On the Sign of the Optimum Marginal Income Tax

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1. INTRODUCTION

A well-known result of Mirrlees (1971; proposition 3) says that the optimal marginal rate of income tax is non-negative throughout the scale, for the model he considers and given only a mild regularity condition on preferences. That is, the burden of taxation unambiguously increases with earnings. This result is very useful. The model to which it applies is admittedly special (identical leisure/consumption preferences), but it is reassuring to know that no further specialisation of assumptions is required to reach such a basic conclusion: incentive effects from taxation will never turn the desired pattern of redistribution on its head, at any level of income.

The above result, as we shall see, is indeed a property of income taxes rather generally, but its proof in Mirrlees (1971) implicitly adopts a very special case, namely additive separability of preferences, which is only later in the same paper explicitly adopted and discussed. Our purpose here is to derive a rather more general set of conditions that ensure the required non-negativity, or in fact strict positivity, of marginal tax at all interior levels of income: these essentially amount to non-inferiority of leisure, apart from the above-mentioned regularity condition on preferences used by Mirrlees, assumption A.1 below. The central result of the paper is theorem 1 at the close of Section 4, whose derivation is based on other results, some with separate interest, obtained earlier on in the paper.

2. BACKGROUND

Let us recall the main features of Mirrlees' income-tax model. These are that individual consumption/leisure preferences are all identical, and that labour can be measured in 'efficiency units' homogeneous in production, so that differences across individuals only arise from the different contents of effective labour in their clock-hour of work. Hence preferences in terms of the quantities relevant in production, namely consumption and income, can be expressed by a one-parametric family of utility functions:

$$u(c,y;n) \equiv U(c,y/n) = U(c,L), \qquad (1)$$

where y and n are relative gross income and relative wage, c net income, and L the time spent at work.

Apart from capturing individual preferences, the above u is taken to represent the utilitarian government's favoured cardinalization of these preferences, its views on distribution. This function (hence the 'primitive' function U(.)) is assumed to be strictly concave and to have $u_c > 0$, $u_y < 0$. The parameter n is distributed in the population with bounded density $f(n)$, the end-points of whose support we denote by $\underline{n}$, $\bar{n}$.

On the face of a net-income schedule $c = \xi(y)$ imposed by the government, individuals maximize utility. A pair of allocation functions $c(n)$, $y(n)$ thus arise, in terms of which it is useful to conduct the analysis. Ignoring corners of optima, individual maximization implies that:

$$u_c c' + u_y y' = 0 \qquad (2)$$

(Seade (1977)). This acts as a constraint on the allocations that can be achieved. Also, a revenue (or linearized-production) constraint is to be met:

$$\int_{\underline{n}}^{\bar{n}} [wy(n) - c(n)] f(n) dn \geq R, \qquad (3)$$

where w is the wage for an efficiency unit in terms of numeraire (consumption), and R is the government's revenue requirement.

Necessary conditions for a maximum of $\int ufdn$ subject to (2) and (3) are that

$$\mu' u_c + \mu u_{cn} = (u_c - \lambda) f \qquad (4)$$

and

$$\mu' u_y + \mu u_{yn} = (u_y + \lambda w) f, \qquad (5)$$

where the constant λ and the function $\mu(n)$ are the multipliers for (3) and (2), respectively, in the corresponding Lagrangean. These, supplemented by (2), (3) and transversality conditions $\mu(\underline{n}) = \mu(\bar{n}) = 0$ characterize the optimum (Mirrlees (1971), Seade (1977)). For the sake of analysis, however, it is convenient to eliminate μ' from these expressions and define

$$s(c,y;n) \equiv -u_y/u_c \qquad (6)$$

(= $dc/dy|_{\bar{u}}$), to obtain

$$(w-s) = -\mu u_c s_n/\lambda f. \qquad (7)$$

This gives the marginal tax to be paid by an n-man: the excess of producers' pay w for the marginal efficiency-unit of work provided, over the net pay actually received, s. A percentual tax rate can be obtained dividing this equation through by the shadow wage w.

