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THE STABILITY OF COURNOT REVISITED

Jesus Seade\*

Warwick Economic Research Paper No. 121, November 1977.

Revised January 1979

**WARWICK ECONOMIC RESEARCH PAPERS**

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## ABSTRACT

It is shown that under certain previously neglected conditions, not implausible and consistent with profit-maximisation, the Cournot oligopoly solution is unstable: a saddle point in  $R^n$  whose unstable manifold has dimension one, a line, and is contained in  $R_{++}^n \cup R_{--}^n \cup 0$ . That is, output-deviations from equilibrium are (or become) all of the same sign: boom or collapse for the whole industry.

Also, (i) a more general 'quasi-Cournot' model with non-zero (but well-defined and parametric) conjectural variations is shown to have similar behaviour as Cournot, (ii) Hahn's [3] range of cases where stability obtains is widened somewhat and (iii) the problems of regularity and linearisability of Cournot equilibria are briefly considered.

## 1. Introduction

The stability properties of the Cournot model of oligopoly aroused considerable interest back in the early sixties, culminating in a paper by Frank Hahn [3], who found certain sufficient conditions for stability of the model, under the standard adjustment system. These were that the marginal-cost curve for individual firms should not fall faster (if at all) than market demand, and that the marginal revenue of each producer should fall, at given output of his, were the remaining producers to expand their collective output. The meaning of the first of these conditions is transparent, and it in fact seems an acceptable assumption for most cases of interest. The second condition, however, is stronger than it might at first sight look, for it excludes a class of interesting cases, corresponding to certain perfectly reasonable demand configurations, only on account of the shape of these demands, basically their curvature, despite the fact that they are compatible with market equilibrium.

The present paper extends the analysis of stability into the cases just mentioned. Sufficient conditions are established, consistent with profit-maximisation and hence a priori of interest, under which the Cournot solution to the oligopoly problem is unstable (Sec. 3): a saddle point in  $R^n$  convergent on a hyper-surface (Sec. 4), so that the unstable manifold has dimension one: a divergent line of flow (orbit) which acts as an attractor for nearly all paths. This path, moreover, leaves from equilibrium into the strictly positive and negative orthants (in the space of divergencies of firms' outputs from equilibrium values). That is, if instability arises it does so in the form of a boom or collapse

homogeneously across the industry. The conditions for instability, a simple example will suggest, are perhaps special but by no means peculiar. Cournot equilibria turn out to be less universally-stable than a reading of the literature might suggest.

The above describes, in essence, our main results. Also, (i) we shall briefly extend these results to a more general "quasi-Cournot" model, with non-zero but parametric conjectural variations (Sec. 5) and (ii) Hahn's range of cases where stability holds will be widened somewhat (in appendix 2). Two technical points, local uniqueness (regularity) of equilibria and linearisability (non-degeneracy) of the flow, are taken up in appendices. In [6] we use the present stability results to show, à la correspondence principle, that certain 'perverse' effects of entry into Cournot equilibrium only arise under unstable equilibria and can be ruled out, while others (such as each firm's output rising with entry) are consistent with the wider class of stable cases just referred to.

## 2. Framework

Let homogeneous output  $Y$  be produced by  $n$  possibly different firms, producing  $y_i$  at a cost  $c_i(y_i)$  each, and collectively facing inverse demand  $p = p(Y)$ . Each firm selects its own output taking total production by all others,  $Q_i = Y - y_i$ , as given. Assuming for convenience twice-differentiability of cost and demand functions, profit-maximisation at an interior optimum requires  $y_i^*$  to satisfy

$$p(y_i^* + Q_i) + y_i^* p'(y_i^* + Q_i) - c_i'(y_i^*) = 0. \quad (1)$$

We shall assume  $p'(\cdot) < 0$ , as well as strict concavity of the profits function for each firm  $i$ ,  $\Pi^i \equiv y_i p(Y) - c_i(y_i)$ , relative to its own output, so that

