that all equilibrium agendas in a setting with single-peaked preferences and a Condorcet consistent voting rule lead to the outcome of the median of the voters' peaks.

5.4 Voting by Successive Elimination and Equilibrium Agendas

The voting procedure of voting by successive elimination is defined as follows. Consider some agenda $a \in A$ and let $a = (x_1; \ldots; x_k)$. In the successive elimination procedure, a vote is first taken to eliminate either x_k or x_{k-1}. The `winning' alternative from the first round is compared to x_{k-2}, and a vote is taken to eliminate either surviving alternative from the first vote or x_{k-2}, and so on. After $(k-1)$ comparisons, the last surviving alternative is declared to be the voting outcome.

At each stage, the elimination of one alternative is according to majority voting. This is well-specified when $T(P)$ is complete. However, in cases where there are ties under the majority preference relation, either resulting from personal indifferences or from an even number of voters, $T(P)$ is not complete. In this case, voting by successive elimination needs to be more completely specified.

We do so as follows. At each stage allow individuals to vote for one of the two alternatives or to abstain (in the case where they may be indifferent). In case of a tie in the voting between alternatives x_i and x_j, x_i is elected if and only if x_i comes before x_j in the ordering of voting $(i < j)$. This favors alternatives proposed earlier in the agenda under ties, which is a natural way to break ties (given that they have not already been broken under $T(P)$).

At the last stage of voting, if the voting boils down to a comparison of x and y where x precedes y in the successive elimination procedure, then x wins if not $yT(P)x$ and y wins otherwise.

However, in order to determine the eventual voting outcome, it is also necessary to describe how voters act. We first examine the case where they vote strategically at each stage, and so focus on the sophisticated voting outcome of this binary voting procedure. This is the outcome under the iterative elimination of weakly dominated strategies that has been well-studied (see Shepsle and Weingast (1984) for the algorithm identifying the outcome). [15]

Let $S(a;P)$ denote the sophisticated voting outcome under voting by successive elimination on agenda a.

The Banks Set

The Banks set associated with the tournament $T(P)$, denoted $BS(P)$, is defined by

$$BS(P) = \cup_{a \in A^m} S(a;P).$$

Thus, the Banks set is the set of sophisticated voting outcomes under voting by successive elimination under all possible full agendas.

There are situations, however, where some orderings of X are not relevant. For instance, X may contain a distinguished alternative x_1 which acts as the status quo. In many legislative procedures, the status quo is treated as if it were the first proposal in the agenda. Recall that our tie-breaking rule in case $T(P)$ is not complete naturally privileges the status quo against the amendments. We generalize the definition of the Banks set in the following way.

Given any $a \in A^k$, let

$$BS(a;P) = \cup_{a' \in A(a) \cap A^m} S(a';P).$$

Equilibrium Agendas and Voting by Successive Elimination

Given that voting by successive elimination is a Pareto efficient voting rule, we have the following corollary of Theorem 2.

---

[15] The Shepsle-Weingast algorithm was defined for the case where $T(P)$ is complete. Our procedure of breaking possible ties in the majority preference relation coming earlier in the ordering a ensures that the sophisticated outcome can be derived from a straightforward variation on the algorithm derived by Shepsle and Weingast.

Corollary 2 Consider a collection of sets of continuation equilibria $\{CE_V(a;P)\}_{a\in A}$ (satisfying (CE1) and (CE2)) and any profile of preferences $P \in \mathcal{P}^*$. For all $a \in A$,

$$\bigcup_{a' \in CE_V(a;P)} S(a';P) \subset BS(a;P);$$

and if consistency (CE3) is also satisfied, then

$$\bigcup_{a' \in CE_V(a;P)} S(a';P) = BS(a;P).$$

Note that the result above also holds if we set the starting agenda a to be the emptyset.[16]

Corollary 2 states that not only does the Banks' set capture the set of outcomes that could arise from arbitrary full length agendas, but that these are also precisely the set of potential equilibrium outcomes when the agendas are endogenous.

While Corollary 2 provides a precise characterization of equilibrium agenda outcomes for an important voting procedure, it is still useful to show that this characterization completely ties down the outcome in some interesting cases. We now show this in the context of an interesting "pork barrel politics" setting. In particular, even though in some cases the top cycle of the majority voting relation may be very large, the Banks set, and thus the set of equilibrium agenda outcomes, can be a singleton.

5.5 Voting over Projects

Ferejohn, Fiorina, and McKelvey (1987) consider the following model. N is a set of legislators (with n odd), each of whom has a project for their constituency. The projects have value only for their constituents, but the cost of a project, if it is undertaken, is split evenly among all constituencies.[17] Ferejohn, Fiorina, and McKelvey assume that projects have different costs, so as to ensure that $T(P)$ is complete, but that is not assumed here (as we can extend their result given our procedure for breaking ties).

---

[16] An easy way to see this is simply to extend the set of alternatives to include some x_0 such that all alternatives are preferred to x_0 by all agents under P, and then set $a = \{x_0\}$ and then apply the theorem as it stands.

[17] This assumption is not necessary. All that matters is that the legislators agree about the relative rankings of how costly (in terms of how much they each pay) different projects are.

So, this is a model of pure "pork-barrel" politics. Here the set of alternatives X is simply a list of which projects are undertaken, and so $X = \{0,1\}^n$. Voting over an agenda is done by sophisticated voting by successive elimination.

Given this setting, legislators' preferences take a specific form. Their favorite alternative is to have their own project undertaken and no other projects undertaken. Beyond the decision concerning a legislator's own project, he or she simply prefers to minimize the costs of the other projects undertaken. The critical freedom in the preferences is in the relative costs of projects, which determines which projects a legislator might tolerate being undertaken in conjunction with his or her own, before the cost becomes so high that he or she would prefer to have none built at all.

