ON RANDOM PREFERENCES, FUTURE FLEXIBILITY AND THE DEMAND FOR ILLIQUID ASSETS (REVISED)

Clive D. Fraser

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On Random Preferences, Future Flexibility and the Demand for Illiquid Assets (Revised)[1]

Synopsis:

Goldman's model of flexibility and portfolio choice with random preferences is extended to the case of two assets alternative to money. The asset highest yielding to maturity, but least flexible, is ex ante divisible but ex post indivisible. This asset is always chosen alongside money initially, irrespective of whether or not it is redeemable prematurely. When it is not so redeemable, transparent restrictions on asset prices are found to ensure that the other non-money asset is also held initially, and with a positive probability of premature redemption. When it is prematurely redeemable, these same restrictions ensure positive probability of this occurring.

Clive D. Fraser
Department of Economics
University of Warwick
Coventry CV4 7AL
England

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I: Introduction and Summary

Perhaps the most important and basic problem which arises in monetary theory is that of justifying the coexistence of money, with zero yield, and alternative higher yielding assets in an individual portfolio. The explanations which have been offered for this phenomenon fall into two broad categories: those which emphasize the randomness of assets' returns and those which emphasize the randomness of the investor's preferences or future investment opportunities, and hence, the importance of the ease with which he can convert an asset to cash (its flexibility). Into the first category can be placed the contributions of Markowitz ([10]) and subsequent researchers such as Tobin ([13]) and Arrow ([1]). In the second category, a more recent departure, is included the work of Goldman ([16]), ([17]), Taylor ([12]) and Baldwin and Meyer ([2]).

While both approaches essentially reduce to justifications for liquidity preference, and thus, their demarcation line is not always crystal clear in practice,[2/] the second seems a more primitive, inclusive and satisfactory approach for at least two reasons. Firstly, it subsumes consideration of both the traditional transaction costs ([3]) and precautionary ([7]),([12]), motives for holding money. For example, both Goldman ([17] and Taylor ([12]) emphasize the uncertainty associated with the individual's ordering of intertemporal consumption bundles in order to examine in great detail his choice at the margin between money and an alternative higher yielding asset where precisely these considerations are important. Secondly, for many assets, the nominal yield if held to maturity or any given premature redemption date is known ex ante. One only need mention certain life insurance policies or pension schemes. What the individual with a given portfolio normally cannot anticipate with precision are the circumstances which will actually lead him to redeem prematurely those assets which would have been highest yielding if held to maturity. It is thus the possibility of such premature redemption at a different (usually penal) rate which induces ex ante randomness of returns on assets other than money, and thus, encourages diversification. Here, therefore, randomness of preferences is logically prior to randomness of assets' returns.

In this paper we will persevere with the second approach in order to explore some aspects of the individual's choice at the margin between a standard interest bearing bond and an ex ante higher yielding, but less flexible, asset in the presence of money. This introduces new, interesting and important features. For example, while the assumption that there are positive costs associated with premature redemption of each of the non-money assets remains sufficient to ensure that money is held in the optimal portfolio, "plunging" into money and the ex ante highest yeilding asset to the exclusion of the standard bond, or infinite indebtedness via initial sales of the standards, now cannot be ruled out a priori. This last is a possibility which slips away when the choice of assets is restricted to money and one other. Technically, this is particularly important within the context of the consumer's lifetime allocation process. Here, as Yaari ([14]; e.g., p.137, p.146) and others have recognized, in the presence of assets with the characteristics to be introduced below, one wants to be sure that the individual who is not explicitly constrained period-by-period, but is subject only to an overall lifetime solvency constraint, will not incur infinite indebtedness in any period with the result that his problem possesses no bounded unconstrained solution. Again, the extension to two non-money instruments enables us to trichotomize the individual's portfolio as comprising money-held partly for transactional, and partly for precautionary purposes - standard bonds - held purely for precautionary reasons - and the highest yielding asset, held purely for "speculative" purposes. "Plunging" would now correspond to a situation where the individual can somehow always satisfy all his precautionary needs by holding only cash balances alongside the highest yielding asset. The standard bond is therefore squeezed from his portfolio, although he might issue debt and hold negative amounts.

One interesting and important feature of the highest yielding asset (called "actuarials" for short) [3/] which we will consider is that it is ex ante perfectly divisible but ex post indivisible. Thus, if premature redemption of holdings of it are permitted and desired by the individual, it must be redeemed in entirety at the penal rate. Such ex post indivisibility adds another important dimension to the notion of "illiquidity" and is characteristic of rather a wide range of both real and financial assets, for example housing and insurance policies. Friend and Blume ([5], p.911) have noted that it is precisely this characteristic which makes such assets not fit easily into the convenient dichotomy of riskless and risky assets, and thus, that "some other mode of analysis (than that of the conventional mean-variance or random returns framework) is required to handle this type of asset".

The model which we will use is inspired by the investment-consumption models of Goldman ([6]) and Taylor ([12]). It is, however, more general than the two, not only because it expands the asset set from two to three dimensions for the reasons given above, but also because we expand the set of trading opportunities by permitting additional purchases of standard bonds when and if the individual revizes his portfolio. Also, unlike Taylor's model, it does not restrict the individual's utility function to be of the Stone-Geary class with randomness entering only through the subsistence component. It is less general than that of Goldman([7]) in that this considers a continuum of assets. However, it is more general in that Goldman there employs a Leontief-type utility function. More for expositional ease than of strict necessity, our restriction here is to the class of homothetic utility functions.

The model formally has three, but effectively only two, periods. Period 0 is the initial portfolio decision period. Period 1 is the possible portfolio revision and first consumption period. Period 2 is the terminal liquidation and consumption period.

We will use this model and the above specification of assets principally to demonstrate the following : (a) provided a bounded solution to the investment-consumption problem exists, the individual will <u>always</u> choose initially to hold some of the ex ante highest yielding asset. (Thus, the individual will always "speculate".) This is despite the possibility that future circumstances might arise which force its premature redemption at the penal rate when this is possible. This feature of the optimal solution probably partly accounts for the fact, recorded in Diamond ([4], p.282), that 20% of insurance-type assets are redeemed within two years of being purchased; (thus) (b) The individual will never adopt a "wait and see" attitude - i.e. he will never hold only money while waiting for uncertainty about future circumstances, and hence preferences, to be resolved, despite the intermediate period's trading opportunities which we introduce; (c) While "plunging" cannot always be excluded a priori, sufficient conditions - involving the configuration of premature redemption rates and initial asset prices - can be found to ensure that positive amounts of the standard bond are held in the individual's optimal portfolio; (d) When the highest yielding asset is prematurely redeemable, virtually these same conditions are sufficient to ensure that there is a positive probability of its premature redemption.

Section II of the paper outlines the formal model of ex ante and ex post trading opportunities, random preferences, and consumer maximization. III presents the results. IV concludes with a discussion of possible extensions and additional implications.

II: A Formal Model

(i) Trading opportunities

Suppose the individual with initial monetary endowment W possesses the option of investing in period 0 in: infinitely divisible standard bonds (B) priced at $\rho \leq 1$ per unit, yielding a gross return 1 in period 2; an actuarial bond (A_1) infinitely divisible ex ante, but not at all ex post (i.e., at any potential time of redemption the amount of actuarial bonds initially purchased must be redeemed in toto, or not at all). If, ex ante, the desired gross yield in actuarial bonds is A_2^a, this amount can be secured by a premium $A_1(A_2^a) \leq A_2^a$. For simplicity, we posit a linear relationship between (gross) yield and premium: $A_1(A_2^a) = \kappa A_2^a$, $\kappa \varepsilon (0,1)$, $A_2^a \geq 0$. Should the individual wish to surrender the actuarial bonds prematurely (on an all-or-nothing basis), at the start of Period 1 it has a surrender value $\tau A_1(A_2^a) = \tau\kappa A_2^a$, $\tau \varepsilon [0,1)$. [4/] In our three-period world, the individual is not permitted to purchase actuarial bonds after period 0. Should the individual wish to redeem any proportion (≤ 1) of his initial holdings of standard bonds, this can be done only during Period 1 for a per unit price of $\delta_1\rho . \delta_1\rho \leq \rho \leq 1$. The individual cannot, however, borrow once his initial set of decisions has been made although, should he discover that, in the light of his revised preferences, he possesses more cash than now desired, he can lend any excess by investing in additional standard bonds, priced at $\delta_2\rho$, $\rho \leq \delta_2\rho \leq 1$, and yielding a gross return of 1 delivered in Period 2. (The case where the individual cannot invest in first-period bonds is formally equivalent to $\delta_2\rho = 1$.) To reflect the institutional constraint that selling and buying prices of first-period bonds differ to the extent of transaction costs, at least ("the middle-man takes his cut"), we posit $\delta_1\rho \leq \delta_2\rho$. Money (M) "costs" unity per unit, and yields unity gross. To conceptualize more concretely the monetary consequences of flexibility, we assume that ex ante per unit asset yields to maturity are inversely-related to premature redemption yields, thus $0 < \kappa \leq \rho \leq 1$, $\tau \leq \delta_1 \leq 1$ (and naturally, $\tau_1\kappa \leq \delta_1\rho \leq 1$), hence "money" is the perfectly flexible asset.