$$(p' + y_i p'') + (p' - c_i'') < 0, \quad \forall y_i, Q_i, i, \quad (2)$$

with  $p(\cdot)$  and  $c(\cdot)$  evaluated at  $(y_i + Q_i)$ . Under this assumption (plus boundedness of the  $y_i$ 's and other minor points) not only does (1) become sufficient for an optimum for firm  $i$ , but existence of a Cournot equilibrium is also ensured<sup>1/</sup>, that is, existence of two vectors  $\bar{y} = (\bar{y}_i)$  and  $\bar{Q} = (\bar{Q}_i)$  which solve (1) and (2), for all  $i$ , with  $\bar{Q}_i = \sum_j \bar{y}_j - \bar{y}_i$ . This equilibrium may or may not be unique (uniqueness is difficult to establish: see Friedman [1, p. 71]), but this need not bother us here: our main purpose is to show certain equilibria to be unstable, a property which holds globally if it does locally, in sharp contrast with the more familiar discussions of stable equilibria, where it is critical to determine whether the domain of the convergent adjustment is 'small' or 'large'. Things would get trickier, on the other hand, if equilibria happened not to be regular, that is not even locally unique, isolated. This, one can dismiss as a non-generic, 'unlikely' occurrence<sup>2/</sup>, although that is often a risky stand to take: there may be endogenous forces picking up the odd one from all possible events. We turn to this problem in the appendix and establish regularity of all equilibria we shall be interested in in this paper.

### 3. Unstable Cournot equilibria

Let the system be in a disequilibrium position, with each output  $y_i$  being continuously adjusted, with some delay, towards its current optimal value, that is, the  $y_i^*$  which solves (1) taking current output by others as given. This  $(y_i^*)$  is unique by (2), even though equilibrium itself may not be. Let this adjustment follow

$$\dot{y}_i = K_i z_i, \quad K_i > 0, \text{ each } i, \quad (3)$$

where  $z_i \equiv y_i^* - y_i$ . This could easily be replaced by any sign-preserving  $C^1$  function of  $z_i$ . The question is whether this process tends towards an equilibrium  $\bar{y}$  as defined above. Hahn [3] shows that the system does converge if the following conditions are met:

Assumption A:  $(p' - c_i^*) < 0 \quad \forall y_i, Q_i, i$ , that is, marginal costs should not fall "too" rapidly, if at all;

Assumption B1:  $(p' + y_i p'') < 0 \quad \forall y_i, Q_i, i$ , which says that marginal revenue for a producer should be a decreasing function of the other producers' output or, equivalently, that marginal revenue be steeper than market demand.<sup>3/</sup> Hence these two assumptions, together, require each term in brackets in (2) to be negative. This is not merely unnecessary for (2) to hold: it actually leaves out important possibilities. A simple example will illustrate this. Consider identical firms facing constant marginal cost and isoelastic demand  $p = AY^{-1/\epsilon}$ . B1 becomes  $\epsilon > \frac{1}{n-1}$ , while (2) requires only  $\epsilon > \frac{1}{2n-1}$ . If  $n = 10$ , for example, B1 does not leave out much - only price elasticities in the interval  $(1/21, 1/10)$ . But when  $n$  is lower the critical values for  $\epsilon$

become higher and their range widens, becoming a substantial (1/3, 1) for the duopoly case ! And of course, demand curves with more interesting shapes could rather more easily violate these stability conditions and perhaps get the industry into disarray, or at least the model into trouble. It is, therefore, of interest to look into the cases excluded by B1. As an alternative to it, consider

Assumption B2 :  $(p' + y_i p'') > - \frac{1}{n} (p' - c_i'')$  at equilibrium  $\bar{y}_i, \bar{Q}_i$ ;  $\forall i$ , where, under A (which must hold when B2 does, if (2) is to be met), the last quantity is positive for each  $i$ . We shall later interpret this condition. Notice that we only need to impose it at (and, by continuity of all the functions involved, in a neighbourhood of) equilibrium  $\bar{y}, \bar{Q}$ , given our earlier remark on 'local' being 'global' instability. Notice also that B1 and B2 give disjoint conditions on demands: the range in between is considered in the appendix and shown to yield stable equilibria too, as B1 does.

