"The Relationship Between Shareholding Concentration and Shareholder Voting Power in British Companies : A Study of the Application of Power Indices for Simple Games"

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This paper is circulated for discussion purposes only and its contents should be considered preliminary.

1. Introduction

The relationship between shareholding concentration and shareholder voting power and the question of corporate control has long been recognised as being of central importance in the economies of the firm and has given rise to a large literature. Despite this, however, and the fact that quite sharp differences in perspective exist in this literature, relatively little work has been done on actually attempting to measure, in a theoretically rigorous way, the quantitative significance of empirically-observed differences in concentration on the distribution of power.

On the other hand the literature on game theory is replete with theoretical examples of the application of the theory of simple games to shareholder voting. Although methods for applying this theory to real-world voting situations exist, little work appears to have been done on this particular question, although applications have been made to problems in political science.[1/]

This paper is an investigation of the empirical application of the method of power indices for simple games to shareholder voting using data for a sample of British companies collected by Collett and Yarrow, previously analysed by them (1976) and by Cubbin and Leech (1983). The paper has three main aims:

(i) to establish the feasibility of computing power indices using observed shareholding distributions;

(ii) to form a view about patterns of distribution of shareholder voting power in typical British companies;

(iii) to compare the results of empirical application of the two main indices of power (the Shapley-Shubik index and the Normalised Banzhaf index).

In Sections 2, 3 and 4 the theory of power indices for simple games is described. Section 5 describes the probabilistic interpretation which is the foundation of empirical approximation algorithms, while the computational aspects are described in Sections 6 and 7. Section 8 describes the empirical results.

2. Simple Games

An arbitrary n-person game is defined by a set of players $N = \{1,2,\ldots,n\}$ and a characteristic function $V(.)$ which associates with each subset or coalition $T \subseteq N$ a number $V(T)$. The characteristic function V satisfies two conditions:

(i) $V(\phi) = 0$; (ii) $V(S \cup T) \geq V(S) + V(T)$ for all sets $S, T \subseteq N$.

For this game the Shapley Value is a well-known solution concept. This is defined as a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ whose elements are defined by the formula:

$$(1) \qquad \gamma_i = \sum_{T \subseteq N - \{i\}} \frac{t!\ n-t-1!}{n!} \left[V(T \cup \{i\}) - V(T)\right], \quad i = 1,\ldots,n,$$

where t is the cardinality of T, etc. The Shapley value for player i is a weighted average of the marginal contributions of that player to coalitions over all coalitions of which he is not a member. The weight for each coalition T is the number of orderings of the remaining $n-1$ players given T as a proportion of the number of ordering of all n players.

Present concern is with simple games[2/] in which coalitions are partitioned into those which are winning and those which are losing. The characteristic function is dichotomous with $V(T) = 1$ if T is winning and $V(T) = 0$ if it is losing. Voting games are of this type and an important class is that in which individual players have different numbers of votes, weighted majority games, A weighted majority game, Γ, with n players is written:

$$\Gamma = [q; p_1, p_2, \ldots, p_n]$$

where q is a quota and $p_1, p_2, \ldots, p_n$ are weights. In the present application the weights are normalised to sum to 1. A coalition T is winning if $\sum_{i \varepsilon T} p_i \geq q$ and losing if $\sum_{i \varepsilon T} p_i < q$.

3. Power Indices

For a simple game the value is a vector of power indices which measure the voting power of each player. Two power indices are considered: (i) the Shapley-Shubik index, and (ii) the Banzhaf index.[3/] Both are based on the concept of swing.

A swing for player i is a coalition, $S \subseteq N - \{i\}$ such that $V(S) = 0$ but $V(S \cup \{i\}) = 1$, i.e. $\sum_{j \in S} p_j < q$ and $\sum_{j \in S} p_j + p_i \geq q$. The power index for player i is the relative number of swings for that player. The two approaches differ in the way swings are counted.

(I) The Shapley-Shubik Power Index for Γ is the Shapley value. Specifically, it is a vector γ whose i^{th} element is

(2) $$\gamma_i = \sum_S \frac{s!\ n-s-1!}{n!}, \quad i = 1,\ldots,n,$$

the summation being taken over all swings for i

(II) The Banzhaf Power Index is based on counting the number of swings for each player. Let η_i be the number of swings for i and let $\bar{\eta} = \Sigma \eta_i$. Two measures of power based on the vector $\eta = (\eta_1,\ldots,\eta_n)$ employ different normalisations.

(IIi) The Normalised Banzhaf Index is a vector β whose i^{th} element is:

(3) $$\beta_i = \eta_i / \bar{\eta}, \quad i = 1,\ldots,n,$$

i.e. the number of swings for player i as a proportion of the total number of swings for all players.

(IIii) The <u>Banzhaf Swing Probability</u> is a vector β' whose i^{th} element is

(4) $$\beta'_i = \eta_i/2^{n-1}, \quad i = 1,\dots,n,$$

i.e. the number of swings for i as a proportion of the number of potential swings (the number of coalitions, which do not include i). (Note that

(5) $\beta_i = \beta'_i/\Sigma\beta'_i$.)

4. Properties

The Shapley-Shubik index has the property that $\Sigma\gamma_i = 1$ and can therefore be thought of as apportioning total voting power among the players. This property is shared by the Normalised Banzhaf index.

The Shapley-Shubik index also has a simple interpretation as the probability of a swing for each player given a certain model of random coalition formation. The Banzhaf swing probability shares this property although the coalition model assumed is different. The Swing Probabilities do not, however, sum to unity.

Specifically, the model of coalition formation underlying the Shapley-Shubik index is one in which players are added sequentially in the build-up of the grand coalition, N. All n! orderings of the players are regarded as equally likely and the weight assigned to a given swing is equal to the number of ways of ordering the players given the swing. This approach is therefore based on counting permutations and coalitions of different size are given different weights.

On the other hand, the model of coalition formation assumed by the Banzhaf approach has no regard to orderings of players. Each swing is counted just once and the weight attached to a swing of a given size is the number of combinations of players. Each coalition is treated as equally likely and every coalition is given the same weight regardless of its size.

Which power index is better will depend on which of these alternative coalition models is more appropriate in the context of analysis.

The Shapley-Shubik index has the advantage of possessing attractive mathematical properties in that it uniquely satisfied Shapley's axioms.[4/] However the coalition model on which it is based has been criticised by Brams (1975) among others, with the implication that mathematical elegance is achieved at the expense of behavioural implausibility. It has been suggested that the assumption that all orderings are equally probable is unduly strong in a model of power in a legislature.[5/] Treating coalitions as equally probable may be a less strong assumption.[6/]

In empirical studies of the distribution of power in voting bodies, however, the two indices have tended to agree fairly closely.[7/] It is not difficult, though, to construct examples in which there is sharp disagreement. One case, which is of importance to analysis of shareholder voting, is the game in which there is a single player with weight $p_1 = a$ and $n-1$ minor players each with weight $p_i = (1-a)/(n-1)$, $i = 2,\dots,n$. It is well known that in the limit as $n \to \infty$, $\gamma_1 \to a/(1-a)$, $\beta_1 \to 1$ and $\beta_1' \to 1$.[8/] Thus if 25 percent of the stock is held in a single bloc while the remainder is distributed equally in a large number of individual holdings, the Shapley-Shubik power index for the large bloc is 1/3 while the Banzhaf index is 1 (Dubey and Shapley, 1979).

5. Probabilistic Voting

An alternative probabilistic basis for both the Shapley-Shubik index and the Swing Probability can be given which does not require a model of coalition formation. This approach also leads to simple algorithms for computing power indices based on reasonable approximations. On the other hand, applying the coalitional definitions directly would be prohibitively expensive, even for low-dimensional problems.

Let S be a swing for player i. The Shapley-Shubik index for i is

$$(2) \qquad \gamma_i = \sum_S \frac{s!\, n-s-1!}{n!} .$$

The term inside the summation sign is a Beta function, which can be written:

$$(6) \qquad B(s+1,n-s) = \frac{s!\, n-s-1!}{n!} = \int_0^1 \pi^s (1-\pi)^{n-s-1} d\pi .$$

The integral can be interpreted as a probability. Letting π be the probability that player $j \neq i$ votes the same way as i (player i is assumed to vote strategically rather than randomly), then the expression under the integral sign in (6), $\pi^s(1-\pi)^{n-s-1}$ is the probability of the swing S occurring given π. Summing over all swings gives the probability of a swing given π,

$$(7) \qquad f_i(\pi) = \sum_S \pi^s (1-\pi)^{n-s-1}$$

Hence, substituting (6) and (7) into (2),

$$(8) \qquad \gamma_i = \sum_s \int_0^1 \pi^s (1-\pi)^{n-s-1} \, d\pi = \int_0^1 f_i(\pi) d\pi \, .$$

The Banzhaf Swing Probability assigns equal probability to each outcome and therefore is obviously obtained from (7) on assuming players vote indifferently. Hence, setting $\pi = 1/2$ we have,

$$(9) \qquad \beta_i' = f_i(1/2) .$$

The theory underlying this approach is given by Owen (1972, 1975). Straffin (1977,1979) has suggested an explicit probabilistic model of behaviour to underlie this interpretation. The voting probability π is assumed to be a random variable with some distribution on (0,1). This variable represents the strength of support in a vote on some particular issue. Since there are many possible issues and the analysis is of power in an abstract, strategic sense unrelated to issues, it is assumed that this distribution is uniform. Two assumptions about the choice of π from this distribution are:

(i) Homogeneity. A value π is selected at random by all players $j \neq i$.

(ii) Independence. The mean $\pi = 1/2$ is chosen by all players $j \neq i$.

The independence assumption leads to the swing probability (9) $\beta_i' = f_i(E(\pi)) = f_i(1/2)$ and is equivalent to assuming an "average" issue on which players are indifferent, making each coalition equally likely. This assumption is the same as that employed by Cubbin and

Leech (1983) who justified it in terms of a hypothetical standard situation in which control of the company had become an issue on which shareholders were equally divided.[9/]

On the other hand, the homogeneity assumption leads to the Shapley-Shubik index. This assumption says that on any issue players will share a common degree of support and therefore a common voting probability π. Given π, player i's probability of affecting the outcome is $f_i(\pi)$. However allowing for all possible degrees of support means defining the power index as the average of $f_i(\pi)$ over the distribution of π. Then, $\gamma_i = Ef_i(\pi)$ which is given by (8).

It is clear from this interpretation that the Shapley-Shubik index employs stronger assumptions than the Banzhaf approach since (i) is more restrictive than (ii). The Shapley-Shubik index requires an assumption about the shape of the whole distribution of π - that it is uniform - while the Banzhaf indices require only that it have a mean of 1/2. The requirement of a uniform distribution means that all possible degrees of support in votes be given equal weight, an assumption which can be criticised on grounds similar to those of Brams' criticisms described above. It is arguably more reasonable to assume voting probabilities to be clustered around the mean. Sraffin (1977) provides generalisations of this.

6. Computation : Complete Data

Computation of both power indices requires the evaluation of the probability of a swing, $f_i(\pi)$, given π. The Banzhaf Swing Probability is then given by (9) and the Normalised Banzhaf index by (5). The Shapley-Shubik index is calculated by integrating (7) as in (8).

In order to calculate (7), the probability of a swing for i given π, suppose player $j \neq i$ casts p_j votes with probability π. The number of votes cast by j is a random variable x_j, with a dichotomous distribution with mean $E(x_j) = \pi p_j$ and variance $var(x_j) = \pi(1-\pi)p_j^2$.

Votes cast by different players are assumed to be independent and the x_j's are independently distributed. Let the total number of votes cast by all players be X_i . Then,

(10) $$E(X_i) = \pi \sum_{j \neq i} p_j = \pi(1-p_i) = \mu_i(\pi)$$

(11) $$Var(X_i) = \pi(1-\pi) \sum_{j \neq i} p_j^2 = \pi(1-\pi)(H-p_i^2) = \sigma_i^2(\pi)$$

where $H = \sum_{i=1}^{n} p_i^2$.

For large n and provided that no p_j is much larger than the others, X_i has a normal distribution. Let the standard normal distribution function be $\Phi(.)$. Then,

$$Pr[X_i < a] = \Phi\left(\frac{a-\mu_i(\pi)}{\sigma_i(\pi)}\right), \quad 0 < a < 1.$$

Therefore,

(12) $$f_i(\pi) = Pr[q - p_i < X_i < q]$$

$$= \Phi\left(\frac{q-\mu_i(\pi)}{\sigma_i(\pi)}\right) - \Phi\left(\frac{q-p_i-\mu_i(\pi)}{\sigma_i(\pi)}\right) .$$

The Shapley-Shubik index can be found by numerically integrating (12) with respect to π, on setting $q = 1/2$.