Of course, our specification of assets as it stands leaves us open to the valid charge that the positive initial demand for money is guaranteed by stipulating penalties for the premature redemption of the other assets. This is a deficiency shared with most similar models and we indicate how it might be circumvented in the conclusions.

The individual makes his initial portfolio decision according to the ex ante budget constraint

$$W \geqq \rho B^a + \kappa A_2^a + M^a \qquad (II.1)$$

where, if C_i^a, i=1,2, are, respectively, his ex ante planned consumption in the first and second periods, then $C_1^a = M^a$, $C_2^a = B^a + A_2^a$. The individual holds a diversified portfolio, possibly involving positive amounts of both second period bonds, not only in order to take advantage of the highest yielding bond, but also as a precautionary measure in case, ex post, circumstances associated with his realized, initially random, preferences should cause him to seek to consume more than he originally planned for and he wishes to mitigate the consequences, in terms of foregone first and second period consumption, associated with having to redeem the actuarial bond at the penal rate. Similar reasoning, vis-à-vis the standard bond, applies to the holding of money which, by our assumption, is undominated. Making the term of borrowing (by selling standard bonds in period zero) of one period duration from period zero is a sufficient way of imposing institutional constraints upon infinite borrowing to finance investment in the ex ante highest yielding bond, given that further borrowing during Period 1 is not permitted. The individual will not borrow money to hold as money, would not seek to borrow to invest in the standard bond (by definition), nor would he seek to borrow until his money holdings are exhausted. No consumption occurs during period 0. [5/]

Non-satiation ensures that (II.1) holds with strict equality.

(ii) Randomness of Preferences

We maintain the same assumptions on the individual's preferences as in Goldman's earlier paper ([6] (A.1)-(A.4)): the individual possesses a (bounded) intertemporal utility function $U(C_1,C_2,\omega)$, where ω is the generic argument referring to the underlying ex ante unforeseen, but foreseeable, eventualities which result in the randomness of the individual's preferences. $\omega \varepsilon [0, 1]$ and possesses density $f(\omega)$.

Also: (A.1) $U(\cdot)$ is strictly concave and homothetic in C_1 and C_2 and continuous in ω ;

Letting

$$R(\theta,\omega) \triangleq \frac{U_1(C_1,C_2,\omega)}{U_2(C_1,C_2,\omega)}$$

where $\theta = C_2/C_1$, also assume :

(A.2) $\forall\omega,\ R(\theta,\omega)\ \varepsilon\ (0,\ \infty)$ for $\theta\ \varepsilon (0,\ \infty)$;

(A.3) $\forall\omega,\ R(\infty,\omega) = \infty,\ R(0,\omega) = 0$;

(A.4) $\forall\theta$, $R(\theta,\omega)$ is continuous in ω and $\exists\,\omega_i,\omega_j$, s.t. $R(\theta,\omega_i) \neq R(\theta,\omega_j)$.

(iii) Ex post Opportunities

For any initial portfolio decision $(A_2^a, B^a, M^a = W-\rho B^a - \kappa A_2^a)$ in period 0 , the individual's ex post trading opportunities are represented in figure (1). Letting $C_i(\omega)$, i=1,2 , refer to the ex post optimal consumption decisions in the light of realized preferences and initial portfolio decision, we can formally represent these opportunities and the resulting ex post optimal decisions by the following six alternative regimes of behaviour (see Figure 1) :

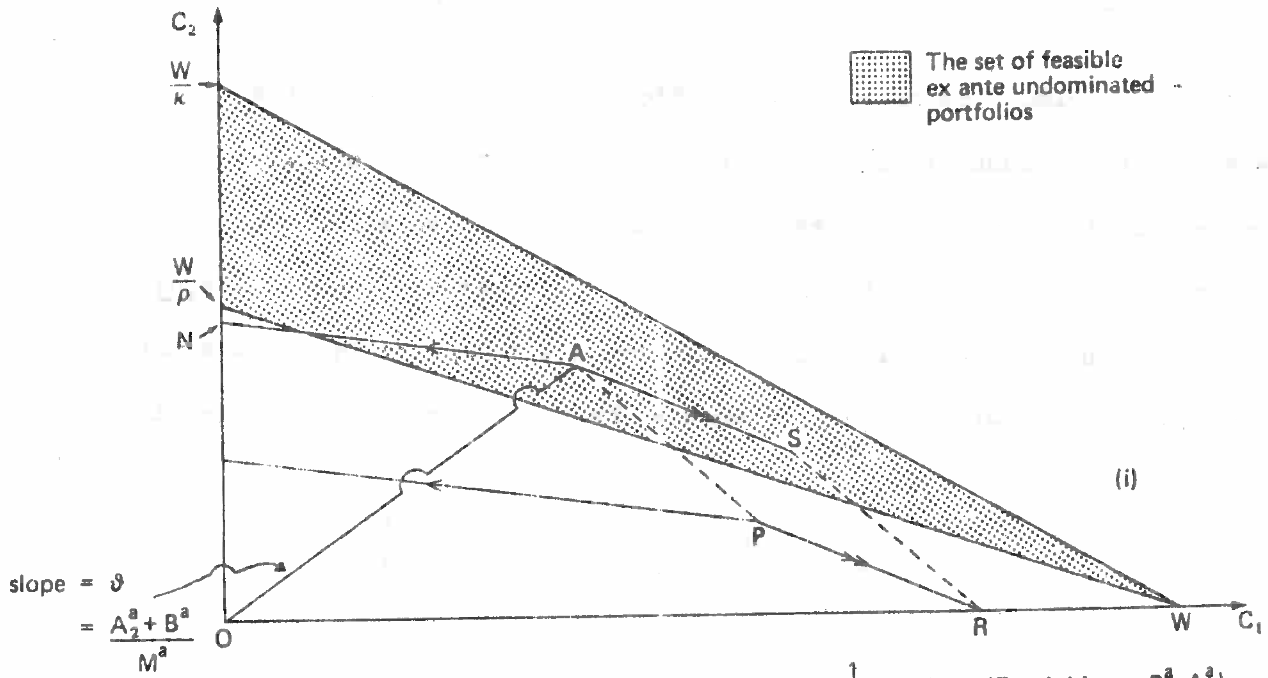
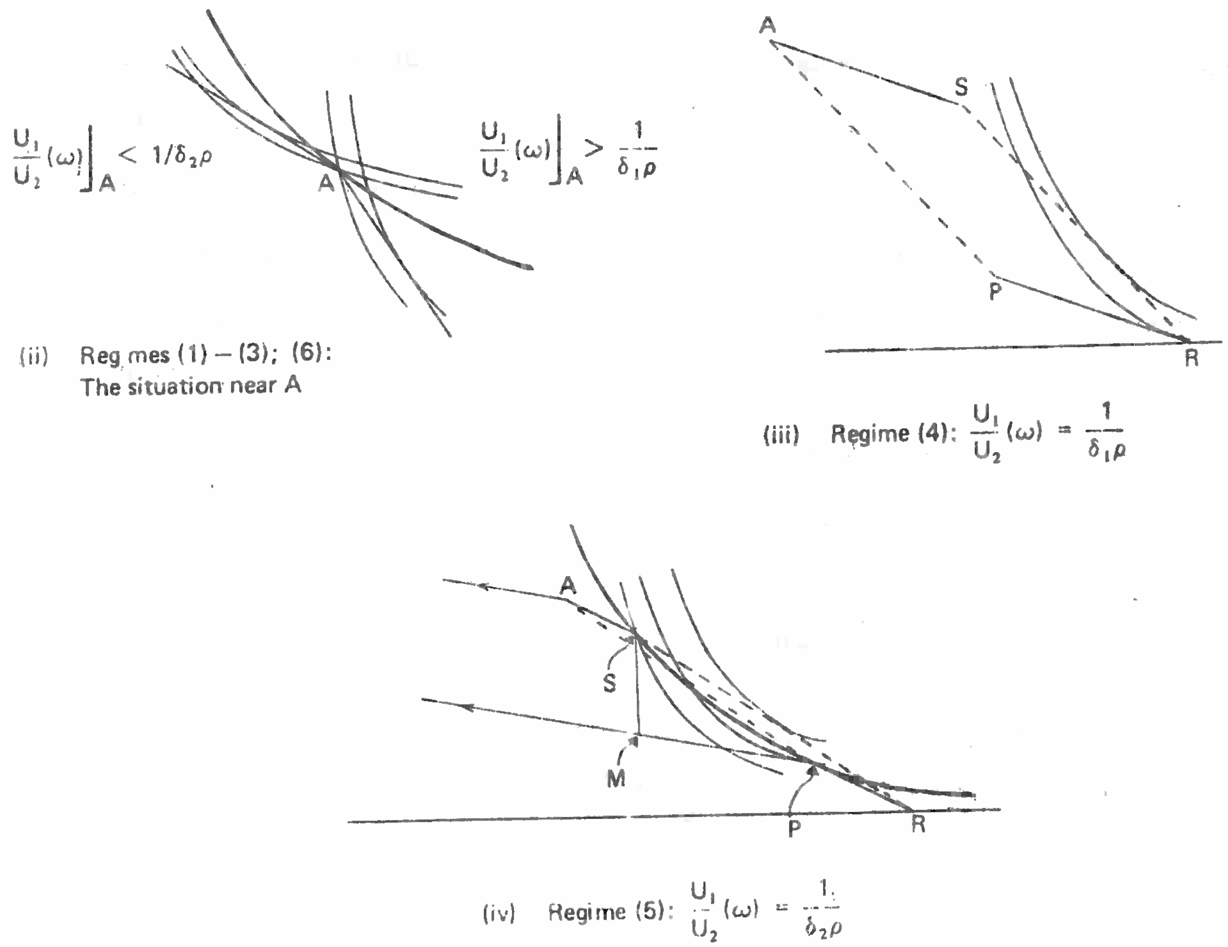


