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Inconsequential arbitrage*

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Abstract

We introduce the concept of inconsequential arbitrage and, in the context of a model allowing short-sales and half-lines in indifference surfaces, we prove that inconsequential arbitrage is sufficient for existence of equilibrium. With a slightly stronger condition of local nonsatiation than required for existence of equilibrium and with a mild uniformity condition on arbitrage opportunities, we show that the existence of a Pareto-optimal allocation implies inconsequential arbitrage, implying that inconsequential arbitrage is necessary and sufficient for existence of an equilibrium. By further strengthening our nonsatiation condition, we obtain a second welfare theorem for exchange economies allowing short sales. To further understand inconsequential arbitrage, we introduce the notion of exhaustible arbitrage and we show that any inconsequential arbitrage is

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exhaustible. We also compare inconsequential arbitrage to the conditions limiting arbitrage of Hart (1974) and Werner (1987), as well as to the conditions recently introduced by Dana, Le Van, and Magnien (1999) and Allouch (1999). For example, we show that the condition of Hart (translated to a general equilibrium setting) and the condition of Werner are equivalent. We then show that the Hart/Werner conditions imply inconsequential arbitrage. To highlight the extent to which we extend Hart and Werner, we construct an example of an exchange economy in which inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions do not hold. Finally, under additional conditions on the model, we show that if agents' indifference surfaces contain no half lines, then inconsequential arbitrage, the Hart/Werner conditions, the Dana, Le Van, and Magnien condition, and Allouch's condition are all equivalent - and in turn, equivalent to the existence of equilibrium.

1 Introduction

We introduce the condition of *inconsequential arbitrage*. This condition ensures that, from the viewpoint of existence of equilibrium, arbitrarily large arbitrage opportunities are inconsequential. We then show that in an exchange economy allowing short sales and half-lines in indifference surfaces, inconsequential arbitrage is sufficient for existence of equilibrium (Theorem 1). With a slightly stronger condition of local nonsatiation than required for existence of equilibrium and with a mild uniformity condition on arbitrage opportunities, we show that the existence of a Pareto-optimal allocation implies inconsequential arbitrage. Thus, it follows that inconsequential arbitrage is necessary and sufficient for existence of an equilibrium - as well as necessary and sufficient for the existence of a Pareto optimal allocation (Theorem 3). By further strengthening our nonsatiation condition, we obtain a second welfare theorem, generalizing the second welfare theorem in Page and Wooders (1996a).[1]

Prior papers in the literature on existence of equilibrium in models with consumption sets unbounded from below - models with short sales - have focused first on establishing sufficient conditions for existence of equilibrium and second, on necessary conditions. Typically, to show that a condition limiting arbitrage is necessary for existence of equilibrium, the prior literature has required that there be no half-lines in indifference surfaces ((Werner (1987), Page and Wooders (1993), Dana, Le Van and Magnien (1999)[2]) or that, at most, one agent have half lines in indifference

[1]Within the context of a model allowing local satiation, Hurwicz (1996) proves a second welfare theorem assuming that consumption sets are bounded from below. Here, within the context of a model *not* allowing local satiation, we carry out our proof of a second welfare theorem allowing short sales, and hence allowing consumption sets to be *unbounded* from below.

[2]See Dana, Le Van and Magnien (1999) for an illuminating discussion of other related work, and in particular, for an excellent discussion of the various conditions limiting arbitrage and compactness of the utility possibility set. According to Werner, a relationship between his condition limiting

surfaces (Page and Wooders (1996a)). In addition, many papers in the literature establishing sufficient conditions for existence of equilibrium have required that arbitrage opportunities be invariant or uniform with respect to endowment (i.e., the consumption at which an arbitrage transaction would begin). Such uniformity is required, for example, by Hart (1974) and Werner (1987) even for their sufficiency results. In the current paper, none of these restrictions are required.

Conditions limiting arbitrage found in the literature generally fall into three broad categories:

(i) *conditions on net trades*, for example, Hart (1974), Page (1987), Nielsen (1989), Page and Wooders (1993,1996a,b), Allouch (1999) - including the condition of inconsequential arbitrage introduced here;
(ii) *conditions on prices*, for example, Green (1973), Grandmont (1977,1982), Hammond (1983), and Werner (1987).
(iii) *conditions on the set of utility possibilities (namely, compactness)*, for example, Brown and Werner (1995), Dana, Le Van, and Magnien (1999).

We compare inconsequential arbitrage to the conditions limiting arbitrage introduced by Hart (1974) and Werner (1987), as well as to the conditions recently introduced by Dana, Le Van, and Magnien (1999) and Allouch (1999). For example, we show that the condition of Hart (translated to a general equilibrium setting) and the condition of Werner are equivalent (Theorem 6), and more importantly, we show that the Hart/Werner conditions imply inconsequential arbitrage (Theorem 7). We also show that under the assumption of no half-lines in indifference surfaces, the Hart/Werner conditions and inconsequential arbitrage are equivalent. In order to highlight the extent to which we extend Hart (1974) and Werner (1987), we construct an example (example 3 below) of an exchange economy in which inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions do not hold. Inconsequential arbitrage, however, has some limitations. In order to make these more clearly understood, we also construct an example (example 4 below) of an exchange economy in which inconsequential arbitrage does not hold, nor do the Hart/Werner conditions. But an equilibrium does exist. Finally, under additional conditions on the model, we show that if agents' indifference surfaces contain no half lines, then inconsequential arbitrage, the Hart/Werner conditions, the Dana, Le Van, and Magnien condition, and Allouch's condition are all equivalent - and in turn, equivalent to the existence of equilibrium.

In a recent paper, Allouch (1999) has shown that inconsequential arbitrage implies compactness of the set of utility possibilities. Moreover, Dana, Le Van, and

arbitrage opportunities and compactness was first remarked by J-F. Mertens in private conversation. The equivalence of the closely related Page-Wooders condition of "no unbounded arbitrage" opportunities, compactness of utility possibilities, and the existence of a Pareto-optimal point was established in Page and Wooders (1996a,b).

Magnien (1999) have shown that if local satiation is *ruled out* (as we do here) then compactness of utility possibilities implies existence. Allouch (1999) also introduces a new condition, bounded arbitrage, and in a model more general than the one developed here, shows that bounded arbitrage implies existence. Allouch (1999) also shows that if local satiation is ruled out then his condition of bounded arbitrage is equivalent to compactness of utility possibilities. However, if local satiation is allowed, this conclusion must be weakened: bounded arbitrage implies compactness of utility possibilities - it is an open question whether the reverse implication still holds.

Inconsequential arbitrage builds on the condition of no unbounded arbitrage in Page and Wooders (1993, 1996a,b) (see also Page (1987)). That condition ensures that no group of agents can make mutually compatible, unbounded and utility-nondecreasing trades. The result of Page and Wooders (1993,1996a) that existence of equilibrium implies no unbounded arbitrage requires that no group of agents can make mutually compatible, unbounded and utility *increasing* trades. In general, no unbounded arbitrage implies inconsequential arbitrage. While the condition of no unbounded arbitrage focuses on expanding utility nondecreasing or increasing trades, inconsequential arbitrage focuses on contracting net trades without decreasing utility. This change, from the focus on expanding trades to the focus on decreasing trades, enables us to treat economies with half-lines in indifference surfaces. It is the ability to contract net trades in directions determined by arbitrage without decreasing utility that allows for existence of an equilibrium in the presence of half-lines in indifference surfaces.

To further understand inconsequential arbitrage, we introduce the notion of exhaustible arbitrage. Roughly, an arbitrage is exhaustible if, starting from any allocation, expanding trades in the direction given by the arbitrage eventually ceases to be utility increasing. Here we show that any inconsequential arbitrage is exhaustible (Theorem 2). It is an open question whether the reverse implication is true.

The approach taken in this paper can also be viewed as an outgrowth of Hart (1974), showing existence of equilibrium in an asset-market model with unbounded short sales. Studying his paper suggested the "back up" argument used in our proofs and the modification of no unbounded arbitrage leading to inconsequential arbitrage. In essence, we are extending the Hart (1974) model of an asset market and his condition limiting arbitrage opportunities to a general equilibrium model. Hart's condition is stated explicitly in terms of the structure of asset returns; this is not required in our work. Our framework and results include Hart's as a special case.

For an excellent exposition of the different notions of arbitrage in the prior literature, we refer the reader to Dana, Le Van, and Magnien (1999), including a discussion of Chichilnisky (1997) and prior papers.[3] Another survey, including asset

[3]For further discussion see Monteiro, Page, and Wooders (1997, 1998, 1999).

market models in addition to general equilibrium models, is provided in Page and Wooders (1999). For excellent treatments of arbitrage and existence of equilibrium in infinite dimensional settings see Brown and Werner (1995) and Dana, Le Van, and Magnien (1997). Extending the concept of inconsequential arbitrage to infinite dimensional settings is an open problem.

Outline of the paper:

2 An economy with short sales

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ denote an *exchange economy*. Each agent j has *consumption set* $X_j \subset \mathbb{R}^L$ and *endowment* ω_j. The j^{th} agent's preferences $P_j(\cdot)$ over X_j are specified via a *utility function* $u_j(\cdot) : X_j \to \mathbb{R}$ as follows

$$P_j(x_j) := \{x \in X_j : u_j(x) > u_j(x_j)\}.$$

The *weak preferred set* is given by

$$\widehat{P}_j(x_j) := \{x \in X_j : u_j(x) \geq u_j(x_j)\}.$$

The set of *individually rational allocations* at endowments $\omega = (\omega_1, ..., \omega_n)$ is given by

$$A(\omega) = \{(x_1, ..., x_n) \in X_1 \times ... \times X_n : \sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \text{ and } x_j \in \widehat{P}_j(\omega_j) \text{ for all } j\}.$$

We shall maintain the following assumptions throughout: for each agent $j = 1, ..., n$,

[A-1] $u_j(\cdot)$ is upper semicontinuous and quasi-concave.

[A-2] $\omega_j \in X_j$ and X_j is closed and convex.

We shall also maintain the following nonsatiation assumption:

[A-3] *Local nonsatiation at rational allocations.*
For any rational allocation $(x_1, ..., x_n) \in A(\omega)$,
$P_j(x_j) \neq \emptyset$ and $clP_j(x_j) = \widehat{P}_j(x_j)$.

In **[A-3]**, "*cl*" denotes closure. Thus, at a rational allocation $(x_1, ..., x_n) \in A(\omega)$, $P_j(x_j)$ is nonempty and x_j is in the boundary of the preferred set $P_j(x_j)$ for each agent j.

Given prices $p \in \mathcal{B} := \{p' \in \mathbb{R}^L : \|p'\| \leq 1\}$ the *budget set* for the *jth* agent is given by

$$B(\omega_j, p) = \{x \in X_j : \langle x, p \rangle \leq \langle \omega_j, p \rangle\},$$

and the interior of the budget set relative to X_j is given by

$$F(\omega_j, p) = \{x_j \in X_j : \langle x_j, p \rangle < \langle \omega_j, p \rangle\}.$$

An *equilibrium* for the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ is an $(n+1)$ -tuple of vectors $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ such that:

(i) $(\overline{x}_1, ..., \overline{x}_n) \in A(\omega)$;

(ii) $\overline{p} \in \mathcal{B}\backslash\{0\}$; and

(iii) for each j, $\langle \overline{x}_j, \overline{p}\rangle = \langle \omega_j, \overline{p}\rangle$ and $P_j(\overline{x}_j) \cap B(\omega_j, \overline{p}) = \emptyset$.

If $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ is an equilibrium, then

$$\text{for each } j,\ x_j \in P_j(\overline{x}_j) \text{ implies that } \langle x_j, \overline{p}, \rangle > \langle \overline{x}_j, \overline{p}, \rangle = \langle \omega_j, \overline{p}\rangle .$$

We say that $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ is a *quasi-equilibrium* if

(i) $(\overline{x}_1, ..., \overline{x}_n) \in A(\omega)$;

(ii) $\overline{p} \in \mathcal{B}\backslash\{0\}$; and

(iii) for each j, $\overline{x}_j \in B(\omega_j, \overline{p})$ and $P_j(\overline{x}_j) \cap F(\omega_j, \overline{p}) = \emptyset$.