7. Computation : Incomplete Data

Shareholding data are often available only in the form of the upper tail of the size distribution. Measures of concentration are dominated by the largest holdings and these are all that are required for an analysis of voting power provided enough of the tail is observed. Since we do not observe all the data (and sometimes we have no knowledge of n) we cannot compute the indices directly as in the previous section. However they can be computed within narrow bounds by making limiting assumptions about the distribution of the smaller holdings. It is assumed that the largest k holdings are observed: $p_1, p_2, \ldots, p_k$ where $p_1 \geq p_2 \geq \ldots \geq p_k \geq p_i$ for all $i > k$.

Limit 1. Most Concentrated Distribution

The non-observed smaller holdings are assumed to be as highly concentrated as possible. Let m be the largest integer no greater than $(1-C_k)/p_k$ where $C_k = \sum_{i=1}^{k} p_i$. The non-observed holdings are assumed to consist of m holdings of size p_k and one holding, p_{k+m+1}, of size $p_{k+m+1} = 1-C_k-mp_k$. The game is therefore:

$$\Gamma_1 = |1/2; p_1, \ldots, p_k, \ldots p_k, \ldots, p_{k+m+1}|$$

computing power indices for Γ_1 using the method described in section 6 gives lower bounds on power indices for the largest holdings.

Limit 2. Least Concentrated Distribution

The non-observed holdings are assumed to be equal and held by an arbitrarily large number of players. Let the number of these minor players be r. Then $p_i = (1-C_k)/r$, for all $i > k$.

Consider first the Shapley-Shubik index. The random variable X_i has variance

$$\sigma_i(\pi)^2 = \pi(1-\pi) \sum_{\substack{j=1 \\ j \neq i}}^{k} p_j^2 + \pi(1-\pi) \frac{(1-C_k)^2}{r} . \qquad (13)$$

As $r \to \infty$ the variance of the sum of the minor weights (the second term on the RHS) goes to zero. The mean of X_i remains unchanged. The algorithm is therefore the same as in Section 6 after setting $H = \sum_{i=1}^{k} p_i^2$.

The Banzhaf Swing Probability is computed on the same basis by the method described in Section 6. However, the Normalised Banzhaf index requires the calculation of the normalising constant. However it is unnecessary to do this since use can be made here of a limit theorem due to Dubey and Shapley (1979).

Denote the game in which the minor weights are all equal by $\Gamma_2 = \left[1/2; p_1,\ldots,p_k,\frac{1-C_k}{r},\ldots,\frac{1-C_k}{r}\right]$ and the swing Probability and Normalised Banzhaf index by $\beta_i'(\Gamma_2)$ and $\beta_i(\Gamma_2)$, $i=1,\ldots,k$, respectively. Consider the game $\Gamma_3 = [1/2 - (1-C_k)/2; p_1,\ldots,p_k]$, in which there are k players (the major players only), with Banzhaf power indices $\beta_i'(\Gamma_3)$ and $\beta_i(\Gamma_3)$. Then Dubey and Shapley show that, under appropriate conditions (which are non-trivial),[10/] $\lim_{r\to\infty} \beta_i'(\Gamma_2) = \beta_i'(\Gamma_3)$ and $\lim_{r\to\infty} \beta_i(\Gamma_2) = \beta_i(\Gamma_3)$ for all $i = 1,\ldots,k$.

Thus upper bounds on the Swing Probability and Normalised Banzhaf index can be found by applying the algorithm to the game Γ_3.

8. Empirical Application to British Companies

Power indices have been calculated for the leading shareholdings in some 85 companies in the engineering, electrical engineering/electronics, food and textile/clothing industries, taken from the top 400 of The Times list ofleading British companies by sales in 1970/1 [11/]. The data consisted of at least one hundred observations for each distribution and observations were confined to holders of record. There was no attempt to identify blocs of shares held by different nominees but with the same beneficiary which would be important in a more thorough analysis of voting power. The analysis is therefore limited to the formal distribution of voting power as revealed by quantitative data taken from share registers. The methods described above in Section 7 were used for each distribution. Illustrative results for three companies are shown in Table 1.

Table 1 shows the results obtained for three randomly selected companies, EMI, William Press and Gill and Duffus, for which the numbers of holdings observed are 265, 154 and 146 respectively. Column (1) contains the weights, p_i, column (2) the Shapley-Shubik indices γ_i, and column (3) the corresponding power ratios, $\rho_i = \gamma_i/p_i$, expressed as percentages. The remaining columns contain the results for the Banzhaf indices.

Calculations of the Swing Probabilities and the Shapley-Shubik indi were carried out using both the limiting assumptions described in Section 7 but the results were in close agreement. It is clear from this that the fact

TABLE 1(a) : EMI

i	Weight p_i	Shapley-Shubik Index γ_i	ρ_i	Banzhaf Index β_i(Min)	β_i(Min)/p_i	β_i(Max)	β'_i(Min)	β'_i(Max)
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	8.90	9.74	109.48	11.53	129.53	20.87	94.30	94.43
2	2.54	2.59	101.62	2.52	99.15	4.57	20.63	20.66
3	1.26	1.26	100.27	1.22	97.53	2.22	10.01	10.03
4	1.05	1.05	100.06	1.03	97.38	1.86	8.39	8.40
5	0.93	0.93	99.94	0.91	97.30	1.64	7.42	7.43
6	0.89	0.89	99.89	0.86	97.27	1.56	7.05	7.06
7	0.84	0.84	99.85	0.82	97.25	1.48	6.67	6.68
8	0.77	0.77	99.78	0.75	97.21	1.36	6.13	6.14
9	0.75	0.75	99.76	0.73	97.20	1.33	6.00	6.01
10	0.64	0.64	99.64	0.62	97.15	1.12	5.06	5.07
20	0.40	0.40	99.41	0.39	97.07	0.71	3.19	3.19
265	0.00*	0.00*	99.02	0.00*	97.02	0.01	0.04	0.04
Total	53.88	54.49		55.25		100.00		

Approximation error : 0.17

* Less than 0.005

TABLE 1(b) : William Press

	Weight	Shapley-Shubik Index		Banzhaf Index					
i	p_i (1)	γ_i (2)	ρ_i (3)	β_i(Min) (4)	β_i(Min)/p_i (5)	β_i(Max) (6)	β_i'(Min) (7)	β_i'(Max) (8)
1	9.45	10.28	108.86	10.78	114.12	15.14	60.80	60.93
2	7.09	7.50	105.73	7.52	106.03	10.55	42.40	42.48
3	5.97	6.23	104.36	6.17	103.33	8.66	34.81	34.88
4	1.86	1.86	99.83	1.82	97.93	2.55	10.27	10.28
5	1.74	1.73	99.71	1.70	97.86	2.39	9.60	9.62
6	1.52	1.51	99.49	1.49	97.75	2.09	8.39	8.40
7	1.47	1.46	99.44	1.44	97.73	2.02	8.11	8.13
8	1.40	1.39	99.37	1.37	97.70	1.92	7.73	7.74
9	1.34	1.33	99.31	1.31	97.67	1.84	7.40	7.42
10	1.32	1.31	99.29	1.29	97.66	1.81	7.27	7.28
20	0.78	0.77	98.76	0.76	97.48	1.07	4.29	4.30
154	0.02	0.02	98.03	0.02	97.39	0.03	0.13	0.13
Total	70.49	71.37		71.26		100.00		

Approximation error: 0.31

TABLE 1(c) : Gill and Duffus

i	Weight p_i (1)	Shapley-Shubik Index γ_i (2)	ρ_i (3)	Banzhaf Index β_i(Min) (4)	β_i(Min)/p_i (5)	β_i(Max) (6)	β'_i(Min) (7)	β'_i(Max) (8)
1	26.65	35.81	134.38	32.36	121.43	36.55	99.70	99.70
2	4.16	4.04	97.16	3.86	92.78	4.36	11.89	11.89
3	3.95	3.83	96.96	3.66	92.71	4.13	11.27	11.27
4	3.49	3.37	96.53	3.23	92.57	3.65	9.95	9.95
5	1.83	1.74	95.03	1.69	92.23	1.91	5.20	5.20
6	1.82	1.73	95.02	1.67	92.23 ?	1.89	5.16	5.16
7	1.66	1.57	94.88	1.53	92.20	1.73	4.71	4.71
8	1.38	1.31	94.64	1.27	92.17	1.44	3.92	3.92
9	1.18	1.12	94.47	1.09	92.15	1.23	3.36	3.36
10	1.18	1.12	94.47	1.09	92.15	1.23	3.32	3.32
20	0.81	0.77	94.15	0.75	92.12	0.85	2.31	2.31
146	0.01	0.01	93.48	0.01	92.10	0.01	0.03	0.03
Total	87.57	93.45		88.55		100.00		

Approximation error: 5.07

that only the upper tail of the distribution is observed in the data is not a limiting factor in the analysis. This can be seen by comparing columns (7) and (8) which contain the respective Banzhaf Swing Probabilities, β_i'. For the Shapley-Shubik indices the correspondence is even closer and only one set of values is reported. There is, however, a marked difference between the Normalised Banzhaf indices in the two cases, reported in columns (4) and (6) and labelled β_i(min) and β_i(max). The values of β_i(max) given in column (6) are always much greater than both those for β_i(min) in column (4) and γ_i in column (2). This is true for every distribution analysed and results from use of the limit theorem for the Banzhaf index described above which gives all power to the major players in the limit as the number of minor players (corresponding to unobserved holdings) goes to infinity (and their individual holdings go to zero). Column (5) reports the power ratio based on β_i(min).

The approximation error is a measure of the error in the approximation used to compute the Shapley-Shubik index. Theoretically, this index must satisfy $\Sigma\, \gamma_i = 1$. The approximation error is defined for the most concentrated limiting case, as $100(\sum_{i=1}^{k+m+1} \gamma_i - 1)$. It can be seen that, for two of these companies, this error is very small (under one percent) while for the other case it is quite large (over 5 percent).

The distribution of the approximation errors over the shareholding distributions studied is presented in Table 2. In the great majority of cases it is very small (in 62 cases less than 0.1 percent) but in a number of cases it is quite large (in 4 cases greater than 10 percent). In all cases where the error is large there is a single, very large holding much bigger than the others and the normality assumption is therefore invalid. For example in all cases where p_1 is between 20 and 30 percent the approximation error is between 4 and 10 percent. In all cases where p_1 exceeds 30 percent the approximation error exceeds 10 percent. This is not a problem with regard to computing the Shapley-Shubik index and Swing Probability for the largest holding in such cases but it does invalidate the calculations of the indices for the other holdings. It does, however, invalidate the calculations of all the Normalised Banzhaf indices since they require the calculation of the normalising constant which depends on the Swing Probabilities for all holdings. In the results described below we have arbitrarily rejected all cases where the approximation error is greater than 2 percent.

TABLE 2 : Approximation errors

Percent error	number
<0.1	62
0.1 - 1.0	17
1.0 - 5.0	5
5.0 - 10.0	4
>10.0	4
	92

Looking at the results in Table 1(a), for EMI, we see that the Shapley-Shubik index is greater than the weight, $\gamma_i > p_i$, for the largest four holdings and is less than the weight for the remainder. The largest holding, 8.9 percent, has a power ratio of 109.48 and therefore a 9.48 percent increase in power. The smallest holding observed, number 254, has a power ratio of 99.02. The Normalised Banzhaf index for the most concentrated case, β_1(min), however, gives a power ratio greater than 100 to only one holding. The increase in power of this holding, however, is much greater, at 29.53 percent. The other Normalised Banzhaf index gives power to every holding observed. This is a feature common to all the shareholding distributions analysed and for this reason little confidence is placed in these indices.

In Table 1(b), William Press, both γ_i and β_i(min) show power greater than weight for three holdings with again the latter showing a greater inequality in the distribution of power.

In Table 1(c) the Shapley-Shubik index shows a 34.38 percent excess of power over weight for the largest holding of 26.65 percent. In this case the computed value of β_1(min) is less than γ_1 but this should be discounted because of the large approximation error affecting the normalising constant for the former.

Figure 1 shows the relationship between the power ratios, 100 γ_i/p_i and 100 β_i/p_i, and weight, p_i, for the largest 20 holdings for each of six representative distributions.[12/] Figures 1(a) and 1(b) are graphical representations of the results for EMI and William Press contained in Tables 1(a) and 1(b). (The results for Gill and Duffies are discarded because of the large approximation error.) The other distributions shown have been chosen as representative of the remaining companies in terms of shareholding concentration and the type of power distributions obtained. In terms of concentration a highly dispersed distribution is that of Courtaulds in Figure 1(c). Figure 1(d) illustrates a slightly more concentrated case, that of Fitch Lovell while one of the most concentrated is Nottingham Manufacturing ordinary (shown in Figure 1(e).

