THE ILLYRIAN FIRM AND FELLNER'S UNION-MANAGEMENT MODEL

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PETER J. LAW

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This paper is circulated for discussion purposes only and its contents should be considered preliminary.

Abstract

It is argued that the theory of the Illyrian firm (that is, the firm which is assumed to maximise income per worker) can be interpreted as a special case of the general union-management model advanced by Fellner (1947) in his classic analysis of bilateral monopoly. Within a framework suggested by this interpretation the nature of the preference map of the labour-managed firm is shown to play a critical rôle in determining comparative static responses to changes in demand and fixed costs, with the result that these responses may be quite different from those of the Illyrian firm.

The Illyrian Firm and Fellner's Union-Management Model[1]

There is now a considerable literature on the Illyrian firm (that is, the firm which is assumed to maximise income per worker), and it has been argued that the analysis may have relevance for the labour-managed or co-operative enterprise. Significant contributions to this literature have been made by Domar (1966), Vanek (1970), Meade (1972,1974) and others but the seminal paper is generally recognised to be that of Ward (1958).

The purpose of the present paper is twofold. First, while in no way reducing the importance of Ward's analysis, it is argued that the theory of the Illyrian firm can be interpreted as a special case of the general model of union-management relations advanced by Fellner (1947) in his classic analysis of bilateral monopoly. Secondly, within a framework suggested by this interpretation, the nature of the preference map of the labour-managed firm is shown to play a critical rôle in determining comparative static responses to changes in demand and fixed costs, with the result that these responses may be quite different from those of the Illyrian firm.

1. Fellner's Model and the Illyrian Firm

The discussion will focus on the case in which the firm's non-labour inputs are fixed and, following Ward (1958), it is assumed that all workers are homogeneous and that the firm's labour input can be varied only by varying the number of workers. The Illyrian firm is assumed to maximise income per worker, y, where

(1) $$y = (PX - F)/L$$

In equation (1) P denotes the selling price of the firm's output X, F represents fixed costs and L denotes the number of workers. As the firm sells its product on an imperfect market

$$(2) \qquad P = P(X, \alpha)$$

where $\partial P/\partial X < 0$ and α is a shift parameter such that $\partial P/\partial \alpha > 0$. The firm's production function is denoted by

$$(3) \qquad X = X(L)$$

The first order condition for a maximum of y with respect to L is easily obtained. As it is supposed that the firm operates in a manner to maximise income per worker the labour force will be adjusted until the marginal revenue product of labour is equal to income per worker,

$$(4) \qquad (P + X \;\partial P/\partial X)\; \partial X/\partial L = y$$

From (1) income per worker may also be called the average net revenue product of labour. Thus the optimal labour force is defined, ceteris paribus, by the equality of the marginal revenue product with the average net revenue product of labour.

Now consider Fellner's model of union-management relations. He views the basic models of bilateral monopoly as not directly applicable in this context because (1947, p. 509) "the behavior of the suppliers of labor services cannot be interpreted in terms of cost functions (in the usual sense of the term)." Instead he assumes the existence of an indifference curve map in wage-employment space summarising the preferences of the union among different wage-employment combinations. (Note that if all the indifference curves are rectangular hyperbolas the implicit maximand of the union is the aggregate wage bill of those employed, an objective which has received some discussion in the literature on economic models of trade unions.) The interaction of the union indifference curve map and the average and marginal revenue product schedules of the firm produces a whole range of possible wage bargaining outcomes. In figure 1 conventional "inverse U-shaped" average revenue product (A.R.P.) and marginal revenue product of labour (M.R.P.) curves are drawn along with the system of indifference curves summarising the union's wage-employment preferences. Fellner assumes that there is a "lowest possible indifference curve" below which the union cannot be pushed and this plays an important rôle in setting a lower limit to the range of possible wage bargaining outcomes. The lowest possible indifference curve is represented by I_0 in figure 1.

Two cases can be distinguished in Fellner's analysis according to whether the amount of employment as well as the wage rate is included in the labour contact.