If $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ is a quasi-equilibrium, then

$$\text{for each } j,\ x_j \in P_j(\overline{x}_j) \text{ implies that } \langle x_j, \overline{p}, \rangle \geq \langle \overline{x}_j, \overline{p}, \rangle = \langle \omega_j, \overline{p}\rangle .$$

Note that every equilibrium is a quasi-equilibrium.[4]

3 Inconsequential arbitrage

We define the j^{th} *agent's arbitrage cone at endowments* $\omega_j \in X_j$ as the closed convex cone containing the origin given by

$$R(\widehat{P}_j(\omega_j)) = \{y_j \in \mathbb{R}^L : \text{for } x_j' \in \widehat{P}_j(\omega_j) \text{ and } \lambda \geq 0,\ x_j' + \lambda y_j \in \widehat{P}_j(\omega_j)\ \} .$$

Thus, if $y_j \in R(\widehat{P}_j(\omega_j))$, then for all $\lambda \geq 0$ and all $x_j \in \widehat{P}_j(\omega_j)$, $x_j + \lambda y_j \in X_j$ and $u_j(x_j + \lambda y_j) \geq u_j(\omega_j)$. The agent's arbitrage cone at ω_j, then, is the recession cone corresponding the weak preferred set $\widehat{P}_j(\omega_j)$ (see Rockafellar (1970), Section 8). Equivalently, $y_j \in R(\widehat{P}_j(\omega_j))$ if and only if y_j is a cluster point of some sequence $\{\lambda^k x_j^k\}_k$ where the sequence of positive numbers $\{\lambda^k\}_k$ is such that $\lambda^k \downarrow 0$, and where for all k, $x_j^k \in \widehat{P}_j(\omega_j)$ (see Rockafellar (1970), Theorem 8.2).

An *arbitrage* at $\omega = (\omega_1, ..., \omega_n)$ is an n-tuple of net trades $y = (y_1, ..., y_n)$ such that y is the limit of some sequence $\{(\lambda^k x_1^k, \ldots, \lambda^k x_n^k)\}_k$ with $\lambda^k \downarrow 0$ and $x^k = (x_1^k, \ldots, x_n^k) \in A(\omega)$ for all k. We shall denote by $arb(\omega)$ the set of all arbitrages at ω. Also, we shall denote by $arbseq^{\omega}(y)$ the set of all sequences $\{\lambda^k x^k\}_k$ with $\lambda^k x^k \to y$, $\lambda^k \downarrow 0$, and $x^k \in A(\omega)$ for all k.

We say that an arbitrage $y = (y_1, \ldots, y_n) \in arb(\omega)$ is in the *back-up set at* ω, denoted by $bus(\omega)$, if for each sequence $\{\lambda^k x^k\}_k \in arbseq^{\omega}(y)$, there exists an $\varepsilon > 0$ such that for all k sufficiently large and all agents j,

$$x_j^k - \varepsilon y_j \in X_j \text{ and } P_j(x_j^k - \varepsilon y_j) \subseteq P_j(x_j^k).$$

[4]Note that it is without loss of generality that we can restrict prices to the unit ball $\mathcal{B}$.

An economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies *inconsequential arbitrage at* ω if

$$arb(\omega) \subseteq bus(\omega). \tag{1}$$

In words, an arbitrage $y \in arb(\omega)$ is inconsequential (i.e., is contained in the back-up set at endowments $bus(\omega)$) if for sufficiently large allocations $x \in A(\omega)$ in the $y = (y_1, \ldots, y_n)$ "directions" from the endowment ω, each agent j can reduce his consumption by a small amount in the $-y_j$ direction without reducing his utility.

4 Main sufficiency result

Our main result states that in an economy allowing short-sales and half-lines in indifference surfaces, inconsequential arbitrage at endowments is sufficient for existence of a quasi-equilibrium.

Theorem 1 *(Inconsequential arbitrage implies the existence of a quasi-equilibrium). Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an economy satisfying* ***[A-1]-[A-3]****. If the economy satisfies inconsequential arbitrage at endowments* $\omega = (\omega_1, ..., \omega_n)$*, then there exists a quasi-equilibrium* $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$*. Moreover, if the quasi-equilibrium* $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ *is such that for each agent* j*,*

(i) $\inf_{x \in X_j} \langle x, \overline{p} \rangle < \langle \omega_j, \overline{p} \rangle$ *, and*

(ii) $P_j(\overline{x}_j)$ *is open relative to* X_j*,*

then $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ *is an equilibrium.*

Condition (i) in Theorem 1 will hold automatically if for each agent j, $\omega_j \in intX_j$.[5] Condition (ii) will hold automatically if for each agent j the utility function, $u_j(\cdot)$, is continuous.

5 Increasing cones, exhaustible arbitrages, and inconsequential arbitrages

A set closely related to the jth agent's arbitrage cone at x_j is the increasing cone at x_j given by

$$I_j(x_j) := \left\{ y_j \in R(\widehat{P}_j(x_j)) : \forall \lambda \geq 0, \exists\, \lambda' > \lambda \text{ such that } u_j(x_j + \lambda' y_j) > u_j(x_j + \lambda y_j) \right\}.$$

[5]Here *int* denotes interior.

We say that an arbitrage $y = (y_1, ..., y_n) \in arb(\omega)$ is *exhaustible* at ω, if for all rational allocations $x = (x_1, ..., x_n) \in A(\omega)$ and all agents j,

$$y_j \in R(\widehat{P}_j(\omega_j))\backslash I_j(x_j).$$

Thus, an arbitrage y fails to be exhaustible at ω if for some rational allocation $x \in A(\omega)$ and some agent j, $y_j \in I_j(x_j)$. We shall denote by $ea(\omega)$ the set of all arbitrages that are exhaustible at ω. Note that if $y_j \in R(\widehat{P}_j(\omega_j))\backslash I_j(x_j)$, then there exists a $\lambda_{x_j} \geq 0$ such that for $\lambda \geq \lambda_{x_j}$, $u_j(x_j + \lambda y_j)$ is nonincreasing in λ. Thus, there does not exist a $\lambda > \lambda_{x_j}$ such that $u_j(x_j + \lambda y_j) > u_j(x_j + \lambda_{x_j} y_j)$, and thus at $x_j + \lambda_{x_j} y_j$ all arbitrages in the y_j direction are exhausted.

Theorem 2 *(Inconsequentiality implies exhaustibility). Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]-[**A-2**]. If y is an inconsequential arbitrage at ω (i.e., $y \in bus(\omega)$), then y is exhaustible at ω (i.e., $y \in ea(\omega)$). Thus,*

$$bus(\omega) \subseteq ea(\omega).$$

Remark 1 *We have then,*

$$bus(\omega) \subseteq ea(\omega) \subseteq arb(\omega).$$

Thus, if the economy satisfies inconsequential arbitrage, we have

$$bus(\omega) = ea(\omega) = arb(\omega).$$

6 Pareto optimality and inconsequential arbitrage

6.1 The equivalence of inconsequential arbitrage, existence of a Pareto-optimal allocation, and existence of equilibrium

When all equilibrium allocations are Pareto-optimal, then existence of equilibrium implies inconsequential arbitrage. It is well known that when indifference surfaces are not "thick", then all equilibria are Pareto optimal. By assumption [**A-3**], thickness is ruled out at rational allocations and guarantees that all equilibria are Pareto-optimal. Moreover, if we strengthen our assumption of nonsatiation and if we assume that arbitrage cones are uniform across rational allocations, then the existence of a Pareto-optimal allocation implies that the economy satisfies inconsequential arbitrage.[6] Thus, we add to our list of assumptions the following:

[6]A rational allocation $(x_1, \ldots, x_n) \in A(\omega)$ is *Pareto-optimal* if there does not exist another rational allocation $(x'_1, \ldots, x'_n) \in A(\omega)$ such that $u_j(x'_j) \geq u_j(x_j)$ for all agents j and for at least one agent j', $u_{j'}(x'_{j'}) > u_{j'}(x_{j'})$.

[A-4] *Nonsatiation off the bus.*
If $y \in arb(\omega)\backslash bus(\omega)$, then for each rational allocation $x = (x_1, ..., x_n) \in A(\omega)$
there is at least one agent j such that for some $\lambda_j > 0$
$$x_j + \lambda_j y_j \in P_j(x_j).$$

[A-5] *Uniformity of arbitrage cones across rational allocations.*
For all rational allocations $x = (x_1, ..., x_n) \in A(\omega)$
$R(\widehat{P}_j(x_j)) = R(\widehat{P}_j(\omega_j))$ for all agents j.

Also, in order to guarantee that every quasi-equilibrium is an equilibrium, we add the following assumptions to our list as well:

[A-6] $\omega_j \in intX_j$ for all agents j.
[A-7] For all rational allocations $x = (x_1, ..., x_n) \in A(\omega)$,
$P_j(x_j)$ is open relative to X_j for all agents j..

We have the following Theorem:

Theorem 3 *(The equivalence of existence of a Pareto-optimal allocation, inconsequential arbitrage, and existence of equilibrium). Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**–**[A-3]**, **[A-4]**-**[A-5]**, and **[A-6]**-**[A-7]**. Then the following statements are equivalent:*
(1) $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ has a rational, Pareto-optimal allocation.
(2) $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies inconsequential arbitrage.
(3) $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ has an equilibrium.

Remark 2 ***(a)*** *It can also be shown that nonemptiness of the core implies inconsequential arbitrage, as shown in Page and Wooders (1993,1996a) for no unbounded arbitrage. The notion of the core involved is that typically used in economics. Since existence of a Pareto-optimal allocation is weaker than nonemptiness of the core, we focus on Pareto-optimal allocations. To relate inconsequential arbitrage to the partnered core, stronger conditions on the economic model are required – specifically, strictly convex preferences; see Page and Wooders (1996b) and, for the partnered core of a game, Reny and Wooders (1996).*

(b) *Rather than assume **[A-5]**, instead we could have made the following assumptions:*

*[**A-8**] Uniformity of increasing cones across rational allocations.*
For all rational allocations $x = (x_1, ..., x_n) \in A(\omega)$
$I_j(x_j) = I_j(\omega_j)$ for all agents j.
*[**A-9**] $bus(\omega) = ea(\omega)$.*

*Together [**A-8**] and [**A-9**] imply [**A-5**].*

6.2 A second welfare theorem

By strengthening assumption **[A-4]** we can prove a second welfare theorem for the exchange economy described above. This result generalizes the second welfare theorem established in Page and Wooders (1996a). The strengthening of **[A-4]** required is the following:

[A-4]* *Uniform nonsatiation off the bus.*
Given any rational allocation $x = (x_1, ..., x_n) \in A(\omega)$,
if $y \in arb(x) \backslash bus(x)$, then for each rational allocation $x' = (x'_1, ..., x'_n) \in A(x)$
there is at least one agent j such that for some $\lambda_j > 0$
$x'_j + \lambda_j y_j \in P_j(x'_j)$.

Here,

$$A(x) = \{(x'_1, ..., x'_n) \in X_1 \times ... \times X_n : \sum_{j=1}^{n} x'_j = \sum_{j=1}^{n} x_j \text{ and } x'_j \in \widehat{P}_j(x_j) \text{ for all } j\}.$$

Theorem 4 *(A second welfare theorem). Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^{n}$ be an economy satisfying **[A-1]**–**[A-3]**, **[A-4]***-**[A-5]**, and **[A-7]**. If the economy satisfies inconsequential arbitrage, then for each rational, Pareto-optimal allocation $x = (x_1, ..., x_n) \in A(\omega)$ with $x_j \in intX_j$ for all j, there is a price vector $p \in \mathcal{B} \backslash \{0\}$ such that $(x_1, ..., x_n, p)$ is an equilibrium relative to some endowment.*

7 Examples

To better understand the examples presented below, we begin by introducing the notion of *proximal trading volume.* Let

$$H(y) := \left\{ z \in \mathbb{R}^L : \langle z, y \rangle = 0 \right\},$$

and for $(x, y) \in \widehat{P}_j(\omega_j) \times \mathbb{R}^L$ let

$$z(x, y) := \text{the orthogonal projection of } x \text{ onto } H(y).$$

Figure 1 below summarizes the basic geometric relationships.