I will now show that B2 guarantees instability of equilibrium. Consider any point  $(y_i) \in \mathbb{R}^n$  such that

$$(z_i) > 0 \quad \text{or} \quad (z_i) > 0. \frac{4}{4} \quad (4)$$

That is, all firms' output-divergencies are of one sign, so that (from (3)) the industry as a whole is expanding or contracting, perhaps following a change in conditions that affected all firms in a similar way: a demand rise or a change in inputs' prices, say. Points where (4) holds clearly occur in every neighbourhood of equilibrium - they do so with positive measure.

We ask whether a path starting from one such point has  $(z_i) \rightarrow 0$  or not. For this, let  $k$  be such that

$$|K_k z_k| \leq |K_i z_i| \quad \forall i, \quad (5)$$

Maybe  $|K_k z_k| = 0$  of course. Suppose for concreteness, in (4), that  $(z_i) > 0$ . Then, from  $z_i \equiv y_i^* - y_i$ ,

$$\dot{z}_k = \dot{y}_k^* - \dot{y}_k. \quad (6)$$

Differentiating (1) to compute  $\dot{y}_k^*$ , gives

$$\dot{y}_k^* = -q_k \sum_{i \neq k} K_i z_i \quad (7)$$

where  $q_k \equiv (p' + y_k^* p'') / (2p' + y_k^* p'' - c_k'')$ ,  $(8)$

with  $p(\cdot)$  and  $c_k(\cdot)$  evaluated at  $(y_k^* + q_k)$ , as in (1). It is easy to check that B2 is equivalent to

$$q_i < -\frac{1}{n-1} \quad \forall i, \quad (9)$$

while B1 corresponds to  $q_i > 0$ . Going back to (6),

$$\dot{z}_k = - (K_k z_k + q_k \sum_{i \neq k} K_i z_i) \quad (\text{Using (3)} \\ \text{and (7)})$$

$$> - (K_k z_k - \frac{1}{n-1} \sum_{i \neq k} K_i z_i) \quad (\text{by (9) and} \\ (K_i z_i) > 0)$$

$$= - \frac{n}{n-1} (K_k z_k - \frac{\sum_i K_i z_i}{n})$$

$$\geq 0. \quad (\text{by (5)})$$

That is, whenever  $(z_i) > 0$ ,  $\min_i (K_i z_i)$  is strictly increasing.

Similarly, if  $(z_i) < 0$ , then  $\max_i (K_i z_i)$  is strictly decreasing.

We have proved

Theorem 1. Under B2, and for the adjustment mechanism (3), the Cournot oligopoly solution is unstable.

#### 4. Qualitative dynamics of unstable Cournot equilibria

It may be said that, as far as applications are concerned, it is immaterial whether an equilibrium's instability takes the form of a source (of divergent paths only) or of a saddle point of some sort - in either case all but a zero-measure set of flows diverge from equilibrium. The question, however, requires an answer if only to complete our theoretical picture of Cournot dynamics, as well as for one's amusement with the exercise which will provide us with a new, rather robust example of a saddle point in positive economics. Moreover, knowledge of exactly how Cournot equilibrium may fail to be stable will shed additional light on our economic understanding of the dynamics of the model.

Linear flows in  $\mathbb{R}^n$ ,  $\dot{x} = Ax$ , are very nice in that one has a complete and rather simple catalogue of all their possible, qualitatively different dynamical structures. Apart from solutions with closed-cyclical components in some directions, which are non-generic (and will anyhow be allowed for below), the general solution is the hyperbolic flow [4]: a generalised saddle point, which can be decomposed uniquely into two constituent manifolds, one whose flow is stable (a sink) while the other has 'fully' unstable flow (a source), and whose dimensions ( $> 0$ ) add up to  $n$  (the two extreme cases being stable and fully unstable equilibria). There is no other possible behaviour.