(a) Where the contract specifies only the wage rate management (to maximise profits) will determine the employment level to equate the marginal revenue product of labour with the wage. If the union enjoys maximal bargaining power and the firm is completely lacking in bargaining power that wage rate is established which corresponds, in Figure 1 to the tangency of a union indifference curve with the marginal revenue product curve. In other words that point on the M.R.P. schedule which is most favourable to the union in the sense of being on the highest possible union indifference curve represents the bargaining outcome. (See Fellner (1947, p. 512) for a minor qualification to this statement).

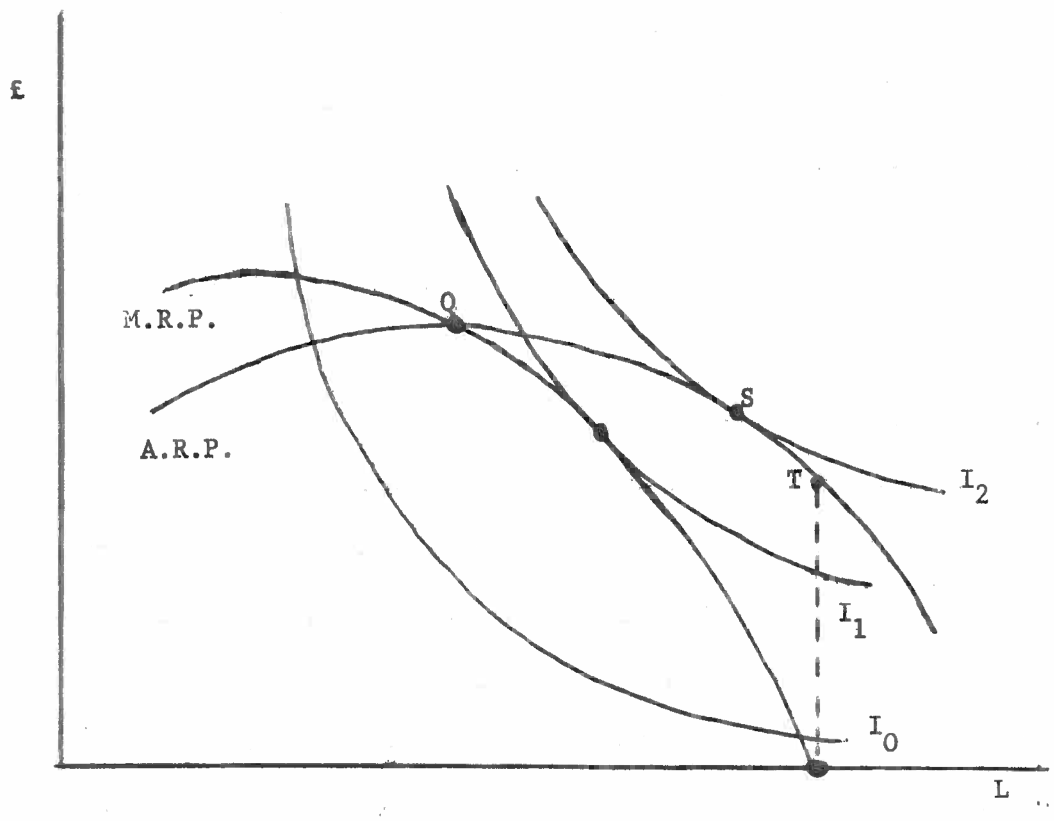

Figure 1

Where, on the other hand management enjoys maximal bargaining power that wage will be established which corresponds to the (lower) intersection of the lowest possible indifference curve with the M.R.P. schedule. In this way two wages are established which represent the upper and lower limits of a wage bargaining range. The wage which is agreed depends on the relative bargaining power of union and management.