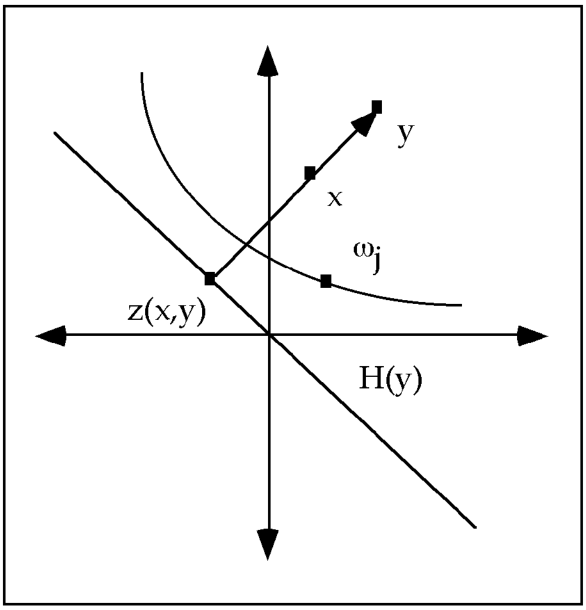

Figure 1

Referring to Figure 1, first note that the vector y is orthogonal to the hyperplane $H(y)$. Second, note that $x \in \widehat{P}_j(\omega_j)$ and that, in this case, $z(x,y) \notin \widehat{P}_j(\omega_j)$.[7] Thus, starting at the vector $z(x,y)$ and moving in the y direction, after a finite distance we arrive at the indifference surface containing endowment ω_j. Moving farther in the y direction, we arrive at the point x , the vector projected onto $H(y)$.

For the jth agent define the function

$$t_j(\cdot,\cdot) := \widehat{P}_j(\omega_j) \times \mathbb{R}^L \to [0,\infty]$$

as follows:

$$t_j(x_j,y_j) := \inf\{t \geq 0 : u_j(z(x_j,y_j) + t'y_j) \text{ is nonincreasing in } t' \text{ for } t' \geq t\}^{8}$$

Note that for all $(x_j,y_j) \in \widehat{P}_j(\omega_j) \times \mathbb{R}^L$, $z(x_j,y_j) + t_j(x_j,y_j)y_j \in \widehat{P}_j(\omega_j)$. Note also that for any $\varepsilon > 0$ and $(x_j,y_j) \in \widehat{P}_j(\omega_j) \times \mathbb{R}^L$, if $z(x_j,y_j) + t_j(x_j,y_j) \cdot y_j - \varepsilon \cdot y_j \in X_j$, then

$$u_j(z(x_j,y_j) + t_j(x_j,y_j) \cdot y_j - \varepsilon \cdot y_j) < u_j(z(x_j,y_j) + t_j(x_j,y_j) \cdot y_j).$$

We define *proximal trading volume*, $ptv(x,y)$, at $x = (x_1,\ldots,x_n) \in A(\omega)$ in directions $y = (y_1,\ldots,y_n) \in arb(\omega)$ as follows:

$$ptv(x,y) := \max\{t_j(x_j,y_j) : j = 1,\ldots,n\}.$$

[7]It is possible, of course, that $z(x,y) \in \hat{P}_j(\omega_j)$.

[8]If $y_j = 0$, then $H(y_j) = \mathbb{R}^L$, $z(x_j,y_j) = x_j$, and $t_j(x_j,y_j) = 0$.

We say that the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ has *bounded proximal trading volume* if along each arbitrage sequences $\{\lambda^k x^k\}_k \in arbseq^\omega(y)$, proximal trading volume $ptv(x^k, y)$ is bounded.

Theorem 5 *(Sufficiency of bounded proximal trading volume for inconsequential arbitrage). Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]-[A-2]**. If the economy has bounded proximal trading volume, then the economy satisfies inconsequential arbitrage.*

Example 1: In this example $bus(\omega)$ is a proper subset of $ea(\omega)$, inconsequential arbitrage does not hold, and proximal trading volume is unbounded. Consider an economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^2$ in which two commodities are traded. Agent 1 has consumption set $X_1 = \{(x_{11}, x_{12}) : x_{11} \geq 1 \text{ and } x_{12} \geq 0\}$ and Leontief preferences which kink along the curve given by $f(x) = \ln x$, for $x \geq 1$. Agent 2 has consumption set

$$X_2 = \{(x_{21}, x_{22}) : x_{21} \leq 0 \text{ and } x_{22} \leq 0\}$$

(i.e., the southwest quadrant) and preferences given by the utility function $u_2(x_{21}, x_{22}) = |x_{22}|$. Thus, agent 2 has straight line indifference curves as depicted in Figure 2.

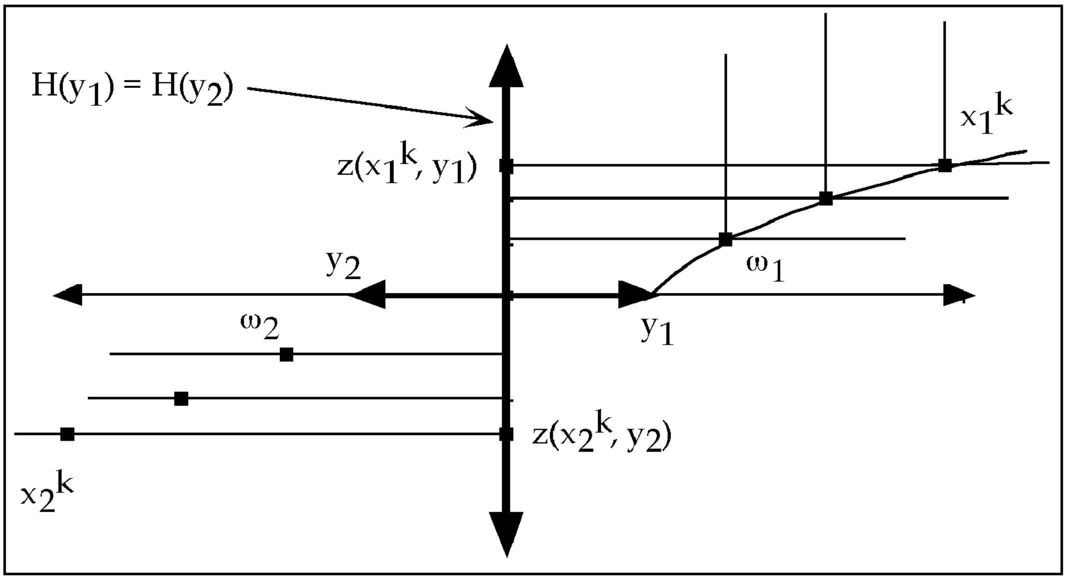

Figure 2

Agent 1 has endowment $\omega_1 = (2, \ln 2)$ while agent 2 has endowment $\omega_2 = -\omega_1$. Now consider the sequence of rational allocations given by $\left\{(x_1^k, x_2^k)\right\}_k$ with $(x_2^k - \omega_2) = -(x_1^k - \omega_1)$ where the sequence of agent 1's consumption vectors $\left\{x_1^k\right\}_k$ moves along the curve $f(x) = \ln x$, for $x \geq 1$ (see Figure 2). The sequence $\left\{(\lambda^k x_1^k, \lambda^k x_2^k)\right\}_k$ with $\lambda^k = \frac{1}{\|x_1^k\|}$ converges to $y = (y_1, y_2) = ((1,0), (-1,0))$. Thus, $y \in arb(\omega)$ and it is clear from Figure 2 that $y \in ea(\omega)$. Note, however, that $y \notin bus(\omega)$. In particular, because

$$x_1^k = z(x_1^k, y_1) + t_1(x_1^k, y_1) \cdot y_1$$

for small $\varepsilon > 0$, $u_1(x_1^k - \varepsilon y_1) < u_1(x_1^k)$ (i.e., $P_1(x_1^k) \subset P_1(x_1^k - \varepsilon y_1)$) for all k. Thus, inconsequential arbitrage is not satisfied. Moreover, note that proximal trading volume along $\left\{(x_1^k, x_2^k)\right\}_k$ in directions $y = (y_1, y_2) = ((1,0),(-1,0))$ is given by $ptv(x^k, y) = x_{11}^k$. Thus, proximal trading volume is unbounded. In this example $bus(\omega) = \{((0,0),(0,0))\}$, while

$$ea(\omega) = \{((0,0),(0,0))\} \cup \{((\gamma,0),(-\gamma,0)) : \gamma > 0\}.$$

Example 2: In the following variation on example 1, the economy has bounded proximal trading volume. Thus, inconsequential arbitrage is satisfied and $bus(\omega)$ is equal to $ea(\omega)$. Consider an economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^2$ in which two commodities are traded. Agent 1's preferences and consumption set are as before. But now agent 2 has consumption set

$$X_2 = \{(x_{21}, x_{22}) : x_{21} \leq 0 \text{ and } x_{22} \geq 0\}$$

(i.e., the northwest quadrant) and preferences again given by the utility function $u_2(x_{21}, x_{22}) = |x_{22}|$. Thus, agent 2 has straight line indifference curves in the northwest quadrant as depicted in Figure 3.

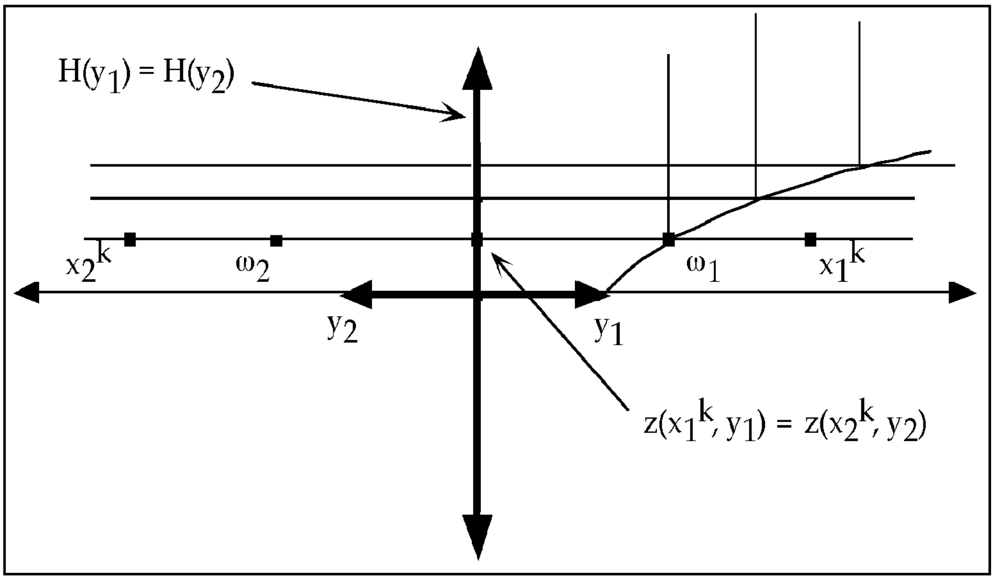

Figure 3

Agent 1 has endowment $\omega_1 = (2, \ln 2)$ while agent 2 has endowment $\omega_2 = (-2, \ln 2)$.

