SPURIOUS PERIODICITY IN

INAPPROPRIATELY DETRENDED TIME SERIES

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1. Introduction

Econometric analysis of time series data is frequently preceded by regression on time to remove a trend component in the data. The resulting residuals are then treated as a stationary series to which procedures requiring stationarity, such as spectral analysis, can be applied. The objective is often to investigate the dynamics of transitory movements in the system, for example, in econometric models of the business cycle. When the data does consist of a deterministic function of time plus a stationary error then regression residuals will clearly be unbiased estimates of the stationary component. However, if the data is generated by (possibly repeated) summation of a stationary and invertible process then the series cannot be expressed as a deterministic function of time plus a stationary deviation, even though a least squares trend line and the associated residuals can always be calculated for any given finite sample. In a recent paper, Chan, Hayya, and Ord (1977)(hereafter CHO) were able to show that residuals from linear regression of a realization of a random walk (the summation of a purely random series) on time have autocovariances which for given lag are a function of time and therefore that the residuals are not stationary. Further, CHO established that the expected sample autocovariance function (the expected autocovariances for given lag averaged over the time interval of the sample) is a function of sample size as well as lag and therefore an artifact of the detrending procedure. This function is characterized by CHO in their Figure 1 as being effectively linear in lag (although the exact function is a fifth degree polynomial) with the rate of decay from unity at the origin depending inversely on sample size. The first differences of a random walk are, of course, stationary with zero autocovariance at all lags. They concluded that "the low frequency portion of the spectrum will be exaggerated and the high frequency portion attenuated" relative to the appropriate first difference transformation.

The objective of this paper is to show that after the expected sample autocovariance function given by CHO is corrected for several errors in the values of coefficients and is examined over a greater range of lags it is seen to imply strongly pseudo-periodic behavior in the time trend residuals. The corresponding spectral density function has a single peak at a period corresponding to .83 of the number of observations in the sample. The distribution of the peak in sample power spectra is studied in a Monte Carlo experiment and is shown to have a mean period corresponding to .65 of sample length with a standard deviation of .21 of sample size. Our results suggest that inappropriate detrending of time series will tend to produce apparent evidence of periodicity which is not in any meaningful sense a property of the underlying system. They further suggest that the dynamics of econometric models estimated from such data may well be wholly or in part an artifact of the trend removal procedure. Since the random walk model is widely accepted as a valid representation of stock market prices we illustrate the phenomenon of pseudo-periodicity by applying time trend regression to the Standard and Poor's 500 Stock Index. We also offer some evidence about the effect of serial correlation in first differences on the distribution of peaks in sample power spectra.

2. The Expected Sample Autocovariance Function and Approximate Expected Autocorrelation and Spectral Density Function for Time Trend Residuals from a Random Walk

The autocovariance function for the residuals produced by regression of a realization of a random walk on time has been derived by CHO (their expression 3.10) and for given lag it is a function of time and the number of observations in the sample. The residuals are therefore nonstationary, but it is straightforward to derive the expected value of the sample autocovariance function as the average across the sample of the autocovariances for given lag. The expression given by CHO (3.13) for this averaged autocovariance contains several numerical errors which materially affect its shape. The corrected expression in the notation of CHO is

$$
\begin{aligned}
\overline{\mathrm{cov}}(s,n) &= \frac{1}{N}\sum_{t=-n+s}^{n} \mathrm{cov}(\hat{\varepsilon}_t, \hat{\varepsilon}_{t-s}) \\
&= \{(32n^5 + 80n^4 + 40n^3 - 20n^2 - 12n) \\
&\quad + s(152n^4 - 304n^3 - 150n^2 + 2n + 6) \\
&\quad + s^2(180n^3 + 270n^2 + 90n) \\
&\quad + s^3(-68n^2 - 68n - 9) \\
&\quad + s^5(3)\} \frac{\sigma^2}{60n(n+1)(2n+1)^2}
\end{aligned} \tag{2.1}
$$

where $N = 2n + 1$ is the length of the sample and σ^2 is the variance of the steps $\{\varepsilon_t\}$ in the underlying random walk. We follow CHO in approximating the expected sample autocorrelation function by $\bar{\rho}(s,n) \equiv \overline{\mathrm{cov}}(s,n)/\overline{\mathrm{cov}}(0,n)$ which can be written as

$$
\begin{aligned}
\bar{\rho}(s,n) &= 1 + \left(\frac{s}{n}\right)\left(\frac{-152 - 304n^{-1} - 150n^{-2} + 2n^{-3} + 6n^{-4}}{32 + 80n^{-1} + 40n^{-2} - 20n^{-3} - 12n^{-4}}\right) \\
&\quad + \left(\frac{s}{n}\right)^2\left(\frac{180 + 270n^{-1} + 90n^{-2}}{32 + 80n^{-1} + 40n^{-2} - 20n^{-3} - 12n^{-4}}\right) \\
&\quad + \left(\frac{s}{n}\right)^3\left(\frac{-68 - 68n^{-1} - 9n^{-2}}{32 + 80n^{-1} + 40n^{-2} - 20n^{-3} - 12n^{-4}}\right) \\
&\quad + \left(\frac{s}{n}\right)^5\left(\frac{3}{32 + 80n^{-1} + 40n^{-2} - 20n^{-3} - 12n^{-4}}\right).
\end{aligned} \tag{2.2}
$$