An interesting aspect of the Ferejohn, Fiorina, and McKelvey (1987) model is the importance of a status quo. The status quo is that no projects are undertaken. Applying our equilibrium approach to this model is of particular interest as it shows how the status quo can tie down equilibrium agendas, and illustrates why we have been careful to defined continuation equilibrium concepts that allow for a status quo. It also shows that the conclusions reached by Ferejohn, Fiorina, and McKelvey (1987) without an equilibrium analysis, are robust to an equilibrium formulation.

Let $X^*(P)$ denote the set of $x \in X$ that (i) undertake exactly $\frac{n+1}{2}$ projects, (ii) are as cheap as any other choice of exactly $\frac{n+1}{2}$ projects, and (iii) are such that $xT(P)0$.

Corollary 3 Consider any profile of admissible preferences $P \in \mathcal{P}$ and collection of sets of continuation equilibria $\{CE_V(a;P)\}_{a \in A}$ (satisfying (CE1) and (CE2)) in the extension of the Ferejohn, Fiorina, and McKelvey setting where some projects may have identical costs.

$$\bigcup_{a' \in CE_V(a_0;P)} S(a';P) = \begin{cases} X^*(P) & \text{if } X^*(P) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Here the equilibrium agendas result in collections of projects corresponding to majorities of minimal size and which choose the cheapest projects. This minimal winning size is an interesting qualitative feature which has been extensively discussed in various areas of political science since Riker (1962). The proof of Corollary 3 appears in the appendix.

5.6 Sincere Voting and an Absence of Chaos

The previous results show that equilibrium conditions on agendas can make narrow predictions. The results concerning the Banks set and voting by successive elimination were constrained to sophisticated voting. As much of the literature on chaos theorems (e.g., McKelvey (1979)) was restricted to sincere voting we show that the same is true there. In particular, we show that even in situations where the top cycle is large (even the whole set of alternatives), considering only equilibrium agendas still narrows the set of predictions in well-defined ways.

While the setting we consider in this section is a finite one (see the next section for the infinite case), we can still see the essence of chaos theorems in the following way. Consider sincere voting by successive elimination, where when asked to compare any two alternatives, voters vote for the one that they prefer, not anticipating the outcome of the votes yet to come in the sequence.[18] [19]

The critical observation is that for any $x \in TC(T(P))$ and any k, there exists an agenda $a \in A^k$, such that $V(a;P) = x$, where V is sincere voting by successive elimination. In particular, setting $k = m$, any x in the top cycle can be reached by at least one full length agenda (in fact at least two).[20]

[18] One might also term this myopic voting. Note, however, that this corresponds to sophisticated voting under the following alternative voting rule. That is important, as otherwise there would a schizophrenia between sophisticated (forward looking) agenda formation and myopic voting, and this exercise would only serve as a comment on the chaos literature. The closely related voting procedure for which this is sophisticated is as follows. On an agenda $a = (x_1;\dots;x_K)$, select x_1 unless a majority votes to move on to x_2; then select x_2 unless a majority votes to move on to x_3, and so forth. Sophisticated voting on this rule can be solved as follows. If one gets to the last decision of whether or not to select x_{K-1} or move on, then the vote will be a sincere vote between x_K and x_{K-1}. Anticipating this, the previous vote is a sincere vote between x_{K-2} and the sincere winner between x_K and x_{K-1}. Rolling this back up the voting tree, this is solved exactly as a sincere vote by successive elimination.

[19] Note that sophisticated behavior in voting by successive elimination can preclude some alternatives from the top cycle as ever being equilibrium outcomes as we already saw in Corollary 2.

[20] A recipe is as follows. Find an ordering of the K alternatives in the top cycle $x = x_1; x_2;\dots;x_K$, such that $x_i T(P) x_{i+1}$ for each $i < K$. Such an ordering always exists. Consider any agenda where the top cycle alternatives maintain this relative ordering and the other alternatives fall in any place. Sincere voting by successive elimination will lead to x. The second variation is to switch the position of x_K and x_{K-1}, which does not affect

This means that if we are not able to do any selection over agendas, then any alternative in the top cycle can be an outcome.

The following example, however, illustrates that our definition of equilibrium selects from the agendas. Here only a subset of the top cycle alternatives are equilibrium outcomes, even though all alternatives (other than a unanimously bad status quo) are in the top cycle. Thus, the notion of equilibrium does preclude alternatives and make selections from the top cycle.[21]

Example 4

Voters' preferences are :

- $x_5 P_1 x_2 P_1 x_3 P_1 x_4 P_1 x_1 P_1 x_0$
- $x_4 P_2 x_5 P_2 x_1 P_2 x_2 P_2 x_3 P_2 x_0$
- $x_3 P_3 x_4 P_3 x_5 P_3 x_1 P_3 x_2 P_3 x_0$

The induced tournament $T(P)$ is that

- x_5 beats x_0, x_1, x_2, and x_3,
- x_4 beats x_0, x_1, x_2, and x_5,
- x_3 beats x_0, x_1, and x_4,
- x_2 beats x_0 and x_3,
- x_1 beats x_0 and x_2.

Note that here $BS(\{x_0\}; P) = \{x_3; x_4; x_5\}$ and $TC(X; P) = \{x_1; x_2; x_3; x_4; x_5\}$; and also that both x_4 and x_5 Pareto dominate x_1.

Under sincere voting by successive elimination, the agendas (with a status quo of x_0) that can lead to an outcome of x_1 are those that follow the ordering

the outcome.

[21] In light of Proposition 1, equilibria under sincere voting by successive elimination will always end up in the top cycle, and so the example shows it can end up being a strict subset that is selected.

of the index of the alternatives without gaps, starting at x_0, except possibly that the last two alternatives may be switched.[22]

None of these are equilibrium agendas when the status quo is $a = \{x_0\}$ (i.e., none of these are in $CE_V(\{x_0\}; P)$). Thus, x_1 is not an equilibrium agenda outcome when V is sincere voting by successive elimination.