In this example,

$$A(\omega) = \{((2+\gamma, \ln 2), (-2-\gamma, \ln 2)) : \gamma \geq 0\}.$$

Given the simple structure of the set of rational allocations, it follows that all sequences of rational allocations $\{x^k\}_k := \{(x_1^k, x_2^k)\}_k$ such that $(\lambda^k x_1^k, \lambda^k x_2^k) \rightarrow (y_1, y_2)$, for some sequence $\lambda^k \downarrow 0$, are of the form

$$\{(x_1^k, x_2^k)\}_k = \{((2+\gamma^k, \ln 2), (-2-\gamma^k, \ln 2)) : \gamma^k \geq 0\}_k.$$

As a result, for all $\{(\lambda^k x_1^k, \lambda^k x_2^k)\}_k \in arbseq^{\omega}(y)$, with $y = (y_1, y_2) = ((\gamma, 0), (-\gamma, 0))$, $\gamma > 0$, the corresponding projection sequences

$$\{(z(x_1^k, y_1), z(x_2^k, y_2))\}_k \subset H(y_1) \times H(y_2)$$

are of the form

$$(z(x_1^k, y_1), z(x_2^k, y_2)) = ((0, \ln 2), (0, \ln 2)) \text{ for all } k.$$

From this it follows immediately that proximal trading volume along any such sequence $\{(x_1^k, x_2^k)\}_k$ is given by $ptv(x^k, y) = 2$. Thus, the economy satisfies inconsequential arbitrage and $bus(\omega) = ea(\omega)$. Given the simple structure of rational allocations, it can be verified by inspection that

$$arb(\omega) = bus(\omega) = \{((0,0),(0,0))\} \cup \{((\gamma, 0), (-\gamma, 0)) : \gamma > 0\}.$$

8 Other Conditions Limiting Arbitrage: Some Comparisons

In this section, we compare inconsequential arbitrage to the conditions limiting arbitrage introduced by Hart (1974) and Werner (1987), as well as to the conditions recently introduced by Dana, Le Van, and Magnien (1999) and Allouch (1999). Conditions limiting arbitrage found in the literature generally fall into three broad categories:

(i) *conditions on net trades*, for example, Hart (1974), Page (1987), Nielsen (1989), Page and Wooders (1993,1996a,b), Allouch (1999) - including the condition of inconsequential arbitrage introduced here;
(ii) *conditions on prices*, for example, Green (1973), Grandmont (1977,1982), Hammond (1983), and Werner (1987).
(iii) *conditions on the set of utility possibilities (namely, compactness)*, for example, Brown and Werner (1995), Dana, Le Van, and Magnien (1999).

8.1 A Comparison to Hart (1974) and Werner (1987)

We begin by showing that the condition of Hart (translated to a general equilibrium setting) and the condition of Werner are equivalent, and more importantly, we show that the Hart/Werner conditions imply inconsequential arbitrage. In order to highlight the extent to which we extend Hart (1974) and Werner (1987), we construct an example (example 3 below) of an exchange economy in which arbitrage cones are not globally uniform. However, increasing cones are uniform across rational allocations (i.e., [**A-8**] holds) and $bus(\omega) = ea(\omega)$ (i.e., [**A-9**] holds). Thus, there is nonsatiation off $bus(\omega)$ (i.e., [**A-4**] holds). In addition, inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions do not hold. In order to more clearly understand the limitations of inconsequential arbitrage, we also construct an example (example 4 below) of an exchange economy in which inconsequential arbitrage does not hold, nor do the Hart/Werner conditions. But an equilibrium does exist.

8.1.1 The Equivalence of Hart's Condition and Werner's Condition

In order to show that Hart's condition (translated to a general equilibrium setting) and Werner's condition are equivalent we must first introduce some new definitions and assumptions.

A set closely related to the *jth* agent's arbitrage cone at ω_j is the *lineality space* at ω_j given by

$$L(\widehat{P}_j(\omega_j)) := -R(\widehat{P}_j(\omega_j)) \cap R(\widehat{P}_j(\omega_j)).$$

Note that if $y_j \in L(\widehat{P}_j(\omega_j))$, then for all $\lambda \in \mathbb{R}$ and all $x_j \in \widehat{P}_j(\omega_j)$, $x_j + \lambda y_j \in X_j$ and $u_j(x_j + \lambda y_j) \geq u_j(x_j)$. Under assumption [**A-1**], if $y_j \in L(\widehat{P}_j(\omega_j))$, then net trades in the directions y_j or $-y_j$ starting at the consumption vector ω_j are utility constant.

Werner (1987) makes the following assumptions concerning lineality spaces and arbitrage cones (i.e., recession cones) in his model:

[**W-1**] *Global uniformity of arbitrage cones.* For each agent j,

$$R(\widehat{P}_j(x)) = R(\widehat{P}_j(\omega_j)) := R_j \textit{ for all } x \in X_j.$$

[**W-2**] *Existence of useful trades.* For each agent j,

$$R_j \backslash L_j \neq \emptyset.$$

Here L_j denotes the lineality space corresponding to the recession cone R_j. Together, [**W-1**] and [**W-2**] play the same role in Werner's (1987) proof of existence as does our assumption of nonsatiation at rational allocations, [**A-3**] in our proof of Theorem 1. Note that Werner's assumption [**W-1**] implies our assumption [**A-5**]. Note also that if agents' utility functions are concave then [**W-1**] holds automatically. Finally, note that under [**W-1**], if $y_i \in L_i$ then for all $\lambda \in \mathbb{R}$ and all $x_j \in X_j$,

$x_j + \lambda y_j \in X_j$ and $u_j(x_j + \lambda y_j) = u_j(x_j)$. In the terminology of Werner (1987), under **[W-1]** a "commodity bundle" $y_j \in L_j$ is *useless.*

Werner (1987) also introduces the *no half lines* condition.[9] The *jth* agent's utility function satisfies the no half lines condition if for each $x_j \in X_j$, there *does not exist* a nonzero vector of net trades y_j such that $u_j(x_j + \lambda y_j) = u_j(x_j)$ for all $\lambda \geq 0$.

[W-3] *No half lines.* Each agent's utility function satisfies the no half lines condition.

Note that **[W-3]** implies that the set of exhaustible arbitrages consists of an n-tuple the zero vectors (i.e., $ea(\omega) = \{(0, \ldots, 0)\}$). Thus, **[W-3]** implies that $bus(\omega) = ea(\omega) = \{(0, \ldots, 0)\}$.

The *dual cone corresponding to* the arbitrage cone R_j, denoted by $\Lambda(R_j)$, is given by

$$\Lambda(R_j) := \{p \in \mathbb{R}^L : \langle y, p \rangle \geq 0 \text{ for all } y \in R_j\}.$$

The *positive dual cone corresponding to* R_j, denoted by $\Lambda^+(R_j)$, is given by

$$\Lambda^+(R_j) := \{p \in \mathbb{R}^L : \langle y, p \rangle > 0 \text{ for all } y \in R_j \backslash L_j\}.$$

If $p \in \Lambda^+(R_j)$, then the price vector p assigns a positive value to any vector of *useful* net trades $y_j \in R_j \backslash L_j$.

Under assumption **[W-2]** it follows from Theorem 2.1 in Yu (1974) that

$$\Lambda^+(R_j) = ri\Lambda(R_j)$$

where "ri" denotes the relative interior.

Now consider an exchange economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfying assumptions **[A-1]**-**[A-2]** and **[W-1]**-**[W-2]**. This economy satisfies *Hart's condition* if

$$\begin{array}{c} \text{whenever } \sum_{j=1}^n y_j = 0 \text{ and } y_j \in R_j \text{ for all } j, \text{ then} \\ y_j \in L_j \text{ for all } j. \end{array} \tag{2}$$

Moreover, the economy satisfies *Werner's condition* if

$$\cap_j \Lambda^+(R_j) \neq \emptyset. \tag{3}$$

An economy satisfies Hart's condition, if all n-tuples of mutually compatible net trades representing potential unbounded arbitrages for the individual agents are useless (i.e., are utility constant). The economy satisfies Werner's condition, if the

[9] Werner (1987) states that if each agent's utility function satisfies the *no half lines condition*, then his condition is necessary as well as sufficient for existence. Page and Wooders (1996a) show that if all but one agent's utility function satisfies the no half lines condition, then Werner's condition is necessary and sufficent for existence.

set of prices assigning a positive value to all n-tuples of useful net trades (i.e., net trades representing potential unbounded arbitrages) is nonempty.

Our next result states that Hart's condition (a condition on net trades) is equivalent to Werner's condition (a condition on prices). Theorem 6 below is obtained by specializing Corollary 16.2.2 in Rockafellar (1970) to cover the exchange economy developed here.

Theorem 6 *(Hart's condition is equivalent to Werner's). Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]-[**A-2**] and [**W-1**]-[**W-2**]. Then Hart's condition (2) is satisfied if and only if Werner's condition (3) is satisfied.*

We shall refer to the conditions (2) and (3) above as the Hart/Werner conditions. Two interesting variations on the Hart/Werner conditions are the *overlapping expectations* condition of Hammond (1983) (also see Green (1973), Grandmont (1977, 1982))[10] and the *no unbounded arbitrage* condition of Page (1987). In an asset market setting, under a set of assumptions mildly stronger than those used in Hart (1974) (including the assumption of no half lines), Hammond shows that his condition is equivalent to Hart's condition. Also in an asset market setting, under a set of assumptions similar to Hammond's (also including the assumption of no half lines), Page shows that his condition given by

$$\begin{gathered}\text{if } \textstyle\sum_{j=1}^n y_j = 0 \text{ and } y_j \in R_j \text{ for all } j, \text{ then} \\ y_j = 0 \text{ for all } j,\end{gathered} \tag{4}$$

is also equivalent to Hart's condition as well as Hammond's. In general, without the no half lines assumption, $\{0, \ldots, 0\}$ is a proper subset of the lineality space L_j. Thus in general, Page's condition (4) implies the Hart/Werner conditions. Page's condition has been used by Nielsen (1989) to prove existence in an exchange economy model with short sales slightly more general than Werner's, and more recently, Page and Wooders (1993, 1996a,b) have refined Page's condition and shown it to be necessary and sufficient for compactness of the set of rational allocations, as well as for existence and nonemptiness of the core. Dana, Le Van, and Magnien (1999) and Allouch (1999) provide an excellent discussions of the different notions of arbitrage found in the general equilibrium literature. Page and Wooders (1999) also provide a discussion of notions of arbitrage, including a discussion of the various notions of arbitrage found in the literature on asset markets.

8.1.2 Hart/Werner Conditions and Inconsequential Arbitrage

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an exchange economy satisfying assumptions **[A-1]-[A-2]** and **[W-1]-[W-2]**. Given the discussion above concerning lineality spaces, it is easy

[10] As far as we know, Hammond (1983) introduced the terminology "overlapping expectations." The condition overlapping expectations is a much weakened form of Green's "common expectations."

to see that

$$L_1 \times \cdots \times L_n \subseteq bus(\omega).$$

This fact together with Theorem 7 imply that if the economy satisfies the Hart/Werner conditions, then any arbitrage $y = (y_1, ..., y_n) \in arb(\omega)$ is contained in $L_1 \times \cdots \times L_n$. Thus, Hart's net trades condition or Werner's price condition implies inconsequential arbitrage. If, in addition, the economy satisfies no half lines, **[W-3]**, then $bus(\omega) = \{0, \ldots, 0\}$, and therefore, $L_1 \times \cdots \times L_n = bus(\omega)$. Thus, assuming no half lines, inconsequential arbitrage implies the Hart/Werner conditions. Stated formally, we have,

Theorem 7 *(Hart/Werner imply inconsequential arbitrage). Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an exchange economy satisfying assumptions* ***[A-1]-[A-2]*** *and* ***[W-1]-[W-2]****.*

$$\left.\begin{array}{l} \text{If } \sum_{j=1}^{m} y_j = 0 \text{ and } y_j \in R_j \text{ for all } j, \\ \text{implies, } y_j \in L_j \text{ for all } j, \\ \text{or} \\ \text{if } \cap_j \Lambda^+(R_j) \neq \emptyset, \end{array}\right\} \text{then } arb(\omega) \subseteq bus(\omega).$$

If, in addition, the economy satisfies ***[W-3]****, no half lines, then*

$$\cap_j \Lambda^+(R_j) \neq \emptyset \text{ if and only if } arb(\omega) \subseteq bus(\omega).$$

8.1.3 A Summary of Results

In this subsection, we summarize our results and those of Werner.