It is apparent that for large samples $\bar{\rho}$ becomes a polynomial in (s/n) and will therefore take on the same numerical value at whatever lag corresponds to a given fraction of sample length. Maxima and minima will likewise occur at fixed functions of sample length and at the same values of $\bar{\rho}$. The exact function is plotted in Figure 1 for sample length 101(n = 50). Note that the autocorrelations decline monotonically until a value of -.28 is reached at lag 34 (that is .34 of

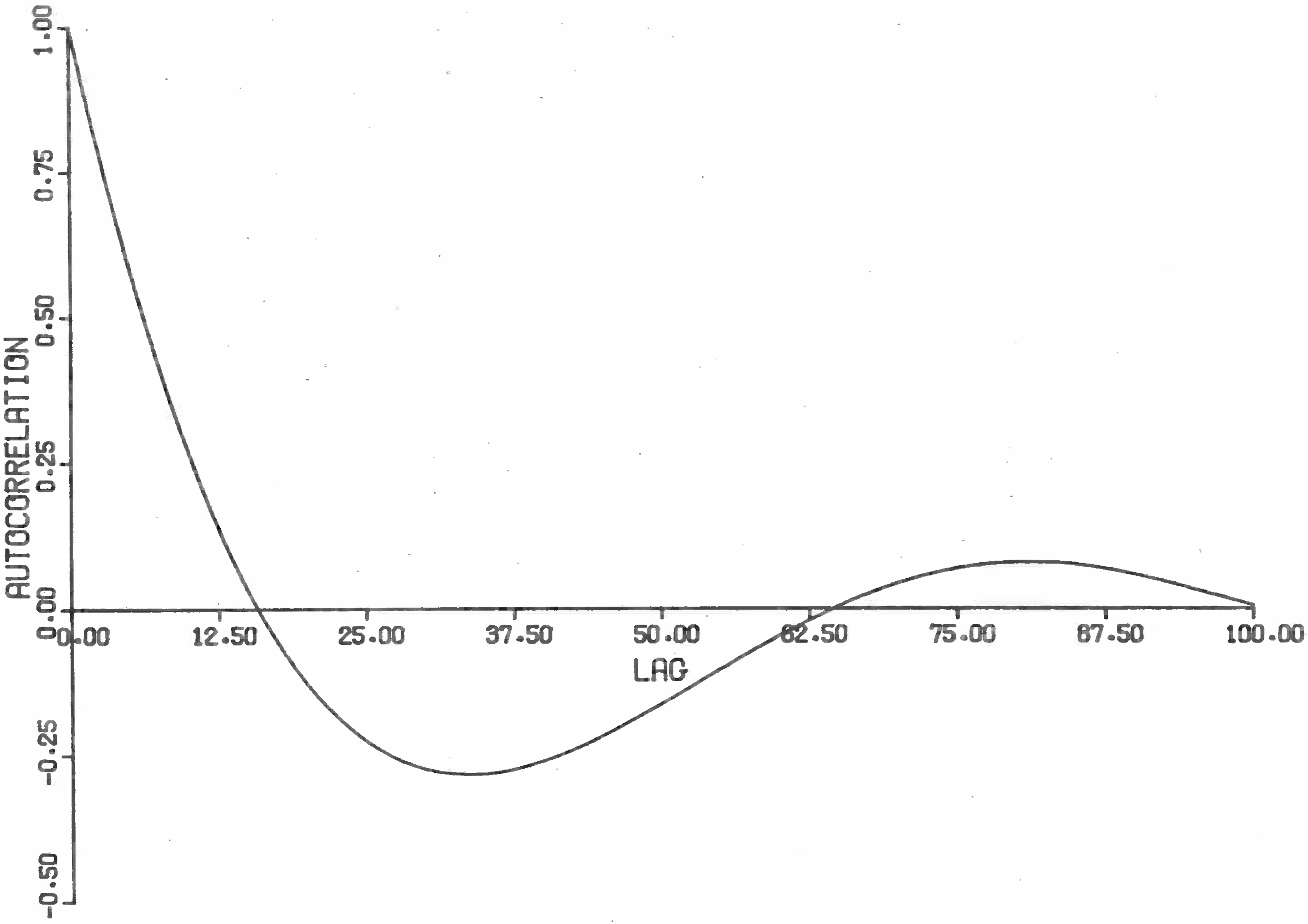

FIG. 1 THEORETICAL AUTOCORRELATIONS FOR DETRENDED RANDOM WALK

sample length), then increase until a value of .08 is reached at lag 81 (.80 of sample length) and finally decline toward zero. For comparison, in a sample of length 11 ($n = 5$) the minimum occurs at lag 4 (.36 of sample length) with a $\bar{\rho}$ value of -.29 and the second maximum at lag 9 (.82 of sample length) with a value of .08. For practical purposes then the shape of the $\bar{\rho}$ function is effectively independent of sample size. The incorrect expression for $\bar{\rho}(s,n)$ given by CHO does not decline as rapidly at low lags, the slope for small values of (s/n) being roughly -3.5 for the CHO expression as opposed to -4.75 for the corrected expression. The CHO expression reaches a first minimum of +.09 at about .33 of sample length and a second maximum of +.22 at about .66 of sample length before declining toward zero. The figures provided by CHO show only the linear decline which characterizes the function for small values of (s/n). The corrected function is more strongly non-linear even at low lags since the coefficient of $(s/n)^2$ is now larger. Calculated values for the CHO and the corrected $\bar{\rho}(S)$ at selected lags are compared in Table 1 for a sample size of 101.