First, it is easily checked that $\{x_0; x_1\}$ and $\{x_0; x_2; x_1\}$, are not continuation equilibrium agendas (i.e., stopping once they are reached), as adding x_5 will lead to an outcome of either x_4 or x_5 which are unanimously preferred to x_1; and so (CE2) is violated. Thus they could not be equilibrium agendas beginning at x_0. We can also check that the agenda $\{x_0; x_1; x_3; x_2\}$ is not a continuation equilibrium. If either x_5 or x_4 is added one obtains either x_3 as the only equilibrium outcome.[23] Then it cannot be an equilibrium to stop, as voters 1 or 3 would gain by proposing either x_4 or x_5.

The agendas that remain to be checked that might lead to x_1 are those in $A(\{x_0; x_1; x_2\})$. Note that for any $a' \in A(\{x_0; x_1; x_2; x_3\})$, the outcome is x_1, while for any $a \in A(\{x_0; x_1; x_2; x_5\})$ the outcome x_4 or x_5. Thus, consistency (CE3) implies that if x_1 is an equilibrium outcome following $\{x_0; x_1; x_2\}$, then also x_4 or x_5 is an equilibrium outcome following $\{x_0; x_1; x_2\}$, and that x_1 can only come from proposing x_3 next. Also, note that x_3 is not an outcome under any agenda in $A(\{x_0; x_1; x_2\})$ as it loses to x_2, and also x_2 and x_0 are never outcomes under any agendas in $A(\{x_0; x_1; x_2\})$. Then by (CE3) it follows that x_1 is not an equilibrium outcome following $\{x_0; x_1; x_2\}$, and those equilibrium outcomes are a subset of $\{x_4; x_5\}$.

The example uses the fact that agendas that lead to x_1 must have x_1 in one of the first three places in the agenda. This always leaves additional alternatives that can be proposed that would lead to other outcomes, and the preference for some of these other outcomes prevents the specific agendas leading to x_1 from being equilibrium agendas.

Thus, chaos is avoided and we have predictions that we end up inside a strict subset of the top cycle.

In fact, we also have a "lower bound" on the set of possible outcomes

[22] Explicitly, the agendas leading to an outcome of x_1 are $\{x_0; x_1; x_2; x_3; x_5; x_4\}$, $\{x_0; x_1; x_2; x_3; x_4; x_5\}$, $\{x_0; x_1; x_2; x_3; x_4\}$, $\{x_0; x_1; x_2; x_4; x_3\}$, $\{x_0; x_1; x_2; x_3\}$, $\{x_0; x_1; x_3; x_2\}$, $\{x_0; x_1; x_2\}$, $\{x_0; x_2; x_1\}$, and $\{x_0; x_1\}$.

[23] By reasoning similar to that above, it is easily checked that if x_5 is added next, then x_4 would be also added next in equilibrium.

of sincere voting under sequential elimination - Claim 1 in the appendix implies that all Pareto optimal elements in the top cycle can be supported as outcomes of continuation equilibria.

Finally, we show that equilibrium agendas under sincere voting under sequential elimination can lead to Pareto inefficient outcomes.

Example 5

Let $X = \{x_0; x_1; x_2; x_3; x_4; x_5\}$. The status quo is x_0. There are 3 individuals, with preferences given below.

- $x_1 P_1 x_2 P_1 x_5 P_1 x_3 P_1 x_4 P_1 x_0$
- $x_5 P_2 x_3 P_2 x_4 P_2 x_1 P_2 x_2 P_1 x_0$
- $x_4 P_3 x_1 P_3 x_5 P_3 x_2 P_3 x_3 P_1 x_0$

The induced tournament $T(P)$ is :

- x_1 beats x_0, x_2, x_3, and x_5,
- x_2 beats x_0 and x_3,
- x_3 beats x_0 and x_4,
- x_4 beats x_0, x_1 and x_2,
- x_5 beats x_0, x_2, x_3 and x_4.

Note that x_2 is Pareto dominated by x_1.

Let us argue that $a = (x_0; x_2; x_3; x_4; x_1)$ which results in x_2 is in $CE_V(\{x_0\}; P)$. Since adding x_5 makes no difference to the outcome, this is an equilibrium agenda once a is reached. Moving back, a is an equilibrium continuation of $(x_0; x_2; x_3; x_4)$. If instead x_5 is added so to get $(x_0; x_2; x_3; x_4; x_5)$, then it will not be an equilibrium to stop as the outcome would be x_5 and agent 1 would prefer to add x_1 so that the outcome would again be x_2. Thus, all equilibrium continuations of $(x_0; x_2; x_3; x_4)$ lead to x_2.

Next, note that $a' = (x_0; x_2; x_3; x_5; x_4; x_1)$ results in x_5, which is voter 2's favorite. Thus, we know that it is possible to reach $(x_0; x_2; x_3)$. Then under (CE3), voter 1 is willing to propose x_4 expecting the continuation of a leading to x_2, given that there is another continuation equilibrium leading to x_5. As argued above, we then have a as an equilibrium continuation once $(x_0; x_2; x_3; x_4)$ has been reached.

Thus x_2 is an equilibrium outcome when the status quo is x_0.

5.7 Handling Infinities

Our discussion so far has focused on a finite set of alternatives X. We now demonstrate how our analysis works in more general settings where the set of alternatives may be infinite. An important first remark is that the definitions we have for continuation equilibria, (CE1)-(CE3), can be applied directly to the infinite case without modification.