To rid ourselves from μ in (7), it is customary to integrate the first-order condition obtained from the variation of the numeraire, namely (4). This yields

$$\mu(n) = \int_{\underline{n}}^{n} \frac{u_c-\lambda}{u_c} f(n')\exp\left\{-\int_{n'}^{n}(u_{cn}/u_c)dn''\right\} dn' \qquad (8)$$

(plus a term $\{\exp(-\int_{\underline{n}}^{n}(u_{cn}/u_c)dn')\}\mu(\underline{n})$ which vanishes by $\mu(\underline{n}) = 0$). But notice that, without having to change numeraire in the rest of the analysis we can solve for μ by integration of (5) instead, i.e. of the equation of variation for the non-numeraire good. We thus obtain the alternative expression

$$\mu(n) = \int_{\underline{n}}^{n} \frac{u_y+\lambda w}{u_y} f(n') \exp\left\{-\int_{n'}^{n}(u_{yn}/u_y)dn''\right\}dn', \qquad (9)$$

which will be rather useful below.

ii. The above analysis requires that income be monotonic in the wage: in fact strictly monotonic except at possible corners of the tax schedule, which we are ignoring. Otherwise variations of the consumption by two different individuals n and n' $(n \neq n')$ who have $y(n) = y(n')$ could not be taken independently, as was implicitly done in the analysis above. The following assumption captures this:

<u>Assumption A.1</u>: $s_n \equiv \partial s(c, y; n)/\partial n < 0 \quad \forall (c, y, n).$

This simply says that different people's indifference curves through any given point (c, y) are flatter the higher their relative wage. The assumption is equivalent to (B) in Mirrlees (1971; p.182), namely that the function $V(c, L) \equiv -LU_L/U_c$ has $V_L > 0$,[1/] but put in this form it is less apparent what the condition represents or why it is required. Under this assumption monotonicity of demands under any budget set is ensured: y and c are non-decreasing functions of the individual wage (Mirrlees (1971, T1)), and in fact strictly increasing except where maximization occurs at corners of the budget set: this is clear from an indifference-curve diagram and can be formalized (e.g. by differential-equation methods). I am ignoring corners: these are discussed in some detail in Seade (1980) and it can be checked that our results below hold for points of differentiability (allowing for corners elsewhere), where marginal taxes are defined. Hence, at such points and under A.1,

$$dc/dn > 0, \quad dy/dn > 0 . \tag{10}$$

The requirement in A.1 is rather weak. It in particular generalizes (i) non-inferiority of consumption and (ii) the restrictive but in related contexts (e.g. Sheshinski (1972)) oft-adopted assumption of positively-sloped labour supplies.

The first of these cases can be shown differentiating $s(c,y;n) = - U_L/nU_c$ (from (1)) partially with respect to n, which yields

$$s_n = -\frac{s}{n} + \frac{Ls}{nU_c}\left(U_{cL} - \frac{U_c}{U_L}U_{LL}\right). \qquad (11)$$

The term in brackets is non-positive if and only if consumption is non-inferior, so that the latter is sufficient for $s_n < 0$ (using $s > 0$).

For the second case mentioned we must momentarily impose linear taxation for the thought-experiment to make sense: $c = \alpha + \beta nL$. Writing $L(\alpha, \beta; n)$ for the (optimal) labour supply, the assumption is that $\partial L/\partial\beta > 0$. This is equation (11) in Sheshinski (1972), which is imposed in that paper specifically to prove (mid p.300) that in the optimum $\beta < 1$, the linear counterpart to our aim in this paper. By standard comparative-statics demand-analysis, under the above linear constraint, we obtain

$$\frac{\partial L}{\partial\beta} = \frac{n\{U_c + L(U_{cL} - U_L U_{cc}/U_c)\}}{(U_L/U_c)\{2U_{cL} - U_L U_{cc}/U_c - U_c U_{LL}/U_L\}}. \qquad (12)$$

Strict quasiconcavity makes the denominator of (12) positive, so that for $\partial L/\partial\beta > 0$ the numerator must be positive too. Together, these yield

$$0 < -U_L + \frac{LU_L}{U_c}\left(U_{cL} - \frac{U_c}{U_L}U_{LL}\right) = -n^2 U_c s_n, \qquad (13)$$

hence $s_n < 0$ generalises this case as well.