The corrected expected autocovariance function has the appearance of a damped sine wave which is indicative of pseudo-periodic behavior in the residual series with a period equal to .80 of the length of the sample or equivalently a frequency of $1.25N^{-1}$. The sample spectral density function corresponding to $\bar{\rho}(s,n)$ is defined as $SDF(f,n) = 1 + 2 \sum_{s=1}^{N-1} \bar{\rho}(s,n)\cos(2\pi fs)$, $0<f<\frac{1}{2}$, and is plotted in Figure 2 for $N = 51$ and 101 over $0<f<.10$. It has its maximum at frequency .024 for $N = 51$ and at .012 for $N = 101$ each corresponding to $1.25N^{-1}$. Note that as the sample size is increased the value at the maximum of the spectral density increases and the entire distribution becomes more concentrated at lower frequencies. It is also possible to show that the value of the function at frequency $f = 0$ is identically zero, while for the corresponding function implied by the CHO expression the value at $f = 0$ is of order n with

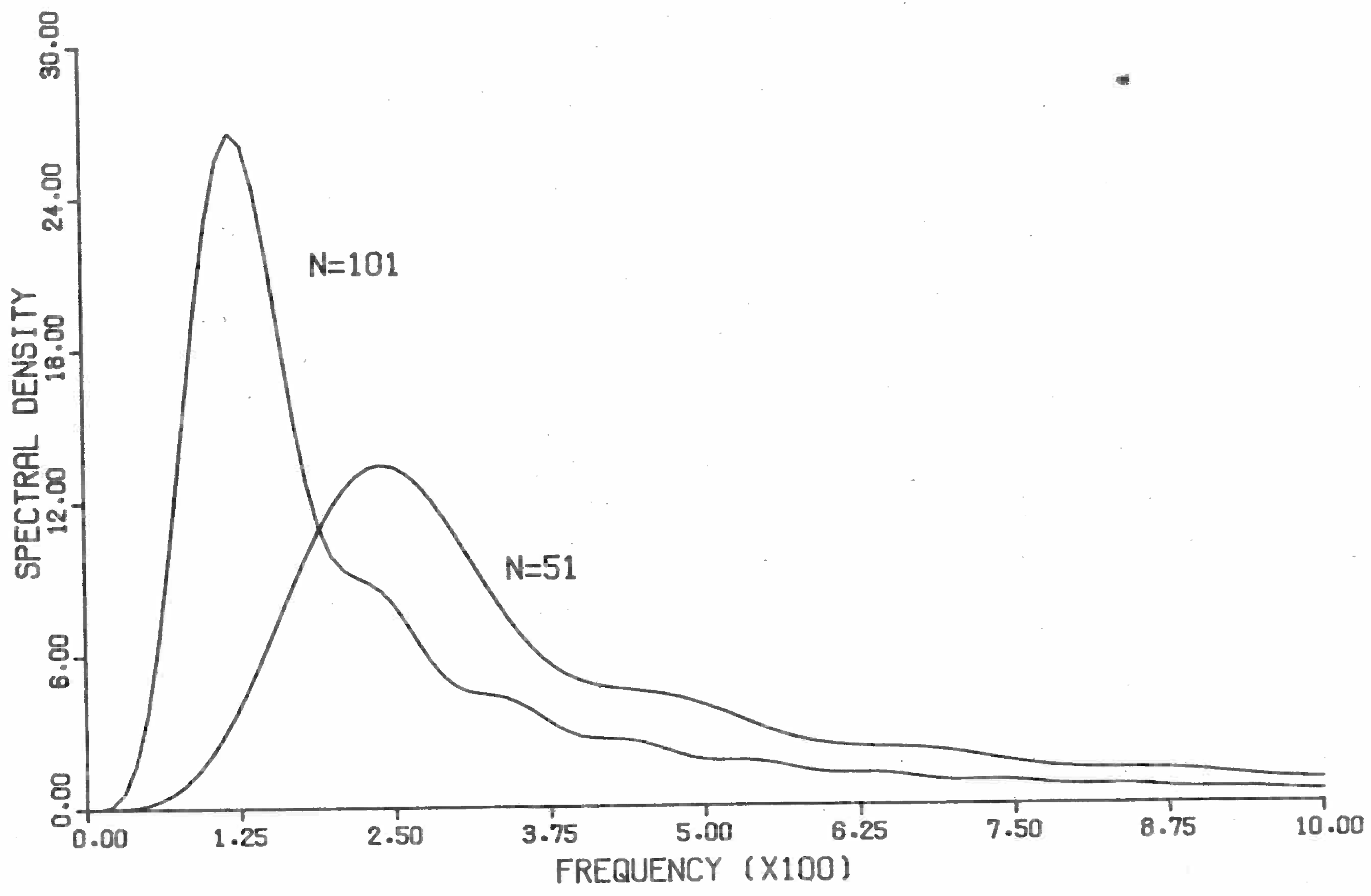

FIG. 2 THEORETICAL SPECTRAL DENSITY FUNCTIONS FOR DETRENDED RANDOM WALK

TABLE 1

APPROXIMATION TO EXPECTED SAMPLE AUTOCORRELATION FUNCTION

FOR SAMPLE OF LENGTH N = 101 (n = 50): CORRECTED AND CHO EXPRESSION

Lag s	1	2	3	4	5	10	20	30	50	75
$\bar{\rho}(S)$	.91	.82	.74	.66	.58	.26	-.13	-.28	-.16	.07
CHO	.93	.87	.81	.75	.70	.47	.19	.10	.16	.20

leading term (5/6)n and thus rises roughly in proportion to sample size. The density function values at $f = \frac{1}{2}$ in both cases decrease roughly in proportion to sample size, leading terms being $(5/2)n^{-1}$ and $(15/8)n^{-1}$ for corrected and CHO versions respectively. The graphical characterization displayed by CHO in their Figure 1 is of monotonic decline from the origin while the corrected function rises from zero at the origin to a peak at frequency $1.25N^{-1}$ and then declines. Their graphical characterization is also not correct for the CHO density function expression which in fact has a prominent peak at a frequency of about $1.37N^{-1}$ in addition to a peak at the origin.

Sample estimates of spectral density functions are usually computed in practice from a sample autocorrelation function which is truncated at a lag considerably smaller than the maximum (N-1) lags computable. The effect of truncation of the $\overline{\rho}(s,n)$ function on the corresponding spectral density function is to reduce the prominence of the major peak and to introduce secondary peaks at higher frequencies which are not present in the complete function. The secondary peaks are spaced at frequency intervals equal to the inverse of the lag at which the $\overline{\rho}(S)$ function is truncated. For a sample of length 101 the primary peak is still in evidence using as few as 25 lags in the autocorrelation function and has shifted from a period of 83 to one of 62 but disappears when only 16 lags are used. In effect, the nonlinearity of $\overline{\rho}(s,n)$ and therefore its pseudo-periodicity is decreasingly evident at very low lags where the function is roughly exponential, resembling that of a first order autoregression. The spectral density function in that case of course also resembles that of a first order autoregression except for weak secondary peaks as noted above.