However, there are new challenges that arise in applying the definition of equilibria in infinite settings, which we will address below. One challenge is whether or not to define voting rules on infinite sequences of alternatives, and if it is done, how to do it. There are different ways that this might be done and the specific choice of how to do it is usually specific to the setting in question. Another challenge is to establish existence of equilibrium sets. In the finite case existence was straightforward as we could follow a simple backward induction argument. In the infinite case the issue is more subtle and will require using some characteristics of the setting being analyzed. A third challenge is that even when collections of sets of agenda equilibria can be shown to exist, it may still be hard to get a handle on a characterization of them as, again, a simply backward induction approach is precluded.

Nevertheless, despite these challenges the definitions turn out to be quite manageable in several ways as we now show.

Consider an infinite X. Let $A = \cup_k A^k$ be now the set of arbitrary length finite agendas.[24]

[24] Here we could extend a voting rule V to be defined over infinite agendas, but it is not necessary. For the interested reader, one way of defining V over infinite agendas is as follows. Consider an infinite a, and let a_k be the agenda consisting of the first k proposed alternatives. If there exists some K such that $V(a_k; P) = V(a_K; P)$ for all $k \geq K$, then define $V(a; P) = V(a_K; P)$. Have some rule for assigning $V(a; P)$ otherwise, such as fixing a status quo x and if voting never resolves itself then the status quo stays in place.

Given a voting rule V, say that an agenda $a \in A$ is maximal at P if $V(a'; P) = V(a; P)$ for all $a' \in A(a)$. Denote the set of maximal agendas for V and P that are the continuation of some a by $M_V(a; P)$.

The analogue of Theorem 2 now follows.

First, we show that when the set of maximal agendas is nonempty, then there exists a natural set of continuation equilibria.

Lemma 1 Consider an infinite X, a profile of preferences $P \in P^*$, and a Pareto efficient voting rule V such that $M_V(a; P)$ is nonempty for all agendas $a \in A$. Then there exists a collection of sets of continuation equilibria $\{CE_V(a; P)\}_{a \in A}$ satisfying (CE1)-(CE3), which is to set $CE_V(a; P) = M_V(a; P)$ for each a.

Lemma 1 leaves open the question of when $M_V(a; P)$ is nonempty for all agendas. This is easy to check in some cases as when there is a Condorcet winner, and can also be verified in some settings such as the three person divide-the-dollar game analyzed by Penn (2001). We leave the exploration of more subtle conditions guaranteeing nonemptyness for future research.

Now we can establish the analog of Theorem 2 for the infinite case.

Theorem 3 Consider an infinite X, a profile of preferences $P \in P^*$, and a Pareto efficient voting rule V such that $M_V(a; P)$ is nonempty for all agendas $a \in A$. For any collection of sets of continuation equilibria $\{CE_V(a; P)\}_{a \in A}$ (satisfying (CE1) and (CE2)), and any finite $a \in A$,

$$\cup_{a' \in CE_V(a;P)} V(a'; P) \subset \cup_{b \in M_V(a;P)} V(b; P);$$

and if consistency (CE3) is also satisfied, then

$$\cup_{a' \in CE_V(a;P)} V(a'; P) = \cup_{b \in M_V(a;P)} V(b; P).$$

The proof of Theorem3 is provided in the appendix. Here, we provide the basic intuition. The proof of Theorem 2 exploited the possibility of backward induction from agendas $a \in A^m$. Notice that if a is a maximal agenda, then all $b \in A(a)$ can essentially be ignored. Hence, maximal agendas play the same role in the infinite setting that agendas in A^m play in the finite environment.

6 Discussion of the Definition of Equilibrium

Proposals to Stop the Agenda or Seconds to Continue an Agenda

Some procedures may allow an individual to propose a motion that voting take place immediately on the existing agenda. This motion is voted "yes" or "no", and a majority support can stop the existing agenda. Alternatively, a procedure may require at least two agents to support a proposal in order to add it to the agenda.

If either of these variations are present, it makes no difference to the analysis, at least under sophisticated voting by successive elimination. Let us offer a heuristic argument for why Corollary 6 extends in this way.

We argue by induction. It is clearly true starting at some full length agenda. Suppose it is true starting at agendas of length at least $k + 1$. Consider an existing agenda $a \in A^k$, $S(a; P) = x$, and individual i proposes the motion that voting take place immediately. If i's motion is defeated, then her proposal is irrelevant. On the other hand, if i's motion is accepted, then x becomes the final outcome. This implies that a majority prefers x to any outcome that can be obtained by some further continuation equilibrium, which from the induction step and the corollary corresponds to the outcome of some $a' \in A(a) \cap A^m$. If x already corresponds to such an outcome, then the claim is true. If not, then by the Shepsle-Weingast algorithm, there must be some alternative $y \notin a$ such that y is preferred by a majority to x and such that y is the outcome under a continuation equilibrium $a' \in A(a) \cap A^m$. This, however, implies that a majority would vote to continue rather than stop at x, which would be a contradiction. Thus the claim is true.

The argument for having a second agent move a proposal to make it part of an agenda is analogous, noting that if a majority prefer y to x, then at least two agents must prefer to follow the continuation equilibrium leading to y rather than stopping at x.

Modifications of Consistency

The notion of consistency (CE3) is one that produces a large set of equilibria relative to those which might be considered (witness Theorem 2 and Claim 1). We now consider a more stringent form of rationalizability, that in turn corresponds to a different form of consistency that includes fewer continuation equilibria.

We say that an agenda $a' = (a; x; \ldots) \in C_V^+(a; P)$ is strongly rationalizable if there exists $i \in N$ such that for any $y \notin a$ and $y \neq x$ there exists some $a'' \in CE_V((a; y); P)$ such that $V(a'; P) R_i V(a''; P)$, and if $a \in CE_V(a; P)$ then also $V(a'; P) R_i V(a; P)$.