3. SIGNING THE TAX: AN INTERMEDIATE RESULT

Under A.1, the sign of the tax in (7) is the sign of μ. Mirrlees' (1971; p.185) argument goes as follows: u_c, it is argued, is a decreasing function of c, hence the marginal utility of net income c is decreasing: $dc/dn > 0 \Rightarrow du_c/dn < 0$. (Mirrlees' discussion is in fact put in terms of weak inequalities throughout). This, if warranted, in turn ensures that $\mu(n) > 0 \;\forall\; n \in (\underline{n}, \bar{n})$. This is because for $\mu(\underline{n}) = \mu(\bar{n}) = 0$ to be met, the integrand in (8) hence the expression $(u_c - \lambda)$ must change sign at least once or be identically zero, and from $u_c > 0$ and $du_c/dn < 0$ there must be precisely one such change of sign, from positive to negative. Hence the truncated integral (8), leaving only a negative (or including only a positive) element from the full-domain integral $\mu(\bar{n}) = 0$, must be positive throughout the interior of $[\underline{n}, \bar{n}]$. The well-known and rather useful result follows from here that given A.1 alone, total tax payments are an increasing function of income in the optimum [Mirrlees (1971; Proposition 3)].

The above argument, however, is conditional upon the assertion that u_c is decreasing in c as we move along the tax schedule (with y adjusted accordingly), i.e. that $dc/dn > 0 \Rightarrow du_c/dn < 0$. This is true, directly, only under additive separability, which is implicit in the above analysis but explicitly introduced only in a later section of Mirrlees (1971). More generally, the implication does not hold, since effort and income are not kept constant as n and c increase.

Differentiating $u_c = U_c(c, y/n)$,

$$du_c/dn = U_{cc}c' + U_{cL}(y'/n - y/n^2)$$

$$= (U_{cc} - U_c U_{cL}/U_L)c' - U_{cL}y/n^2, \qquad (14)$$

using (2), the first-order condition for individuals, which in terms of the U-function reads $U_c c' + U_L y'/n = 0$. Hence sufficient conditions for the required non-positivity of du_c/dn are A.1 (hence $c' > 0$) plus

$$(U_{cc} - U_c U_{cL}/U_L) \leq 0, \qquad (15)$$

$$U_{cL} \geq 0. \qquad (16)$$

In fact, du_c/dn will be strictly negative if either of these inequalities is strict, but this is something that we do not need to impose separately: (15) and (16) can both hold only if one of them holds with slack. We therefore have

> Proposition 1. If A.1, (15) and (16) hold, that is income is non-decreasing in the wage, leisure is non-inferior, and leisure and consumption are non-complements in the Edgeworth ('cardinal') sense, then the optimum income-tax function is strictly increasing at all (observed) levels of income.

A result similar to this one (in weak form) is derived by Sadka (1976; T1), using (15), (16) and a stronger version of A.1, namely normality of consumption. Mirrlees (1971; P3) assumes A.1 and, implicitly, additive separability (as a statement both on preferences <u>and</u> on the cardinalization adopted) which directly implies (15) and (16). Sheshinski (1972) and Romer (1976) prove related results for the linear case under the assumption of forward-sloping labour supplies, by individuals and only by the aggregate respectively.

4. THE MAIN DISCUSSION

The first two assumptions used in the above result are reasonable ones: they come from individual preferences alone, and they are weak. The third, however, is not a good assumption: it is strong (it excludes for example, the Cobb-Doublas $c^{\alpha}(1-L)^{\beta}$!), and relates to both individual and the government's preferences in some obscure way, for the sign and size of U_{cL} depend on the cardinalization $U(\cdot)$ adopted. I shall now follow a different line of argument in search of a more satisfactory set of conditions ensuring a declining u_c, dependent 'almost' only on concavity of $U(\cdot)$.

We firstly need the following lemma, which in fact applies more generally to (and is thus stated in terms of) the non-linear-tax model of Seade (1977) and others, i.e. allowing for many goods and for more general structures of preferences than are being considered here.