3. The Sampling Distribution of Peaks in Sample Spectral Density Function

The corrected theoretical results of CHO imply that one would expect to find a predominant peak at a low frequency in the sample spectrum of residuals obtained by regression of a random walk on time but do not lead readily to a description of the sampling distribution of the period or frequency at which the predominant peak occurs. We have conducted a Monte Carlo experiment to provide some idea of what this distribution looks like. Realization of length 101 observations were generated by a random walk process, regressed on time, and the sample autocorrelations and spectrum calculated. For each realization the spectrum was searched to find its maximum on a grid of .001 intervals starting at .010, the lowest frequency of practical interest. This was repeated for 500 independent realizations.

In 20 cases out of 500 the maximum was recorded at f = .010. The highest frequency at which the maximum was recorded was at .065 corresponding to a period of 15. The value of the spectral density in that case was 15.22 which would be regarded as highly significant by an investigator having the null hypothesis of white noise. The mean frequency at which maxima occurred was .018 corresponding to a period of 56, while the mean period at which maxima occurred was 66 corresponding to a frequency of .015. The standard deviation for peak frequencies was .008 and for peak periods 21.5. The average sample autocorrelation function reached a low point of -.23 at lags 29 through 34 and a second maximum of +.06 at lags 78 through 87 which compare with -.28 at lag 34 and +.08 at lag .81 for the expression $\overline{\rho}(S)$ given by (2.2) which approximates the expected value of $\rho(S)$ by the ratio of expected autocovariances. The average sample spectral density reaches its peak at the same frequency as the theoretical approximation (.012) but at a lower value, 21 compared with 27.

The results of the sampling experiment suggest then that we would expect apparent periodicity to be encountered at something around two-thirds of the length of the sample if a random walk describes the true nature of the data being detrended. Further, in many samples the predominant peak will occur at a period equivalent to half or less of sample length.

4. Spurious Periodicity in Stock Market Prices

An extensive literature supports the hypothesis that the log of stock market prices is well characterized as a random walk. For example, Granger and Morgenstein (1963) studied the spectrum of first differences of a stock price index and concluded that it is essentially flat as the hypothesis implies. To see if time trend residuals from stock price index display the spurious low frequency periodicity suggested by our analysis we examine monthly closing values for the Standard and Poor's 500 Index. The Mónte Carlo evidence suggests that we should find a prominent peak at a frequency equal to about $1.82N^{-1}$. If this peak appears in the sample spectrum and is indeed spurious, then extention of the sample by addition of more data should be accompanied by a shift in the peak to a proportionately lower frequency. Genuine periodicities should, in contrast, persist as the record is lengthened.