In contrast, non-linear flows are tricky, and may in principle yield nearly any dynamic configuration : semi-stable equilibria (e.g.  $\dot{x} = x^2$ ), monkey-saddles (the simplest : four alternating stable/unstable lines intersecting at  $\underline{0}$  in  $\mathbb{R}^2$  ; can you see the reason for the name ?), etc. Their catalogue, however, reduces precisely to the linear one whenever the non-linear flow is diffeomorphic to a linear one at the origin, and can therefore be linearised locally. This requires not simply differentiability, but also that the origin be a nondegenerate critical point: that the Jacobian of the flow be nonsingular there. The reason is simple: every linearisation is justified, explicitly or otherwise, by recourse to the implicit function theorem, whose applicability must therefore be checked first. I shall leave this for an appendix, and proceed now with our main argument, assuming qualitatively linear behaviour near the origin, as described above.

The conventional way to identify a saddle point would be to linearise the equations of motion at equilibrium and show the corresponding Jacobian matrix to have the right numbers of eigenvalues with negative and positive real parts. The method is, however, untidy and hard to apply in most cases, which explains why one is usually content to establish stability alone even if only for some cases (sufficient conditions), for which alternative methods are available, without seeking to establish where exactly instability starts (necessary conditions for stability) or the precise nature of the unstable equilibria involved. We shall here take an alternative tack, instead of the 'eigenvalues' approach, proving directly the existence and dimensionality of the constituent stable and unstable manifolds of the total flow.

Consider

$$\sum_i K_i z_i = 0, \quad (10)$$

with  $(K_i)$  as defined in (3). Since  $(K_i) \gg 0$ , (10) is the equation of a hyper-plane coming onto the origin from all orthants in  $\mathbb{R}^n$  other than  $\mathbb{R}_+^n$  and  $\mathbb{R}_-^n$ . On (10), sign  $z_i = - \text{sign } \sum_{j \neq i} K_j z_j$ . Now, recalling that

$$\dot{z}_i = \dot{y}_i^* - \dot{y}_i = -q_i \sum_{j \neq i} K_j z_j - K_i z_i$$

and that, under B2,  $(q_i) \ll 0$  (eqn. (9)), it follows that, for all points on (10),

$$\text{sign } \dot{z}_i = - \text{sign } z_i. \quad (11)$$

Hence all  $z_i$ 's in (10) strictly fall in absolute value if this is not zero, or remain zero otherwise. Clearly, convergence of these points does not follow directly from here, for (10) will generally not remain holding along the ensuing path. We have not found, that is, a convergent  $(n-1)$ -manifold explicitly, but we shall be able to deduce its existence based on this "local convergence" of points (10).

From T1, it follows that the flow has at least one divergent component: an unstable manifold of dimension  $\geq 1$ . In the proof of T1 it was shown, moreover, that every point in  $\mathbb{R}_+^n$  or  $\mathbb{R}_-^n$  (including the walls of these orthants, but of course not the origin) moves strictly into the interior of these orthants and remains there. This by itself

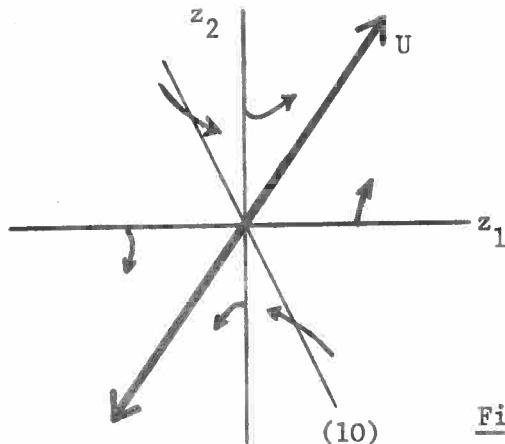


Figure 1

suggests the divergent attractor to be one-dimensional: an unstable "45° - line" in  $\mathbb{R}^n$ . Fig. 1. illustrates the situation as we so far know it.