Figure 1: Ex Post Budget Possibilities with $1/\tau\kappa > \frac{1}{\delta_1\rho} > 1/\delta_2\rho$ (For Arbitrary B^a, A_2^a)



1) $C_1(\omega) = M^a$, thus $C_2(\omega) = B^a + A_2^a = \dfrac{B^a(\kappa-\rho) + W-M^a}{\kappa}$

and the individual remains at A in Figure 1. This situation arises when , w.r.t. Figure 1 (ii), the individual's highest attainable ex post indifference curve passing through the initial allocation point (A) has an MRS with modulus within the interval $[1/\delta_2\rho\ ,\ 1/\delta_1\rho]$ at A. It is thus not possible for him to improve his situation by reallocating consumption between the two periods now, hence, his portfolio of assets;

2) $C_1(\omega) < M^a, \rightarrow\ C_2(\omega) = B^a + A_2^a + \left(\dfrac{M^a - C_1(\omega)}{\delta_2\rho}\right)$

$$= \frac{B^a(\kappa-\rho)\delta_2\rho + M^a(\kappa-\delta_2\rho) + W\delta_2\rho - C_1(\omega)\kappa}{\kappa\delta_2\rho},$$

and the individual trades up his ex post budget constraint along AN of $|\text{slope}|\ 1/\delta_2\rho$. C_2/C_1 increases ex post. This situation occurs when, w.r.t. Figure 1 (ii) again, the individual's highest attainable ex post indifference curve has an $|\text{MRS}| < 1/\delta_2\rho$ at A. His position can be improved , therefore, by moving left from A to a point of tangency along AN. This necessitates the purchase of additional first period bonds. All the other regimes which follow can be understood by an extension of the same type of argument:

3) $M^a < C_1(\omega) \leq \text{Min.}(M^a + \tau\kappa A_2^a\ ,\ M^a + \delta_1\rho B^a)$,

the individual redeems the necessary amount of standard bonds, equal to $(C_1(\omega) - M^a)$ in value, and $\dfrac{C_1(\omega) - M^a}{\delta_1\rho}$ in number, thus

$$C_2(\omega) = A_2^a + B^a - \frac{C_1(\omega) - M^a}{\delta_1\rho} = \frac{B^a(\kappa-\rho\ )+W-M^a}{\kappa} + \left(\frac{M^a-C_1(\omega)}{\delta_1\rho}\right)$$

$$= \frac{B^a(\kappa-\rho)\delta_1\rho + M^a(\kappa-\delta_1\rho) + W\delta_{1\rho} - C_1(\omega)k}{\kappa\delta_1\rho},$$

and the individual trades down from A along AS with $|\text{slope}| = 1/\delta_1\rho$. C_2/C_1 decreases;

4) $C_1(\omega) \geqq \text{Max.}(M^a + \tau\kappa A_2^a,\ M^a + \delta_1\rho B^a)$, implying the individual <u>must</u> redeem the actuarial bond, (assuming that both categories of bond were originally held) plus the requisite amount of standard bonds. $\frac{C_1(\omega) - M^a - \tau\kappa A_2^a}{\delta_1\rho}$ of the latter are redeemed, thus

$$C_2(\omega) = B^a - \left(\frac{C_1(\omega)-M^a- \tau\kappa A_2^a}{\delta_1\rho}\right) = \frac{B^a(\delta_1-\tau)\rho+M^a(1-\tau)+\tau W-C_1(\omega)}{\delta_1\rho}$$

and the individual trades down from A, beyond P, along PR with $|\text{slope}| = 1/\delta_1\rho$; C_2/C_1 decreases.

5) $C_1(\omega)\ \varepsilon\ I(A) = (\text{Min.}(M^a+ \tau\kappa A_2^a, M^a+\delta_1\rho B^a),\ \text{Max.}(M^a+\tau\kappa A_2^a, M^a+\delta_1\rho B^a))$, if $L(A) \nRightarrow \phi$.

If $M^a+\tau\kappa_2^a > M^a+\rho\delta_1 B^a$, the individual <u>must</u> redeem the actuarial bond and then reinvest the difference between this plus M^a and $C_1(\omega)$ in the first-period standard bonds. Thus

$$C_2(\omega) = B^a + \left(\frac{C_1(\omega)-M^a-\tau\kappa A_2^a}{\delta_2\rho}\right) = \frac{B^a(\delta_2+\tau)\rho-M^a(1-\tau)+C_1(\omega)}{\delta_2\rho}.$$

The individual trades down from A to P , then up along PM with $|slope| = 1/\delta_2 \rho$; C_2/C_1 decreases;

6) $C_1(\omega) \in I(A)$, $M^a + \rho\delta_1 B^a > M^a + \tau\kappa A_2^a$. Then the individual redeems the requisite amount of standard bonds, $\dfrac{C_1(\omega)-M^a}{\delta_1\rho}$; thus

$$C_2(\omega) = A_2^a + B^a - \left(\frac{C_1(\omega)-M^a}{\delta_1\rho}\right)$$

$$= \frac{B^a(\kappa-\rho)\delta_1\rho + M^a(\kappa-\delta_1\rho) + \delta_1\rho W - \kappa C_1(\omega)}{\kappa\delta_1\rho}$$

and the individual trades down from A along the line AS with $|slope| = 1/\delta_1\rho$, C_2/C_1 decreases.

Regime (6) differs from Regime (3) only insofar as (in obvious notation), $C_1^6(\omega) > C_1^3(\omega)$ $(>M^a)$ and, relative to Regime (5), the premature redemption rule : that the individual redeems first the bond which has the least opportunity cost in terms of foregone Period 1 consumption, and only then the less flexible bond should the money raised by the former's sale prove insufficient, needs to be brought into operation.

The ex post indivisibility of the actuarial bond introduces a non-convexity into the individual's ex post budget set. This manifests itself in the fact that, w.r.t. Figure 1 (iii), e.g., points along SR are not available to the individual. This raises the possibility of a multiple solution occurring where the indifference contour for a particular realization of ω just touches S and is either tangential to PM or just touches P in Figure 1 (iv). However, this causes no additional complexity because, for any initial portfolio allocation, such a multiple solution could only occur on an ω set of measure zero.

Dominance (non-satiation) means that a point such as P in Figure 1 (i) will not be considered in the set of ex post opportunities while S is available.

Finally, the assumptions on preferences ensure that $C_i(\omega)>0, i=1,2.$

Some Additional Notation

Let $\theta = \frac{C_2^a}{C_1^a} = \frac{A_2^a+B^a}{M^a} = \frac{A_2^a+B^a}{W-\kappa A_2^a-\rho B^a}$. For any choice M^a and B^a

(and hence θ and A_2^a), define the sets $\Lambda^i(\theta) = \{\omega |$ ex post, the individual falls within regime $i\}$, $i=1,\ldots,6$. Thus (again in obvious notation),

$$\Lambda^1(\theta) = \{\omega | \frac{U_1}{U_2}(\omega)\Big|_A \in [1/\delta_2\rho, 1/\delta_1\rho]\}$$

$$\Lambda^2(\theta) = \{\omega | \frac{U_1}{U_2}(\omega)\Big|_A < \frac{1}{\delta_2\rho}\};$$

$$\Lambda^3(\theta) = \{\omega | \frac{1}{\tau\kappa} > \frac{U_1}{U_2}(\omega)\Big|_A > \frac{1}{\delta_1\rho} \text{ and } (C_1(\omega)-M^a) \leq \delta_1\rho B^a\};$$

$$\Lambda^4(\theta) = \{\omega | \frac{U_1}{U_2}(\omega)\Big|_A > \frac{1}{\delta_1\rho} \text{ and } (C_1(\omega)-M^a) \in (\tau\kappa A_2^a, \delta_1\rho B^a+\tau\kappa A_2^a), \tau\kappa A_2^a > \delta_1\rho B^a\};$$

$$\Lambda^5(\theta) = \{\omega | U_{\frac{1}{U_2}(\omega)}\Big|_A > \frac{1}{\delta_1\rho} \text{ and } (C_1(\omega)-M^a)\varepsilon(\delta_1\rho B^a , \tau\kappa A_2^a) \};$$

$$\Lambda^6(\theta) = \{\omega | U_{\frac{1}{U_2}(\omega)}\Big|_A > \frac{1}{\delta_1\rho} \text{ and } (C_1(\omega)-M^a)\varepsilon(\tau\kappa A_2^a , \delta_1 \rho B^a)\}.$$

Finally, we let $\frac{C_2}{C_1}(\omega) = \theta^{(i)}, \ \omega \ \varepsilon \ \Lambda^i(\theta), \ i=1,\ldots,6.$

Of course, these $\theta^{(i)}$ are only of significance notationally. With money and <u>two</u> other assets, it is obviously the actual portfolio composition which is important in determining ex post opportunities.