(b) In Fellner's second case the amount of employment as well as the wage rate are specified in the labour contract, that is, the contract contains an all-or-none clause. Because of this, when the union enjoys maximal conceivable bargaining power it may attain a higher indifference curve than in case (a). Thus a wage-employment combination characterised by the tangency of a union indifference curve with the A.R.P. schedule results. It is easily shown that this tangency lies vertically above the intersection of the marginal curve to that indifference curve with the M.R.P. schedule.(2) Similarly the lower limit is set by the wage-employment combination on I_0 which lies vertically above the intersection of the marginal curve to I_0 with the marginal revenue product curve.(3) A Paretian contract curve can be traced out between the two limits and any move from that locus would make at least one party worse off. The equilibrium position will again depend on the relative bargaining strength of union and management.

Fellner's second case can be related to the worker-managed firm. The A.R.P. schedule may be interpreted as an average net revenue product curve and indeed it is sensible to do so since point S would otherwise indicate bankruptcy as non-labour costs (which are fixed in this case) are not being covered. Point S which represents one extreme of the bargaining range might now be interpreted as a position of "market syndicalist" equilibrium since it corresponds to maximal union bargaining power and zero bargaining power on the part of management. Indeed at this point the union _is_ the firm; this limiting position on the bargaining range formally represents the emergence of a worker-managed firm with preferences defined by (5) below. The equilibrium at S results from the maximisation of

$$(5) \qquad U = U(y, L) \qquad\qquad U_y, U_L > 0$$

where U is the utility function of the union-firm and y is of course identical with the average net revenue product of labour. The first order condition for a maximum of (5) yields

$$(6) \qquad U_y \; \partial y/\partial L \; + \; U_L \; = \; 0$$

whence it is easily seen that the slope of the indifference curve which represents the subjective rate of substitution between income per worker and employment must equal the slope of the average net revenue product curve. The second order condition may be written

$$(7) \qquad U_y \; \partial^2 y/\partial L^2 \; + \; (U_L^2 \, U_{yy} - 2U_L \, U_y \, U_{Ly} + U_y^2 \, U_{LL})/U_y{}^2 \; < \; 0$$

The left hand side of the above inequality will hereafter be denoted by D. It is clear that $D < 0$ for figure 1 since the average net revenue product schedule, y, is strictly concave and the indifference curves are convex to the origin.

Fellner's model is also capable of generating an equilibrium of the Illyrian type. As he recognises, it is quite possible that the union may have horizontal indifference curves implying that higher income per head is always desirable regardless of the consequences for employment. In this case if maximal bargaining power is enjoyed by the union equilibrium will occur at point Q. At Q the marginal revenue product and the average net revenue product of labour are equal and (4) is satisfied. Ward's Illyrian firm thus has an equilibrium corresponding to the upper limit on Fellner's bargaining range when the union has horizontal indifference curves.

Vanek (1970) and others have argued that the labour-managed firm may have advantages over its capitalist counterpart with respect to the optimisation of effort by the work force and "production-mindedness". If this is indeed the case the interpretation of the Illyrian firm as a special case of Fellner's union-management model may be qualified by Vanek's argument (1970 ,p. 244), that

> "The solidarity and unity of spirit of a labour union which in a capitalist environment is expended (wasted ?) predominantly in generating the countervailing power .... can now more constructively be used in running the enterprise and working to the greatest advantage of everyone."

It might also be argued that the preference map would undergo change in the transition to worker-management. Nevertheless a utility function such as (5) with indifference curves which are convex to the origin may have some relevance to the worker-managed firm. Unlike the simple income per worker maximand (5) implies that a reduction in membership with no change in income per worker reduces enterprise utility, which seems a reasonable assumption for a labour-managed firm. Moreover the converse implication that the worker-managed enterprise gains utility if employment is expanded at a given income level seems appropriate particularly where there is unemployment and the worker-managed firm takes some social responsibility for expanding employment opportunities. An alternative interpretation of (5) might be possible where workers or the state appoint managers who, under certain conditions, enjoy enough discretionary power to pursue their own goals. Income per worker may remain important and other managerial goals might be appropriately proxied by number of workers employed - at least in the context of a short-run static model. Managerial salaries might depend, in part at least, on the number of workers in the enterprise.