<u>Definition:</u> $[n^o, n^1]$ <u>is a maximal range of single-signedness of</u> $\mu(n)$ <u>if the latter has a single sign or is zero in</u> (n^o, n^1) <u>and either it changes sign at</u> n^o (<u>alt.at</u> n^1), <u>or</u> $n^o = \underline{n}$ (<u>alt.</u> $n^1 = \bar{n}$).

Lemma 1. Consider an optimum income tax or set of non-linear taxes and let $[n^o, n^1]$ be a maximal range of single-signedness of $\mu(n)$. The marginal utilities of 'goods' (or minus marginal utilities of 'bads') need all be not lower or all not higher at n^1 than at n^o : it will be the former/latter depending on whether μ is non-positive/non-negative in the given range $[n^o, n^1]$.

Remark: The sign of distortions, given by (7) in the present case, is essentially determined by the sign of μ. What the lemma implies is that tax funds will be channelled towards one or other end of a one-sign range not only depending on where the marginal utility of numeraire is higher, which could in principle behave in a rather unpredictable way in view of cross effects with other goods, but on where the marginal utility of each good is higher. A monotonicity assumption on the marginal utility of a singled-out good is therefore essentially a monotonicity assumption on deservingness generally, on which one can more easily have an opinion.

Proof. The lemma holds independently of whether there is bunching at either end-point or corners at interior points of the schedule, but I am ignoring these complications, assuming interiority and differentiability $\forall n$. Suppose $[n^o, n^1]$ is a maximal range of non-negative μ. That is, in particular, $\mu(n) \geq 0 \; \forall n \in (n^o, n^1)$, with $\mu(n^o) = \mu(n^1) = 0$. It follows that μ is non-decreasing (non-increasing) 'near' n^o (resp. n^1) and hence that the integrand of (8) has the appropriate sign at those points, i.e. that $u_c \geq \lambda$ at n^o and $u_c \leq \lambda$ at n^1. Thence $u_c^o \geq u_c^1$ in obvious notation. Similarly, $u_c^o \leq u_c^1$ if $\mu(n) \leq 0$ in $[n^o, n^1]$. But exactly the same argument applies to the intergrand of (9), or more generally to expressions equivalent to (8) or (9) obtained from the variation of non-numeraire goods. Arguing as before, it follows that for a maximal range of say non-negative μ, $(u_y + \lambda w)/u_y$ must be ≥ 0 (≤ 0) at n^o (at n^1), and since $u_y < 0$, the implication in the lemma follows. Similarly for other 'goods' as in (8) or 'bads' as in (9), and similarly for ranges of non-positive μ. ||

We now apply this result to the structure of utility of the present model, given by equation (1) above. We wish to derive conditions to ensure strict positivity of the marginal income tax at all interior levels of income. Suppose therefore that, to the contrary, $\mu(n) \leq 0$ $\forall$ $n \in (n^o, n^1)$, where $[n^o, n^1]$ is a maximal range of single-signed μ. From the lemma, and since $u_y < 0$, it follows that $u_c^1 \geq u_c^o$ and $u_y^1 \leq u_y^o$. That is, expressing these inequalities in terms of the function $U(c,L)$,

$$U_c(c^1, L^1) \geq U_c(c^o, L^o) \tag{17}$$

and

$$U_L(c^1, L^1)/n^1 \leq U_L(c^o, L^o)/n^o, \tag{18}$$

using $u_c = U_c$, $u_y = U_L/n$. By $n^1 > n^o \geq 0$, 2/ (18) further implies

$$U_L(c^1, L^1) \leq (n^1/n^o)\, U_L(c^o, L^o)$$

$$< U_L(c^o, L^o). \tag{19}$$

On the other hand, from individual maximization, utility is non-decreasing in the wage:

$$U(c^1, y^1/n^1) \geq U(c^o, y^o/n^1)$$

$$\geq U(c^o, y^o/n^o)$$

(in fact '>' if $y^o > 0$) so that

$$U(c^1, L^1) \geq U(c^o, L^o). \tag{20}$$

We thus require, for a non-positive interior marginal tax, that

$$U_c^1 \geq U_c^o, \quad U_L^1 < U_L^o \quad \text{and} \quad U^1 \geq U^o. \tag{21}$$