(iv) <u>Individual Maximization</u>

Using the above notation, for any initial choice (M^a,B^a), thus $(\theta,\theta^{(i)}, i=2,\ldots,6)$ made by the individual, we have:

$$\begin{aligned}
&\text{(i)} \quad \omega \ \varepsilon \ \Lambda^1(\theta) \rightarrow C_1(\omega) = M^a, \ C_2(\omega) = B^a + A_2^a = \frac{W-M^a+B^a(\kappa-\rho)}{\kappa};\\
&\text{(ii)} \quad \omega \ \varepsilon \ \Lambda^2(\theta) \rightarrow C_1(\omega) = \frac{M^a(\kappa-\delta_2\rho)+\delta_2\rho\left[B^a(\kappa-\rho)+W\right]}{\left[1+\theta^{(2)}\delta_2\rho\right]\kappa}, \ C_1(\omega) = \theta^{(2)}C_1(\omega);\\
&\text{(iii)} \quad \omega \ \varepsilon \ \Lambda^3(\theta) \rightarrow C_1(\omega) = \frac{M^a(\kappa-\delta_1\rho)+\delta_1\rho\left[B^a(\kappa-\rho)+W\right]}{\left[1+\theta^{(3)}\delta_1\rho\right]\kappa}, \ C_2(\omega) = \theta^{(3)}C_1(\omega);\\
&\text{(iv)} \quad \omega \ \varepsilon \ \Lambda^4(\theta) \rightarrow C_1(\omega) = \frac{M^a(1-\tau)+\tau W+B^a(\delta_1-\tau)\rho}{\left[1+\delta_1\rho\theta^{(4)}\right]}, \ C_2(\omega) = \theta^{(4)}C_1(\omega);\\
&\text{(v)} \quad \omega \ \varepsilon \ \Lambda^5(\theta) \rightarrow C_1(\omega) = \frac{M^a(1-\tau)+B^a(\delta_2-\tau)\rho+\tau W}{\left[1+\theta^{(5)}\delta_2\rho\right]}, \ C_2(\omega) = \theta^{(5)}C_1(\omega);\\
&\text{(vi)} \quad \omega \ \varepsilon \ \Lambda^6(\theta) \rightarrow C_1(\omega) = \frac{B^a(\kappa-\rho)\delta_1\rho+M^a(\kappa-\delta_1\rho)+W\delta_1\rho}{\left[1+\theta^{(6)}\delta_1\rho\right]\kappa}, \ C_2(\omega) = \theta^{(6)}C_1(\omega).
\end{aligned} \qquad \text{(II.2)}$$

For any initial portfolio choice, the individual's expected utility is given by

$$EU = \Sigma_i \int_{\Lambda^i(\theta)} U[C_1, C_2, \omega] \; f(\omega)d\omega \qquad \text{(II.3)}$$

which he is assumed to seek to maximize by the appropriate choice of initial portfolio subject to (II.2) (and where, of course, $\bigcup_i \Lambda^i(\theta) = [0,1]$).

Ex post, for realised ω, he chooses $C_i(\omega)$ to maximise $U[C_1, C_2, \omega]$ subject to the constraints imposed by the initial portfolio choice, and thus, the possible regimes of behaviour outlined above.

For ease of comparison with Hool ([8]) and Goldman ([6]), we will initially consider the first order necessary conditions (FONC) for B^a and M^a for the maximization of EU ((II.3)) w.r.t. (II.2). In the sequel we will primarily be concerned with those for A^a and B^a, in which case (II.2) will have to be modified slightly. The former FONC are:

$$\text{(i)} \quad \frac{\partial EU}{\partial M^a} = \int_{\Lambda^1(\theta)} \left\{ U_1 - U_2 \left(\frac{1}{\kappa}\right) \right\} f(\omega)d\omega \; +$$

$$+\int_{\Lambda^2(\theta)} \left\{ \frac{U_1(\kappa-\delta_2\rho)}{[1+\theta^{(2)}\delta_2\rho]\kappa} + \frac{U_2\theta^{(2)}(\kappa-\delta_2\rho)}{[1+\theta^{(2)}\delta_2\rho]\kappa} \right\} f(\omega)d\omega + \int_{\Lambda^3(\theta)} \left\{ \frac{U_1(\kappa-\delta_1\rho)}{[1+\theta^{(3)}\delta_1\rho]\kappa} + \frac{U_2\theta^{(3)}(\kappa-\delta_1\rho)}{[1+\theta^{(3)}\delta_1\rho]\kappa} \right\}$$

$$f(\omega)d\omega + \int_{\Lambda^4(\theta)} \left\{ \frac{U_1(1-\tau)}{1+\delta_1\rho\theta^{(4)}} + \frac{U_2\theta^{(4)}(1-\tau)}{1+\delta_1\rho\theta^{(4)}} \right\} f(\omega)d\omega + \int_{\Lambda^5(\theta)} \left\{ \frac{U_1(1-\tau)}{1+\theta^{(5)}\delta_2\rho} + \frac{U_2\theta^{(5)}(1-\tau)}{1+\theta^{(5)}\delta_2\rho} \right\}$$

$$f(\omega)d\omega + \int_{\Lambda^6(\theta)} \left\{ \frac{U_1(\kappa-\delta_1\rho)}{[1+\theta^{(6)}\delta_1\rho]\kappa} + \frac{U_2\theta^{(6)}(\kappa-\delta_1\rho)}{[1+\theta^{(6)}\delta_1\rho]\kappa} \right\} f(\omega)d\omega = 0;$$

(ii) $$\frac{\partial EU}{\partial B^a} = \int_{\Lambda^1(\theta)} \left\{ U_2 \left(\frac{\kappa-\rho}{\kappa}\right) \right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^2(\theta)} \left\{ \frac{U_1 \delta_2 \rho(\kappa-\rho) + \theta^{(2)} U_2 \delta_2 \rho(\kappa-\rho)}{[1+\theta^{(2)} \delta_2 \rho]\kappa} \right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^3(\theta)} \left\{ \frac{U_1 \delta_1 \rho(\kappa-\rho) + U_2 \theta^{(3)} \delta_1 \rho(\kappa-\rho)}{[1+\theta^{(3)} \delta_1 \rho]\kappa} \right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^4(\theta)} \left\{ \frac{U_1 (\delta_1-\tau)\rho + U_2 \theta^{(4)} (\delta_1-\tau)\rho}{[1+\theta^{(4)} \delta_1 \rho]} \right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^5(\theta)} \left\{ \frac{U_1 (\delta_2-\tau) + U_2 \theta^{(5)} (\delta_2-\tau)\rho}{[1+\theta^{(5)} \delta_2 \rho]} \right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^6(\theta)} \left\{ \frac{U_1 (\kappa-\rho)\delta_1 \rho + U_2 \theta^{(6)} (\kappa-\rho)\delta_1 \rho}{[1+\theta^{(6)} \delta_1 \rho]\kappa} \right\} f(\omega)\,d\omega \overset{(\leq)}{=} 0.$$

(The weak inequality in the latter would hold were $\hat{B}^a$ restricted to be non-negative; the domination of a portfolio involving money - even by itself - over any other excluding money means the former holds with strict equality.) Noting that

$$\omega\epsilon\Lambda^2(\theta) \rightarrow \frac{U_1}{U_2}(\omega) = 1/\delta_2\rho; \quad \omega\epsilon\Lambda^3(\theta) \rightarrow \frac{U_1}{U_2}(\omega) = 1/\delta_1\rho; \quad \omega\epsilon\Lambda^4(\theta) \rightarrow \frac{U_1}{U_2}(\omega) = 1/\delta_1\rho;$$

$$\omega\epsilon\Lambda^5(\theta) \rightarrow \frac{U_1}{U_2}(\omega) = 1/\delta_2\rho; \quad \omega\epsilon\Lambda^6(\theta) \rightarrow \frac{U_1}{U_2}(\omega) = 1/\delta_1\rho,$$

these FONC reduce to:

$$\frac{\partial EU}{\partial M^a} = \int_{\Lambda^1(\theta)} \left\{U_1 - U_2\left(\frac{1}{\kappa}\right)\right\} f(\omega)\,d\omega + \int_{\Lambda^2(\theta)} \left\{U_1 - U_2\left(\frac{1}{\kappa}\right)\right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^3(\theta)} \left\{U_1 - U_2\left(\frac{1}{\kappa}\right)\right\} f(\omega)\,d\omega + \int_{\Lambda^4(\theta)} \left\{U_1 - \frac{\tau}{\delta_1\rho} U_2\right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^5(\theta)} \left\{U_1 - \frac{\tau}{\delta_2\rho} U_2\right\} f(\omega)\,d\omega + \int_{\Lambda^6(\theta)} \left\{U_1 - \left(\frac{1}{\kappa}\right) U_2\right\} f(\omega)\,d\omega = 0 \ (i);$$

$$\frac{\partial EU}{\partial B^a} = \int_{\Lambda^1(\theta)} \left\{U_2\left(\frac{\kappa-\rho}{\kappa}\right)\right\} f(\omega)\,d\omega + \int_{\Lambda^2(\theta)} \left\{U_2\left(\frac{\kappa-\rho}{\kappa}\right)\right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^3(\theta)} \left\{U_2\left(\frac{\kappa-\rho}{\rho}\right)\right\} f(\omega)\,d\omega + \int_{\Lambda^4(\theta)} \left\{U_2\left(\frac{\delta_1-\tau}{\delta_1}\right)\right\} f(\omega)\,d\omega$$

$$+ \int_{\Lambda^5(\theta)} \left\{U_2\left(\frac{\delta_2-\tau}{\delta_2}\right)\right\} f(\omega)\,d\omega + \int_{\Lambda^6(\theta)} \left\{U_2\left(\frac{\kappa-\rho}{\kappa}\right)\right\} f(\omega)\,d\omega \overset{(\leq)}{=} 0 \ (ii)$$