Figure 1(a)

SHAPLEY&BANZHAF POWER RATIOS VS S/HOLDING SIZE:EMI

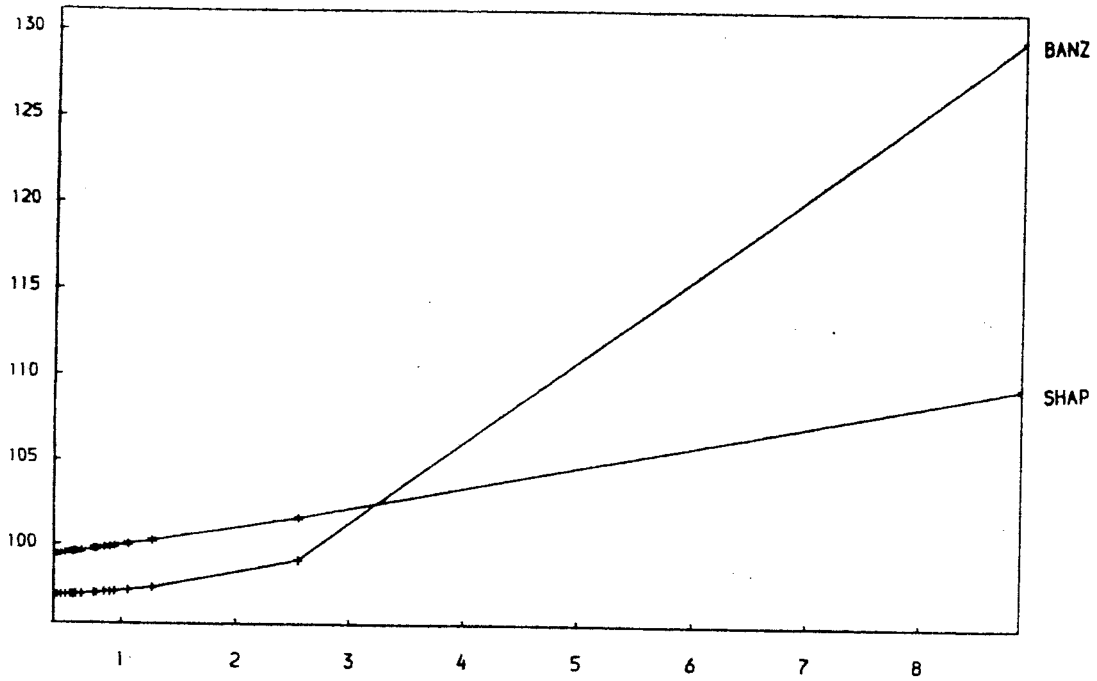


Figure 1(b)

SHAPLEY & BANZHAF POWER RATIOS VS S/HOLDING SIZE:WILLIAM PRESS

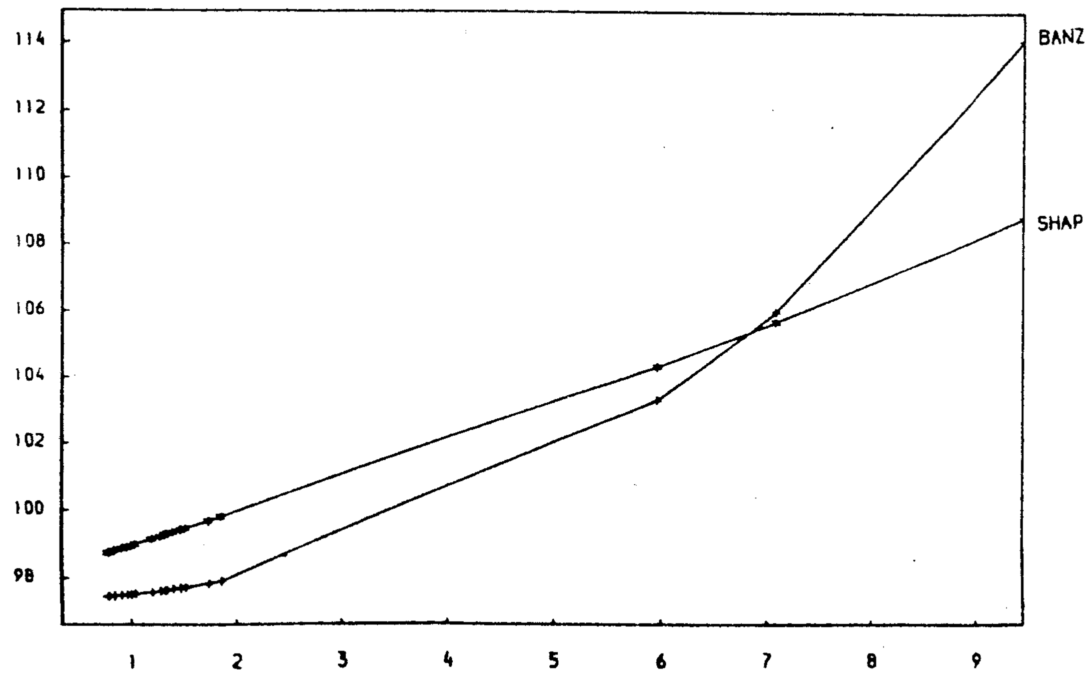


Figure 1(c).

SHAPLEY & BANZHAF POWER RATIOS VS S/HOLDING SIZE: COURTAULDS

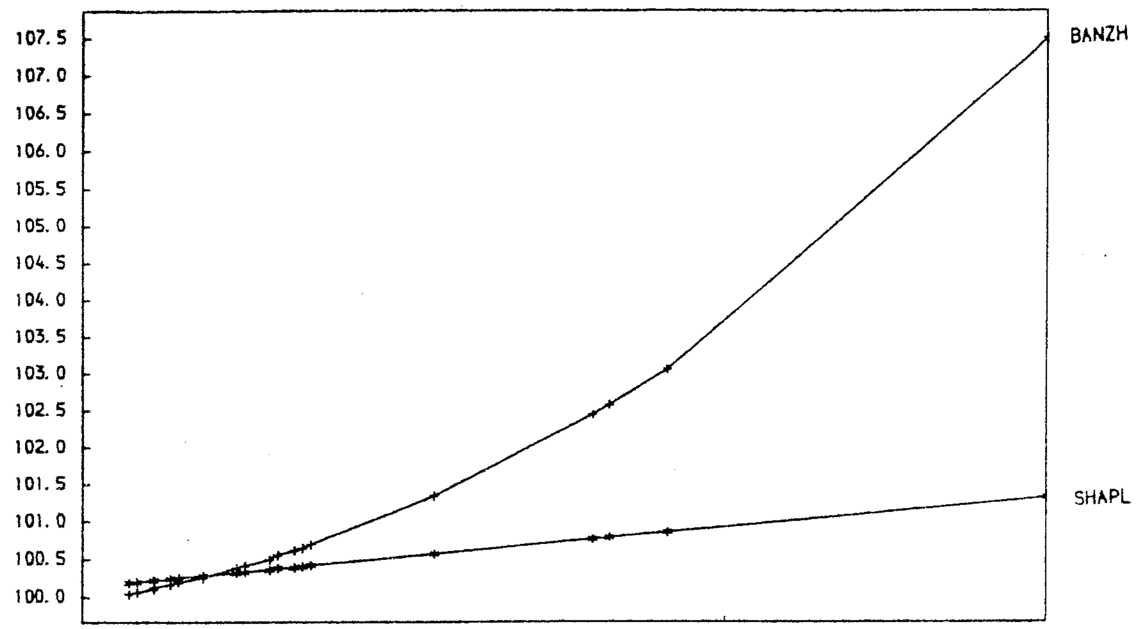


Figure 1(d)

SHAPLEY &BANZHAF POWER RATIOS VS S/HOLDING SIZE:FITCH LOVELL

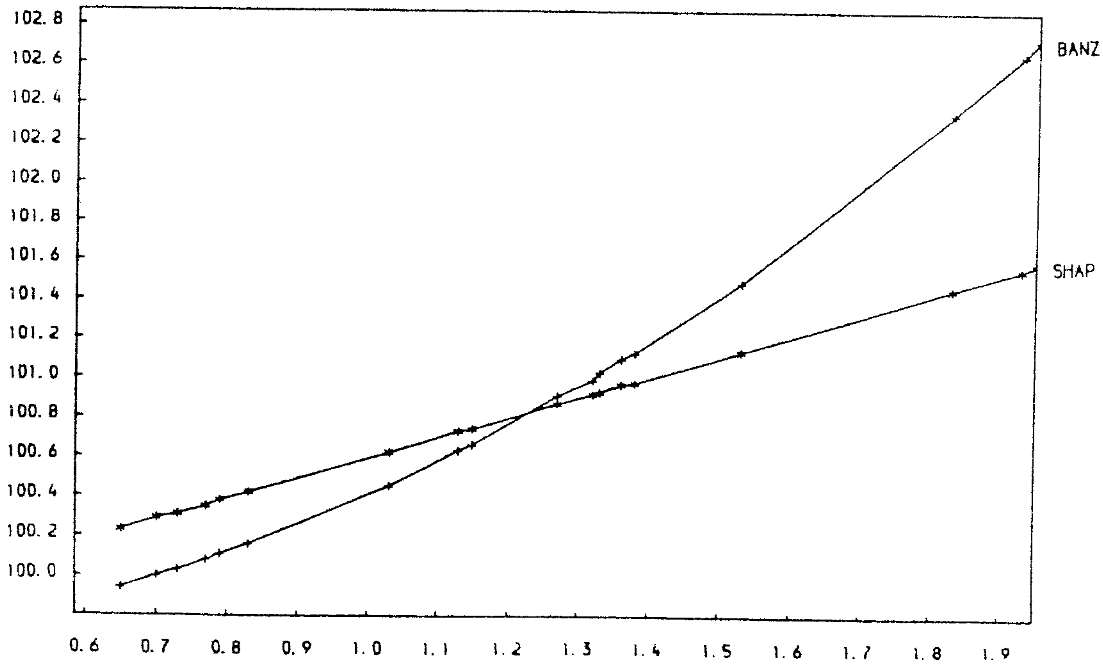


Figure 1(e)

SHAPLEY&BANZHAF VALUES VS S/HOLDING SIZE:NOTT MANUF ORDINARY

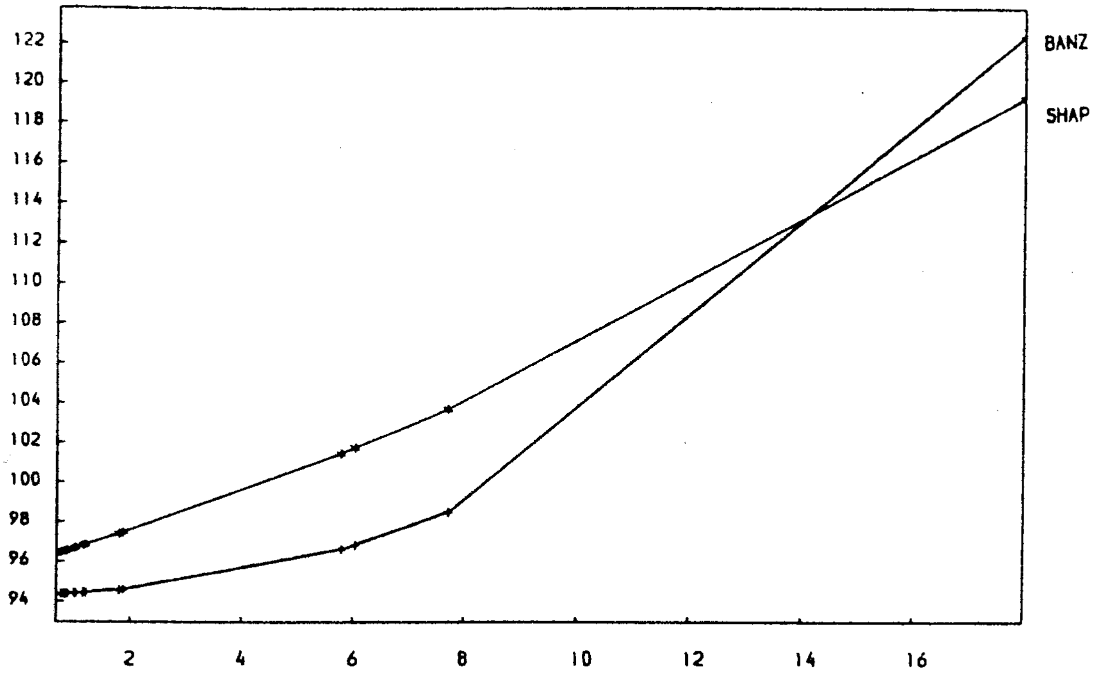
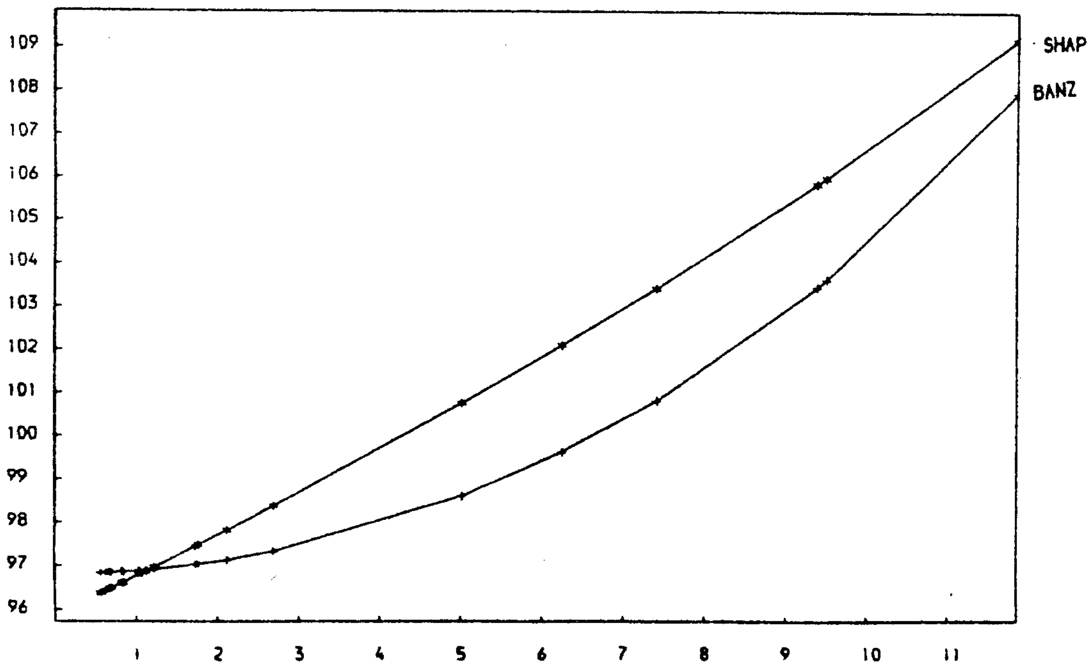