That is, that it be possible for both marginal utilities of 'goods' (e.g. of leisure, namely $-U_L$) to increase, one of them strictly, as a not-lower-utility point is reached. This possibility is ruled out, under suitable conditions, using the following result of Dixit and Seade:

> Lemma 2 [Dixit and Seade (1980)]. Let $v : R^n \to R$ be twice continuously differentiable, monotone increasing, and strictly concave. Given a point x^1 in R^n, we can find a neighbourhood N of it and another point x^2 in N such that $v(x^2) > v(x^1)$ and grad $v(x^2) \gg$ grad $v(x^1)$, if and only if at least one commodity is inferior in N.

In fact the inequalities in this lemma are all strict while some of those in (21) are not. If only one (or no) inequality in (21) were to hold without slack, we would be able to find neighbouring (c,L)-points with all three inequalities strict, by continuity of U and its derivatives. This we can rule out assuming non-inferiority of consumption and leisure and applying the lemma. On the other hand both weak inequalities in (21) will hold as equalities if and only if a path of constant U_c and constant U exists between the two points in question, which can easily be checked to amount to income-independence (not normality nor inferiority) of leisure on that path. Hence under strict normality of leisure and non-inferiority of consumption, (21) will not arise. We can thuse state the following:

Proposition 2. If consumption and leisure are normal, the optimum income-tax function is strictly increasing at all (observed) levels of income.

(A.1, required for the analysis generally, is implied by non-inferiority of consumption and therefore not mentioned separately).

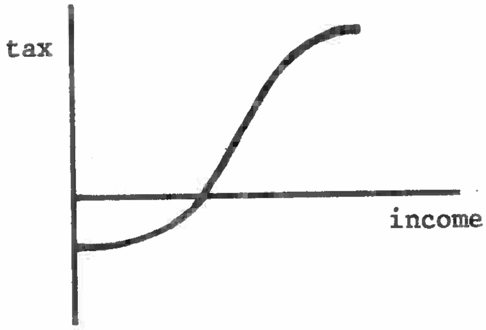
Let us now take stock of the possibilities that arise. If both goods are normal the marginal tax, wherever it exists, is strictly positive, by P2. Furthermore if either good is income-independent, i.e. not normal nor inferior, normality of the other good and positivity of U_{cL} must both follow, so that in either case the conditions for P1 are met. Similarly if consumption is actually inferior (but A.1 is reimposed), P1 again applies.

We can therefore bring P1 and P2 together in the following, main result of the paper, with the widest conditions we have found to ensure the intuitively so plausible positivity of the marginal income tax.

Theorem 1. If leisure is non-inferior and A.1 holds, the optimum income-tax function is strictly increasing at all (observed) levels of income.

5. CONCLUDING REMARKS

i. The above result is the natural complement to the condition, derived in Seade (1977), that the marginal tax should be zero at both endpoints of the scale (of the tax-base in question, in this case the income scale). This now says that, under very plausible conditions (but given identical preferences!), the distortion should be zero only there, at the endpoints, and positive elsewhere. Hence the characteristic shape of optimum income taxes is that of an S, in fact a strict S in the obvious sense, as in the figure [3/]. All interior points have redistributive taxation.



This is what one would expect in the light of the rationale offered for the no-distortion-endpoint result in the above-mentioned paper (pp.231 f.) A non-zero marginal tax at a given point is only a means to transfer income between the groups strictly above and below that point (vis-à-vis using a flat poll tax), having only a second-order effect on the income of those whose earnings are exactly of that level. In contrast, the distortion has a negative incentive effect precisely and only at the point where it is imposed, on the behaviour of the local n-men. Hence the redistribution gain from a distortion 'near' an endpoint is negligible relative to the deadweight costs, for either the group 'below' or the group 'above' an endpoint is empty: no payers left to capture tax from, at the top, nor beneficiaries of a locally-redistributive policy at the bottom. But by the same token that redistributive effect on a distortion does not vanish anywhere else, at each point in the interior of the schedule. At such points, and if redistribution itself is desirable at the margin as it is usual to assume (falling marginal utility of consumption), a positive distortion is in order. Checking whether this redistributive motive (in the egalitarian direction!) would necessarily arise given 'concave utilitarianism' alone was in effect the main problem tackled in this paper, for the central, special case of the model of Mirrlees (1971).