(II.4) (FON C)

Further manipulation reveals

$$\frac{\Sigma_i \int_{\Lambda^i(\theta)} U_1 f(\omega)\,d\omega}{\Sigma_i \int_{\Lambda^i(\theta)} U_2 f(\omega)\,d\omega} \overset{(\leq)}{=} \frac{1}{\rho} \qquad (i);$$

$$\left(\frac{\kappa-\rho}{\kappa}\right) \Sigma_{i \neq 4,5} \int_{\Lambda^i(\theta)} U_2 f(\omega)\,d\omega + \left(\frac{\delta_1-\tau}{\delta_1}\right) \int_{\Lambda^4(\theta)} U_2 f(\omega)\,d\omega + \left(\frac{\delta_2-\tau}{\delta_2}\right) \int_{\Lambda^5(\theta)} U_2 f(\omega)\,d\omega \overset{(\leq)}{=} 0 \ (ii)$$

(II.5) (FON C)

We may note that (II.5(i)) is formally identical to Goldman's FONC ([6, (3), p.208]) characterizing the optimal ex ante choice of money balances, except insofar as we have expanded the choice of assets and trading opportunities, and thus ex post optimal regimes of behaviour open to the individual, and we also explictly allow for the possibility of a corner solution for B^a if constrained to be non-negative (i.e., if no initial period borrowing is permitted by invoking the assumption that the borrowing term is of one period duration). More interesting, perhaps, is the formal equivalence of this, together with (II.5(ii)), characterizing the demand for the standard (and, implicitly, the actuarial) bond, with Hool's FON C ([8, (2.6)-(2.8), p.77]) characterizing the representative individuals maximizing choices in a temporary general equilibrium model involving fixed preferences, money, commodities, and a standard bond whose return is random ex ante. The suggestive implications of the latter observation will be pursued below in the sequel. Here we only develop the interpretation of the latter of these FONC [(II.5(i) is merely analogous to the individual's expected MRS between first and second period consumption.]

Taking the coefficients of this expression in turn, $(\kappa-\rho)$ is the per-unit price differential between actuarial and standard bonds in period zero, while $(\frac{1}{\kappa})$ is the gross per unit return on actuarial bonds held to maturity, thus $(\frac{1}{\kappa})$ $(\kappa-\rho)$ represents the opportunity loss, in terms of foregone period two's consumption, from having purchased an incremental unit of standard bonds, in the event that (in regimes i, $i \neq 4,5$) it was not in fact necessary to redeem the latter. Similarly, $\frac{(\delta_1-\tau)\rho}{\delta_1\rho}$ equals the per unit "gain" from the premature redemption of a unit of standard rather than actuarial bond. Thus, if the latter must be redeemed, there is a per unit opportunity loss, $(\delta_1-\tau)$, in terms of foregone period 1 consumption, from not having had more standard bonds to redeem rather than actuarials in the portfolio. Finally, $\frac{(\delta_2-\tau)\rho}{\delta_2\rho}$ equals the per unit opportunity loss arising from an initial investment of ρ in actuarial bonds in the initial period which has then to be converted to first period standard bonds and which would have yielded an identical sum in the 2nd period — $1/\delta_2\rho$ equals the gross 2nd period return from an investment of $\delta_2\rho$ in standard bonds in the first period. (II.5(ii)) therefore just expresses the fact that *ex ante* there should be no perceived residual advantage to be obtained from portfolio reallocation at the margin, thus these opportunity losses and gains net to zero (or a non-positive sum).

The second-order sufficiency conditions (SOSC) for the maximization are

$$\frac{\partial^2 EU}{\partial M^{a2}} < 0 \text{ (i)}; \quad \frac{\partial^2 EU}{\partial B^{a2}} < 0 \text{ (ii)}; \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 EU}{\partial M^{a2}} & \frac{\partial^2 EU}{\partial B^a \partial M^a} \\ \frac{\partial^2 EU}{\partial B^a M^a} & \frac{\partial^2 EU}{\partial B^{a2}} \end{vmatrix} > 0 \text{ (iii)}, \quad \text{(II.6)}.$$

By strict concavity of $U[\cdot]$ these can be assumed to be satisfied (((i) and (ii) (II.6)) can be verified straightforwardly by tedious differentiation).

III: Results

We will now proceed to work with the FONC for A_2^a and B^a. These are (in their most useful forms):

$$\frac{\partial EU}{\partial A_2^a} = \int_{\Lambda^1(\theta)} \{U_2-\kappa U_1\}f(\omega)d\omega + \int_{\Lambda^2(\theta)} \{U_1(\delta_2\rho-\kappa)\}f(\omega)d\omega + \int_{\Lambda^3(\theta)} \{U_1(\delta_1\rho-\kappa)\}f(\omega)d\omega$$

$$+ \int_{\Lambda^4(\theta)} \{U_1(\tau-1)\kappa\}f(\omega)d\omega + \int_{\Lambda^5(\theta)} \{U_1(\tau-1)\kappa\}f(\omega)d\omega + \int_{\Lambda^6(\theta)} \{U_1(\delta_1\rho-\kappa)\}f(\omega)d\omega \leq 0 \quad (i);$$

(III.1)

$$\frac{\partial EU}{\partial B^a} = \int_{\Lambda^1(\theta)} \{U_2-\rho U_1\}f(\omega)d\omega + \int_{\Lambda^2(\theta)} \{U_1(\delta_2-1)\rho\}f(\omega)d\omega + \int_{\Lambda^3(\theta)} \{U_1(\delta_1-1)\rho\}f(\omega)d\omega$$

$$+ \int_{\Lambda^4(\theta)} \{U_1(\delta_1-1)\rho\}f(\omega)d\omega + \int_{\Lambda^5(\theta)} \{(\delta_2-1)\rho U_1)\}f(\omega)d\omega + \int_{\Lambda^6(\theta)} \{U_1(\delta_1-1)\rho\}f(\omega)d\omega \leq 0 \quad (ii)$$

(We use the inequality forms to indicate the possibility that the individual might otherwise wish to sell either category of bond in the most general case if explicitly constrained).

Varians of the SOSC given above apply.

(i) The Demand for Actuarials

As $\tau = \delta_1 = \delta_2$, $\rho = \kappa$, and $\rho = 1$ together imply that all assets are equivalent *ex ante*, we initially assume $\kappa < \rho < 1$ and $\delta_2 > 1 > \delta_1 > \tau$, and thus, that there is a strict inverse relationship between ex ante per unit yields to maturity and ex post premature redemption rates. Assuming, for the moment, that a bounded, consistent, solution to the individual's problem exists, we then have the following results (R):

R.1: The individual possessing ex ante random preferences satisfying (A1)-(A4), if confronted with the ex ante trading possibilities outlined above, apart from that borrowing is also proscribed, will never adopt a "wait-and-see" attitude; i.e., the individual will never choose to hold only money in his portfolio while waiting for the riskiness of his preferences to be resolved.

Proof: Suppose $\hat{A}_2^a$, $\hat{B}^a = 0$, thus $\Lambda^i(\theta) = \emptyset$, $\forall i \neq 2$, $M^a = W > 0$, and $\Lambda^2(\theta) = \{\omega | \omega \varepsilon [0,1]\}$. (In other words, the individual has to invest in first period bonds alone, or carry excess money balances forward, as the case may be, after the riskiness associated with his preferences are resolved). Then the FONC (III.1) reduce to

$$\left.\begin{aligned} \frac{\partial E^u}{\partial B^a} &= \int_0^1 \{U_1(\delta_2 - 1)\rho\} f(\omega)\, d\omega \leq 0 \qquad \text{(i)}, \\ \frac{\partial EU}{\partial B^a} &= \int_0^1 \{U_1(\delta_2\rho - \kappa)\} f(\omega)\, d\omega \leq 0 \qquad \text{(ii)} \end{aligned}\right\} \quad \text{(III.2)}.$$

But $(\delta_2\rho - \kappa) > (\delta_2 - 1)\rho > 0$, and $U_1 > 0$, thus (III.2) are immediately contradicted and the result follows. Q.E.D.

Both Goldman ([16]) and Taylor ([12]) explicitly proscribed trading by lending in first period bonds, although Goldman ([7, p.265]) speculated about the possibility of relaxing this. Were first period lending proscribed here and the individual had to carry excess money forward to period 2, the coefficients of the U_1 terms in (III.3) would be, respectively, $(1-\kappa)$ and $(1-\rho)$. As $(1-\kappa) > (1-\rho) > 0$, R1 would continue to apply.