To carry out this kind of experiment we divided the 192 month period January 1961 through December 1976 into two 96 month periods and detrended the data in logs for each subperiod separately. The sample spectral densities for each using all 95 computable sample autocorrelations are displayed in Figure 3. Each exhibits a highly significant peak at a frequency close to the expected value of .019 which corresponds to a period of 53 months. When the combined sample of 192 months is detrended the sample spectrum is the one displayed in Figure 4. The predicted frequency for the spurious peak is .009. What is observed is a very low frequency peak at .006, a second and less prominent peak at .013, and a third very prominent peak at .020. Lengthening of the sample has evidently resulted in separation of the peak at .019 into two peaks, one presumably the spurious peak and the other possibly a genuine periodicity. Note that .021 might be thought of as a "Presidential frequency" since it corresponds to the 48 months between Presidential elections. To see if this peak as well as the

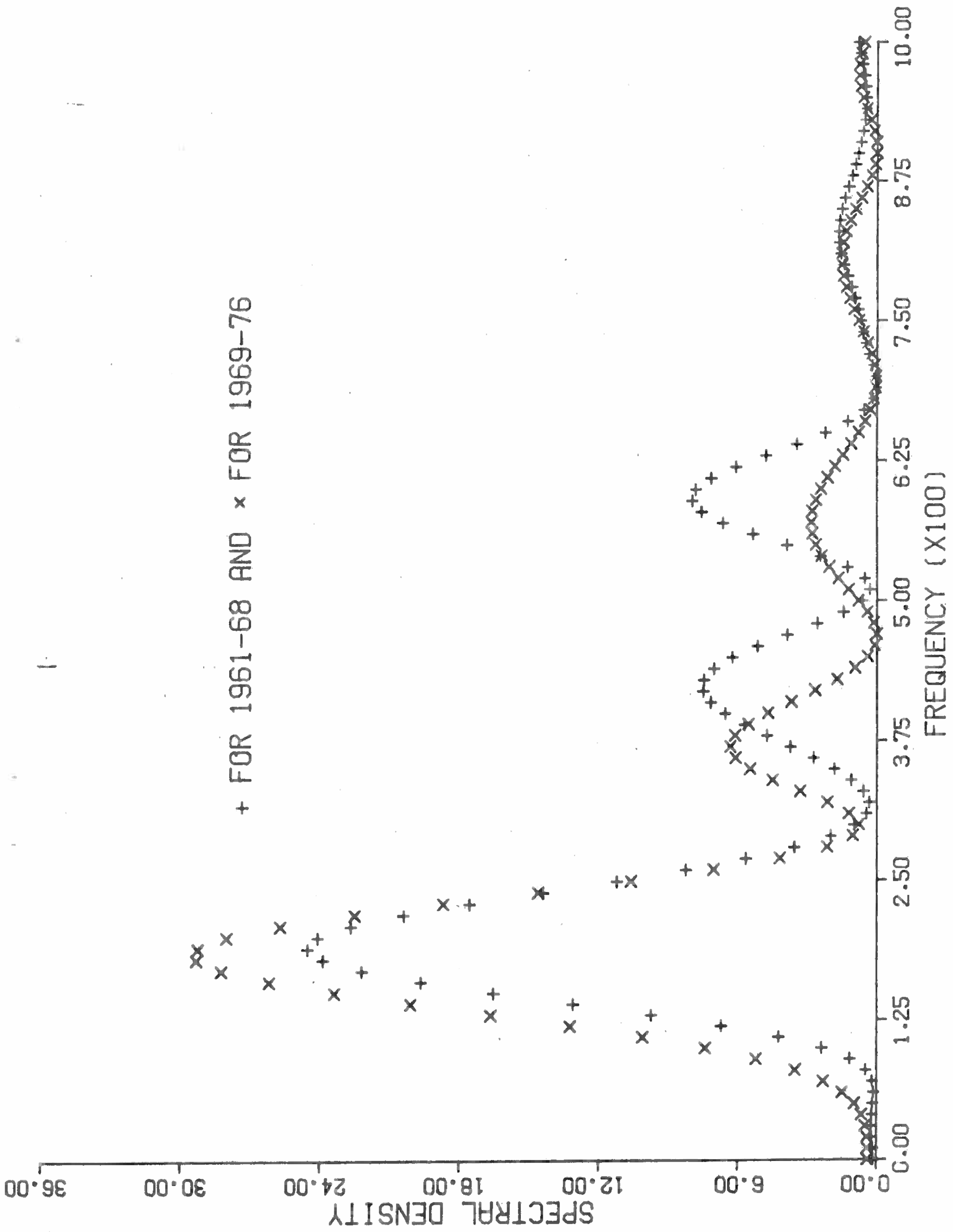

FIG. 3 SAMPLE SPECTRA FOR DETRENDED STOCK PRICES

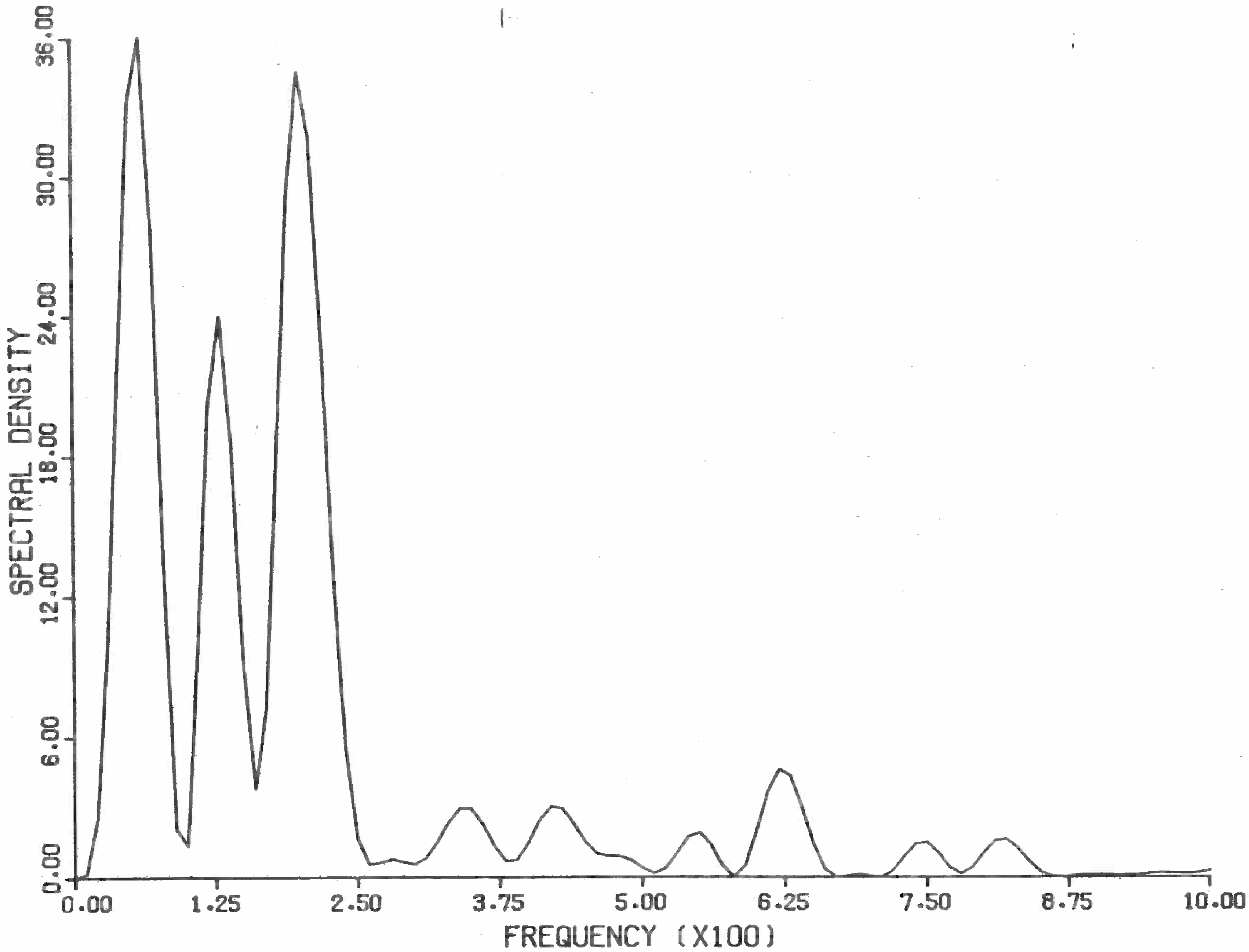

FIG. 4 SAMPLE SPECTRA FOR DETRENDED STOCK PRICES, 1961-1976

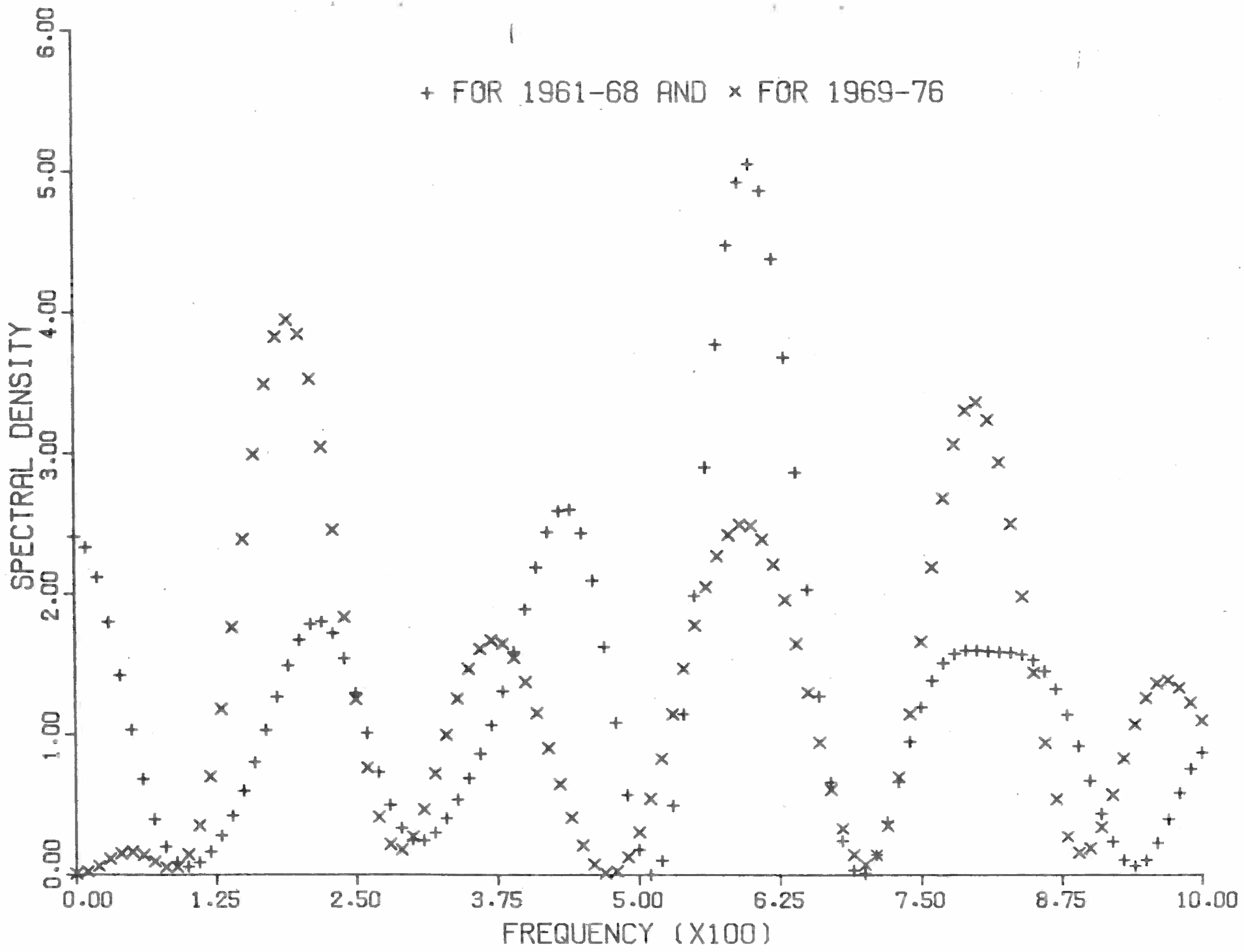

FIG. 5 SAMPLE SPECTRA FOR FIRST DIFFERENCES OF STOCK PRICES