Strong rationalizability only allows for an agenda $(a; x; \ldots)$ which is a continuation of a to be supported only if there is some agent who does not prefer all equilibrium continuations of $(a; y)$ to those of $(a; x)$. The idea being that an agent who prefers all continuations of $(a; y)$ to those of $(a; x)$ would not propose x, but would instead propose y (or possibly some other alternative). This differs from rationalizability, in that rationalizability allows some i to propose x if there is some alternative continuation that the agent finds worse; but this does not consider the fact that the agent might prefer to propose something else in y's place.

We say that a collection of sets of continuation equilibria is strongly consistent if it satisfies[25]

(CE4) (Strong Consistency) If $a' \in C_V^+(a; P)$ is strongly rationalizable, then $a' \in CE_V(a; P)$. Conversely, if $a' = (a; x; \ldots) \in CE_V(a; P)$ and either $a \in CE_V(a; P)$ or $a'' = (a; y; \ldots) \in CE_V(a; P)$ for some $y \neq x$, then a' is strongly rationalizable.

Note that from Theorem 1 we know that for Pareto efficient rules continuation equilibria satisfying strong consistency (CE4) always are a subset of those satisfying consistency (CE3). Example 1 is easily seen to be one where this is a strict subset. However, that is an inefficient voting rule. The following example shows that the selection may be strict even for sophisticated voting by successive elimination, where strong consistency results in a strict subset of the Banks' set.

Example 6

Let $X = \{x_1; x_2; x_3; x_4; x_5\}$ and $N = \{1; 2; 3\}$.
The preference profile is:

[25] When we modify (CE3) to (CE4), we might also consider adding another condition, which was implied under (CE1), (CE2) and (CE3), but not under (CE1), (CE2) and (CE4). The condition is (5) If $(a; x; \ldots) \in CE_V(a; P)$ then $CE_V((a; x); P) \subseteq CE_V(a; P)$. This is irrelevant in the discussion below.

- $x_1 P_1 x_3 P_1 x_2 P_1 x_4 P_1 x_5$
- $x_5 P_2 x_3 P_2 x_4 P_2 x_1 P_2 x_2$
- $x_2 P_3 x_4 P_3 x_5 P_3 x_1 P_3 x_3$

Then, the induced tournament $T(P)$ is

- x_4 beats x_1 and x_5.
- x_1 beats x_2 and x_3.
- x_2 beats x_4 and x_5.
- x_3 beats x_2 and x_4.
- x_5 beats x_1 and x_3.

Then, $BS(\{x_0\}; P) = X$. We want to show that if $CE(\{x_0\}; P)$ satisfy (CE1),(CE2) and strong consistency, then $EO_V(a; P) = \{x_1, x_2, x_5\}$.

First, note that if $a \in C_S^+(\{x_0, x_1\}; P)$, then $S(a; P) = x_4$. For if $a \in A(\{x_1\})$, the possible outcomes are in $\{x_5, x_4\}$. But since x_4 beats x_5, (CE2) implies that $S(a; P) = x_4$ if $a \in C_S^+(\{x_0, x_1\}; P)$.

Analogously, the following are true.

- If $a \in C_S^+(\{x_0, x_2\}; P)$, then $S(a; P) = x_1$.
- If $a \in C_S^+(\{x_0, x_5\})$, then $S(a; P) = x_2$.
- If $a \in C_S^+(\{x_0, x_3\})$, then $S(a; P) = x_5$.
- If $a \in C_S^+(\{x_0, x_4\}; P)$, then $S(a; P) = x_3$.

The proof is completed by showing that no one wants to propose x_1 or x_4 initially.

This must be true since 1 prefers to propose x_2 initially. This guarantees choice of x_1, which is 1's most preferred element in X. Similarly, 2 and 3 prefer initial proposals of x_3 and x_5 respectively.

References

Austen-Smith, D. (1987), "Sophisticated Sincerity: Voting over Endogenous Agendas", American Political Science Review, 81, pp 1323-29.

Austen-Smith, D. and J.S. Banks, (1999), Positive Political Theory I : Collective Preference, University of Michigan Press, Ann Arbor.

Banks, J.S. (1985), \Sophisticated Voting Outcomes and Agenda Control," Social Choice and Welfare, 1, pp 295-306.

Banks, J.S. and F. Gasmi (1987), \ Endogenous Agenda Formation in Three-Person Committees", Social Choice and Welfare, 4, pp 133-152.

Baron, D. and J. Ferejohn (1989) \Bargaining in Legislatures," American Political Science Review, vol. 83, pp 1181-1206.

Besley, T. and S. Coate (1997), \An Economic Model of Representative Democracy, Quarterly Journal of Economics, 112, pp 85-114.

Chwe, M. S-Y. (1994), \Farsighted Coalitional stability", Journal of Economic Theory, 63, pp 299-325.

Denzau, A., W. Riker, and K.A. Shepsle (1985), \Farquharson and Fenno: Sophisticated Voting and Home Style," American Political Science Review, vol. 79, pp 1117-1135.

Dutta, B., M.O. Jackson and M. Le Breton (1998), \Voting by Successive Elimination and Strategic Candidacy," forthcoming in the Journal of Economic Theory.

Dutta, B., M.O. Jackson and M. Le Breton (2001), \Strategic Candidacy and Voting Procedures," Econometrica, Vol. 69, pp 1013{1037.

Farquharson, R. (1969), Theory of Voting, New Haven: Yale University Press.

Ferejohn, J., Fiorina, M. and R.D. McKelvey (1987), "Sophisticated Voting and Agenda Independence in the Distributive Politics Setting", American Journal of Political Science, 31, pp 169-194.

Groseclose, T. and Krehbiel, K. (1993), "On the Pervasiveness of Sophisticated Sincerity", Chapter 10 in: W.A. Barnett, M.J. Hinich, and N.J. Scho¯eld (eds), Political Economy: Institutions, Competition, and Representation, Cambridge University Press, pp 247-277.