R2: $\forall \rho > \kappa$, $\hat{A}_2^a > 0$: The individual will always choose to hold the ex ante highest yielding asset.

Proof: Letting the subscript c denote maximizing values in the constrained case, suppose $\hat{A}_2^{ac} = 0$. Then $\Lambda^4(\theta) = \emptyset = \Lambda^5(\theta)$, and the FONC reduce to

$$\frac{\partial EU}{\partial B^{ac}} = \int_{\Lambda^1(\theta)} \{U_2 - \rho U_1\} f(\omega)d\omega + \int_{\Lambda^2(\theta)} \{U_1(\delta_2 - 1)\rho\} f(\omega)d\omega + \int_{\Lambda^3(\theta)} \{U_1(\delta_1 - 1)\rho\} f(\omega)d\omega$$

(i);

$$+ \int_{\Lambda^6(\theta)} \{U_1(\delta_1 - 1)\rho\} f(\omega)d\omega \leq 0 \quad \text{(i)}; \qquad \text{(III.3)}$$

$$\frac{\partial EU}{\partial A^{ac}} = \int_{\Lambda^1(\theta)} \{U_2 - \kappa U_1\} f(\omega)d\omega + \int_{\Lambda^2(\theta)} \{U_1(\delta_2 \rho - \kappa)\} f(\omega)d\omega + \int_{\Lambda^3(\theta)} U_1(\delta_1 \rho - \kappa)\} f(\omega)d\omega$$

$$+ \int_{\Lambda^6(\theta)} \{U_1(\delta_1 \rho - \kappa)\} f(\omega)d\omega \leq 0 \qquad \text{(ii)};$$

By R.1 and Kuhn-Tucker complementary slackness, these two cannot hold with strict inequality simultaneously thus, by assumption, (III.(i)) holds with equality, and $\hat{B}^{ac} > 0$. Now, $\rho > \kappa$, $\delta_2\rho-\kappa > (\delta_2-1)\rho$, and $\delta_1\rho-\kappa > (\delta_1-1)\rho$, thus all the terms of (III.3)(ii) strictly exceed the corresponding terms of (III.3(i)), $\forall\rho > \kappa$. But this in turn $\Rightarrow \frac{\partial EU}{\partial B^{ac}} < 0$, $\Rightarrow \hat{B}^{ac} = 0$ also, which would contradict R.1 and the initial supposition, thus $\hat{A}_2^a > 0$. Q.E.D.

The intuitive basis of R2 should be relatively clear: for any initial endowment $W, \kappa, \rho,$ and $f(\omega)$, the individual possesses at least the minimal option of holding all his wealth in money, bar the greatest investment in actuarials $\bar{A}_2^a = \min.C_2(\omega)$, $\omega\epsilon[0,1]$, which just ensures that he will not be called upon to redeem the actuarials in any eventuality, but might be required to either invest in first period bonds or carry excess money forwards. By our assumptions on preferences, $\bar{A}_2^a = \min.C_2(\omega) > 0$, $\omega\epsilon[0,1]$ clearly exists. It is also very easy to see, however, that such a minimal option is unlikely to be efficient:

R.3. If $\delta_1\rho-\kappa \geqq 0$, any bounded, consistent, solution to the individual's problem must involve a positive probability of premature redemption of either actuarials or/and standards.

<u>Proof:</u> Suppose otherwise, thus $\Lambda^i(\theta) = \emptyset$, $i \neq 1,2$, hence the FONC reduce to

$$\frac{\partial EU}{\partial A_2^a} = \int_{\Lambda^1(\theta)} \{U_2-\kappa U_1\}f(\omega)d\omega + \int_{\Lambda^2(\theta)} \{U_1(\delta_2\rho-\kappa)\}f(\omega)d\omega = 0 \qquad \text{(i)}$$

(III.4).

$$\frac{\partial EU}{\partial B^a} = \int_{\Lambda^1(\theta)} \{U_2-\rho U_1\}f(\omega)d\omega + \int_{\Lambda^2(\theta)} \{U_1(\delta_2-1)\rho\}f(\omega)d\omega \leqq 0 \qquad \text{(ii)}$$

Firstly, note that as $\rho > \kappa$ and $(\delta_2-1)\rho \not> 0$, $\frac{\partial EU}{\partial A^a} > \frac{\partial EU}{\partial B^a}$ unambiguously, thus if our supposition is correct and

$$\left.\frac{\partial EU}{\partial A^a}\right|_{\Lambda^i(\theta)=\emptyset, i\neq 1,2} = 0$$

(as it must, by R2, for any bounded, consistent solution), $\frac{\partial EU}{\partial B^a} < 0$ and the standard bond will never be held in positive amounts in the individual's portfolio. (Indeed, were borrowing by period zero sales of standards permitted, the individual would borrow up to the limit of any solvency constraint). Let us therefore examine (III.4(i)). By definition of $\Lambda^1(\theta)$, $\frac{U_1}{U_2}(\omega)\varepsilon\left[\frac{1}{\delta_2\rho}, \frac{1}{\delta_1\rho}\right]$, $\omega\varepsilon\Lambda^1(\theta)$; i.e., most crucially, $1/\delta_1\rho \geqq \frac{U_1}{U_2}(\omega)$, $\Rightarrow U_2 \geqq \delta_1\rho U_1$, thus

$$U_2-\kappa U_1 \geqq \delta_1\rho U_1-\kappa U_1 = U_1(\delta_1\rho-\kappa)$$

$$\geqq 0 \text{ iff } (\delta_1\rho-\kappa) \geqq 0 \quad (\text{as } U_1 > 0).$$

But if $\delta_1\rho-\kappa \geqq 0$, as $\delta_2\rho-\kappa > 0$ ($\because \delta_2 \geqq 1, \rho > \kappa$), this is sufficient to ensure

$$\left.\frac{\partial EU}{\partial A^a}\right|_{\Lambda^i(\theta) = \emptyset,\ i \neq 1,2} > 0$$

unambiguously, contradicting (III.4(i)) for any bounded, consistent, solution, while

$$0 \lesseqgtr \left.\frac{\partial EU}{\partial B^a}\right|_{\Lambda^i(\theta) = \emptyset,\ i \neq 1,2} < \left.\frac{\partial EU}{\partial A_2^a}\right|_{\Lambda^i(\theta) = \emptyset,\ i \neq 1,2} .$$

As $\delta_2\rho-\kappa > 0$, by continuity there also exists $\kappa > \delta_1\rho$ so that $\left.\frac{\partial EU}{\partial A_2^a}\right|_{\Lambda^i(\theta) = \emptyset,\ i \neq 1,2} > 0$ remains true. These partials indicate that the individual could obtain an increase in EU by moving from this minimal option, and his EU-maximizing portfolio must therefore encompass the non-zero probability of premature redemption of at least the holdings of the actuarials. Q.E.D.

Note that the R3 does not actually guarantee positive initial holdings of standards as well as actuarials when $\delta_1\rho-\kappa \gtreqless 0$. This is because the individual might hold sufficient actuarials to ensure the possibility of their premature redemption and then, should this eventuality arise, use some of the proceeds of their sale either to carry forward or to buy first period bonds to finance the reduced period 2 consumption which is now desired.

Further inspection of the FONC (III.1) reveals that, firstly, unless at least one of $\Lambda^4(\theta) \neq \emptyset$, or $\Lambda^5(\theta) \neq \emptyset$ holds, $\frac{\partial EU}{\partial A_2^a}$ unambiguously; secondly, if $(\delta_1\rho-\tau\kappa) < (\rho-\kappa)$, $\Lambda^5(\theta) \neq \emptyset$, otherwise $\frac{\partial EU}{\partial B^a} < \frac{\partial EU}{\partial A_2^a}$ unambiguously; thirdly, if $\delta_1\rho-\kappa \geq 0$, unless at least one of $\Lambda^4(\theta) \neq \emptyset$, $\Lambda^5(\theta) \neq \emptyset$, $\frac{\partial EU}{\partial A^a} > 0$ unambiguously.

The second observation is particularly interesting. It suggests that if the per unit premature redemption price differential between the standard and the actuarial bonds is less than the initial purchase price differential, unless there always exists some realization of ω for which the individual will need to redeem the actuarial prematurely while not requiring all its proceeds for any initial portfolio allocation, he will never hold the standard bond in positive amounts in his portfolio. Obviously, this is because under this configuration of price paramaters there will then be no expected opportunity loss vis-a-vis standards from holding only actuarials alongside money in his portfolio. The result is not tautological because there is nothing in the model which guarantees $\Lambda^5(\theta) \neq \phi$ for all initial portfolio allocations.

(ii) The Demand for Standards

Thus far, we have not actually proved that $\hat{B} > 0$ is either possible or likely. Nor have we made any explicit use of whether $\tau \gtreqqless 0$ or of the consequences of $\tau = 0$, although $\tau > 0$ has been implicit in our listing of the available regimes. We can address both issues simultaneously by showing that when $\tau = 0$, and $\delta_1\rho - \kappa \gtreqqless 0$, a bounded consistent solution must involve a positive probability of premature redemption of standards, thus some positive amount of standards must be held in the first place. This extreme case of $\tau = 0$ could correspond to the economically relevant situation where an individual is "locked" into a pension scheme or, perhaps, a house tied to his occupation.