lower frequency peaks can be related to structure present in the first differences the sample spectra for first differences were computed and are displayed in Figure 5. For the 1961-1968 subperiod no significant peak is evident near the Presidential frequency; however, a significant peak is apparent around .019 in the 1969-1976 subperiod. The effect of linear regression detrending was evidently to add a spurious peak near .02 in the earlier subperiod and to greatly reinforce the peak present in the second subperiod. This reinforcement also carries over to the spectrum for the detrended combined sample in what we can only interpret as a phenomenon akin to resonance which we speculate may also account for the peak at .013 which does not show up in any of the other spectra.

5. The Effect of Serial Correlation in First Differences on the Sample Spectrum

Analysis of the pure random walk case has indicated that as sample length increases the sample spectrum of time trend residuals will become increasingly concentrated around a peak at a frequency which decreases in proportion to the inverse of sample size. Descriptively, the residuals will tend to exhibit cycles of increasing length and increasing amplitude around a fitted trend line as the length of the sample is increased. Clearly, periodicities due to underlying autocorrelation in first differences will be of fixed period and amplitude and, hence, will contribute relatively less to total variance in longer samples. In the mixing of actual periodicities with the spurious periodicity associated with regression detrending we would expect as a rough characterization that positive autocorrelation in first differences would reinforce the spurious low frequency peak and negative autocorrelation would dilute it. This is confirmed in a second Monte Carlo experiment in which thirty independent samples of length 101 observations were generated by the integrated process

$$X_t = X_{t-1} + D_t$$

$$D_t = \phi D_{t-1} + \varepsilon_t \qquad (5.1)$$

using successively 0, +.3, -3, +.8, -.8 as values for ϕ. The random number generator was restarted with the same seed for each successive value of ϕ so that there was no sampling variation across values of ϕ.

The results of the experiment are summarized in Table 2. They indicate that the spurious peak will dominate the spectrum in a sample of realistic length and for values of ϕ of the magnitude often encountered in practice. They confirm that the effect of underlying positive autocorrelation will be to shift the distribution of the peak towards lower frequencies (longer periods). The average of the sample spectra shows greater concentration of variance around its peak and slight shifting of the peak to lower frequencies. The correspondingly reverse implications are confirmed for the case of negative autocorrelation.