Laslier, J.F. (1997), Tournament Solutions and Majority Voting, Berlin/ Heidelberg: Springer-Verlag.

McKelvey, R.D. (1976), \Intransitivities in Multidimensional Voting Models and some Implications for Agenda Control," Journal of Economic Theory, Vol. 12, pp 472{482.

McKelvey, R.D. (1979), \General Conditions for Global Intransitivities in Formal Voting Models," Econometrica, Vol. 47, pp 1085{1112.

McKelvey, R.D. and Niemi, R.G. (1978), "A Multistage Game Representation of Sophisticated Voting for Binary Procedures", Journal of Economic Theory, 18, 1-22.

Miller, N.R., Grofman, B and S.L. Feld (1990), \Cycle Avoiding Trajectories, Strategic Agendas, and the Duality of Memory and Foresight: An Informal Exposition", Public Choice, 64, pp 265-277.

Ordeshook, P. (1993), "The Development of Contemporary Political Theory", Chapter 4 in: W.A. Barnett, M.J. Hinich, and N.J. Scho¯eld (eds), Political Economy: Institutions, Competition, and Representation, Cambridge University Press, pp 71-104.

Ordeshook, P. and Schwartz (1987), \Agendas and the Control of Political Outcomes", American Political Science Review, 81, pp 179-200.

Osborne, M.J. and A. Slivinski (1996), \A Model of Political Competition with Citizen Candidates," Quarterly Journal of Economics, 111, pp 65-96.

Penn, E.M. (2001), \A Distributive N-Amendment Game with Endogenous Agenda Formation," mimeo: Caltech.

Plott, C. (1967), \A Notion of Equilibrium and its Possibility under Majority Rule, American Economic Review, Vol. 57, pp 787{806.

Riker, W.H. (1962), The Theory of Political Coalitions, New Haven, Yale University Press.

Shepsle, K. and B. Weingast (1984), \Uncovered Sets and Sophisticated Voting Outcomes with Implications for Agenda Institutions," American Journal of Political Science, 28, pp 49-74.

Appendix

Let
$$EO_V(a;P) = \cup_{a' \in CE_V(a;P)} V(a';P).$$

We first state and prove a lemma which will be used repeatedly.

Lemma 2 Consider any $P \in \mathcal{P}$ and $a \in \mathcal{A}$. Suppose that $\cup_{b \in \mathcal{A}(a) \cap \mathcal{A}^m} V(b;P) \subset D(P)$, for some $D(P) \subset X$ such that if $x, y \in X$, and $x \notin D(P); y \in D(P)$ then $\exists i \in N$ such that $yP_i x$. Under (CE1) and (CE2) $EO_V(a;P) \subset D(P)$.

Proof of Lemma 2: We prove this by induction on the cardinality of a. If $a \in \mathcal{A}^m$, then $CE_V(a;P) = \{a\}$, and so the assertion must be true. Suppose that for some $K < m$, the claim is true for each $k > K$ and $a \in \mathcal{A}^k$. We show that the claim is true for $a \in \mathcal{A}^K$.

From the induction hypothesis it follows that $CE_V^+(a;P) \subset D(P)$, and so from (CE1) we only need to show that if $V(a;P) \notin D(P)$, then $a \notin CE_V(a;P)$. Consider any $x \notin a$, and $b \in CE_V((a;x);P) \subset D(P)$. Since $V(a;P) \notin D(P)$, it follows from the properties of $D(P)$ that $V(b;P)P_iV(a;P)$ for some i. (CE2) then implies that $a \notin CE_V(a;P)$, as required. ∎

Proof of Theorem 1: Fix a Pareto efficient V and a profile P.

The proof that $V(a';P)$ is Pareto efficient for any a' in $CE_V(a;P)$ and $a \in \mathcal{A}$ follows directly from Lemma 2, by letting $D(P)$ in the lemma be the set of Pareto efficient alternatives in X.

To complete the proof of the theorem, we show that (CE1), (CE2) and (CE3) can be satisfied if and only if[26]

$$CE_V(a;P) = \begin{cases} PE_V(a;P) & \text{if } V(a;P)R_iV(a';P) \text{ for all } i \text{ and } a' \in C_V^+(a;P) \\ PE_V(a;P) \setminus a & \text{if } V(a';P)P_iV(a;P) \text{ for some } i \text{ and } a' \in C_V^+(a;P). \end{cases}$$

It is straightforward to check if $CE_V(a;P)$ is defined above then (CE1), (CE2) and (CE3) are satisfied. So we show the converse.

Consider $CE_V(a;P)$ satisfying (CE1), (CE2) and (CE3). The proof proceeds by induction. Note that for any $a \in \mathcal{A}^m$, $CE_V(a;P) = \{a\}$ and that by the Pareto efficiency of V the claim follows directly. So, consider some

[26] Note that in the second case it must be that $PE_V(a;P) \setminus a$ is nonempty.

$K < m$ and suppose that the claims are true for any agenda in A^k for any $k > K$, and let us show that they hold for $a \in A^K$.

First, consider the case where $V(a;P)R_iV(a';P)$ for all i and $a' \in C_V^+(a;P)$. In this case it follows from (CE2) that $a \in CE(a;P)$. By the induction step, any $a' \in C_V^+(a;P)$ must result in a Pareto efficient outcome, and so it follows that a' is rationalizable relative to a and so by (CE3), $a' \in CE_V(a;P)$. This implies that $PE_V(a;P) \subset CE_V(a;P)$. Also, from the induction step $V(a';P)$ is Pareto efficient (relative to X) for any $a' \in C_V^+(a;P)$, and so $CE_V(a;P) \subset PE_V(a;P)$. It follows that $CE_V(a;P) = PE_V(a;P)$.