Figure 1(f)

SHAPLEY&BANZHAF RATIOS V S/HOLDING SIZE:BURTON ORDINARY



All the results obtained (all 92 distributions) have the common feature that power is more highly concentrated than ownership for both indices in that the power ratio increases monotonically with weight. This applies to the whole range of holdings observed and not only the top 20 shown in the diagrams.

Comparing the power distributions among the largest 20 holdings a characteristic pattern is revealed in diagrams 1(a) to 1(e) in which the Banzhaf index gives a more unequal distribution of power than the Shapley-Shubik index. The Banzhaf index exceeds the Shapley-Shubik index for one or more of the largest holdings and is less than it for the smaller holdings among the largest 20. The same pattern is revealed for every distribution not illustrated.

There is, however, one exception to this typical pattern, that of Burton ordinary illustrated in Figure 1(f). In this case the ranking of the indices is reversed with the Shapley-Shubik index greater than the Banzhaf index for a group of large holdings and less than it for the remainder. (The approximation error is 0.80 in this case and is taken to be small enough for the normality assumption not to be invalid.) The reason for this reversal of the rankings is unclear. One possible explanation is that the distribution is fairly concentrated with a relatively large number of large holdings. In particular p_2 and p_3 are both large in relation to p_1 and about equal.

When the results are analysed for all the shareholdings observed in the data (not just the largest 20), somewhat more diversity in the results emerges. Although the characteristic pattern remains for 83 distributions (that is $\beta_i > \gamma_i$, $i < j$ for some j, and $\beta_i < \gamma_i$ for all i, $j \leq i \leq k$) there is a group of 8 companies for which the ranking of the indices is reversed for small holdings. For these companies we have $\beta_i > \gamma_i$, $i < j$, some j, $\beta_i < \gamma_i$ for $j \leq i \leq h$ for some $h > 20$ and $\beta_i > \gamma_i$ for all i, $h < i \leq k$.

A stylistic representation of the type of results obtained for all 92 distributions is shown in Figure 2 in which each set of results is regarded as an observation on part of a general pattern. The general pattern is observed in only 8 cases. Since the results are determined entirely by the shareholding distribution, an interesting question is whether it is theoretically or empirically possible to observe other parts of this diagram.

Figure 3 shows plots of the Banzhaf power ratio against the Shapley-Shubik power ratio for the same companies as in Figure 1. They all show the same pattern of the algebraic difference between the former and the latter increasing monotonically and at an increasing rate with the Shapley-Shubik power ratio.

The remaining diagrams all present results in terms of some overall measure for all the distributions in the sample. Figure 4 shows the relationship between the concentration of power and that of ownership, both measured by the Herfindahl index. That is, in Figure 4a, the Herfindahl index of concentration for the Shapley-Shubik index, $\Sigma\gamma_i^2$ is plotted on the vertical axis, and that for shareholding, Σp_i^2, on the horizontal axis. Figure 4b shows the same plot for the Banzhaf index, $\Sigma\beta_i^2$. Both diagrams show the same effect, that power is slightly more

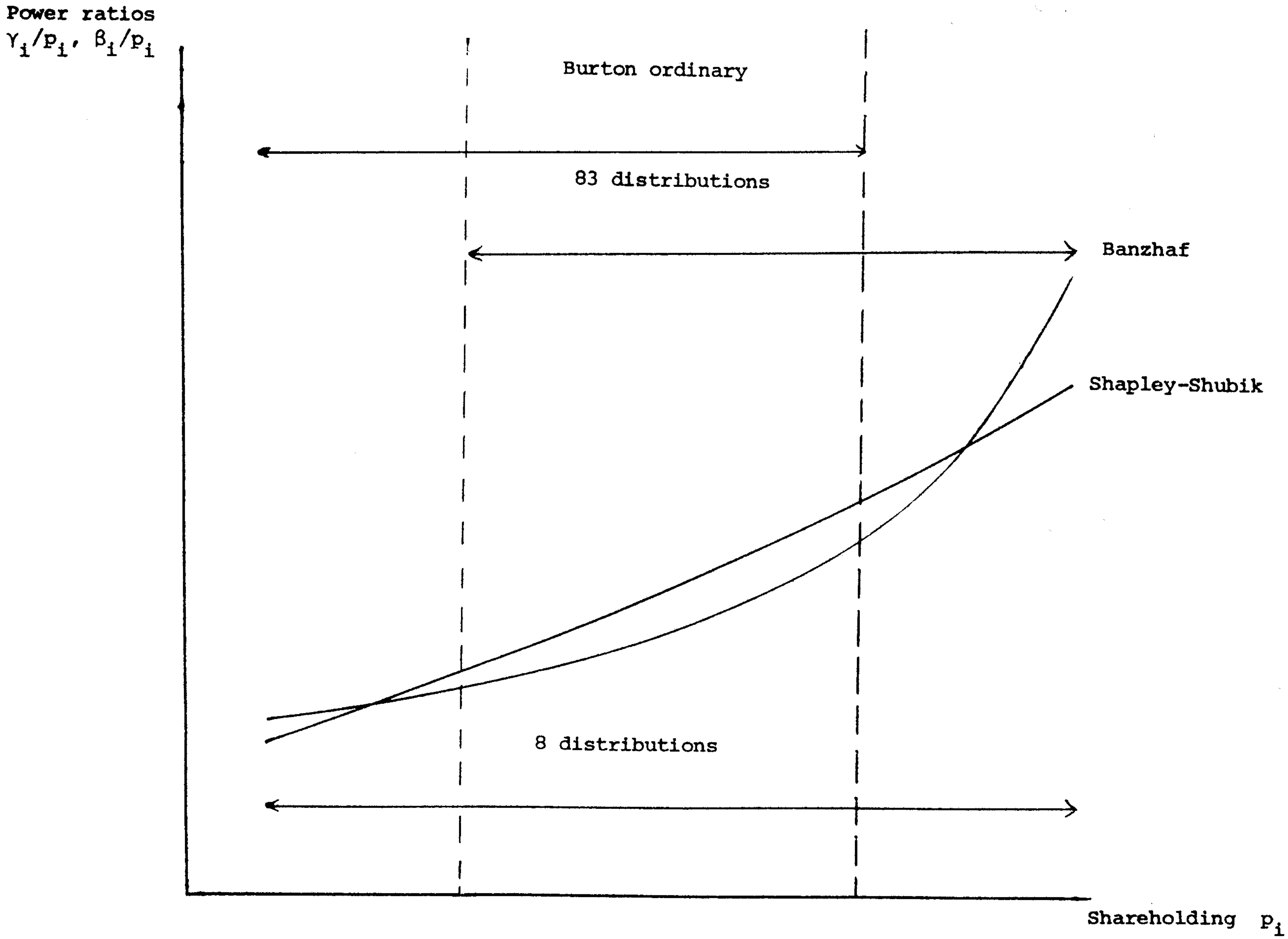


Figure 2. Classification of patterns of power indices

Figure 3(a)

SHAPLEY VS BANZHAF POWER RATIOS: EMI

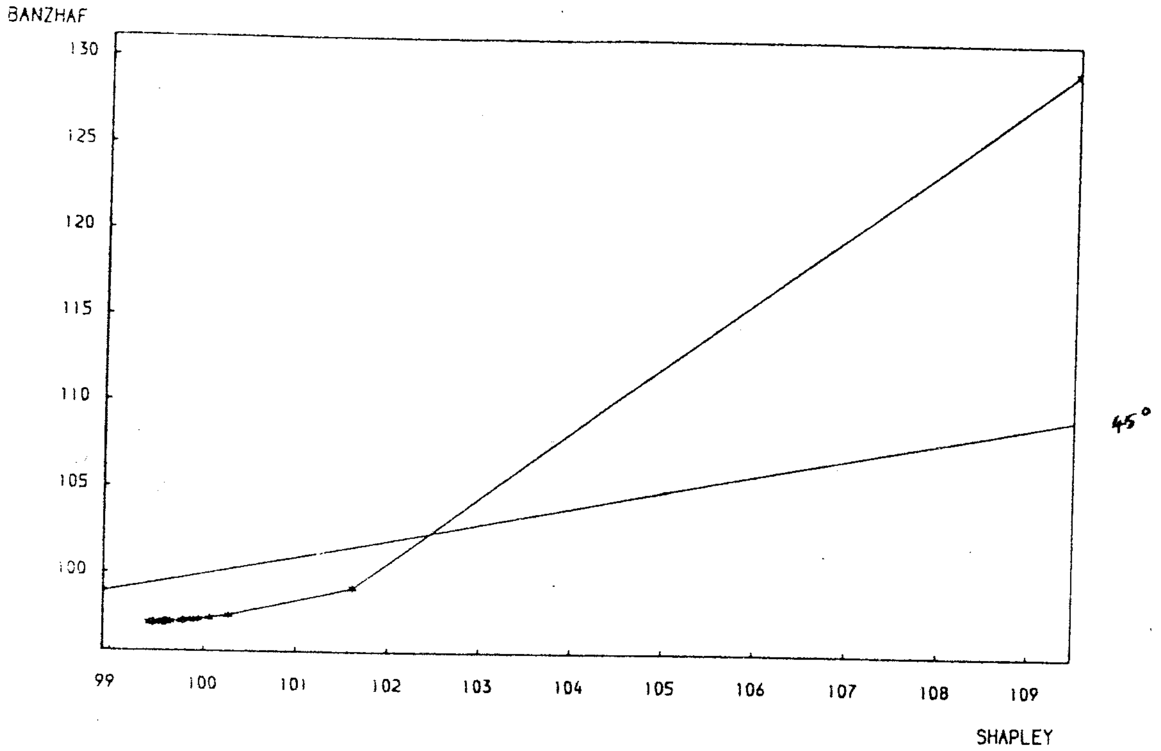


Figure 3(b)

SHAPLEY VS BANZHAF POWER RATIOS: WILLIAM PRESS

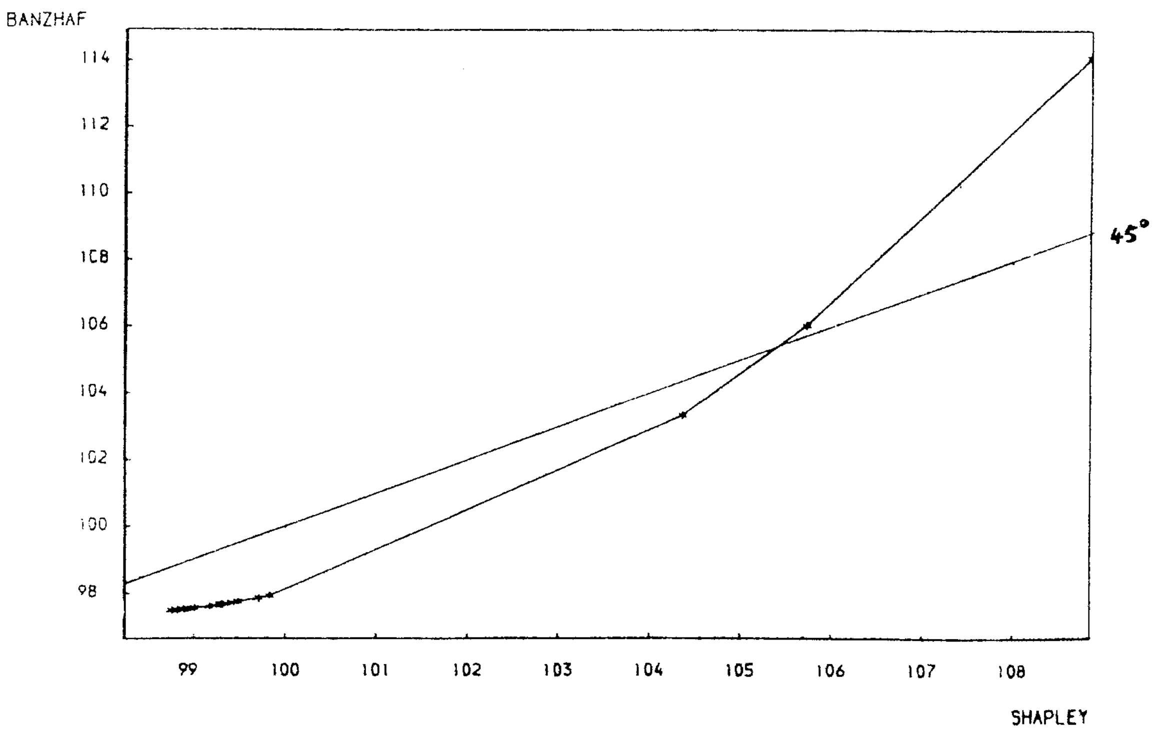


Figure 3(c)