This figure may, however, be deceptive: the saddle-point structure is now obvious for  $\mathbb{R}^2$ , but must be checked carefully for bigger spaces which, besides, contain different kinds of saddle points. Suppose, then, the divergent manifold to have dimension  $m \geq 2$ . This "surface" will necessarily intersect the plane (10) at points other than the origin and in every neighbourhood of the origin.<sup>5/</sup> A contradiction is now immediate : these points will have to move both closer to the origin according to Euclidean norm, as implied by (11), and away from the origin according to the same norm, as trajectories on the unstable manifold of a linearized flow not only do "eventually" diverge from the origin but do so everywhere (in this norm) near enough the origin.<sup>6/</sup> We conclude that the divergent manifold is one-dimensional, a line.

A similar argument<sup>7/</sup> can be shown to exclude purely-cyclical components (i.e. purely imaginary eigenvalues), and it follows that the convergent manifold has dimension  $(n - 1)$ . We can therefore state the following result.

Theorem 2. Under B2 the dynamical structure of Cournot equilibria

in  $\mathbb{R}^n$  (n firms) is the direct-sum composition of an unstable flow on a

line  $U$  (a 1-manifold) and a stable flow on a hyper-surface  $S$  (a

$(n - 1)$ -manifold). Moreover,  $U$  is entirely contained in  $\mathbb{R}_{++}^n \cup \mathbb{R}_{--}^n$

$(\cup \underline{0})$ , and  $S$  has empty intersection with  $\mathbb{R}_+^n \cup \mathbb{R}_-^n \setminus \underline{0}$ .

Notice that, since the attractor (the unstable manifold) is a line through the positive and negative orthants, the specific way in which Cournot equilibria under B2 will explode, is either an expansive run by all firms, or a depressive one leading to the closure of some (or all) firms and the formulation of an entirely fresh oligopoly problem. The situation is as shown in figures 2 and 3, the first of which is a filled-in version of figure 1, showing the relation of the plane (10) to the convergent manifold  $S$ , while the second illustrates the nature of the dynamics in higher dimensions.<sup>8/</sup>

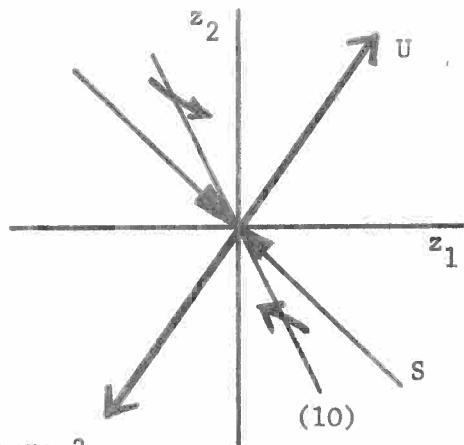


Figure 2

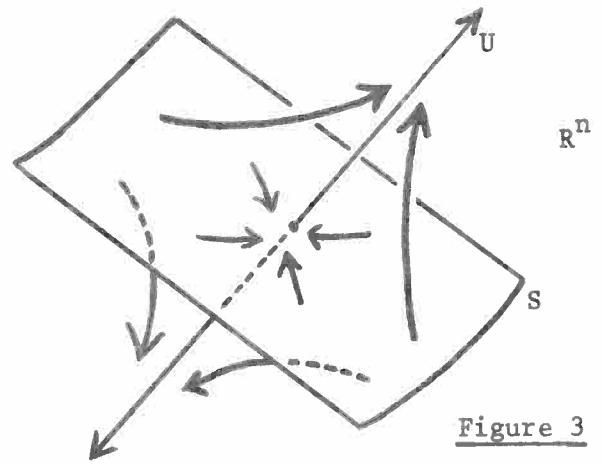


Figure 3

It is in the light of this result that we can best give an economic interpretation to condition B2. Suppose that, starting from equilibrium, all firms were to expand output by one unit. B1 not holding means, as we saw (p.4 and fn. 3), that the effect on firm  $i$ 's marginal revenue (hence marginal profit) of the others' output-expansion is positive, that is,  $\frac{\partial^2 \Pi^i}{\partial y_i \partial y_j} \equiv \Pi_{ij}^i > 0$ , whereas its own expansion lowers the firm's marginal profit, by concavity. If the former effect dominates the latter, marginal profit will have turned positive for each firm (it was

zero at equilibrium) and a self-sustaining expansive run will follow.