A Summary of Sufficiency Results:

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-2]** and **[A-6]**-**[A-7]**:

Werner (1987): Under **[W-1]**-**[W-2]**,

$$\cap_j \Lambda^+(R_j) \neq \emptyset \Rightarrow \text{existence.}$$

Page, Wooders, &Monteiro: Under[**A-3**],

$$arb(\omega) \subseteq bus(\omega) \Rightarrow \text{existence.}$$

Thus, inconsequential arbitrage is sufficient for existence *without uniformity of arbitrage cones* (see example 3 below).[11]

A Summary of Characterization Results:

[11]Werner's assumption **A5** is weaker than our assumption **[A-6]** - see page 1410 in Werner (1987). In fact, our assumption **[A-6]** implies Werner's **A5**. On the other hand, Werner assumes utility functions are continuous while we require only upper semicontinuity.

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-2]** and **[A-6]**-**[A-7]**:

Werner (1987): Under **[W-1]**-**[W-3]**,

$$\cap_j \Lambda^+(R_j) \neq \emptyset \Leftrightarrow \text{ existence.}$$

Page, Wooders, & Monteiro: Under **[A-3]** and **[A-4]**-**[A-5]**,

$$arb(\omega) \subseteq bus(\omega) \Leftrightarrow \text{ existence.}$$

Thus, inconsequential arbitrage is necessary and sufficient for existence even *in the presence of half lines* (see example 3 below).

8.1.4 Examples

Example 3: Here we construct an example of an exchange economy satisfying assumptions **[A-1]**-**[A-3]** in which arbitrage cones are uniform across rational allocations (i.e., **[A-5]** holds) - but *not* globally uniform. Moreover, in our example, increasing cones are uniform across rational allocations (i.e., **[A-8]** holds) and $bus(\omega) = ea(\omega)$ (i.e., **[A-9]** holds). Thus, there is nonsatiation off the $bus(\omega)$ (i.e., **[A-4]** holds). In addition, inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions fail.

Consider an economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^2$, where for each j, $X_j = \mathbb{R}^2$. Agent 1 has endowment $\omega_1 = (2, -1)$ while agent 2 has endowment $\omega_2 = -\omega_1$. Agent 1's utility function is given by

$$u_1(x_1, x_2) = \begin{cases} x_1 & \textit{if} \quad x_1 \leq 0 \quad \textit{or} \quad x_2 \geq -1 \\ -\frac{x_1}{x_2} & \textit{if} \quad x_1 \geq 0 \quad \textit{and} \quad x_2 \leq -1, \end{cases}$$

while agent 2 has utility function given by

$$u_2(x_1, x_2) = x_1 + 2x_2.$$

Figure 4 summarizes the situation. Note that agents 1's utility function has arbitrage cones (i.e., recession cones) that are *not globally* uniform. Thus, this example is not covered by Werner's (1987) model.

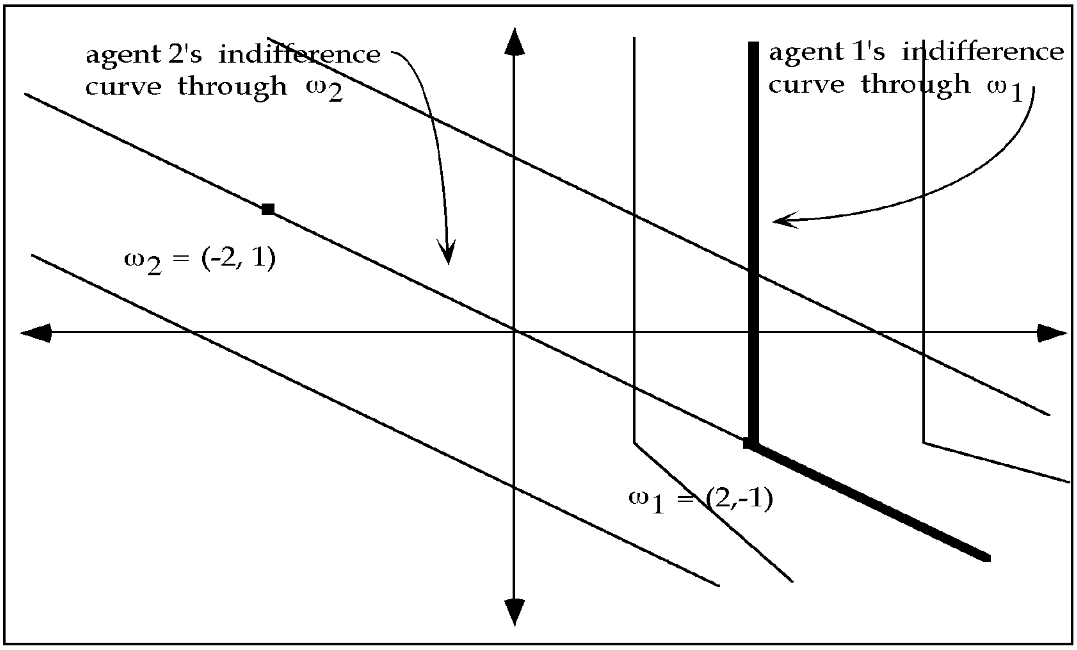

Figure 4

In this example,

$$A(\omega) = \{((2,-1) + \gamma \cdot (2,-1), (-2,1) + \gamma \cdot (-2,1)) : \gamma \geq 0\}.$$

By inspection of Figure 4, it is easy to see that increasing cones are uniform across rational allocations (i.e., [**A-8**] holds). In particular,

$$\begin{gathered} \text{for all } \gamma \geq 0 \\ I_1((2,-1) + \gamma \cdot (2,-1)) = I_1((2,-1)), \\ \text{and} \\ I_2((-2,1) + \gamma \cdot (-2,1)) = I_2((-2,1)). \end{gathered}$$

Moreover, it is easy to see that arbitrage cones are uniform across rational allocations (i.e., [**A-5**] holds) and that local nonsatiation at rational allocations holds (i.e., [**A-3**] holds).

Given the simple structure of the set of rational allocations, it follows that all sequences of rational allocations $\{x^k\}_k := \{(x_1^k, x_2^k)\}_k$ such that $(\lambda^k x_1^k, \lambda^k x_2^k) \rightarrow (y_1, y_2)$, for some sequence $\lambda^k \downarrow 0$, are of the form

$$\{(x_1^k, x_2^k)\}_k = \{((2,-1) + \gamma^k \cdot (2,-1), (-2,1) + \gamma^k \cdot (-2,1)) : \gamma^k \geq 0\}_k.$$

Thus, for all $\{\lambda^k x^k\}_k \in arbseq^{\omega}(y)$, with $y = (y_1, y_2) = (\gamma \cdot (2,-1), \gamma \cdot (-2,1))$, $\gamma > 0$, the corresponding projection sequences

$$\{(z(x_1^k, y_1), z(x_2^k, y_2))\}_k \subset H(y_1) \times H(y_2)$$

are of the form

$$(z(x_1^k, y_1), z(x_2^k, y_2)) = ((0,0),(0,0)) \text{ for all } k.$$

Proximal trading volume along any such sequence $\left\{(x_1^k, x_2^k)\right\}_k$ is given by $ptv(x^k, y) = \|(2,-1)\|$, and thus by Theorem 5 , inconsequential arbitrage is satisfied. In fact, given the simple structure of rational allocations, it can be verified by inspection of Figure 4 that

$$arb(\omega) = bus(\omega) = ea(\omega) = \{((0,0),(0,0))\} \cup \{(\gamma \cdot (2,-1), -\gamma \cdot (-2,1)) : \gamma > 0\}.$$

Finally, note that lineality space (the set of useless net trades) for agent 1 is

$$L(\widehat{P}_1((2,-1)) = \{(0,0)\},$$

while for agent 2, the lineality space is given by

$$L(\widehat{P}_2((-2,1)) = \{\gamma \cdot (-2,1) : \gamma \in \mathbb{R}\}.$$

Thus, in this example, the Hart/Werner conditions are not satisfied.

Example 4: In this our last example, we modify example 3 by changing the endowment of agent 1. As a result, increasing cones are no longer uniform across rational allocations (i.e., [**A-8**] fails), equality of $bus(\omega)$ and $ea(\omega)$ fails (i.e., [**A-9**] fails), and nonsatiation off the $bus(\omega)$ fails (i.e., [**A-4**] fails). Moreover, inconsequential arbitrage does not hold, nor do the Hart/Werner conditions. But an equilibrium does exist. Figure 5, illustrates the problem.

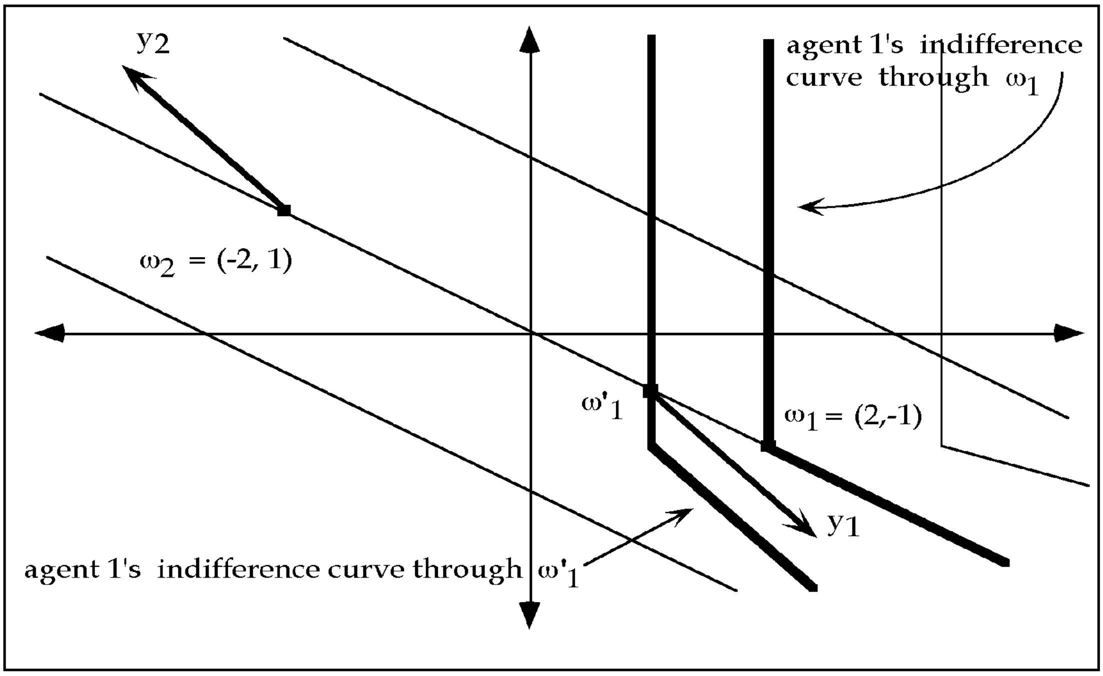

Figure 5

Agent 1's endowment is now ω'_1 (rather than $\omega_1 = (2,-1)$). While agent 2's endowment is as before. Consider the sequence $\{(\lambda^k x_1^k, \lambda^k x_2^k)\}_k \in arbseq^{\omega'}((y_1, y_2))$ given by

$$\{(\lambda^k x_1^k, \lambda^k x_2^k)\}_k = \left\{ \left(\frac{1}{k}(\omega'_1 + k \cdot y_1), \frac{1}{k}(\omega_2 + k \cdot y_2) \right) \right\}_k$$

where $\omega' = (\omega'_1, \omega_2)$ and y_1 and y_2 are given in Figure 5.[12] It is easy to see that

$$y = (y_1, y_2) \notin bus(\omega').$$

Thus, inconsequential arbitrage does not hold.[13] By inspection of Figure 5 it can be verified that increasing cones are no longer uniform across rational allocations, and that $bus(\omega')$ is a proper subset of $ea(\omega')$. In fact, in this example

$$bus(\omega') = \{((0,0),(0,0))\},$$

while

$$ea(\omega) = \{((0,0),(0,0))\} \cup \{(\gamma \cdot (2,-1), \gamma \cdot (-2,1)) : \gamma > 0\}.$$