SHAPLEY VS BANZHAF POWER RATIOS: COURTAULDS

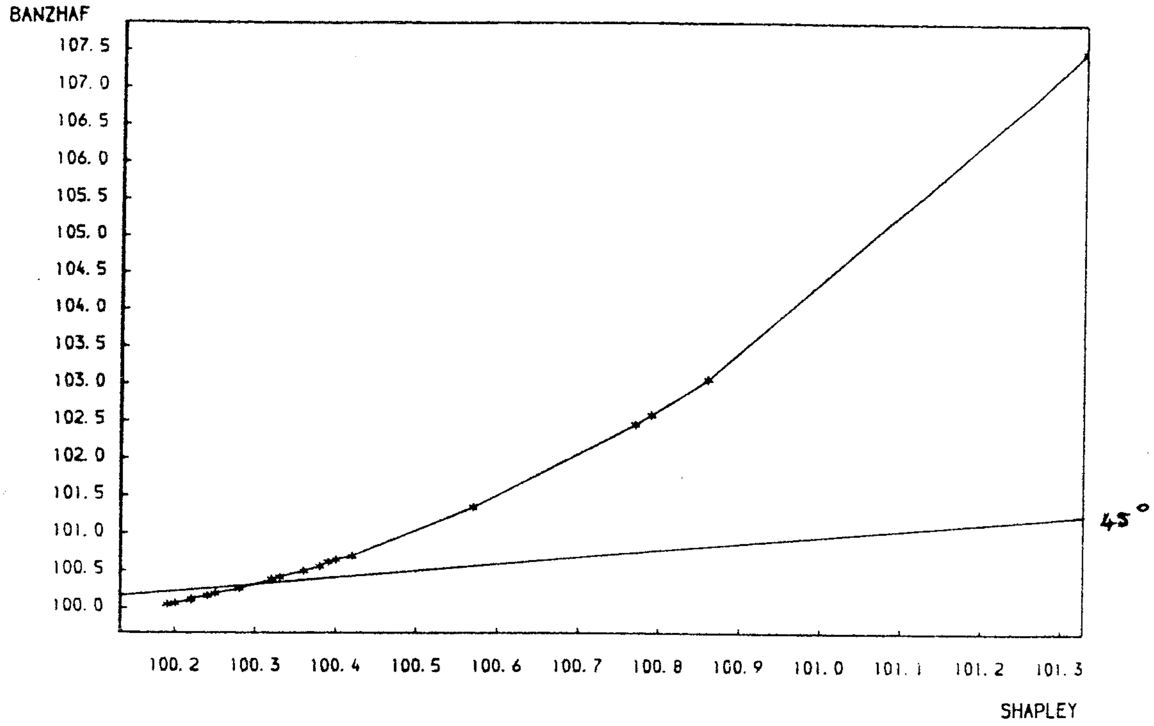


Figure 3(d)

SHAPLEY VS BANZHAF POWER RATIOS: FITCH LOVELL

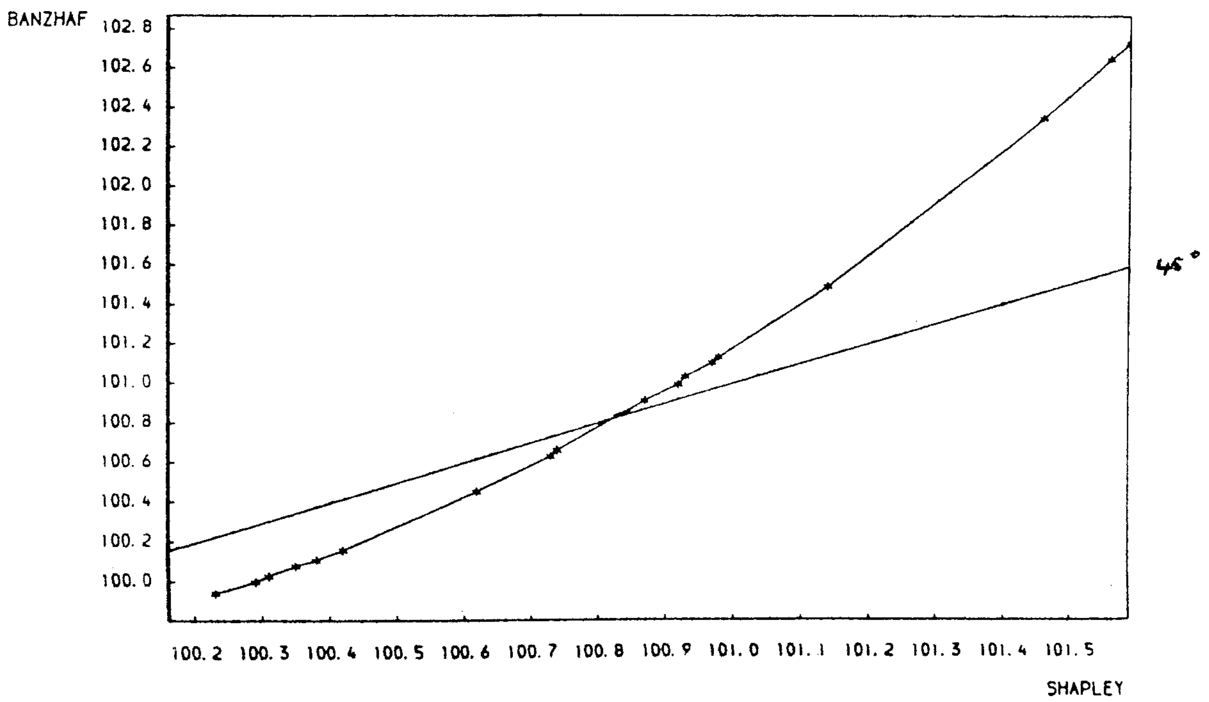


Figure 3(e)

SHAPLEY VS BANZHAF POWER RATIOS: NOTTINGHAM MANUF ORDINARY

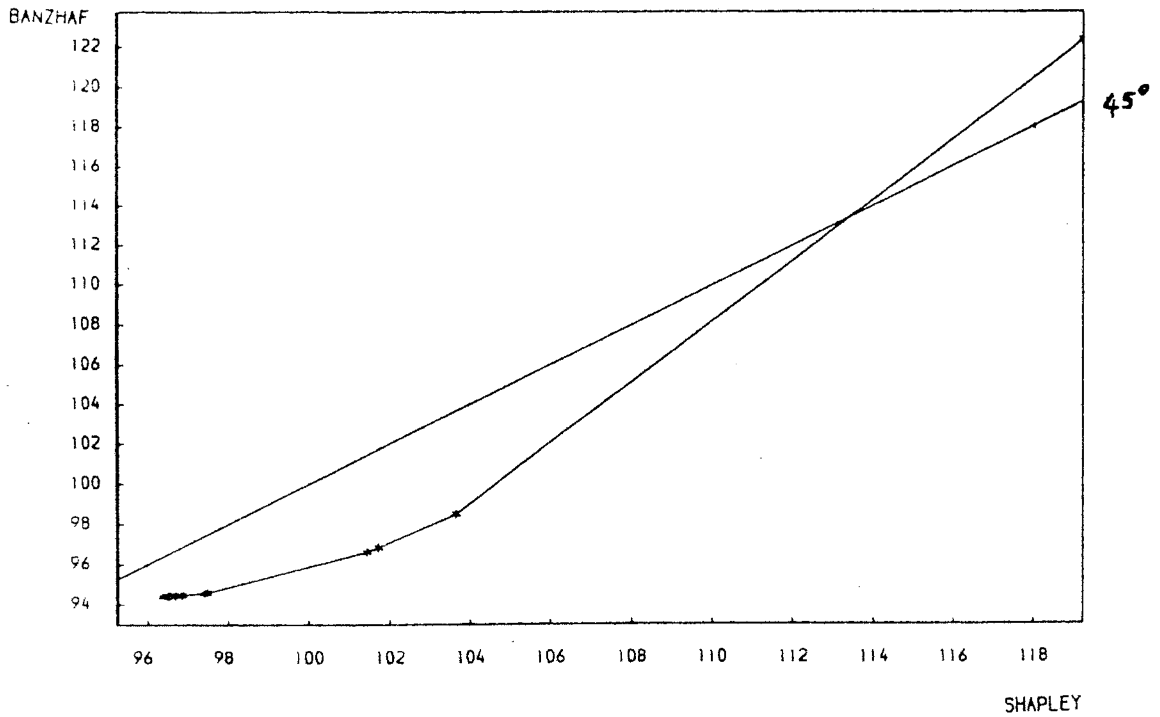
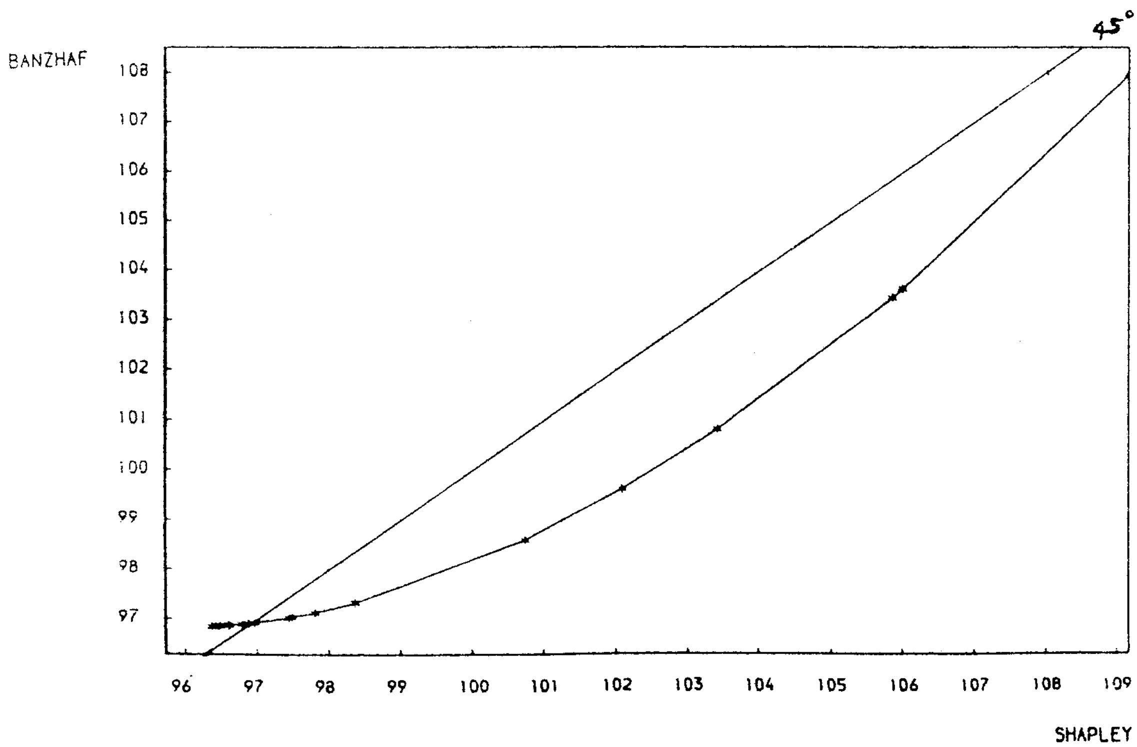


FIGURE 3(f)

SHAPLEY VS BANZHAF POWER RATIOS: BURTON ORDINARY



concentrated than ownership for all the distributions and that this effect increases with ownership concentration. There is more variability in power concentration as measured by the Banzhaf index than the Shapley-Shubik index.