ii. The question remains of whether or not the positive-distortions result would extend to cases not meeting either of the two assumptions made, namely A.1 and non-inferiority of leisure. The former is an essential assumption for the method of solution adopted, a regularity condition whose relaxation is perhaps of little interest as it is a rather weak condition. But intuitively, allowing for income-reversals with the wage, one would expect 'deservingness' at a given level of income to relate to some weighted average of the marginal utilities of local men, so that total tax liability should be higher at income y^o than at $y^1 > y^o$ if consumers at y^o are predominantly 'richer', wage-rate-wise, than those at y^1.

On the other hand, the assumption on leisure plays no role in the analysis except in these results, and is the stronger of the two conditions. A simple argument like that used in the derivation of Proposition 1 turns out not to work. Specifically, seeking to obtain an expression embodying normality of consumption (hence allowing for inferiority of leisure), we proceed as we did in §3 in deriving P1, but now differentiating u_y instead of u_c:

$$du_y/dn = d\{U_L(c, y/n)/n\}dn$$

$$= -(U_L + LU_{LL})/n^2 + (U_L U_{cL} - U_c U_{LL})c'/nU_L \quad (22)$$

after some manipulations, using the first-order condition for individuals $u_c c' + u_y y' = 0$. We thus obtain the 'normality' expression for consumption as expected; but (22) has ambiguous sign in the case that matters: when consumption is normal as inferiority of leisure would require.

Now this indeterminacy of the optimal pattern of redistribution when leisure is inferior may be for good reasons. Suppose for example that leisure is normal for most of the scale but inferior from some point n^o upwards. A lump-sum transfer from lower incomes to this higher inferiority-range would unambiguously increase everybody's supply of labour hence total revenue (given positive taxes at the margin elsewhere), and the increase in the latter could then be fed back into the economy. Thus, even with some degree of inequality aversion (concave preferences) it might pay to effect some such anti-redistributive transfer from bottom to top, which of course calls for a negative marginal tax at and near the point n^o. The possibility for a gain in this form is fairly clear if one starts from an arbitrary initial situation, but conditions for optimality might well exclude such possibility, and we would not advance the conjecture that non-inferiority of leisure be an essential ingredient for the result to hold.

It should finally be added that the remarks in the previous paragraphs are only made for the sake of completeness, to suggest why or how the result might fail to hold if the conditions obtained were removed; this may in turn be of some use in understanding the way the problem works. But for actual tax-design these possibilities can safely be ignored: normality of leisure and monotonicity of income on the wage seem excellent assumptions to make, and marginal rates of income tax should, in the absence of extraneous considerations to the contrary, indeed generally be positive.

I am grateful to Peter Hammond for detailed and valuable comments, and for urging me to look for a counter-example to proposition 1 (without its special assumptions), thus motivating the fourth and main section of the paper. Comments by Vidar Christiansen, Nick Stern and a referee are also acknowledged with thanks. The paper is based on a section (§3.1) of my D.Phil. thesis, presented at Oxford in Michaelmas Term 1979, for which Jim Mirrlees provided most helpful supervision.

Footnotes:

1. Another similar expression used by Mirrlees is defined by $\{\Psi(u,L) = -LU_L(c, L),\ u = U(c, L)\}$, which appears in his main tax-equation (27) (our (7)), although his Proposition 3 (on the sign of the tax) is formulated in terms of V. The equivalence of A.1 with $V_L > 0$ and $\Psi_L > 0$ follows from $V_L = \Psi_L/U_c = n^2 s_n$.

2. There is no problem in allowing $n^o = 0$, in (18) and below. In that case we put $U_L/n^o = -\infty$ (u_y is $-\infty$ for someone with a zero wage!) and the discussion applies all the same.

3. This is only a statement on the 'typical' (or central) shape of the tax curve, which could however have more than two 'bends', depending upon greater detail on the functions involved. For example, marginal taxes would be higher, other things equal, at interior low-density ranges of n, which may occur in a disjoint fashion.

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