This will happen if

$$\sum_j \pi_{ij}^i > 0 \quad \forall i. \quad (12)$$

But  $\pi_i^i = p(Y) + y_i p'(Y) - c_i^i (y_i)$ , and it is easy to check that (12) is, or corresponds precisely to, B2. This gives us an interpretation for B2 as well as a rationale for the instability result under B2.

One would expect, on the face of the previous argument, the converse often to be true too: that  $\{ \sum_j \pi_{ij}^i < 0 \quad \forall i \Rightarrow \text{stability} \}$ . This turns out to be correct in most interesting cases, not generally. Local instability follows (i) under B1 (Hahn), which amounts to the strong requirement that  $\pi_{ij}^i < 0 \quad \forall i, j$ , or under (ii)

$\frac{1}{n-2} (p' - c_i^{\text{II}}) < p' + y_i p'' < \frac{1}{n} |p' - c_i^{\text{II}}| \quad \forall i$ , which overlaps with B1  $[p' + y_i p'' < 0]$  and covers the range of cases in between B1 and B2. The converse (of the closure) of B2  $\equiv$  (12) thus emerges, from among the class of cases where one of these conditions holds for all  $i \frac{9}{9}$ , as the necessary and sufficient condition for stability of Cournot equilibrium.

##### 5. Quasi-Cournot : positive conjectural variations

It is of some interest, in applications, to allow for a certain degree of awareness by producers of their interdependence, due to their reactions to each other's actions. To model this, while still avoiding game-theoretic difficulties, one may assume that each producer expects the others to react to his own policy-changes in a well-defined form, so that  $dQ_i/dy_i$  can be computed. If we further treat this expression as a constant  $\lambda_i - i$ , which may be a natural simplification in certain contexts

such as local-stability analysis, we get a rather simple, convenient extension of Cournot (when  $\lambda_i \equiv 1$ ), which may naturally be called quasi-Cournot behaviour. This is a common way of modelling collusive behaviour in industrial economics, as increases in the  $\lambda$ 's normally have the same effects as expansions in the number of firms [6].

We now consider the following assumptions, generalising their earlier counterparts.

Concavity of  $\Pi^i$  :  $\lambda_i(p' + \lambda_i y_i p'') + (\lambda_i p' - c_i'') < 0$  ;

A' :  $(\lambda_i p' - c_i'') < 0$  ;

B1' :  $(p' + \lambda_i y_i^* p'') < 0$  ; and

B2' :  $(p' + \lambda_i y_i^* p'') > -\frac{1}{n} (\lambda_i p' - c_i'')$  , all  $\forall i$ .

These are in a sense the very same assumptions as before, for we notice that the relevant, perceived demand curve for the individual quasi-Cournot producer has slope  $\lambda_i p'$ , rather than  $p'$  alone, on account of the fact that a unitary expansion of individual output brings about a fall in price that corresponds not to a unitary but to a  $\lambda_i$  change in total output. Assumptions A' and B2', however, become more, and B1' less likely to be met as the  $\lambda_i$ 's increase.

We again assume, without further mention, concavity of profits. It is now routine to check that A' and B1', the modified form of Hahn's assumptions, are sufficient for stability of this model. Similarly, theorems 1 and 2 generalise to

Theorem 3 : under 'B2' and for the adjustment mechanism (3), the quasi-Cournot oligopoly solution is unstable: a saddle point with convergence on a  $(n-1)$ -manifold, whose divergent manifold is a line contained in the positive and negative orthants.