In the next section the reactions of a worker-managed firm to changes in fixed costs and in demand are compared under the Illyrian assumption that income per worker alone is maximised and under the assumption of equation (5) that numbers employed as well as income per worker may enter the enterprise utility function.

2. Changes in Fixed Costs and in Demand

Where labour is the only variable input it is a well known result of Ward (1958) that the Illyrian firm responds to an increase in fixed costs by increasing employment and hence output. Moreover Ward (1958) has shown that an increase in demand, which leaves the slope of the demand curve unchanged in the new equilibrium, will result in a reduction in employment and output if the average product of labour is diminishing. Similarly Vanek (1970, pp. 106-107) and Meade (1974) demonstrate that a demand increase which leaves demand elasticity unchanged will result in a contraction of output and employment.[(5)]

Consider the response of a firm with utility function, (5) in a syndicalist equilibrium at S in Figure 1 to a small change in fixed costs, dF. By differentiation of the equilibrium condition (6) with respect to F the following condition is derived

$$(8) \qquad dL/dF = (L\,U_y\,U_{Ly} - L\,U_L\,U_{yy} - U_y^2)/(U_y\,L^2\,D)$$

The denominator of the expression on the right hand side of (8) is negative but the sign of the numerator is indeterminate. Results are however easily derived for the specific utility function

$$(9) \qquad U = y^a L^b$$

in which case (8) becomes

$$(8)' \qquad dL/dF = y^{a-1} L^{b-2} (b - a)/D$$

Thus, $\qquad dL/dF \gtreqless 0$ as $b \lesseqgtr a$

Note that if $a = 1$ and $b = 0$, y alone is maximised and $dL/dF = -1/(L^2 \ \partial^2 y/\partial L^2) > 0$ which is the result of Ward (1958) mentioned above. In the present case if the "weight" attached to employment (number of workers) in the utility function exceeds or equals that attached to income per worker (that is $b \geq a$) employment and output, contrary to Ward's finding, will be reduced or will remain constant if $dF > 0$. Finally note if $a = b$ the maximand is aggregate workers' income, and output and employment are unaffected by changes in fixed cost; this maximand, as is easily confirmed, implies an initial equilibrium at T in figure 1.

Consider now the response of a firm in syndicalist equilibrium to a demand increase first under Ward's assumption that the slope of the demand curve remains invariant under shift. By differentiation of (6) with respect to α the following expression is derived when $\partial^2 P/\partial X \, \partial\alpha = 0$

$$(10) \qquad \frac{dL}{d\alpha} = \frac{\partial P}{\partial\alpha} \left\{ \frac{X}{L}\left(\frac{U_y}{L} + U_{yy}\frac{U_L}{U_y} - U_{Ly} \right) - \frac{U_y}{L}\frac{\partial X}{\partial L} \right\} \Big/ D$$

The sign of the numerator in the expression on the right hand side of (10) cannot, in general, be determined but again the specific case of (9) may be considered and it is easily shown that

$$(10)' \qquad dL/d\alpha = (\partial P/\partial\alpha) \ X y^{a-1} L^{b-2} \left\{ a(1 - \eta) - b \right\} \Big/ D$$

where $\eta = (\partial X/\partial L)\,(L/X)$ denotes the elasticity of output with respect to employment. Again if $a = 1$ and $b = 0$, y alone is maximised and it can be seen that $dL/d\alpha = (\partial P/\partial \alpha)\; X(1 - \eta)/(L^2\, \partial^2 y/\partial L^2) < 0$ if $0 < \eta < 1$ that is, if the average product of labour is diminishing. In the case of (10)' with $0 < \eta < 1$ it can be concluded that

$$dL/d\alpha \gtreqless 0 \quad \text{as} \quad a \lesseqgtr b/(1 - \eta)$$

Output and employment responses to an increase in demand will thus not always be in the direction indicated by Ward and indeed, in this case $a \leq b$ is a sufficient but not a necessary condition to ensure that the response indicated by Ward does not occur.