Moreover, **[A-4]**, nonsatiation off $bus(\omega')$ fails. For example, take

$$y' = (y'_1, y'_2) = ((2,-1),(-2,1)) \in arb(\omega') \backslash bus(\omega').$$

It is clear from Figure 5, that for rational allocation $((\omega'_1 + y'_1), (\omega_2 + y'_2))$

$$\begin{gathered} u_1(\omega'_1 + y'_1 + \lambda y'_1) = u_1(\omega'_1 + y'_1) \\ u_2(\omega_2 + y'_2 + \lambda y'_2) = u_2(\omega_2 + y'_2) \\ \text{for all } \lambda > 0. \end{gathered}$$

Finally, note that this economy has an equilibrium $(\bar{x}_1, \bar{x}_2, \bar{p})$, for example

$$\begin{aligned} \bar{p} &= \tfrac{1}{\|(1,2)\|} \cdot (1,2), \\ \bar{x}_1 &= \omega'_1 + y'_1, \\ \bar{x}_2 &= \omega_2 + y'_2. \end{aligned}$$

8.2 A comparison to Dana, Le Van, and Magnien (1999) and Allouch (1999)

We begin with some terminology. We say that the preference mapping $x_j \to P_j(x_j)$ defined on X_j is *strictly quasi-concave* if given any $x_j \in X_j$ and $x'_j \in P_j(x_j)$

$$(1-\lambda)x_j + \lambda x'_j \in P_j(x_j) \text{ for all } \lambda \in (0,1].$$

The set of *utility possibilities* at endowments $\omega = (\omega_1, ..., \omega_n)$ is given by

$$U(\omega) = \{(u_1, ..., u_n) \in \mathbb{R}^n : u_j(\omega_j) \leq u_j \leq u_j(x_j) \text{ for some } x = (x_1, \ldots, x_n) \in A(\omega)\}.$$

[12] Observe that $y_2 = -y_1$.

[13] Note that if agent 1's indifference curves had the same shape as his indifference curve through ω_1 then inconsequential arbitrage would be restored. However, if all of agent 1's indifference curves had the same shape as his indifference curve through ω'_1 then inconsequential arbitrage would again fail to hold (i.e., global uniformity is not enough to guarantee that the economy satisfies inconsequential arbitrage).

8.2.1 Dana, Le Van, and Magnien (1999)

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-2]**. Dana, Le Van, and Magnien (1999) assume

Global nonsatiation at rational allocations:
For any rational allocation $(x_1, ..., x_n) \in A(\omega)$, $P_j(x_j) \neq \emptyset$ for all agents j.

They then show (Theorem 1) that

if global nonsatiation at rational allocations is satisfied, and
if agents' preference mappings $x_j \rightarrow P_j(x_j)$
are strictly quasi-concave, then
compactness of utility possibilities $\Rightarrow$ the existence of a quasi-equilibrium.

Note that global nonsatiation at rational allocations and strict quasi-concavity of preference mappings together imply our assumption **[A-3]**, local nonsatiation at rational allocations. However, in an economy satisfying **[A-1]**-**[A-2]** and strict quasi-concavity, Allouch (1999, Proposition 5.2) has shown that inconsequential arbitrage implies compactness of utility possibilities. Thus, inconsequential arbitrage is a stronger condition than compactness of utility possibilities.

8.2.2 Allouch (1999)

Allouch (1999) introduces a new condition, *bounded arbitrage*, defined as follows:

The economy satisfies bounded arbitrage if for all sequences of rational allocations $\{x^n\}_n \subset A(\omega)$ there exists

a subsequence $\{x^{n_k}\}_k$,

a rational allocation $z \in A(\omega)$, and

a sequence $\{z^k\}_k \subset X_1 \times \cdots \times X_n$ converging to z such that

$$z_j^k \in \widehat{P}_j(x_j^{n_k}).$$

Thus, by bounded arbitrage, for every sequence of rational allocations there is a subsequence that is preference-dominated by a sequence converging to a rational allocation.

Allouch shows (Proposition 5.1) that if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies **[A-1]**-**[A-2]**, global nonsatiation at rational allocations, and

if agents' preference mappings $x_j \rightarrow P_j(x_j)$
are strictly quasi-concave, then
bounded arbitrage $\Leftrightarrow$ the set of utility possibilities is compact.

Moreover, Allouch shows (Proposition 5.2) that if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies **[A-1]**-**[A-2]**, and

if agents' preference mappings $x_j \to P_j(x_j)$
are strictly quasi-concave, then
inconsequential arbitrage $\Rightarrow$ the set of utility possibilities is compact.

In fact, Allouch's Proposition 5.2 continues to hold *without strict quasi-concavity.* Thus, we have

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies **[A-1]**-**[A-2]**, then
inconsequential arbitrage $\Rightarrow$ the set of utility possibilities is compact.

However, *without strict quasi-concavity* Allouch's Proposition 5.1 must be weakened as follows:

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies **[A-1]**-**[A-2]**, then
the economy satisfies bounded arbitrage $\Rightarrow$ the set of utility possibilities is compact.

Thus, without strict quasi-concavity bounded arbitrage implies compactness of utility possibilities, but it is an open question whether the reverse implication still holds.

Allouch's main existence result (Theorem 3.1), translated to the model here, is the following:

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies
global nonsatiation at rational allocations and **[A-2]**, where
$P_j(x_j) := \{x \in X_j : u_j(x) > u_j(x_j)\}$, and
$\widehat{P}_j(x_j) := \{x \in X_j : u_j(x) \geq u_j(x_j)\}$
with $u_j(\cdot) : X_j \to \mathbb{R}$ quasi-concave,
and if
(i) $\widehat{P}_j(\omega_j)$ is closed, and
(ii) $x_j \to P_j(x_j)$ is open-valued on $\widehat{P}_j(\omega_j)$, then
bounded arbitrage $\Rightarrow$ the existence of a quasi-equilibrium.

Comparing Allouch's Theorem 3.1 to the following restatement of our Theorem 1, we have shown that:

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies
local nonsatiation at rational allocations and **[A-2]**, where
$P_j(x_j) := \{x \in X_j : u_j(x) > u_j(x_j)\}$, and
$\widehat{P}_j(x_j) := \{x \in X_j : u_j(x) \geq u_j(x_j)\}$
with $u_j(\cdot) : X_j \to \mathbb{R}$ quasi-concave,
and if
$x_j \to \widehat{P}_j(x_j)$ is closed-valued on X_j, then
inconsequential arbitrage $\Rightarrow$ the existence of a quasi-equilibrium.

It should be noted that, in fact, Allouch (1999) proves the existence of a quasi-equilibrium in a model more general than the model developed here. For example, Allouch assumes that agents' preferences are given via a partial preorder, $\prec_j$, and does not require that the induced preference mappings, $x_j \to \{x'_j \in X_j : x_j \prec_j x'_j\}$, be everywhere convex-valued nor that the induced weak preference mappings, $x_j \to \{x'_j \in X_j : x_j \preceq_j x'_j\}$, be everywhere closed-valued. Similar extensions of our results are also possible.

8.3 A fundamental equivalence result

As before, let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ denote an exchange economy where the j^{th} agent's preferences $P_j(\cdot)$ over X_j are specified via a utility function $u_j(\cdot) : X_j \to \mathbb{R}$ as follows

$$P_j(x_j) := \{x \in X_j : u_j(x) > u_j(x_j)\}.$$

But now suppose that the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies the following assumptions for each agent $j = 1, ..., n$:

[A-1]* $u_j(\cdot)$ is *continuous* and quasi-concave.

[A-2]* $\omega_j \in intX_j$ and X_j is closed and convex.

[A-3] *Local nonsatiation at rational allocations.* $\forall$ $(x_1, ..., x_n) \in A(\omega)$, $P_j(x_j) \neq \emptyset$ and $clP_j(x_j) = \widehat{P}_j(x_j)$.

[W-1] *Global uniformity of arbitrage cones.* $R(\widehat{P}_j(x)) = R(\widehat{P}_j(\omega_j)) := R_j$ for all $x \in X_j$.

[W-3] *No half lines.* Each agent's utility function satisfies the no half lines condition.

We have the following equivalence result:

Theorem 8 *(A fundamental equivalence). Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]***, **[A-2]***, **[A-3]**. If the economy satisfies inconsequential arbitrage, then the economy has an equilibrium. Moreover, if the economy satisfies the conditions of* uniformity *and* no half lines, ***[W-1]**, and **[W-3]**, then the following statements are equivalent:*
(1) $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies inconsequential arbitrage.
(2) $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies the Hart/Werner conditions.
(3) $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies Allouch's bounded arbitrage condition.
(4) The set of utility possibilities, $U(\omega)$, is compact.
(5) $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ has an equilibrium.

The equivalence of (1) and (2) follows from our Theorem 7. By Allouch's Proposition 4.3, (2) implies (3), and by Allouch's proposition 5.1, (3) implies (4). Moreover, given assumptions **[A-1]***, **[A-2]***, and **[A-3]**, it follows from Theorem 1 in Dana, Le Van, and Magnien (1999) that (4), compactness of utility possibilities, implies (5), the existence of an equilibrium. Finally, given assumptions **[W-1]** arbitrage cones are uniform, and by Theorem 1 in Monteiro, Page, and Wooders (1997), the no half lines assumption **[W-3]** implies that

$$R_j \backslash L_j = R_j \backslash \{0\} = I_j.$$

Here L_j denotes the j^{th} agent's lineality space corresponding to the arbitrage cone R_j, while I_j denotes the j^{th} agent's increasing cone. Thus, given assumptions **[A-1]***, **[A-2]***, and **[W-1]**-**[W-3]**, it follows from our Theorem 3 that (1) and (5) are equivalent.

9 Proofs

9.1 Bounded economies

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$be an exchange economy satisfying **[A-1]**-**[A-3]**, and let $C(k)$ be a closed ball in $\mathbb{R}^L$ of radius r_k, with $r_k > 1$, centered at the origin such that for each agent j, $\omega_j \in intC(k)$ where *int* denotes interior. We shall assume that $r_k \to \infty$ as $k \to \infty$. Now define

$$P_{kj}(x_j) := P_j(x_j) \cap C(k),$$

$$\widehat{P}_{kj}(x_j) := \widehat{P}_j(x_j) \cap C(k),$$

$$X_{kj} := X_j \cap C(k),$$

$$B_k(\omega_j, p) = \{x_j \in X_{kj} : \langle x_j, p \rangle \leq \langle \omega_j, p \rangle\},$$

$$F_k(\omega_j, p) = \{x_j \in X_{kj} : \langle x_j, p \rangle < \langle \omega_j, p \rangle\},$$

and consider the k-bounded economy

$$E_k := (X_{kj}, \omega_j, P_{kj}(\cdot))_{j=1}^n.$$

The set of rational allocations for the k-bounded economy E_k is given by

$$A_k(\omega) = \{(x_1, ..., x_n) \in X_{k1} \times ... \times X_{kn} : \sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \text{ and } x_j \in \widehat{P}_{kj}(\omega_j) \text{ for all } j\}.$$

An *equilibrium* for the economy E_k is an $(n+1)$-tuple of vectors $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ such that:

(i) $(\overline{x}_1, ..., \overline{x}_n) \in A_k(\omega)$;

(ii) $\overline{p} \in \mathcal{B}\backslash\{0\}$; and

(iii) for each j, $\langle \overline{x}_j, \overline{p}, \rangle = \langle \omega_j, \overline{p} \rangle$ and $P_{kj}(\overline{x}_j) \cap B_k(\omega_j, \overline{p}) = \emptyset$.[14]

A *quasi-equilibrium* for the economy E_k is an $(n+1)$ -tuple of vectors $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ such that:

(i) $(\overline{x}_1, ..., \overline{x}_n) \in A_k(\omega)$;

(ii) $\overline{p} \in \mathcal{B}\backslash\{0\}$; and

(iii) for each j, $\overline{x}_j \in B_k(\omega_j, p)$ and $P_{kj}(\overline{x}_j) \cap F_k(\omega_j, \overline{p}) = \emptyset$.