#### 6. A final remark

The Cournot behavioural assumption, or its positive-conjectural-variations extension of section 5, are often taken to give reasonable descriptions of oligopolistic markets in equilibrium only, and one should be well aware of the difficulties that arise as soon as any dynamics are put into the problem, when these Cournot-like assumptions are continuously falsified during adjustment.

I agree with this view, and for this reason, I do not find discussions of global stability of the Cournot or quasi-Cournot models to be of any interest, for the models themselves should be given up for markets well away from equilibrium. Rather, I would regard the present paper as a study of the local stability properties of Cournot. After all, near enough equilibrium, the Cournot adjustment mechanism is as believable or unbelievable as Cournot equilibrium itself, as it approximates locally, for example, Cournot-like models with a richer kind of consistent-expectations formation. And whenever local instability prevails, as we have found to be the case for certain cases, global instability follows too, whether for the simple Cournot model or for any richer one taking Cournot behaviour around a given equilibrium. In brief, the stability-analysis of Cournot does not tell us much about the dynamics of Cournot, but rather about the observability of its static solution, or whether the Cournot assumption is at all sensible on the face of a given demand

configuration.

Appendix : local uniqueness, linearizability and other stable equilibria

Regularity of equilibrium and non-degeneracy of the flow are formally similar problems, both requiring some form of invertibility of the equilibrium function (forcing  $Q_i = \sum_{j \neq i} y_j$  on (1)) and of the linearization (derivative) of the flow, respectively.

1. Set  $y \equiv (y_i)$ ;  $f_i(y) \equiv p(\sum_j y_j) + y_i p'(\sum_j y_j) - c_i'(y_i)$ ;

and  $f \equiv (f_i)$ .

These  $f_i$ 's are no other

than marginal profits (cf (1)),  $\Pi_i^i$ . A solution  $\bar{y}$  to  $f(y) = 0$

will be isolated if the implicit function theorem applies at  $\bar{y}$ , i.e.

if the Jacobian  $J \equiv |Df_y|$  is non-vanishing, where  $Df_y$  is the derivative of  $f$  at  $\bar{y}$ :

$$Df_y = \begin{bmatrix} r_1 & s_1 & \dots & s_1 \\ s_2 & r_2 & \dots & s_2 \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_n & \dots & r_n \end{bmatrix}, \quad (a1)$$

where  $r_i \equiv f_{ii} = 2p' + y_i p'' - c_i''$  and  $s_i \equiv f_{ij} = p' + y_i p'' (i, j)$ .

Computing the Jacobian of (a1) we find, after some manipulations,

$$J = \left(1 + \sum_i \frac{s_i}{r_i - s_i}\right) \prod_i (r_i - s_i) \quad (a2)$$

$$= \left(1 + \sum_i \frac{p' + y_i p''}{p' - c_i''}\right) \prod_i (p' - c_i''), \quad (a3)$$

which is non-vanishing if either assumptions A and B1 (Hahn) or A and B2

hold  $\forall i$ .

2. The dynamics of the flow are given by (3), which is, in vector form,

$$\dot{y} = K \cdot z = K \cdot (y^*(y) - y) \equiv F(y), \quad (a4)$$

$F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , with  $y^*(y)$  defined by (1) for each  $i$ . Differentiating (1),  $\partial y_i^*/\partial y_i \equiv y_{ii}^* = 0$  and  $\partial y_i^*/\partial y_j \equiv y_{ij}^* = -q_i \quad \forall i \neq j$ . Thus, differentiating (a4), the system becomes, to first order,

$$\dot{y} = DF_y \cdot (y - \bar{y}), \quad (a5)$$

where the derivative  $DF_y$  turns out to have exactly the form (a1), again, but now with  $r_i \equiv -K_i$  and  $s_i \equiv -K_i q_i$ . The following is immediate.