While the sample spectra do then exhibit characteristics traceable to the underlying autocorrelation structure of the first difference process, it seems unlikely that the latter would be detected in practice. For example, consider the comparison in Figure 6 of the averaged sample spectra of detrended data from (5.1) for $\phi = -.3$ with the theoretical spectral density of the first differences. Evidently, the high frequency power present in the theoretical spectrum is obliterated by the low frequency power introduced by regression detrending. It seems plausible to us that a major cost of inappropriate detrending may be that genuine dynamics will often be overlooked. Perhaps this explains in part why Granger (1966) was able to characterize economic time series as having a "typical spectral shape" which was closely akin to that described in this paper as being typical of integrated time series which have been detrended by regression on time.

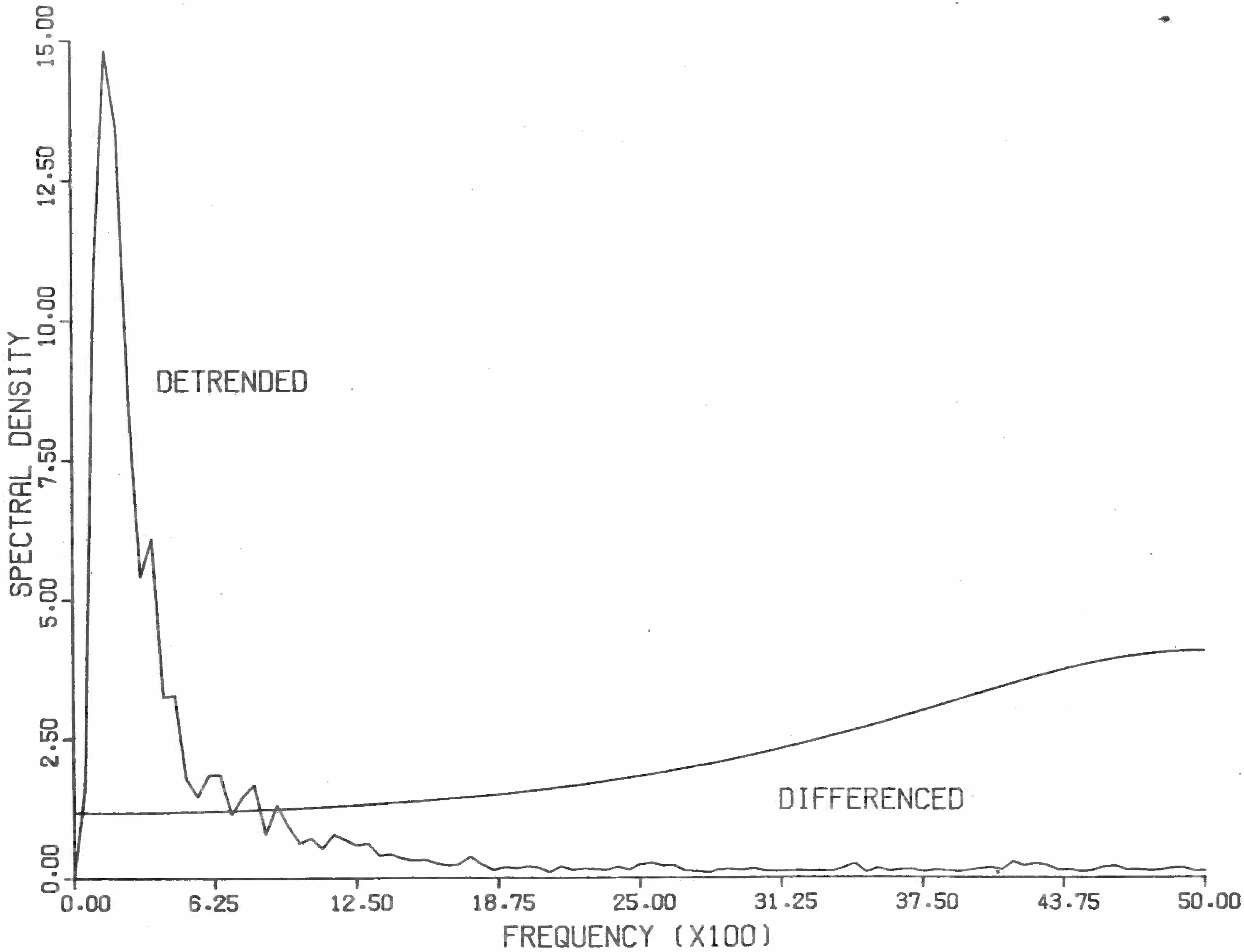

FIG. 6 SDF OF INTEGRATED AR(1) PROCESS,

FIRST DIFFERENCES AND TREND RESIDUALS

TABLE 2

SUMMARY OF MONTE CARLO EXPERIMENT FOR

INTEGRATED PROCESSES WITH AUTOREGRESSIVE DIFFERENCES

	Average of		Average Sample SDF	
ϕ	Peak Freq.	Peak Period	Freq. at Peak	Value at Peak
0.0	.021	57	.014	15.7
0.3	.020	59	.014	16.7
-0.3	.021	57	.014	14.8
0.8	.019	63	.013	22.1
-0.8	.022	56	.014	10.9

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