Finally the response of the firm to an elasticity-preserving demand shift can be analysed. The following expression is derived

$$(11) \qquad \frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha} \left\{ \left(\frac{U_y}{L} + U_{yy} \frac{U_L}{U_y} - U_{Ly} \right) \frac{X}{L} - \frac{U_y}{L} \left(1 + \frac{1}{\epsilon} \right) \frac{\partial X}{\partial L} \right\} \Big/ D$$

where $\epsilon = (\partial X/\partial P)\,(P/X)$ is the (constant) elasticity of demand. The sign of the numerator on the right hand side of (11) is indeterminate and the specific case of (9) is taken. It is easily shown that (11) now becomes

$$(11)' \qquad \frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha} X\, y^{a-1}\; L^{b-2} \left[a \left\{ 1 - \eta \left(1 + \frac{1}{\epsilon} \right) \right\} - b \right] \Big/ D$$

Whence

$$dL/d\alpha \gtreqless 0 \text{ as } a \left\{ 1 - \eta \left(1 + \frac{1}{\epsilon} \right) \right\} \lesseqgtr b$$

If $a = 1$ and $b = 0$, y is maximised and

$$\frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha}\left(\frac{PX}{L} - y\right) \Big/ \left(LP\frac{\partial^2 y}{\partial L^2}\right) < 0.$$

which restates the Vanek-Meade result mentioned above. In the case of (11) or (11)' it is possible that the firm will expand employment and output in response to a demand increase.

3. Concluding Remarks

It has been argued that the equilibrium of the Illyrian firm can be interpreted as a limiting case of Fellner's union-management model with workers possessing maximal bargaining power and income per worker the only component of enterprise utility. Where the number of workers as well as income per worker enter the utility function of the worker-managed enterprise it has been demonstrated that responses to changes in demand and fixed costs may be very different from those of the Illyrian firm.

It should be noted that the appropriateness of the Illyrian maximand of income per worker for analysis of the co-operative or labour-managed firm has been disputed by a number of writers. A full review of alternative models obviously cannot be attempted here. Vanek (1975, pp. 30-33), discusses five different objective functions which have been proposed in the literature on labour-managed firms and Furbotn (1976) has developed a model challenging Ward's maximand and emphasising the importance of the "initial majority of the collective" rather than all workers in determining the policies of the firm. In contrast to Ward's assumptions on hiring and firing a number of writers have argued that employment variability in the short-run, especially in the downward direction, may be limited in the labour-managed firm and indeed employment levels and job security may be arguments in the enterprise utility function. (See McCain (1973) for discussion of some of these issues). Expected length of employment with the firm might be a relevant variable in intertemporal analysis and a more sophisticated model might assume that the enterprise would view employment expansion and contraction in an asymmetric fashion. The analysis of the present paper has been restricted to a particularly simple modification of the Illyrian objective function.

Footnotes:

(1) The author wishes to thank J.A. Brack, K.G. Cowling, N.J. Ireland and K.G. Knight for helpful comments and suggestions.

(2) The equation of the indifference curve tangential to A.R.P. may be written, $y = \bar{U} - f(L)$, and the curve marginal to it has equation $m = d(Ly)/dL$. At point S, $Ly = PX - F$, therefore $m = M.R.P.$

(3) This is easily confirmed by maximising profit, $PX - F$, subject to $w = U_0 - g(L)$, where w represents the wage rate along the lowest possible indifference curve.

(4) Fellner does not do this as he is not primarily concerned with extreme positions on the bargaining range. He does however discuss a number of complications to his model (pp. 516-520) which are not reviewed here.

(5) Domar (1966), Vanek (1970) and Meade (1972) have shown that these results may have to be modified where there are labour supply constraints, more than one variable input or more than one output.

