ON THE SPEED OF ADJUSTMENT IN THE CLASSICAL TATONNEMENT PROCESS : A LIMIT RESULT *

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1. Introduction

As a natural consequence of the development of a long series of theorems concerning the existence of equilibrium, in the general equilibrium framework, the question arose of how to formulate a dynamic price adjustment process, with the property that the prices determined by this process eventually converge to the equilibrium price. In other words, the problem posed was to describe a reasonable adjustment process with respect to which the price equilibrium is stable. The problem was first approached by Walras, who formulated the so-called tatonnement process. Since this process was first proposed a considerable literature discussing the problems raised by it has developed, see e.g. Negishi's (1962) survey article.

The basic idea of the tatonnement process is that, given a price system, each agent informs an auctioneer of his demand and supply of each commodity. Having computed the total excess demand for each commodity the auctioneer then announces a new price vector which is obtained from the previous one by increasing (decreasing) the price of commodities with positive (negative) excess demand. This process is repeated until equilibrium is reached. A fundamental feature is that no agent is allowed to trade any commodity until equilibrium is achieved.

The question now arises. Given this assumption of "no trade out of equilibrium", if the tatonnement process does converge to an equilibrium, can we then say anything about the speed of adjustment? For the no trade assumption to make any sense the convergence should be "fast", that is the organisation of the market prior to trade should not take long. This question has been posed several times in the literature (see Negishi (1962)), but to the best of our knowledge no answer has been offered.

The purpose of this paper is to provide a partial answer to this question by showing how a well-known result in the theory of ordinary differential equations gives, directly, an estimate of the speed of adjustment of the tatonnement process in the limit, (i.e. as the price approaches equilibrium).

2. The basic Model

In this section we shall present the basic model. As our result is closely related to the "neo-classical" stability theorems, we shall here adopt the model which has been used throughout the vast majority of papers concerning price adjustment processes by a tatonnement procedure.

We consider a pure exchange economy with m consumers $(i = 1, \dots, m)$ and ℓ commodities $(h=1, \dots, \ell)$. The commodity space is described by $\mathbb{R}^\ell$. P is defined by $P := \{ x \in \mathbb{R}^\ell \mid x \gg 0\}$ and $\bar{P}$ is the closure of P. Each agent is assumed to have some initial resources $\omega^i = (\omega_1^i, \dots, \omega_\ell^i)$ belonging to $\bar{P}$ and ω, defined as the total initial resources $\omega := \Sigma\, \omega^i$, is assumed to lie in P.

A state $(x^1, \dots, x^m) \in \bar{P}^m$ is a specification of a commodity bundle for each agent, and a state is said to be attainable if $\sum_i x^i = \omega$.

A price system is a linear map $p: \mathbb{R}^\ell \to \mathbb{R}$ which to each commodity bundle $y = (y_1, \dots, y_\ell)$ assigns its value, $p.y$. We shall only consider price vectors $p \gg 0$, where we have used the canonical isomorphism between $\mathbb{R}^\ell$ and $(\mathbb{R}^\ell)^*$. Each agent has a preference - indifference relation $\succsim_i$ (reflexive, transitive, complete), defined on $\bar{P}$, and it is assumed monotone.

The i'th agent's demand set at the price system p is defined as the maximal elements w.r.t. $\succsim_i$ in the budget set $\gamma^i := \{ x \in \bar{P} \mid p.x \le p.\omega^i\}$. If we impose the strong convexity assumption on $\succsim_i (x \sim_i y,\ x \neq y \Rightarrow x <_i tx + (1-t)y,\ 0<t<1)$ the demand set will consist of exactly one point, so the i'th agent's demand is a function ξ^i. $\bar{p}$ is called an equilibrium price vector, if supply equals demand at the price system $\bar{p}$, i.e. $\Sigma\, \xi^i(\bar{p}) = \omega$.
$\xi := \sum_i \xi^i$ is the total demand function and $\zeta := \xi - \omega$ is called the total excess demand function.