The following Lemma is crucial for the proof of Theorem 1.

Lemma 1 Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an exchange economy satisfying **[A-1]**-**[A-3]**. Let $\{x^k\}_k = \{(x_1^k, ..., x_n^k)\}_k \subset A_k(\omega)$ be a sequence of individually rational allocations such that for each k, $\left\|x_j^k\right\| = r_k$ for some agent j (i.e., $\sum_{j=1}^n \left\|x_j^k\right\| \to \infty$ as $k \to \infty$) and consider the sequence $\{\lambda^k x^k\}_k := \{(\lambda^k x_1^k, ..., \lambda^k x_n^k)\}_k$ where $\lambda^k = \left(\sum_{j=1}^n \left\|x_j^k\right\|\right)^{-1}$. Then the following statements are true.

1. For any cluster point $y = (y_1, ..., y_n)$ of the sequence $\{(\lambda^k x_1^k, ..., \lambda^k x_n^k)\}_k$, it holds that
$$\sum_{j=1}^n y_j = 0 \text{ and} \sum_{j=1}^n \|y_j\| = 1.$$

2. For any cluster point $y = (y_1, ..., y_n)$ and subsequence $\{(\lambda^{k'} x_1^{k'}, ..., \lambda^{k'} x_n^{k'})\}_{k'}$ such that $(\lambda^{k'} x_1^{k'}, ..., \lambda^{k'} x_n^{k'}) \to (y_1, ..., y_n)$ it holds that
$$\text{for } k' \text{ sufficiently large, } x_j^{k'} - \varepsilon y \in intC(k') \text{ for all } \varepsilon \in (0, 1].$$

Proof of Lemma 1: First recall that any cluster point $y = (y_1, ..., y_n)$ of the sequence $\{(\lambda^k x_1^k, ..., \lambda^k x_n^k)\}_k$ is such that for each agent j, $y_j \in R(\widehat{P}_j(\omega_j))$. Part (a) of Lemma 1 is a restatement of Lemma 3.3 in Page (1987). To prove part (b), first note that if $y_j = 0$, then $\|x_j^k\| < r_k$ infinitely often (i.e., $x_j^k \in intC(k)$ for infinitely many k). In particular, if $\|x_j^k\| = r_k$ for all k then
$$\frac{1}{n} = \frac{r_k}{n \cdot r_k} = \frac{\|x_j^k\|}{\sum_{j'=1}^n \|x_j^k\|} \text{ for all k.}$$

[14]Recall that $\mathcal{B} := \{p' \in \mathbb{R}^L : \|p'\| \leq 1\}$.

Thus, if $y_j = 0$, we have a contradiction, since

$$\frac{x_j^k}{\sum_{j'=1}^n \|x_j^k\|} \to y_j.$$

Because there is a finite number of agents, we can extract a subsequence

$$\{(\lambda^{k'} x_1^{k'}, ..., \lambda^{k'} x_n^{k'})\}_{k'}$$

such that

$$(\lambda^{k'} x_1^{k'}, ..., \lambda^{k'} x_n^{k'}) \to (y_1, ..., y_n),$$

and such that

$$\text{if } y_j = 0 \text{ then } \|x_j^{k'}\| < r_{k'} \text{ for all } k' \text{ sufficiently large.}$$

To show that for k' sufficiently large $x_j^{k'} - \varepsilon y \in intC(k')$ for all $\varepsilon \in (0, 1]$, consider the following

$$\begin{aligned} \|x_j^{k'} - \varepsilon y\| &\leq \|x_j^{k'} - \varepsilon\lambda^{k'} x_j^{k'}\| + \|\varepsilon\lambda^{k'} x_j^{k'} - \varepsilon y_j\| \\ &= (1 - \varepsilon\lambda^{k'})\|x_j^{k'}\| + \|\varepsilon\lambda^{k'} x_j^{k'} - \varepsilon y_j\| \\ &= \|x_j^{k'}\| + \left(\|\varepsilon\lambda^{k'} x_j^{k'} - \varepsilon y_j\| - \|\varepsilon\lambda^{k'} x_j^{k'}\|\right) \\ &= \|x_j^{k'}\| + \varepsilon\left(\|\lambda^{k'} x_j^{k'} - y_j\| - \|\lambda^{k'} x_j^{k'}\|\right) \end{aligned}$$

If $y_j = 0$, then $\|x_j^{k'} - \varepsilon y\| < r_{k'}$ by the arguments above. Suppose now that $y_j \neq 0$. Since $\|\lambda^{k'} x_j^{k'} - y_j\| \to 0$ and $\|\lambda^{k'} x_j^{k'}\| \to \|y_j\| > 0$, we have for all k' sufficiently large $\left(\|\lambda^{k'} x_j^{k'} - y_j\| - \|\lambda^{k'} x_j^{k'}\|\right) < 0$. Since already $\|x_j^{k'}\| \leq r_{k'}$ we can conclude that for all k' sufficiently large $\|x_j^{k'} - \varepsilon y\| < r_{k'}$ for all $\varepsilon \in (0, 1]$.□

9.2 Proof of Theorem 1

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ an economy satisfying **[A-1]**-**[A-3]** and inconsequential arbitrage.

Part I. Consider the k-bounded economy $E_k := (X_{kj}, \omega_j, P_{kj}(\cdot))_{j=1}^n$, and note the following:

1. Given assumption **[A-3]**, local nonsatiation at rational allocations, the k-bounded economy E_k also satisfies *local nonsatiation at rational allocations.*

2. The mapping $x_j \to \widehat{P}_{kj}(x_j)$ has nonempty, closed, convex values and the weak preference relation defined via the mapping $x_j \to \widehat{P}_{kj}(x_j)$ is transitive.

3. For each agent j, $\omega_j \in X_{kj}$ and X_{kj} is convex and compact.

Given these observations, it follows from Theorem 1 in Bergstrom (1976) that for each k, E_k has a quasi-equilibrium, $(x_1^k, ..., x_n^k, p^k)$.[15]

We now want to show that if for each j, (i) $\inf_{x\in X_j} \langle x, p^k \rangle < \langle \omega_j, p^k \rangle$, and (ii) $P_j(x_j^k)$ is open relative to X_j, then $(x_1^k, ..., x_n^k, p^k)$ is an equilibrium for E_k. First, note that since for all j, $\omega_j \in intC(k)$, the fact that $\inf_{x\in X_j} \langle x, p^k \rangle < \langle \omega_j, p^k \rangle$ for all j implies that $\inf_{x\in X_{kj}} \langle x, p^k \rangle < \langle \omega_j, p^k \rangle$ for all k and j. Second, if $P_j(x_j^k)$ open relative to X_j for all j, then $P_{kj}(x_j^k)$ is open relative to X_{kj} for all k and j. Thus, we have the following for all k and j,

$$\text{if for } k \text{ and } j \text{ (i) } \inf_{x\in X_j} \langle x, p^k \rangle < \langle \omega_j, p^k \rangle \text{ and (ii) } P_j(x_j^k) \text{ open relative to } X_j,$$

then

$$\text{(i)}' \inf_{x\in X_{kj}} \langle x, p^k \rangle < \langle \omega_j, p^k \rangle, \text{ and (ii)}' \; P_{kj}(x_j^k) \text{ open relative to } X_{kj}.$$

Fix k. By (i)$'$ we have for all j some $\widetilde{x}_j^k \in X_{kj}$ such that $\langle \widetilde{x}_j^k, p^k \rangle < \langle \omega_j, p^k \rangle$. Now suppose that for some j, there exists $\widehat{x}_j^k \in P_{kj}(x_j^k) \cap B_k(\omega_j, p^k)$. For $\lambda \in [0,1]$, define $x_j^k(\lambda) := \lambda\widetilde{x}_j^k + (1-\lambda)\widehat{x}_j^k$ and note that for all $\lambda > 0$, $\langle x_j^k(\lambda), p^k \rangle < \langle \omega_j, p^k \rangle$. By (ii)$'$, we have for some $\lambda \in (0,1)$, $x_j^k(\lambda) \in P_{kj}(x_j^k)$. This contradicts the fact that $(x_1^k, ..., x_n^k, p^k)$ is a quasi-equilibrium. Thus, we must conclude that $P_{kj}(x_j^k) \cap B_k(\omega_j, p^k) = \emptyset$ for all j, and therefore, we must conclude that $(x_1^k, ..., x_n^k, p^k)$ is an equilibrium for the k-bounded economy E_k.

Part II. We now want to show that if $(x_1^k, ..., x_n^k, p^k)$ is a quasi-equilibrium (resp., equilibrium) for the economy $E_k := (X_{kj}, \omega_j, P_{kj}(\cdot))_{j=1}^n$ with $x_j^k \in intC(k)$ for each j, then it is a quasi-equilibrium (resp., equilibrium) for the original economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$.

Let $(x_1^k, ..., x_n^k, p^k)$ be a quasi-equilibrium for the economy $E_k := (X_{kj}, \omega_j, P_{kj}(\cdot))_{j=1}^n$ with $x_j^k \in intC(k)$ for each j. We have then $p^k \in \mathcal{B}\backslash\{0\}$, and since $A_k(\omega) \subset A(\omega)$, we also have $(x_1^k, ..., x_n^k) \in A(\omega)$. Finally, we have for all j

$$x_j^k \in B_k(\omega_j, p^k) \text{ and } P_{kj}(x_j^k) \cap F_k(\omega_j, p^k) = \emptyset.$$

Thus, $x_j^k \in B(\omega_j, p^k)$ for all j. We want to show that

$$P_{kj}(x_j^k) \cap F_k(\omega_j, p^k) = \emptyset \text{ implies that } P_j(x_j^k) \cap F(\omega_j, p^k) = \emptyset.$$

Suppose not. Then there exists $z' \in P_j(x_j^k) \cap F(\omega_j, p^k)$. There is a $\delta > 0$ such that

$$\left\| \delta z' + (1-\delta)x_j^k \right\| < k$$

[15] Note that because the weak preference mapping

$$x_j \to \widehat{P}_{kj}(x_j)$$

is defined via a utility function, preferences are complete (see Gay (1979) for a discussion of Bergstrom's results and the problems caused by the absense of completeness).

Since

$$\left\langle \delta z' + (1-\delta)x_j^k, p^k \right\rangle < \left\langle \omega_j, p^k \right\rangle,$$

there is a $z'' \in P_{kj}(x_j^k)$ sufficiently near such that

$$\left\langle \delta z' + (1-\delta)z'', p^k \right\rangle < \left\langle \omega_j, p^k \right\rangle,$$
and
$$\|\delta z' + (1-\delta)z''\| < k.$$

Thus,

$$\delta z' + (1-\delta)z'' \in P_{kj}(x_j^k) \cap F_k(\omega_j, p^k),$$

a contradiction. We must conclude therefore that

$$P_j(x_j^k) \cap F(\omega_j, p^k) = \emptyset.$$

Thus, $(x_1^k, ..., x_n^k, p^k)$ is a quasi-equilibrium for the original economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$.

If for each j, (i) $\inf_{x\in X_j} \left\langle x, p^k \right\rangle < \left\langle \omega_j, p^k \right\rangle$, and (ii) $P_j(x_j^k)$ is open relative to X_j, then we know from part I above that $(x_1^k, ..., x_n^k, p^k)$ is an equilibrium for E_k. By following a line of argument similar to the one in part I above (with the obvious modifications), we can show that if (i) and (ii) hold, then $(x_1^k, ..., x_n^k, p^k)$ is an equilibrium for the original economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$.