Figure 5 shows the plot of the Herfindahl concentration indices against each other. The Banzhaf index gives rise to a slightly more concentrated distribution than the Shapley-Shubik index in all cases but one (Burton ordinary).

Figure 6 shows results for the largest holding of each distribution only. In both diagrams the power ratio is plotted against weight. That is, in Figure 6a, γ_1/p_1 is plotted against p_1. In Figure 6b β_1/p_1 is plotted on the vertical axis. There is a very sharp difference in the appearance of the two scatters. The Shapley-Shubik power ratio is closely related to weight while the Banzhaf index is highly variable around a clear relationship.

The power index for the largest shareholder depends upon the size of holding and the concentration of the remainder of the distribution. Regressions of the power ratios plotted in Figure 6 on these two variables are shown below. The concentration of the distribution excluding p_1 is measured by the appropriate Herfindahl index defined by $H_1 = (H - p_1^2)/(1-p_1)^2$ where $H = \Sigma p_i^2$.

$$(\gamma_1/p_1) = \underset{(0.0003)}{0.9967} + \underset{(0.00005)}{0.0113}\ p_1 - \underset{(0.03)}{1.15}\ H_1$$

$$R^2 = 0.9985$$

$$(\beta_1/p_1) = \underset{(0.0081)}{1.0641} + \underset{(0.00150)}{0.0190}\ p_1 - \underset{(0.89)}{7.14}\ H_1$$

$$R^2 = 0.6708$$

(standard errors are in parentheses. Degrees of freedom equsls 79). These results show a very high degree of explanation of variation in the Shapley-Shubik power ratio in terms of R^2 and a much lower R^2 for the Banzhaf power ratio. The fitted regression show a stronger association between the Banzhaf power ratio and p_1. The Banzhaf power ratio is also much more strongly affected by the concentration variable than the Shapley-Shubik.

Figures 7, 8 and 9 provide other comparisons of the two power indices. Figure 7 is a plot of the power ratio β_1/p_1 against γ_1/p_1. In every case except one (Burton ordinary), the Banzhaf power ratio exceeds the Shapley-Shubik and there is a general positive association, with a correlation coefficient of 0.70. Figure 8 provides a comparison of a measure of the extent to which each distribution redistributes power. The power discrepancy plotted in Figure 8 is the extent to which power is redistributed from the twentieth shareholder to the first. The power discrepancy is defined as the difference between the power ratio for $i = 1$ and for $i = 20$. Thus on the vertical axis the Banzhaf power discrepancy is $\beta_1/p_1 - \beta_{20}/p_{20}$ and the Shapley-Shubik power discrepancy on the horizontal axis is defined as $\gamma_1/p_1 - \gamma_{20}/p_{20}$. For all distributions both power discrepancies are positive and for all except Burton Ordinary the Banzhaf exceeds the Shapley-Shubik power discrepancy. The correlation coefficient is 0.67.

An alternative global description of the results for a particular distribution is the number of holdings for which the power index exceeds the weight. This varies over quite a large range for both indices (for the Shapley-Shubik index between 1 and 85 and for the Banzhaf index between 1 and 22) and the results are plotted in Figure 9. There is a positive correlation and the number given by the Shapley-Shubik index always exceeds the number given by the Banzhaf index.

The general result which emerges is that for all companies power is somewhat more highly concentrated than shareholding. The Banzhaf index tends to give a slightly higher concentration than the Shapley-Shubik with a characteristic pattern of a higher index for the former for a smaller number of large holdings.

A final aspect of the analysis is the relationship between the concentration of power relative to shareholding and the size of the company. Figure 10 shows plots of the power discrepancy defined above against sales for each company (only one distribution for each company has been used). These give correlation coefficients of 0.27 and 0.37 respectively. These figures, however, are affected by a large outlier in both cases. It is clear that there is no evidence of a simple relationship between the power discrepancy and company size as measured by sales.

FIGURE 4a

SHAPLEY HERFINDAHL V WEIGHT HERFINDAHL

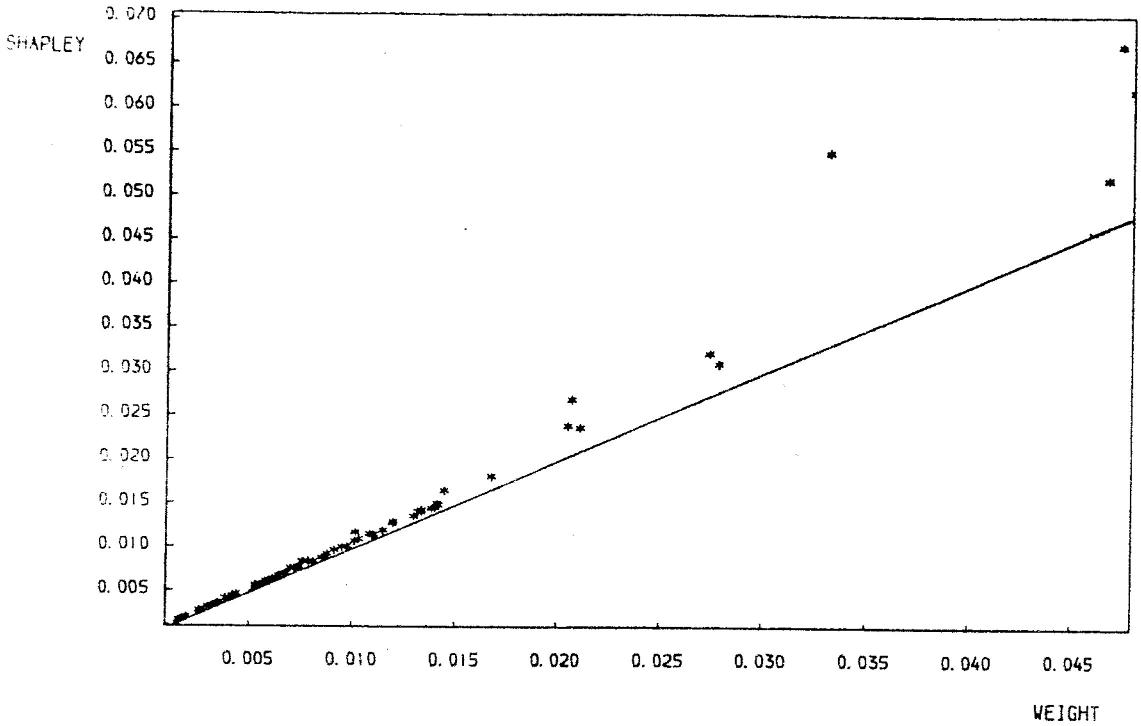


FIGURE 4b

BANZHAF HERFINDAHL V WEIGHT HERFINDAHL

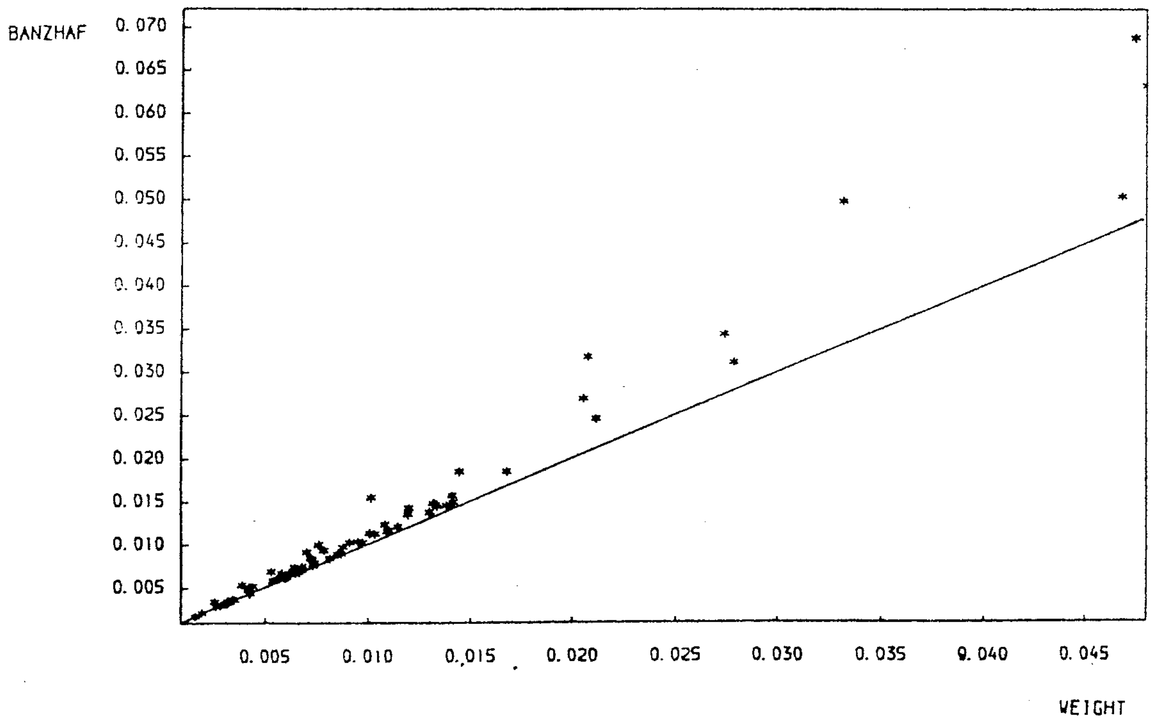


FIGURE 5

PLOT OF BANZHAF HERFINDAHL V SHAPLEY HERFINDAHL

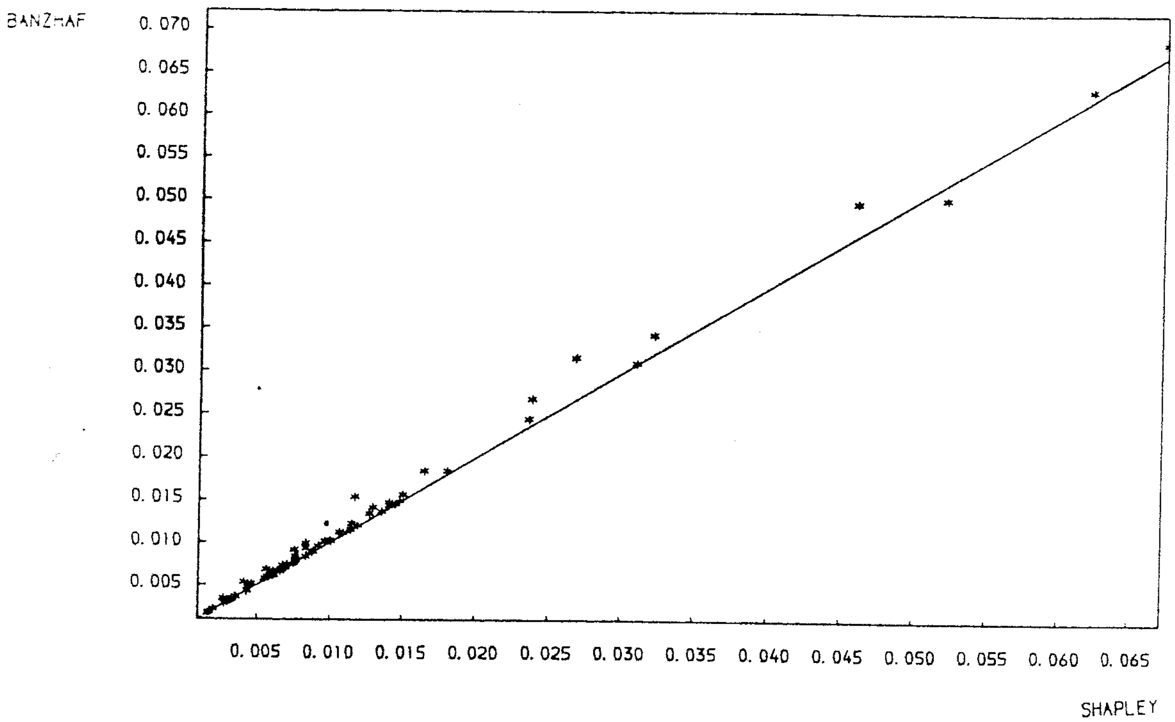


FIGURE 6a

SHAPLEY-SHUBIK INDEX: POWER RATIO vs WEIGHT

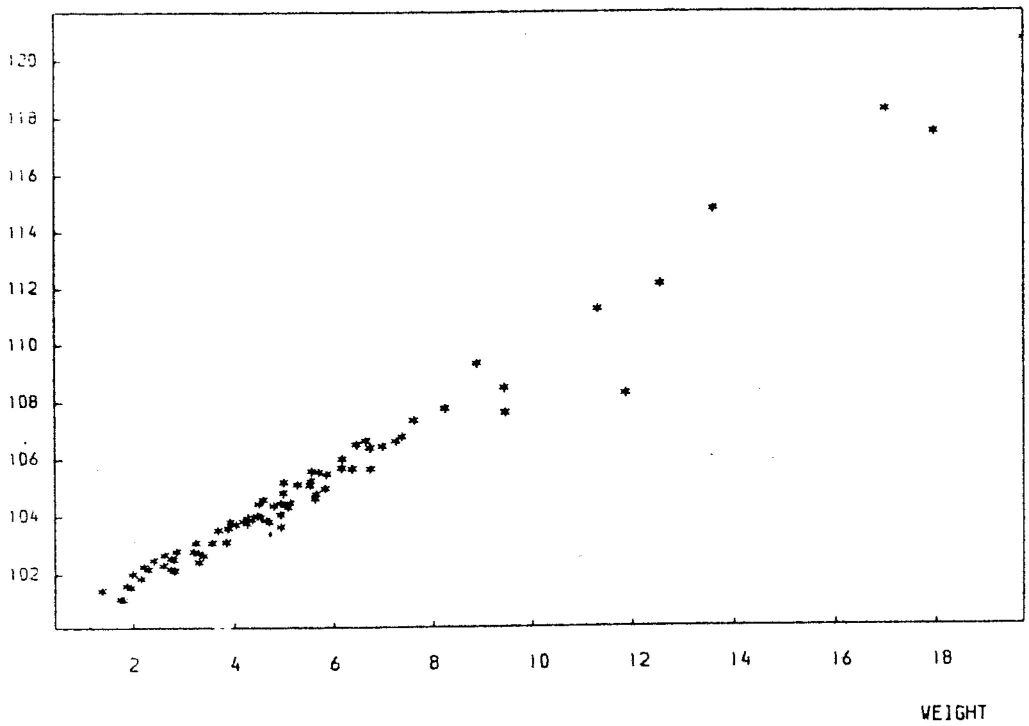


FIGURE 6b

NORMALISED BANZHAF INDEX: POWER RATIO (B1/P1) vs WEIGHT (P1)