Next, consider the case where $V(a';P)P_iV(a;P)$ for some i and $a' \in C_V^+(a;P)$. In this case, by (CE2) $a \notin CE_V(a;P)$. Thus, by nonemptiness (CE1), there is some $a' \in CE_V(a;P)$, where $a' \neq a$. By our induction any $a'' \in C_V^+(a;P)$ is Pareto efficient, and so is rationalizable relative to a', and so by (CE3) $a'' \in CE_V(a;P)$. This implies that $PE_V(a;P) \setminus a \subset CE_V(a;P)$. Also, from the induction step $V(a';P)$ is Pareto efficient for any $a' \in C_V^+(a;P)$, and so $CE_V(a;P) \subset PE_V(a;P) \setminus a$. It follows that $CE_V(a;P) = PE_V(a;P) \setminus a$. ∎

Proof of Claim 1: Take any Pareto efficient x such that $x = V(a;P)$ for some $a = (a_1; \ldots; a_m)$. We prove by induction on k that $a \in CE_V((a_1; \ldots; a_k); P)$ for all $k = 1; \ldots; m$. The proof is obvious for $k = m$.

Now, assume that the assertion holds for $k > K$ where $K < m$, let us show it holds for K. By the Pareto efficiency of V, given any $y \neq V(a;P)$, there exists i such that $V(a;P)R_iy$. By the induction step, $a \in C_V^+((a_1; \ldots; a_K); P)$, and so (CE3) then directly implies that $a \in CE_V((a_1; \ldots; a_K); P)$. ∎

Proof of Theorem 2: We first show the first claim in the theorem. We use Lemma 2. Choose $a \in A$ and any profile $P \in \mathcal{P}^*$. Let $D(P) = \cup_{b \in A(a) \cap A^m} V(b;P)$. We show that $D(P)$ satisfies the conditions of lemma2.

Take any $y \in D(P)$ and $x \notin D(P)$. Since V is Pareto efficient, y is not Pareto dominated by x. Given that $P \in \mathcal{P}^*$, this means that there is $i \in N$ such that yP_ix.

Since $D(P)$ satisfies the required condition from the lemma, it follows that $EO_V(a;P) \subset \cup_{b \in A(a) \cap A^m} V(b;P)$, as claimed in the theorem.

Next, we show that equality holds if consistency is satisfied. Note that since V is Pareto efficient, the outcomes from full length extensions of a must be Pareto efficient. It follows from Claim 1 that (CE1), (CE2) and (CE3) imply that the equilibrium continuation outcomes following a coincide with the outcomes of full length agenda continuations of a. ∎

We now show the claim that the algorithm in Theorem 2 defines a minimal set of consistent continuation equilibria, even when V may not be Pareto efficient.

Consider the following definition of smallness on continuation equilibria. Given two collections of sets of continuation equilibria $\{CE_V(a;P)\}_{a\in A}$ and $\{CE'_V(a;P)\}_{a\in A}$, we say that $\{CE_V(a;P)\}_{a\in A}$ is smaller than $\{CE'_V(a;P)\}_{a\in A}$ if $CE_V(a;P) \subseteq CE'_V(a;P)$ for all $a \in A$.

The minimal set of continuation equilibria may be identified as follows, as we shall prove below.

We define $CE^*_V(a;P)$ by induction on the length of a. Consider $a \in A$ of length k and let

$$C^{*+}_V(a;P) = \cup_{x \notin a} CE^*_V((a;x);P)$$

Then we construct $C^*_{1V}(a;P)$ as follows.

-Either $V(a;P)R_iV(a';P)$ for all $a' \in C^{*+}_V(a;P)$ and for all $i \in N$. Then $C^*_{1V}(a;P) = \{a\}$

-Or $V(a';P)P_iV(a;P)$ for some $a' \in C^{*+}_V(a;P)$ and some $i \in N$. Let $C_i(a;P)$ be the subset of $C^{*+}_V(a;P)$ consisting of the agendas b in $C^{*+}_V(a;P)$ such that $V(b;P)R_iV(a';P)$ for all $a' \in C^{*+}_V(a;P)$. Then $C^*_{1V}(a;P) = \cup_{i\in N} C_i(a;P)$.

Then $C^*_{2V}(a;P)$ is defined as the set

$$\left\{ a' \in C^{*+}_V(a;P) : V(a';P)R_iV(b;P) \text{ for some } b \in C^*_{1V}(a;P) \text{ and some } i \in N \right\}$$

Since the set $C^{*+}_V(a;P)$ is finite, there exists j such that $C^*_{jV}(a;P) = C^*_{j+1V}(a;P)$. Define $CE^*_V(a;P)$ as such a set.

It follows quite easily from the above construction that $\{CE^*_V(a;P)\}_{a\in A}$ is a collection of sets of consistent continuation equilibria.

Next, let $\{CE^{**}_V(a;P)\}_{a\in A}$ be defined inductively by

$$CE^{**}_V(a;P) = \begin{cases} PE_V(a;P) \text{ if } V(a;P)R_iV(a';P)\ \forall i \in N \text{ and } \forall a' \in C^{**+}_V(a;P) \\ PE_V(a;P)\setminus a \text{ otherwise} \end{cases}$$

Claim 2 There exists a unique smallest collection of sets of consistent continuation equilibria which is given by $\{CE^*_V(a;P)\}_{a\in A}$ above, and this coincides with $\{CE^{**}_V(a;P)\}_{a\in A}$.

Proof of Claim 2:

Step 1: $\{CE_V^{**}(a;P)\}_{a\in A}$ is a collection of sets of consistent continuation equilibria and $CE_V^{**}(a;P) = CE_V^{*}(a;P)$ for all $a \in A$ and all $P \in \mathcal{P}$.