When $\tau = 0$ (and thus, the highest yielding asset is totally inflexible) there is a further possibility open to the individual which has not been accounted for so far. This is that he might hold both non-money assets initially and then find himself in a situation where it is ex post optimal to prematurely redeem all his standards (and possibly more, were that but possible). Simultaneously, regimes 4 and 5, which would now involve the individual giving away any positive holdings of the actuarials - which is what their premature redemption now entails - can be ruled out by dominance and non-satiation.

As only holdings of standard bonds are now prematurely tradeable, it is notationally convenient to introduce a symbol, z, to represent transactions in these bonds at the start of period 1. $z > (<) 0$ represents the number of standards bought (sold) at this time. Clearly $z \geqq -B_2^a$. It is also notationally convenient to introduce the function $G(z)$ defined by

$$G(z) = \begin{cases} \delta_1 z & (z < 0) \\ 0 & (z = 0) \\ \delta_2 z & (z > 0) \end{cases}$$

$z = 0$ implies that the individual remains at A. Otherwise, $\rho G(z)$ represents either the revenue derived from premature redemption, or the expenditure upon addition purchases, of the standards at the start of period 1.

With these innovations we have, ex post, $C_1(\omega) = W - \rho B_2^a - \kappa A_2^a - \rho G(z)$ and $C_2(\omega) = B_2^a + A_2^a + z$, $\forall\omega$. Suppressing scripts, for arbitrary A, B, the individual's period 1 problem is to choose z according to

$$\underset{z \geq -B}{\text{Max}}\ U\left[W-\rho B-\kappa A-\rho G(z),\quad B + A + z;\ \omega\right] \qquad \text{(III.5).}$$

Four possible regimes can be identified: (1) the corner solution where the individual redeems all of any initial holding of standards; (2) the individual redeems some but not all of any initial holding of the same; (3) the individual remains where he is; (4) the individual purchases additional standards. Formally,

(i) $$\hat{z} = -B \quad \text{if} \quad -\rho\delta_1 U_1\left[W-\kappa A-\rho B(1-\delta_1), A; \omega\right] + U_2[\cdot] \leq 0 \qquad \text{(III.6)},$$

i.e., by inversion, utilizing (A3)-(A.4), if $\omega \leq \hat{\omega}(W-\kappa A-\rho B(1-\delta_1), A)$ (III.6').

The individual remains where he is $(\hat{z} = 0)$ if

$$\left.\begin{array}{ll} -\rho U_1\left[W-\rho B-\kappa A, A+B; \omega\right] + U_2[\cdot]/\delta_2 \leq 0 & \text{(i)} \\ \text{or} & \\ -\rho U_1\left[W-\rho B-\kappa A, A+B; \omega\right] + U_2[\cdot]/\delta_1 \geq 0 & \text{(ii)} \end{array}\right\} \qquad \text{(III.7)}$$

i.e.,by inversion,if $\omega \leq \bar{\omega}(W-\rho B-\kappa A, A+B)$ or $\omega \geq \underline{\omega}(W-\rho B-\kappa A, A+B)$. Thus,

(ii) $$\hat{z} = 0 \quad \text{if} \quad \omega\varepsilon\left[\underline{\omega}, \bar{\omega}\right] \qquad \text{(III.7')}$$

and

(iii) $$\hat{z} \quad \varepsilon(-B, 0) \quad \text{if} \quad \omega\varepsilon(\hat{\omega}, \underline{\omega}) \qquad \text{(III.8)},$$

(iv) $$\hat{z} \geq 0 \quad \text{if} \quad -\rho\delta_2 U_1\left[W-\kappa A-\rho B, A+B; \omega\right] + U_2[\cdot] \geq 0 \qquad \text{(III.9)}$$

i.e. by inversion, $\hat{z} > 0$ if $\omega\varepsilon(\bar{\omega}, 1]$ (III.9').

If $\omega\varepsilon[0,\hat{\omega}]$, $C_1(\omega) = W-\kappa A-\rho B(1-\delta_1)$, $C_2(\omega) = A$; if $\omega\varepsilon[\underline{\omega}\ \bar{\omega}]$, $C_1(\omega) = W-\kappa A-\rho B$, $C_2(\omega) = A+B$. Let $\theta^{(i)} = (C_2/C_1)(\omega)$, $i = 2,4$, for the other regimes. Then, proceeding as in Section II, we can define the individual's indirect expected utility for arbitrary initial portfolio (A,B) by

$$V(A,B) = \int_0^{\hat{\omega}} U[W-\kappa A-\rho B(1-\delta_1),A;\omega]f(\omega)d\omega + \int_{\hat{\omega}}^{\underline{\omega}} U\left[\frac{W+B(\delta_1-1)\rho+A(\delta_1\rho-\kappa)}{1+\theta^{(2)}\delta_1\rho},\ \theta^{(2)}C_1(\omega);\omega\right]f(\omega)d\omega +$$

$$+\int_{\underline{\omega}}^{\bar{\omega}} U[W-\kappa A-\rho B,A+B,\omega]f(\omega)d\omega + \int_{\bar{\omega}}^{1} U\left[\frac{W+B(\delta_2-1)\rho+A(\delta_2\rho-\kappa)}{1+\theta(\omega)\delta_2\rho},\ \theta^{(4)}C_1(\omega);\omega\right]f(\omega)d\omega \quad \text{(III.10)}.$$

Noting that $\omega\varepsilon[0,\hat{\omega}] \Rightarrow \frac{U_1}{U_2}(\omega) \leq 1/\delta_1\rho$, $\omega\varepsilon(\hat{\omega},\underline{\omega}) \Rightarrow \frac{U_1}{U_2}(\omega) = 1/\delta_1\rho$, $\omega\varepsilon[\underline{\omega}\ \bar{\omega}] \Rightarrow \frac{U_1}{U_2}(\omega)\,\varepsilon\,[1/\delta_2\rho, 1/\delta_1\rho]$, and $\omega\varepsilon(\bar{\omega},1] \Rightarrow \frac{U_1}{U_2}(\omega) = 1/\delta_2\rho$, the first-order conditions for the maximization of V(A,B) w.r.t. the choice (A,B) yield, on simplification,

$$\partial V/\partial A = \int_0^{\hat{\omega}}\{U_2-\kappa U_1\}f(\omega)d\omega + \int_{\hat{\omega}}^{\underline{\omega}}\{U_1(\delta_1\rho-\kappa)\}f(\omega)d\omega + \int_{\underline{\omega}}^{\bar{\omega}}\{U_2-\kappa U_1\}f(\omega)d\omega + \int_{\bar{\omega}}^{1}\{U_1(\delta_2\rho-\kappa)\}f(\omega)d\omega \leq 0 \quad \text{(i)}$$

$$\partial V/\partial B = \int_0^{\hat{\omega}}\{U_1\rho(\delta_1-1)\}f(\omega)d\omega + \int_{\hat{\omega}}^{\underline{\omega}}\{U_1(\delta_1-1)\rho\}f(\omega)d\omega + \int_{\underline{\omega}}^{\bar{\omega}}\{U_2-\rho U_1\}f(\omega) + \int_{\underline{\omega}}^{1}\{U_1(\delta_2-1)\rho\}f(\omega)d\omega \leq 0 \quad \text{(ii)} \quad \text{(III.11)}.$$

It is easy to verify that R1 remains valid in this restricted environment (R1'). We next show that R2 remains valid also (R2'). Suppose otherwise, thus $\hat{A} = 0$, and the individual can never be in the new regime 1. But then $\partial V/\partial A > \partial V/\partial B$ unambiguously, implying $\hat{B} = 0$

also, contradicting (R1'). Thus $\hat{A} > 0$: hence, the ex ante highest yielding asset is always held initially (the individual always "speculates"), irrespective of its flexibility properties.

Theoretically, it still seems possible for $\partial V/\partial A > \partial V/\partial B$ for an arbitrary configuration of price parameters and for all initial portfolios, in which case plunging into actuarials cannot be ruled out if the individual is not explicitly constrained. However, it is now straightforward to show that $\partial_1\rho-\kappa \geqq 0$ is sufficient, but unnecessary, to ensure $\hat{B} > 0$ when $\tau = 0$, and thus, that the individual initially holds a fully diversified portfolio under these circumstances.

R4. If $\partial_1\rho-\kappa \geqq 0$ when $\tau = 0$, $\hat{B} > 0$ unambiguously.

Proof: This essentially replicates the argument in the proof of R3: If $\hat{B} = 0, [0,\hat{\omega}] = \phi$. But then, if $\partial_1\rho-\kappa \geqq 0$, $\partial V/\partial A > 0$ unambiguously, as before, contrading the Kuhn-Tucker conditions for a consistent solution. Thus $[0,\hat{\omega}] \neq \phi$. But this is only possible if $\hat{B} > 0$. Q.E.D.

It is important to stress that the condition $\partial_1\rho-\kappa \geqq 0$ is merely sufficient. By continuity, there exists, again, a range of $\kappa > \partial_1\rho$ for which $\hat{B} > 0$ unambiguously.