Part III. Suppose that the sequence of quasi-equilibria $\left\{(x_1^k, ..., x_n^k, p^k)\right\}_k$ for the for the k-bounded economies $\{E_k\}_k$ is such that for each k there is an agent j with x_j^k on the boundary of $C(k)$. Thus, we are assuming that the case treated in part II above does not arise. We will show that this does not matter: inconsequential arbitrage allows us to construct a quasi-equilibrium (resp. equilibrium) for the bounded economy that fits the case treated in part II. So suppose that the sequence $\left\{(x_1^k, ..., x_n^k, p^k)\right\}_k$ of quasi-equilibria is such that for each k, x_j^k on the boundary of $C(k)$ for some j. This implies that $\sum_{j=1}^n \left\|x_j^k\right\| \to \infty$ as $k \to \infty$. Let $y = (y_1, ..., y_n)$ be a cluster point of the sequence $\left\{(\lambda^k x_1^k, ..., \lambda^k x_n^k)\right\}_k$ where $\lambda^k = \left(\sum_{j=1}^n \left\|x_j^k\right\|\right)^{-1}$. By Lemma 1, $\sum_{j=1}^n y_j = 0$ and $\sum_{j=1}^n \|y_j\| = 1$, and for any subsequence $\{(\lambda^{k'} x_1^{k'}, ..., \lambda^{k'} x_n^{k'})\}_{k'}$ with $(\lambda^{k'} x_1^{k'}, ..., \lambda^{k'} x_n^{k'}) \to (y_1, ..., y_n)$, we have for all k' sufficiently large, $x_j^{k'} - \varepsilon y_j \in intC(k')$ for $\varepsilon \in (0, 1]$. Moreover, by *inconsequential arbitrage*, $y = (y_1, ..., y_n) \in bus(\omega)$. Thus, for all k' sufficiently large

$$x_j^{k'} - \varepsilon y_j \in X_j \text{ and } P_j(x_j^{k'} - \varepsilon y_j) \subseteq P_j(x_j^{k'}),$$

and thus,

$$x_j^{k'} - \varepsilon y_j \in X_{k'j} \text{ and } P_{k'j}(x_j^{k'} - \varepsilon y_j) \subseteq P_{k'j}(x_j^{k'}). \tag{5}$$

Our proof will be complete if we can show that $(x_1^{k'} - \varepsilon y_1, ..., x_n^{k'} - \varepsilon y_n, p^{k'})$ is a quasi-equilibrium for the k-bounded economy $E_{k'}$.

Given that

$$\sum_{j=1}^{n} y_j = 0,$$

the fact that $(x_1^{k'}, ..., x_n^{k'}, p^{k'})$ is a quasi-equilibrium for $E_{k'}$ implies that

(i) $(x_1^{k'} - \varepsilon y_1, ..., x_n^{k'} - \varepsilon y_n) \in A_{k'}(\omega)$,

(ii) $p^{k'} \in \mathcal{B}\backslash\{0\}$, and

(iii) $P_{k'j}(x_j^{k'}) \cap F_{k'}(\omega_j, p^{k'}) = \emptyset$.

Given (iii), $P_{k'j}(x_j^{k'} - \varepsilon y_j) \subseteq P_{k'j}(x_j^{k'})$ implies that $P_{k'j}(x_j^{k'} - \varepsilon y_j) \cap F_k(\omega_j, p^{k'}) = \emptyset$.

To complete the proof, it remains only to show that for agents $j = 1, 2, ..., n$, $x_j^{k'} - \varepsilon y_j \in B_{k'}(\omega_j, p^{k'})$. Given that $y = (y_1, ..., y_n) \in bus(\omega)$ and that $x_j^{k'} - \varepsilon y_j \in X_{k'j}$, this will be true provided $\left\langle x_j^{k'} - \varepsilon y_j, p^{k'} \right\rangle \leq \left\langle \omega_j, p^{k'} \right\rangle$. We will show that for agents $j = 1, 2, ..., n$, $\left\langle y_j, p^{k'} \right\rangle = 0$. Since $\sum_{j=1}^{n} y_j = 0$ it suffices to show that for $j = 1, 2, ..., n$, $\left\langle y_j, p^{k'} \right\rangle \leq 0$. Suppose not. Let $\left\langle y_{j''}, p^{k'} \right\rangle > 0$ for some agent j''. For this agent

$$\left\langle x_{j''}^{k'} - \varepsilon y_{j''}, p^{k'} \right\rangle < \left\langle \omega_{j''}, p^{k'} \right\rangle.$$

By **[A-3]**, there exists $z_{j''} \in P_{k'j''}(x_{j''}^{k'} - \varepsilon y_{j''})$ sufficiently near $x_{j''}^{k'} - \varepsilon y_{j''}$ such that

$$\left\langle z_{j''}, p^{k'} \right\rangle < \left\langle \omega_{j''}, p^{k'} \right\rangle.$$

This contradicts the fact that $P_{k'j}(x_j^{k'} - \varepsilon y_j) \cap F_k(\omega_j, p^{k'}) = \emptyset$ for all $j = 1, 2, ..., n$. We must conclude therefore that for $j = 1, 2, ..., n$, $\left\langle y_j, p^{k'} \right\rangle = 0$, and thus we must conclude that for $j = 1, 2, ..., n$, $\left\langle x_j^{k'} - \varepsilon y_j, p^{k'} \right\rangle \leq \left\langle \omega_j, p^{k'} \right\rangle$ implying that for $j = 1, 2, ..., n$, $x_j^{k'} - \varepsilon y_j \in B_{k'}(\omega_j, p^{k'})$. Thus, $(x_1^{k'} - \varepsilon y_1, ..., x_n^{k'} - \varepsilon y_n, p^{k'})$ is a quasi-equilibrium for the k-bounded economy $E_{k'}$ such that for $j = 1, 2, ..., n$, $x_j^{k'} - \varepsilon y_j \in intC(k')$. ■

9.3 Proof of Theorem 2

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^{n}$ be an economy satisfying **[A-1]-[A-2].** To show that $bus(\omega) \subseteq ea(\omega)$, let $y \in arb(\omega)$ such that $y \notin ea(\omega)$ (i.e., y is not exhaustible). This implies that for some rational allocation $x \in A(\omega)$ and some agent j, $y_j \in I_j(x_j)$. Thus, there exists a strictly increasing sequence of positive numbers $\{t_j^k\}_k$ such that $t_j^k \uparrow \infty$. Moreover, we may suppose without loss of generality that

$$t_j^k = \min\{s \geq 0 : u_j(x_j + s y_j) = u_j(x_j + t_j^k y_j)\}.$$

Thus, if $\varepsilon > 0$

$$u_j(x_j + t_j^k y_j - \varepsilon y_j) < u_j(x_j + t_j^k y_j) \ \ (*)$$

for large k. Consider the sequence $\{\frac{1}{t_j^k}(x_1+t_j^k y_1), \ldots, \frac{1}{t_j^k}(x_j+t_j^k y_j), \ldots, \frac{1}{t_j^k}(x_n+t_j^k y_n)\}_k$. This sequence is contained in $arbseq^{\omega}(y)$. Given $(*)$, this implies that $y \notin bus(\omega)$. ■

9.4 Proof of Theorem 3

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-7]**, and let $(\overline{x}_1, ..., \overline{x}_n) \in A(\omega)$ be a rational, Pareto-optimal allocation. Suppose now that inconsequential arbitrage is *not* satisfied. Thus, for some $y \in arb(\omega)$, $y \notin bus(\omega)$. By **[A-4]**, nonsatiation off the $bus(\omega)$, there is at least one agent j' such that for some $\lambda_{j'}$, $u_{j'}(\overline{x}_{j'} + \lambda_{j'} y_{j'}) > u_{j'}(\overline{x}_{j'})$. By **[A-5]**, uniformity of arbitrage cones across rational allocations,

$$u_j(\overline{x}_j + \lambda_{j'} y_j) \geq u_j(\overline{x}_j) \text{ for all } j.$$

But

$$((\overline{x}_1 + \lambda_{j'} y_1), \ldots, (\overline{x}_{j'} + \lambda_{j'} y_{j'}), \ldots, (\overline{x}_n + \lambda_{j'} y_n)) \in A(\omega)$$

contradicting the Pareto optimality of $(\overline{x}_1, ..., \overline{x}_n)$. ■

9.5 Proof of Theorem 4

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-3]**, **[A-4]***, **[A–5]**, and **[A-7]**. Also, suppose the economy satisfies inconsequential arbitrage. Let $\overline{x} = (\overline{x}_1, ..., \overline{x}_n) \in A(\omega)$ be a rational, Pareto-optimal allocation with $\overline{x}_j \in intX_j$ for all j. Consider the economy $(X_j, \overline{x}_j, P_j(\cdot))_{j=1}^n$. This economy satisfies **[A-1]**-**[A-2]** and must also satisfy inconsequential arbitrage, otherwise, given **[A-4]*** and **[A–5]**, we could arrive at a contradiction of the Pareto optimality of $\overline{x}$ via the arguments used in the proof of Theorem 3. Given **[A-1]**-**[A-3]**,and **[A-7]**, $\overline{x}_j \in intX_j$ for all j implies via Theorem 1 that $(X_j, \overline{x}_j, P_j(\cdot))_{j=1}^n$ has an equilibrium, say $(x_1', ..., x_n', p')$. For each agent j, $u_j(x_j') \geq u_j(\overline{x}_j)$. In fact, we must have $u_j(x_j') = u_j(\overline{x}_j)$ for all j, otherwise, we would contradict the Pareto optimality of $(\overline{x}_1, ..., \overline{x}_n)$. Since $(x_1', ..., x_n', p')$ is an equilibrium, for any x_j'' such that $u_j(x_j'') > u_j(x_j') = u_j(\overline{x}_j)$, $\langle x_j'', p' \rangle > \langle \overline{x}_j, p' \rangle$. Thus, $(\overline{x}_1, ..., \overline{x}_n, p')$ is an equilibrium. ■

9.6 Proof of Theorem 5

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-2]**, and suppose the economy has bounded proximal trading volume. Let $\{\lambda^k x^k\}_k$ be any sequence in $arbseq^{\omega}(y)$. Now define $\gamma(x_j^k, y_j)$ to be the number such that

$$z(x_j^k, y_j) + \gamma(x_j^k, y_j) \cdot y_j = x_j^k.$$

Thus, $\gamma(x_j^k, y_j)$ is a proximal measure of the distance from the projection $z(x_j^k, y_j)$ of x_j^k onto the hyperplane $H(y_j)$ to the vector x_j^k. For $y_j \neq 0$, $\gamma(x_j^k, y_j) \to \infty$. Thus, since $ptv(x^k, y)$ is bounded, for $y_j \neq 0$, $ptv(x^k, y) < \gamma(x_j^k, y_j)$ for k sufficiently large. Given the definition of $ptv(x^k, y)$, this implies that $y \in bus(\omega)$. ■

9.7 Proof of Theorem 6

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-2]** and **[W-1]**-**[W-2]**. As noted in the text, the proof of Theorem 6 follows directly from Corollary 16.2.2 in Rockafellar (1970). ■

9.8 Proof of Theorem 7

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying **[A-1]**-**[A-2]** and **[W-1]**-**[W-2]**. The fact that the Hart/Werner conditions imply inconsequential arbitrage is an immediate consequence of the fact that

$$L_1 \times \cdots \times L_n \subseteq bus(\omega).$$

Under the additional assumption of no half lines, **[W-3]**, the fact that inconsequential arbitrage implies the Hart/Werner conditions is an immediate consequence of the fact that with no half lines

$$L_1 \times \cdots \times L_n = bus(\omega) = \{0, \ldots, 0\}.$$

■

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