This model has the following two features:

1^o $\forall i : \xi^i$ is a positive homogeneous function of degree 0 in p

i.e. $\xi^i(\alpha p) = \xi^i(p)$, $\forall\ \alpha > 0$

2^o Walras' law , i.e. $p.\ \zeta(p) = 0$, $\forall p$.

Next we want to describe the dynamical feature of the model. This is given by the following differential equation

(1) $$\frac{dp}{dt} = f(p)$$

where $f = (f_1, .. , f_\ell)$ is a map defined on P with values in $\mathbb{R}^\ell$, such that sign f_h = sign ζ_h , $h = 1, 2, \ldots , \ell$.

This process describes the adjustment in the prices through time with the interpretation of a tatonnement process as given above in the introduction.

Definition. $\Psi(t;\ p^o)$ is said to be a solution to (1) with initial value p^o, if for $h = 1, \ldots, \ell$

$$\begin{cases} \dfrac{\partial \Psi_h(t;\ p^o)}{\partial t} = f_h(\Psi(t;\ p^o)) , \ \forall t \geq 0 \\ \Psi_h(0;\ p^o) = p^o_h . \end{cases}$$

$\bar{p}$ is an equilibrium for (1) if and only if $\bar{p}$ is an equilibrium price vector, i.e. $\zeta_h(\bar{p}) = 0$, $h = 1, .., \ell$.

<u>Definition</u> An equilibrium $\bar{p}$ is called globally stable, if for every p^o and every solution $\Psi(t; p^o)$ to (1)

$$\lim_{t \to \infty} \Psi(t; p^o) = \bar{p}$$

The adjustment process (1) is called globally stable if for every p^o there exists an equilibrium $\bar{p}$ such that

$$\lim_{t \to \infty} \Psi(t; p^o) = \bar{p}$$

An equilibrium $\bar{p}$ is called locally stable if there exists a neighbourhood U of $\bar{p}$ such that for every $p^o \varepsilon U$ we have

$$\lim_{t \to \infty} \Psi(t; p^o) = \bar{p}$$

for every solution to (1).

The literature concerning tatonnement processes presents a variety of theorems giving more or less similar conditions for which (global or local) stability is present. Here we shall state only one such theorem. This theorem is representative of the existing stability theory and so reasonable to accept as the basic result for the following section.

In order to formulate the theorem we have to define the notion of commodities, which are strongly gross substitutes.

<u>Definition</u> The commodities $h = 1, \ldots, \ell$ are said to be strongly gross substitutes if the total excess demand function $\zeta(p) = (\zeta_1(p) \ldots, \zeta_\ell(p))$ is differentiable at all price vectors $p >> 0$ and

$$\frac{\partial \zeta_h}{\partial p_k} > 0 \text{ for } h \neq k,\ p >> 0$$

<u>Stability Theorem</u> (Uzawa (1961)).

Suppose

1^o For any initial price vector $p^o \gg 0$ the process (1) has a solution $\Psi(t;\ p^o)$ which is uniquely determined by p^o and continuous w.r.t. p^o.

2^o There exists a strictly positive equilibrium price vector.

3^o All commodities are gross substitutes.

Then the equilibrium price vector is uniquely (up to scalar multiplication) determined and the adjustment process (1) is globally stable.

3. The Speed of adjustment in the limit

If the assumptions of the above theorem are satisfied, then it follows that given $\varepsilon > 0$ there exists a t_o such that for $t > t_o$ $\|\Psi(t; p^o) - \bar{p}\| < \varepsilon$.

This formulation immediately leads to one possible way of describing the speed of adjustment, namely by expressing t_o as a function of ε and the functions f_h. How this should be carried out precisely does not seem quite clear. Another approach to the problem would be to compare the convergence of the solution with the convergence of a well-known function, and this is exactly the task of this paper.

Let us start by considering the problem locally in a neighbourhood of an equilibrium. Let $\bar{p}$ be a locally stable equilibrium price vector associated with the tatonnement process

$$\frac{dp}{dt} = f(p) \text{ , sign } f_h = \text{sign } \zeta_h, \ h=1, \ldots , \ell.$$

Making the substitution $q: = p - \bar{p}$ we may instead consider the process

$$(2) \quad \frac{dq}{dt} = Z(q) \text{ , } Z(q): = f(p), \ q: = p - \bar{p}$$

where $\bar{q} = 0$ is a locally stable point.