There are many possible justifications for R4. Perhaps the most simple and succinct runs as follows. For given ρ, the smaller is κ, the greater is the individual's incentive to speculate initially by purchasing actuarials vis-a-vis standards or holding money. But then, the more likely is he to be in need of additional cash at the start of period 1 for any realization of ω. Given that actuarials cannot now be redeemed, this simultaneously enhances demand for standards to act as a precautionary hedge against this eventuality. The larger is ∂_1, hence $\partial_1 \rho$, the more adequately do standards fulfill this precautionary role.

Of course, the situation is not quite as straightforward as this because of the two countervailing influences operating upon the demand for standards arising from income and substitution effects in the model. The first influence, tending to increase the demand, is the precautionary role alluded to above. The second, tending to reduce the demand, is the fact that smaller κ enables the individual to purchase more actuarials for given initial expenditure. The individual can thus maintain larger initial money balances simultaneously, partly obviating the need for precautionary balances of standards.

(iii) An Implication for Precommitment.

Suppose we identify "actuarials" with pensions and imagine the individual's problem in period 0 as being that of choosing the optimal level of precommitment (when $\tau = 0$). Then R2, R2' and R3 carry an obvious implication for a government's policy towards compulsory precommittment. This is that it is unlikely to be either important or difficult to ensure that an individual purchases an adequate pension: he will initially opt to do so anyway, provided that the prospective yield to maturity is sufficiently attractive. Rather, what is difficult and important is to ensure that the individual retains his pension once purchased. This suggests that attention might be directed towards vesting rules, perhaps in the direction of making them more liberal.

This, however, will only be possible if the illiberal vesting rules are not only what enable the ex ante high yields on the pensions in the first place. (cf. Goldman [7, p.275]).

(iv) Comparative Statics

To pursue the comparative statics of the more general models used above would be both tedious and uninformative. Such an exercise, within the context of various specializations of $U\{C_1,C_2,\omega\}$, and of specific applications of the model, will be performed elsewhere.

IV. Conclusion: Extensions and Implications

We have been able to establish fairly transparent sufficient conditions for the individual with random preferences to hold a diversified portfolio incorporating a plausible choice of assets. In this way we have been able to both accommodate and help to explain the large variety of phenomena discussed in the Introduction.

Our modelling of assets has been relatively realistic in that, typically, the nominal yields of those assets which are held to maturity are known ex ante. What is usually not known ex ante, even in nominal terms, is what short-run return an asset will yield if redeemed prematurely. This means that, even with random preferences, there remains some residual risk confronting the individual to be accounted for.

To a first approximation, it might be plausible to assume that τ is known for any particular fixed period to premature redemption, but not δ_1. For example, our "standard" bond could be a Treasury

Bill, or promisory note, whose nominal yield to maturity is certain, but not the holding period yield if it has to be converted prematurely on the market where its capital value is going to depend inversely upon the current ex ante rates being offered on other standard (and other) bonds, in turn dependent upon the vagaries of the economic environment then current.

The importance of all this is that ex ante randomness of premature redemption rates means that the individual's behaviour, for any realization of ω and for any fixed mixed portfolio composition, is still perceived as random ex ante. However, provided he can assign some ex ante probabilistic assessment to the likely relationship between δ_1 and τ, all the ex post theoretical possibilities can, in principle, be outlined with suitable modifications to the alternative regimes of behaviour which involve premature redemption. This is particularly simple if we assume that the ex ante highest yielding bond is also divisible ex post.

More formally, letting superscript $\sim$ denote a random variable, suppose δ_1 possesses a (subjectively) known distribution with density $\pi(\delta_1)$ and (cumulative) distribution $\Pi(\delta_1) = \text{prob.}\ \{\tilde{\delta}_1 \in (0, \delta_1]\} \equiv \int_0^{\delta_1} \pi(s)ds$ (where $\delta_1 = 0$ in a more general intertemporal model would indicate that, when transaction costs are taken into consideration, a premature seller could not find a buyer at any positive price, reminiscent perhaps of some categories of long-dated government debt), and $1-\Pi(\delta_1) = \text{prob.}\ \{\tilde{\delta}_1 \varepsilon (\delta_1, \delta_2]\}$. When δ_1 is random it is unlikely for it to remain necessary that it be bounded above by unity in order to ensure the non-domination of money; however, it remains necessary, and plausible, to assume that the per unit selling price is bounded above by the first period buying price - unless buying and selling of the standard bond in the same period is explicitly prescribed, as it would be in a temporary general equilibrium framework. But were this the case, and were δ_2 also random, this in turn would cause additional complications.

If this latter possibility is neglected, and assuming independence of randomness of $U(\cdot)$ from that of δ_1, it is easy to see that in our revised model with $\tilde{\delta}_1$ and ex post divisibility of actuarials, regimes 1 and 2 remain invariant while in regime 3 the individual will redeem actuarials if $\tau > \tilde{\delta}_1$, which will occur with probability $\Pi(\tau)$, and standards if $\tau < \tilde{\delta}_1$, which occurs with the complementary probability, and so on. In this case, postulating ex post divisibility of the highest yielding bond again reduces the model to an ex post six regimes one.

However, were the actuarial asset individisible ex post, the situation then is not as simple as postulating that the individual redeems the actuarial if $\tau > \tilde{\delta}_1$ and conversely in the converse case. In the former case, if the individual has holdings of the actuarial assets whose first period value is excess to his needs then (i.e., $\tau\kappa A_2^a > C_1(\omega) - M^a$), the individual is faced with the problem of balancing the gain from the premature redemption of the actuarial asset at the higher per unit surrender rate against the loss from having to reinvest any excess in first period standard bonds (where possible), or from carrying excess money forward, rather than redeeming an exact amount of the standard bond at the lower redemption rate. Unless some ad hoc rule of thumb is imposed - such as that of redeeming actuarials only as a last resort - this decision is endogenous and must be analysed separately.[6/]

A final issue of note is that raised by the qualitative similarity between our FONC and those of Hool ([8]) (a similarity which previous authors have obscured, either by restricting the asset set ([6], [12]), or by severely limiting the form of U(·), and thus the scope of consumer maximization ([7])). An implication of this might be that highly aggregative models of consumer intertemporal expected utility maximization based upon price expectation schemes, but fixed preferences, are indistinguishable from models of intertemporal expected utility maximization based upon spot price expectations and random preferences, certainly in terms of their predictions at the empirical level. And this despite the fact that the latter effectively postulate money illusion - unless the random shift in U(.) is a direct consequence of the reduced real value of any nominal level of consumption consequent upon any level of inflation [7/] - while the former most certainly do not. This suggests that it might be fruitful to examine further the probability structure induced by randomness of preferences upon ostensibly deterministic asset returns. This is not least because, at the most elementary level, the possible congruence of the two categories of models discussed is likely to have clear implications for the econometric modelling and testing of either. All the issues raised in this section are the subject of current research.

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Footnotes

1/ An earlier version of this paper appeared as University of York Department of Economics and Related Studies Discussion Paper 48. January 1980. I have had helpful conversations and correspondence with John Driffill, Bob Nobay, Dave Rowan, Peter Simmons and, most especially, Maurice McNamus and Jim Mirrlees. The usual exonerations apply.

2/ For example, works like Stiglitz ([11]) and Lorie ([9]) seem to straddle both approaches.

3/ The denotation "actuarial", though suggestive, should not be taken too literally because, in the general model to be outlined below, we will not be concerned explicitly with such specific issues as risk of death as the source of randomness of preference. Such applications will be pursued elsewhere.

4. All premature redemption rates, and first period standard bond prices, are assumed to be known for certain in nominal terms, and we are not explicitly considering the possibility of inflation. But see the final section.

5/ An intertemporal separability assumption, together with the possibility of two-stage budgeting, would be sufficient to ensure that consumption possibilities in period O can be neglected in this fashion: if period O's non-stochastic utility of consumption $U_O(C_O)$ is separable from that of periods 1 and 2, we can define the indirect expected utility of the latter by $V(\bar{W} - C_O) \triangleq \underset{\{M^a, B^a\}}{\text{Max}}\ EU$ (where EU is defined by (II.3) in the text below) for any aggregate investment $\bar{W}-C_O$ in money and bonds in period O. Then, letting $\bar{W} = W + C_O$, $\Leftrightarrow W = \bar{W} - C_O$, where $\bar{W}$ is the individual's overall endowment, the individual's three period utility is given by: $E_OU = U_O(C_O) + \lambda V(\bar{W} - C_O)$, $\lambda \in (O, 1]$ being a discount of impatience factor if need be. Assuming differentiability of V, maximization of E_OU with respect to C_O yields the intertemporal envelope condition

$$\frac{\partial E_O U}{\partial C_O} = U_O'(C_O) - \lambda V'(W-C_O) = O$$

which, by concavity and invocation of the Inada conditions, is uniquely solvable for interior C_O and hence $W = \bar{W} - C > O$. This extension of the model to explicitly consider period O consumption would be indispensable for certain applications. For example, we might be interested in the effect of randomness of preferences upon the intertemporal distribution of consumption or the effect of lifetime uncertainty (summarized by ω) upon the same.

6/ This topic, more generally one of the optimum realization of capital gains, is the subject of current research.

7/ Both Goldman ([6, p.203]) and Taylor ([12, p.67]) briefly indicate this possibility.