Proposition : If  $-\frac{1}{n-1} < q_i < \frac{1}{n-1} \quad \forall i$ ,  $DF_y$  has a dominant diagonal, and therefore all its eigenvalues have negative real parts, equilibrium is stable.<sup>10/</sup>

This covers the gap of values for  $q_i$  assumptions B1 and B2 leave, i.e.  $-\frac{1}{n-1} < q_i < 0$ , but allowing some  $q_i$ 's to fall in the  $0 < q_i < \frac{1}{n-1}$  interval of the B1 range. This adds to Hahn's class of stable Cournot equilibria, considerably so if  $n$  is low.

The Jacobian (a2), is, in this case,

$$J = (-1)^n \left( 1 + \sum_i \frac{q_i}{1-q_i} \right) \prod_i K_i (1 - q_i). \quad (a6)$$

It is easy to check that if  $q_i < -1/(n-1) \forall i$  (B2), or if  $-1/(n-1) < q_i < 1 \forall i$  (extended B1, with A holding), J does not vanish - it takes the sign of  $(-1)^{n+1}$  in the former case, the opposite sign in the latter. Cournot dynamics has in all these cases a modicum of good behaviour.

Footnotes:

\* I have benefitted from interesting conversations with Avinash Dixit and from a referee's comments. A version of this paper was presented at the 1978 European Meeting of the Econometric Society (Geneva).

1/ See [1;T7.8].

2/ Write  $f(y)$ ,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  for LHS of (1) in vector form. If 0 is a regular value of  $f$  (non-vanishing Jacobian), which happens generically [2; p.35, Stability Theorem], then  $f^{-1}(0)$  is an  $n$ -codimensional submanifold of  $\mathbb{R}^n$ , i.e. the set of equilibria consists of isolated points [2;p.28].

3/ Marginal Revenue  $\equiv MR(y_i, Q_i) = p(y_i + Q_i) + y_i p'(y_i + Q_i)$ .

Hence  $MR_{Q_i} = p' + y_i p''$ , which B1 requires to be negative.

Also,  $MR_{y_i} = 2p' + y_i p'' < p'$  (steeper than  $p(Y)$ ) iff B1 holds.

4/ Notation : for a vector  $x \equiv (x_i)$ , we write  $x > 0$  iff  $x_i \geq 0 \forall i$  and  $x \neq 0$ .

5/ (10) is a separating hyper-plane for  $\mathbb{R}_{++}^n$  and  $\mathbb{R}_{--}^n$ , both of which contain (uncountably many) points of the unstable manifold.

The latter and (10) therefore intersect transversally in a manifold, or have intersection which contains a manifold, whose codimension is the sum of their codimensions,  $[n-m] + [n - (n-1)]$ , i.e. of dimension  $m-1 \geq 1$ . See [2; p.30].

- 6/ See [ 4 ]:T7.1.2 and, reversing time, the discussion on p.149.
- 7/ The composition of a 2-dimensional closed-cyclical flow with the unstable 1-manifold would be a flow on a cylindroid intersecting (10). All points on this intersection would then have to remain 'on the average' the same distance from the origin (the integral of distance-changes all along the intersection be zero), again contradicting (11).
- 8/ It is of interest to contrast the dynamical structure we have found with the well-known saddle point characteristic of optimal accumulation problems. The latter has convergence on a single line and a divergent hyper-surface, that is, the exact converse of the present picture: a reverse-time version of fig. 3.
- 9/ We have not allowed for those cases where, say, B2 holds for some i and B1 for others, a situation that may well arise if differences across firms (esp. in their market shares) are large at the equilibrium point. Little can be said in general for these 'mixed' cases. Heuristic arguments can be given to show that (i) the flow is never cyclical whenever for all firms either B1, B2, or neither of these (i.e. the intermediate range) holds, whereas (ii) both cycles and convergence may or may not obtain in mixed cases, depending on factors like 'how strongly' a given condition holds for each firm and their relative adjustment speeds.
- 10/ See, e.g., [5:T2].

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