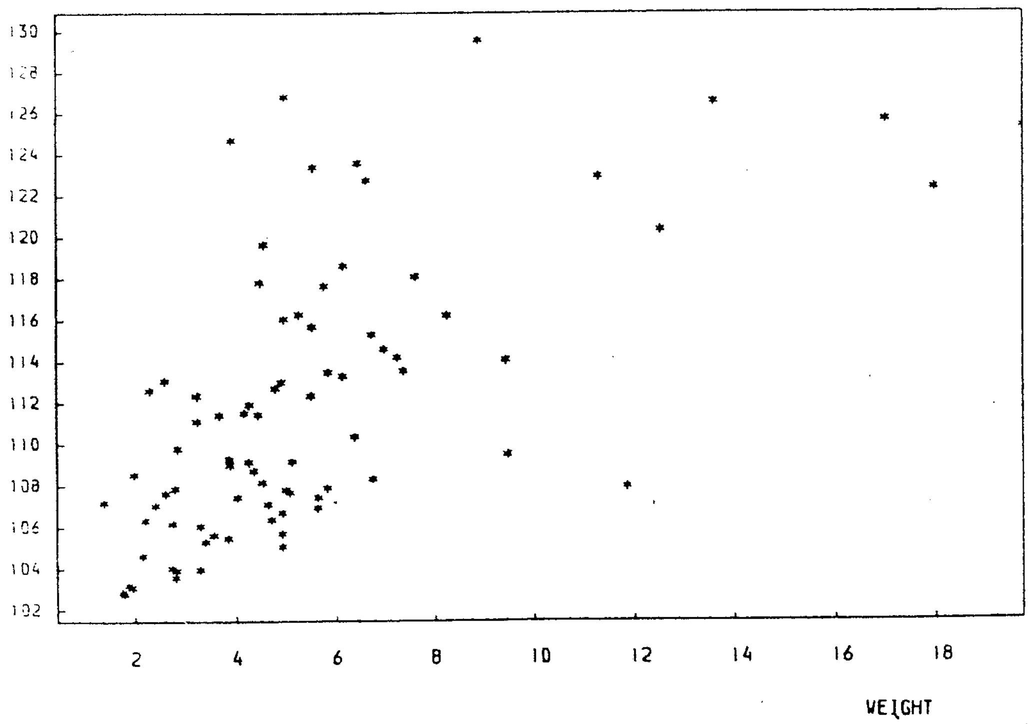
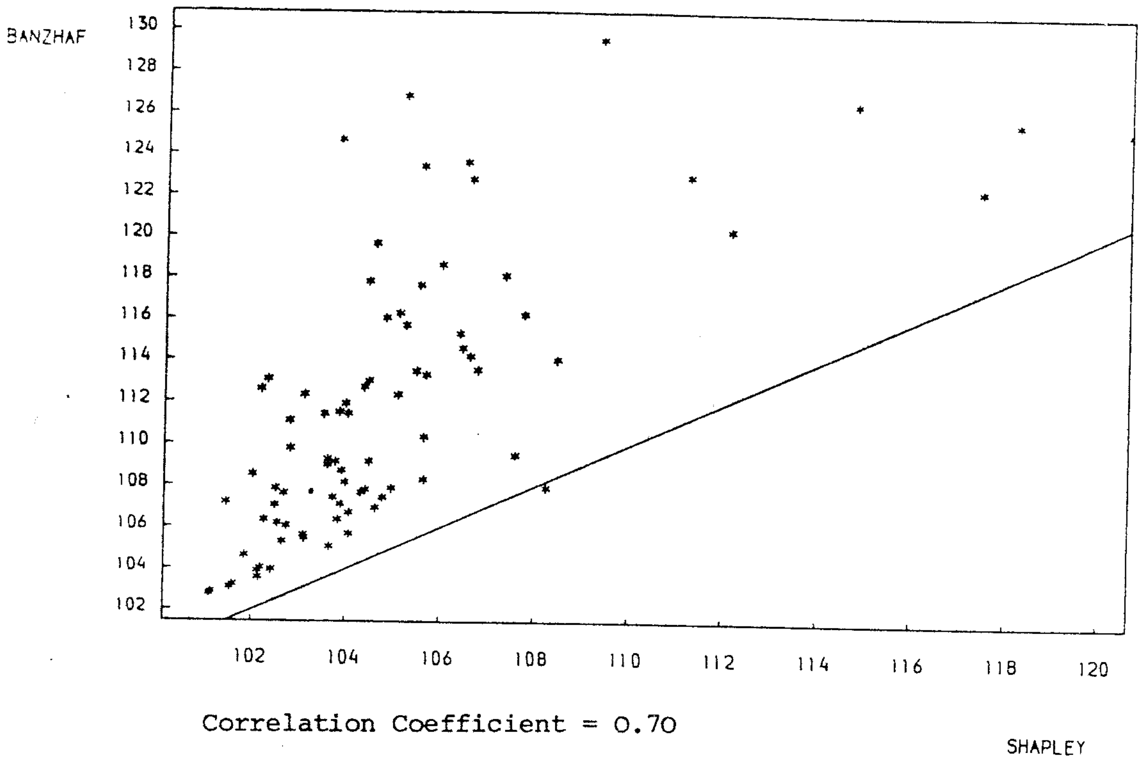


FIGURE 7

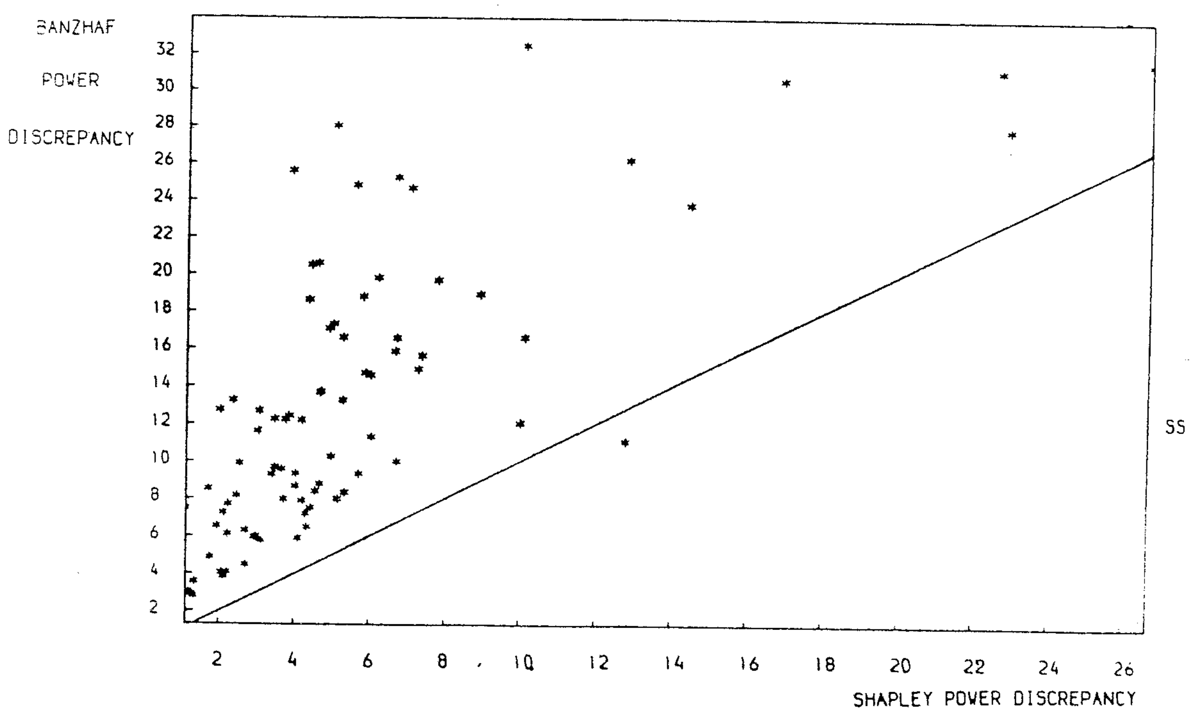
POWER RATIOS: B1/P1 vs Y1/P1



Correlation Coefficient = 0.70

FIGURE 8

BANZHAF POWER DISCREPANCY VS SHAPLEY POWER DISCREPANCY



Correlation Coefficient = 0.67

FIGURE 9 : Number of Holdings with Power Ratio > 1

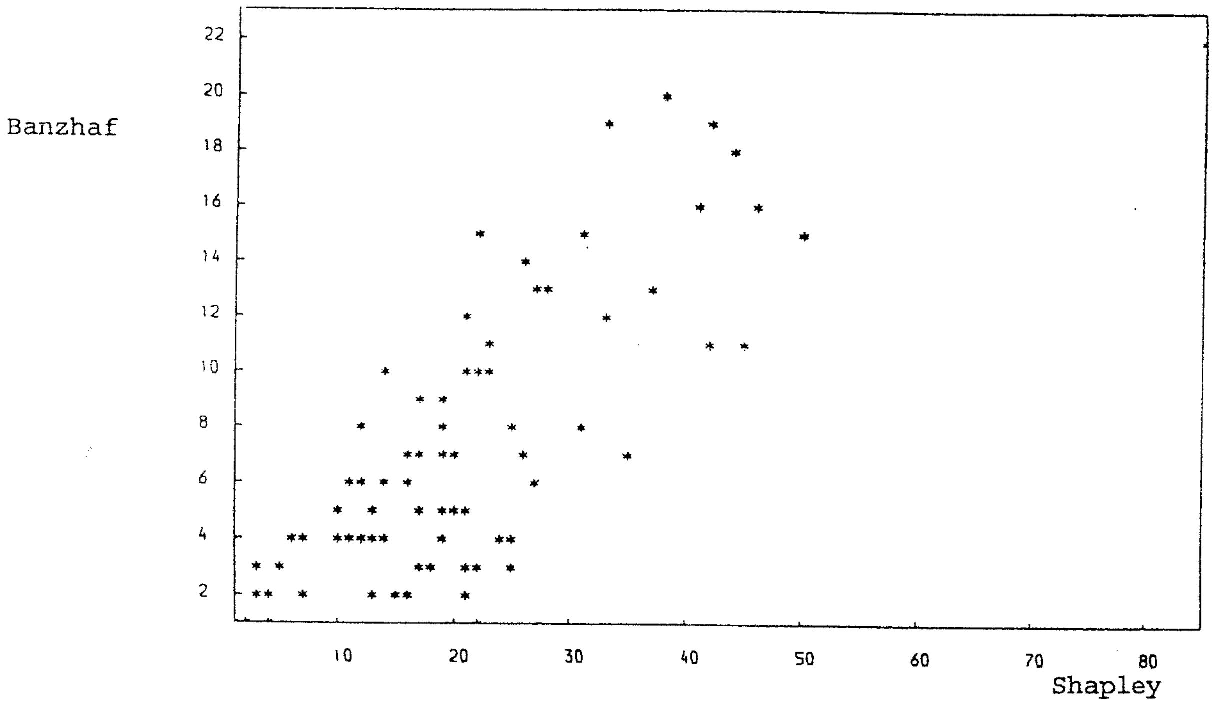
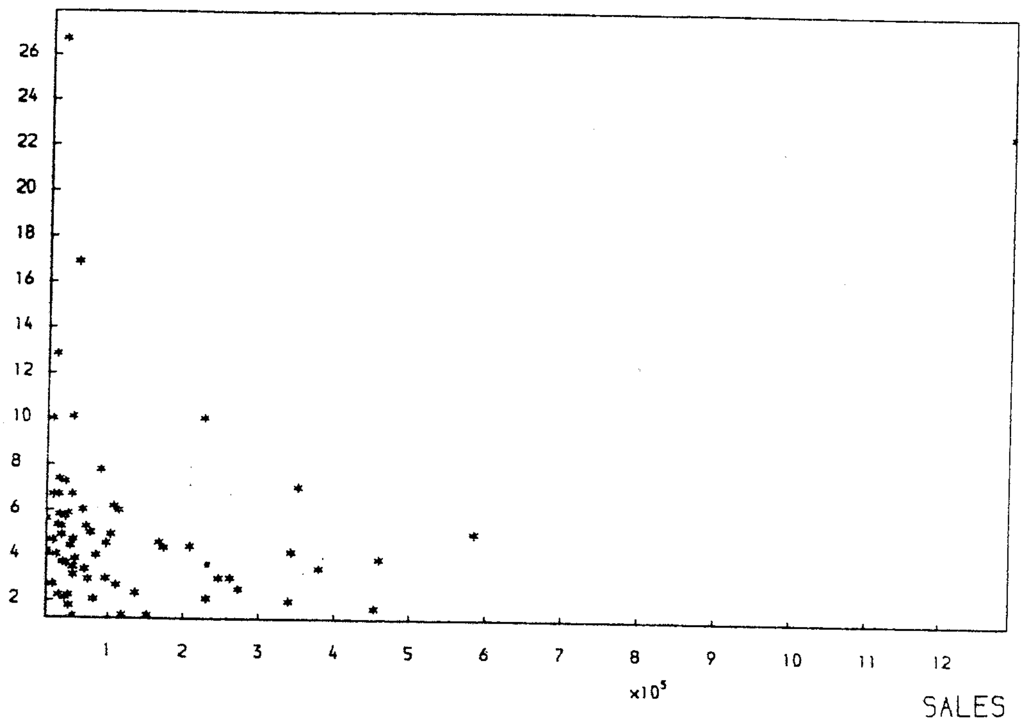