It is straightforward to see that $\{CE_V^{**}(a;P)\}_{a\in A}$ is a collection of sets of consistent continuation equilibria .We prove the above identity by induction over $\#a$. Assume that $CE_V^{**}(a;P) = CE_V^{*}(a;P)$ for all a such that $\#a > K$ and let a be such that $\#a = K$.

Assume first that $a \in CE_V^{**}(a;P)$. Then it follows from the induction hypothesis that $V(a;P)R_iV(a';P)\ \forall i \in N$ and $\forall a' \in C_V^{*+}(a;P) = C_V^{**+}(a;P)$ and therefore from (CE2), $a \in CE_V^{*}(a;P)$. We prove similarly that if $a \in CE_V^{*}(a;P)$, then $a \in CE_V^{**}(a;P)$. In that case, if $b \in CE_V^{**}(a;P)$, then $b \in CE_V^{*}(a;P)$ as there exists at least one $i \in N$ such that $V(b;P)R_iV(a;P)$. Similarly, if $b \in CE_V^{*}(a;P)$, then $b \in CE_V^{**}(a;P)$. Assume indeed on the contrary that there exists $c \in C_V^{**+}(a;P)$ such that $V(c;P)P_iV(b;P)\ \forall i \in N$. Since from the induction hypothesis, $c \in C_V^{*+}(a;P)$; we contradict our construction of $CE_V^{*}(a;P)$.

The proof of equality in the case where $a \notin CE_V^{**}(a;P)$ is similar.

Step 2: Step 2: $\{CE_V^{*}(a;P)\}_{a\in A}$ is the unique smallest collection of sets of consistent continuation equilibria.

This follows from Claim 1 and the characterization of $\{CE_V^{*}(a;P)\}_{a\in A}$ in Step 1. ∎

Proof of Proposition 1:

Both assertions in the proposition follow from Lemma 2 . First, let $D(P) = TC(P)$. Since $TC(P)$ satisfies the requirements of $D(P)$ in the lemma, it follows from that if V is top cycle consistent, then $EO_V(a;P) \subset TC(T(P))$.

To prove the second statement concerning Condorcet consistency, let P be any profile with a Condorcet winner, say x. Then, let $D(P) = \{x\}$. Since $D(P)$ satisfies the requirements of the lemma, the statement follows. ∎

Proof of Corollary 3: Note the following observations: (i) Any y which beats 0 must have at least $\frac{n+1}{2}$ projects built. (ii) Any $x \in X^*(P)$ beats any y such that $yT(P)0$ and $y \notin X^*(P)$ (as then y must involve at least $\frac{n+1}{2}$ projects and yet be more expensive than x).

Using these observations, it follows from (i) than only 0 or some choice of at least $\frac{n+1}{2}$ projects can be the outcome of a full length agenda. From (ii) it follows that only choices in $X^*(P)$ can be the outcome of a full length agenda in $A(a_0)$. This implies that only outcomes in $X^*(P)$ (if it is nonempty) can be the outcomes of full length agendas in $A(a_0)$. Next note that no element in $X^*(P)$ beats any other element in $X^*(P)$, and so the first one appearing in the agenda will be the outcome. This means that each element in $X^*(P)$ is the outcome of at least one full length agenda in $A(a_0)$. The result then follows from Corollary 2. ■

Proof of Lemma 1: Let us show that setting $CE_V(a;P) = M(a;P)$ for each a satisfies (CE1)-(CE3).

It follows from the definition of maximal agenda that if $a \in M_V(a;P)$ then $M_V(a;P) = A(a)$ and moreover, that $M_V(b;P) = A(b)$ for all $b \in A(a)$. Then it easily follows that (CE1)-(CE3) are satisfied starting at any maximal a at P. So, consider $a \in A$ that is not maximal. By the definition of $CE_V(a;P)$, it follows that $C_V^+(a;P) = \cup_{x \notin a} M_V((a;x);P)$. It follows from the definition of maximality that $\cup_{x \notin a} M_V((a;x);P) = M_V(a;P)$. So, $CE_V(a;P) = C_V^+(a;P) = M_V(a;P)$. It then follows directly (noting nonemptiness of M) that (CE1) is satisfied. Next, using Pareto efficiency of V, since $a \notin M_V(a;P)$, there must be $b \in M_V(a;P)$ such that $V(b;P)P_iV(a;P)$ for some i. Since $a \notin M_V(a;P)$, we know that $a \notin CE_V(a;P)$, which then satisfies (CE2) since we have established that $V(b;P)P_iV(a;P)$ for some i and $b \in M_V(a;P) = C_V^+(a;P)$. Finally, note that given Pareto efficiency of V, any $b \in M_V(a;P)$ must be Pareto optimal (if not, some y Pareto dominates $x = V(b;P)$, which implies that $y \notin b$; but then by Pareto optimality $V((b;y);P) \neq x$ which is a contradiction). It then follows that for all distinct pairs $b, c \in M_V(a;P)$, there exist i, j with bR_ic and cR_jb. It then follows that all of $C_V^+(a;P) = M_V(a;P) = CE_V(a;P)$ is rationalizable and that (CE3) holds. ■

Proof of Theorem 3: Consider an infinite X, a Pareto efficient voting rule V and a profile of preferences $P \in \mathcal{P}$ such that $M_V(a;P)$ is nonempty for each $a \in A$.

The remaining part of the proof is identical to that of Theorem 2 after noting that Lemma 2 and Claim 1 remain valid after some slight modification. That is, if $\cup_{b \in A(a) \cap A^m} V(b;P)$ is replaced by $\cup_{b \in A(a) \cap M_V(a;P)} V(b;P)$ in Lemma 2, the modified statement remains true. Similarly, Claim 1 can be modified to show that if V is Pareto efficient and $x = V(b;P)$ for some $b \in M_V(a;P)$ for some a, then $a \in EO_V(a;P)$. The details are left to the reader. ■