In order to get an idea of the kind of result we can obtain, let us first look at a linear adjustment process which has 0 as a stable equilibrium. This process has the form

$$\frac{dq}{dt} = A \cdot q$$

where A is a $\ell \times \ell$ - matrix .

For this differential equation we can explicitly state the solution, which has the form

$$a(t) = K \cdot e^{tA}$$

where $K \in \mathbb{R}^{\ell}$ is a constant determined by an initial value condition.

For a linear process local stability is equal to global stability. Because the real parts λ_i of the eigenvalues of A are negative, we get the existence of a positive constant C such that

$$\forall t \quad \frac{|q(t)|}{e^{at}} < C, \text{ where } \max_i \lambda_i < a < 0,$$

$$|q(t)| := |q_1(t)| + \ldots + |q_{\ell}(t)|$$

Since $e^{at} \to 0$, $|q(t)| \to 0$ for $t \to \infty$, it is reasonable to say that the speed of adjustment of the q- vector is, in the limit, as least as fast as the convergence of the exponential map e^{at} .

The result is not astonishing, because of the form of the solution of the linear differential equation. However, this estimate of the speed of adjustment (in the limit) extends to the general case.

Theorem Suppose the map $f : P \to \mathbb{R}^{\ell}$ is of class C^2 and 1°, 2° and 3° of the stability theorem are satisfied.

Then the solutions p(t) to the tatonnement process

$$\frac{dp}{dt} = f(p)$$

fulfill the following :

1° $\frac{\|p(t) - \bar{p}\|}{e^{at}} \to 0$ as $t \to \infty$ with $\lambda_i < a < 0$, $\forall i$.

2^o $$\frac{\log \|p(t) - \bar{p}\|}{t} = \lambda_i \text{ for some } i ,$$

where $\bar{p}$ is the equilibrium price vector and the λ_i's are the negative real parts of the eigenvalues of $Df(\bar{p})$.

Proof The theorem is a straight forward application of theorem 6.1, Chapter IX in Hartman (1961):

Using the Taylor expansion around $\bar{q} = 0$ we have the tatonnement process written in the form

$$\frac{dq}{dt} = DZ(0).\ q + R(q)$$

where $R(q)$ is of class C^1 and $\frac{R(q)}{\|q\|} \to 0$ as $q \to 0$ (i.e. $R(0) = 0$ and $D_qR(0) = 0$), as $\bar{q} = 0$ is a sink.

This implies that the conditions in Hartman's theorem are satisfied, so 1^o and 2^o follow immediately.

Remark The following example shows that we cannot generally obtain a better estimate of the speed of adjustment, i.e. choose $a = \max_i \lambda_i$

Suppose $f(p) = Df(\bar{p})\ (p-\bar{p}) + R(p-\bar{p})$ with $Df(\bar{p}) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$,

$R(p-\bar{p}) = ((p_2-\bar{p}_2)^{2k+1}, 0)$ (k large) and $\bar{p} >> 0$ the equilibrium price vector.

In a small neighbourhood of $\bar{p}$, the solution to the adjustment process

$$\frac{dp}{dt} = f(p)$$

will behave as the solution to the process

$$\frac{dp}{dt} = Df(\bar{p})\ (p-\bar{p}).$$

As $Df(\overline{p})$ has -1 as an eigenvalue with multiplicity 2, the solution to this process is of the form

$$(p_1(t),\ p_2(t)) = (\overline{p}_1 + K_1 e^{-t},\ \overline{p}_2 + K_2 e^{-t} + K_1 t e^{-t});\quad K_1, K_2 \in \mathbb{R}.$$

Finally we notice that the theorem extends to cover the case with a globally stable system without any further restrictions, as the proof only uses the behaviour of the adjustment process in a neighbourhood of the equilibrium.

References:

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