FIGURE 10a

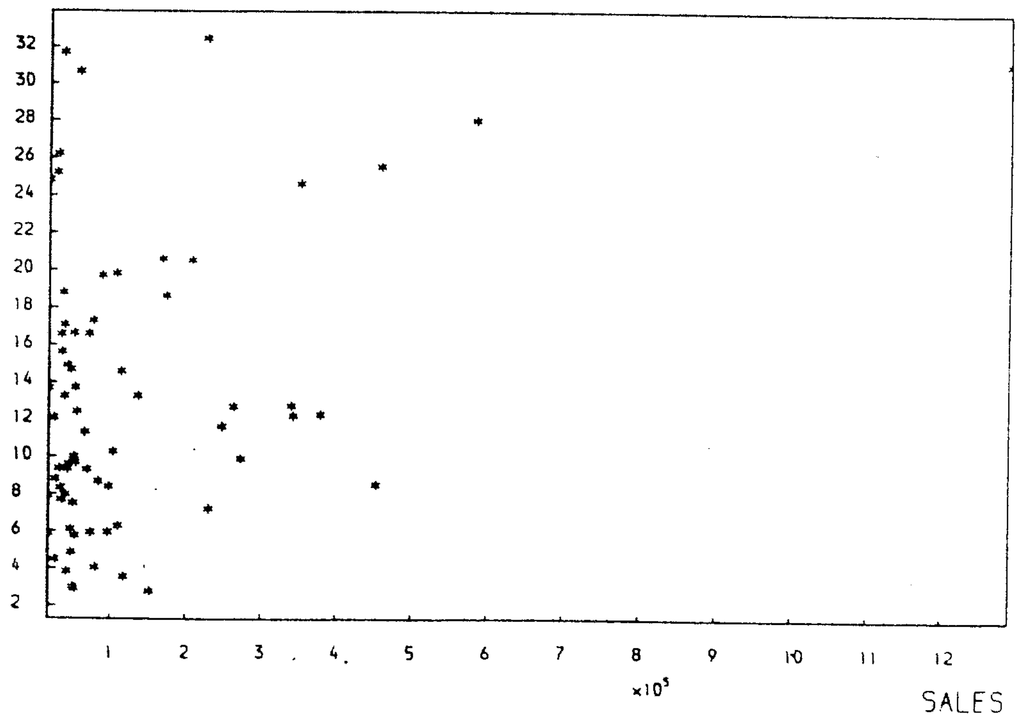
SHAPLEY POWER DISCREPANCY VS SALES



Correlation Coefficient = 0.27

FIGURE 10b

BANZHAF POWER DISCREPANCY VS SALES



Correlation Coefficient = 0.37

9. Summary and Conclusions

This paper has used the method of power indices for simple games to examine the relationship between the distribution of shareholder voting power and ownership concentration in a sample of British companies. Approximation methods based on multi-linear extensions of games have been applied, apparently successfully in most cases, to data on the upper tail of each shareholding distribution. While bounds have been obtained for the Shapley-Shubik index only a lower bound has been obtained for the Banzhaf index. The limit theorem which has been used as the basis of computing an upper bound on the Banzhaf index has proved of little empirical value.

The results obtained show:

(i) All shareholding distributions analysed have the property that voting power is more concentrated than ownership. Both indices show that the share of voting power of the bulk of smaller holdings is somewhat less than their share of ownership while for a number of larger shareholdings their power index exceeds their ownership.

(ii) The characteristic pattern observed for all distributions is that the Shapley-Shubik power ratio is much closer to being a linear function of shareholding than the Banzhaf power ratio.

(iii) In all cases except one the Banzhaf power ratio exceeds the Shapley-Shubik power ratio for a small number of large holdings. In some cases this excess is quite large.

(iv) In terms of the whole distribution (the upper tail of the shareholding size distribution observed), in most cases (83 distributions) the Shapley-Shubik index exceeds the Banzhaf index for the relatively small shareholdings. Hower, in 8 cases the Banzhaf index exceeds the Shapley-Shubik index for both relatively small and very large holdings while the ranking is reversed over an intermediate range. In one case the former exceeds the latter for relatively small holdings and is less than it for the largest holdings.

(v) An examination of the power indices for the largest single holding shows that the Shapley-Shubik power ratio is quite closely correlated with the size of that holding. On the other hand the Banzhaf power ratio for the largest holding is only weakly associated with the size of the holding. A regression of the Shapley-Shubik power ratio for the largest holding on size of holding and a measure of concentration gives an extremely high degree of explanatory power. By contrast the fit of the corresponding regression for the Banzhaf power ratio is much worse although the regression coefficients are larger, indicating a greater sensitivity of the Banzhaf index to the characteristics of the shareholding size distribution.

(vi) An analysis of the respective numbers of shareholdings whose power exceeds their weight, according to the two indices, shows that this number is greater according to the Shapley-Shubik index in every case.

(vii) An analysis of the association between the power discrepancy (a measure of the redistribution of power from the twentieth largest to the largest shareholder) and company sales shows no relationship.

APPENDIX : The sample of Shareholding Distributions

Company and type of shares:	No. of observations
Associated Biscuits Ordinary	129
Associated Biscuits A ordinary	137
Associate Dairies	111
Associated Engineering	140
Associated Fisheries	192
Aveys	123
Babcock and Wilcox	129
Baker Perkins	122
Bassett	163
Bibby	156
BICC	120
Birmid Qualcast	112
Bovril	120
British Electric Traction	124
British Ropes	139
Brockhouse	127
Brooke Bond Liebig	315
John Brown	113
BSA	133
Burton Ordinary	125
Burton A ordinary	165
Cadbury Schweppes	323
Cammel Laird	127
Cavenham	136
Chloride	132
Chubb	147
Clark Chapman	143
George Cohen	124
Courtaulds	197
Delta Metal	117
Dowty	124
Drake and Cubitt	153
Duport	147
B Elliott	117
EMI	265
English Calico	145
Ever Ready	116
Express Daries A Ordinary	128
Fairey	121
Firth Cleveland	108
Fitch Lovell	129
FMC	117
GEC	306
Gill and Duffins	146
GKN	127
Glynwed	127
Haden	138
Hawker Siddeley	132
Alfred Herbert	186
Illingworth Morris Ordinary	119

Company and type of	No. of observations
Illingworth Morris A ordinary	115
George Kent	128
Johnson Matthey	126
F H Lloyd	130
London Merchant Securities	115
J Lyons Ordinary	156
J Lyons A Ordinary	153
Manbre and Garton	237
Mann Egerton	111
Mather and Matt	118
Melbray	107
Morgan Crucible	103
Norcross	152
Northern Dairies	174
Nottingham Manufacturing Ordinary	139
Nottingham Manufacturing A Ordinary	150
Pegler Hattersley	122
Plessey	175
Powell Duffryn	103
William Press	154
Rank Hovis McDougall	290
Renold	126
Reyrolle Parsons	163
Rowntree Mackingtosh	172
Selincourt	109
Simon Engineering	144
Smithfield and Zwanenberg	121
Staveley	154
Stone Platt	122
Swans	152
Tate and Lyle	143
Thorn Electrical	187
Thomas Tilling	146
Tube Investments	145
Unigate	228
Unilever	155
Vickers	118
Thomas Ward	154
Westinghouse	123
Whessoe	114
Woolcombers	145
Wrights	154

Footnotes:

1/ For example the US electoral college, Owen (1975a).

2/ See Shapley (1962).

3/ Shapley and Shubik (1954), Banzhaf (1965, 1968), Owen (1978).

4/ Shapley (1953). The three axioms are: symmetry (equal indices for players of equal weight); efficiency (indices sum to unity over players and therefore provide a distribution of power); additivity (the distribution of power in two independent games is the same as that obtained by evaluating the two games separately).

5/ A player's power depends on the frequency with which he is able to affect the outcome as the last member to join an ordered coalition which becomes minimal-winning. The model of coalition formation assumed is a legislature in which, for a given bill, members may be arranged in order of support. The bill's sponsors, in organising a minimal majority, are assumed to enlist the support of members sequentially beginning with the most supportive. In order to recruit less supportive members, the sponsors must bargain with them and pay a price in terms of amending the bill or supporting other measures. The highest price will be paid to the marginal member or "pivot" who is by definition the least supportive member of the winning coalition. This seems a not-unreasonable model of coalition formation in many applications. But it assumes a single ranking ofmembers and there will in general be many such attitudinal dimensions. In constructing the power index it is necessary to allow for all of them.

By treating all orderings of the n members as equiprobable, the Shapley-Shubik index is implicitly assuming that the number of attutudinal dimensions in the legislature is precisely $n!$ It is clear that, even for small legislative bodies, the number of such rankings is truly enormous. It seems more reasonable to assume that the number of issues on which legislators take up positions is determined independently of the size of the legislature. Moreover, statistical analyses of actual voting bodies have revealed that votes on concrete issues can be expressed in terms of a small number of attitudinal dimensions (see the references quoted by Brams). See also Riker (1964).

6/ The Banzhaf indices are not subject to the criticism made in the previous footnote. They are defined in terms of critical defections from minimal-winning coalitions without regard to their order of formation. In applying power indices to shareholdings the question of which index is the more appropriate can be thought of as depending in part on whether shareholders can be ranked along a very large number of attitudinal dimensions or whether we should treat all coalitionsas equally likely.

7/ See Owen (1975b) for a comparison of the results for the US electoral college.

8/ See Dubey and Shapley (1979) and Shapiro and Shapley (1978). The limiting behaviour of the Shapley-Shubik indices in a stock-holder-voting game with two large holdings has been analysed by Milnor and Shapley (1978).

9/ The measure of power employed by Cubbin and Leech and by Leech (1984, 1985) is the degree of control, α, defined for the largest shareholding bloc only, as the probability of majority support for that holding in an explicit model of probabilistic voting which allows abstentions. Disallowing abstentions gives the relation between the degree of control and the Swing Probability as $\beta_1' = 2\alpha-1$.

10/ These results are based on the assumption that there are no "pitfall" points for which the limiting Banzhaf indices for the major players are zero. A finite number of these points may occur at which the number of minor swings becomes so numerous that the relative number of swings for each major player goes to zero. This problem is assumed to be unimportant empirically.

11/ For every company ordinary shareholdings were analysed. In some cases where there were more than one type of ordinary share each distribution was analysed separately. The total number of shareholding distributions analysed (after excluding two companies in the sample which had a majority shareholding and after amalgamating two ordinary share distributions for each of GEC and Rank Hovis MacDougall) was 92.

12/ The power ratio is taken as the relevant measure of "bias" in voting. It should be noted that for both indices and for all deistributions their simple correlations with shareholdings and with each other are extremely high, in excess of 0.98.

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