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Mathematical Notes

A. Equilibrium Conditions

Maximise

$$(5) \qquad U = U(y, L)$$

The first order condition for a maximum is

$$(6) \qquad U_y \frac{\partial y}{\partial L} + U_L = 0$$

whence

$$(6)' \qquad \frac{\partial y}{\partial L} = - \frac{U_L}{U_y}$$

From (6) the second order condition is derived

$$\frac{\partial^2 U}{\partial L^2} = U_y \frac{\partial^2 y}{\partial L^2} + U_{yy} \left(\frac{\partial y}{\partial L} \right)^2 + 2\, U_{Ly} \frac{\partial y}{\partial L} + U_{LL} < 0$$

Subsitution for $\frac{\partial y}{\partial L}$ from (6)' yields

$$(7) \qquad U_y \frac{\partial^2 y}{\partial L^2} + \frac{1}{U_y^{\,2}} \left(U_L^{\,2}\, U_{yy} - 2\, U_L\, U_y\, U_{Ly} + U_y^{\,2}\, U_{LL} \right) < 0$$

B. The Effect of an Increase in Fixed Costs on Employment

Equation (6) can be written

$$U_y \left\{ \frac{1}{L} \frac{\partial R}{\partial L} - \frac{(R - F)}{L^2} \right\} + U_L = 0$$

The total differential of (6) can now be written

$$\left\{ U_y \frac{\partial^2 y}{\partial L^2} + U_{yy}\left(\frac{\partial y}{\partial L}\right)^2 + 2\, U_{yL} \frac{\partial y}{\partial L} + U_{LL} \right\} dL$$

$$= \left(\frac{U_{Ly}}{L} + \frac{U_{yy}}{L} \frac{\partial y}{\partial L} - \frac{U_y}{L^2} \right) dF$$

Substitution from (6)' yields

$$(8) \qquad \frac{dL}{dF} = \frac{(L\, U_y\, U_{Ly} - L\, U_L\, U_{yy} - U_y^{\,2})}{U_y\, L^2\, D}$$

If

$$(9) \qquad U = y^a\, L^b$$

$$U_y = a\, y^{a-1}\, L^b \; ; \; U_{yy} = (a - 1)\, a\, y^{a-2}\, L^b \; ; \; U_L = b\, y^a\, L^{b-1} ;$$

$$U_{LL} = (b - 1)\, b\, y^a\, L^{b-2} ; \; U_{Ly} = a\, b\, y^{a-1}\, L^{b-1}$$

Substitution of these values in (8) yields

$$(8)' \qquad \frac{dL}{dF} = \frac{y^{a-1}\, L^{b-2}\, (b - a)}{D}$$

In the case where $a = 1$ and $b = 0$ by substitution in (7) and (8)',

$$\frac{dL}{dF} = - \frac{1}{L^2 \frac{\partial^2 y}{\partial L^2}}$$

C. The Effect of an Increase in Demand on Employment

(i) Ward assumption; $\frac{\partial^2 P}{\partial X\ \partial \alpha} = 0$

Equation (6) may be written as follows,

$$U_y \left\{ \frac{1}{L} \frac{\partial X}{\partial L} \left(X \frac{\partial P}{\partial X} + P \right) - \left(\frac{PX - F}{L^2} \right) \right\} + U_L = 0$$

The total differential of (6) can now be written

$$\left\{ U_y \frac{\partial^2 y}{\partial L^2} + U_{yy} \left(\frac{\partial y}{\partial L} \right)^2 + 2\, U_{yL} \frac{\partial y}{\partial L} + U_{LL} \right\} dL$$

$$= - \left\{ U_y \frac{X}{L} \frac{\partial X}{\partial L} \frac{\partial^2 P}{\partial X \partial \alpha} + \frac{\partial P}{\partial \alpha} \left(\frac{U_y}{L} \frac{\partial X}{\partial L} - \frac{U_y X}{L^2} + \right. \right.$$

$$\left. \left. U_{yy} \frac{\partial y}{\partial L} \frac{X}{L} + U_{Ly} \frac{X}{L} \right) \right\} d\alpha$$

With $\frac{\partial^2 P}{\partial X \partial \alpha} = 0$,

(10) $$\frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha} \left\{ \frac{X}{L} \left(\frac{U_y}{L} + U_{yy} \frac{U_L}{U_y} - U_{Ly} \right) - \frac{U_y}{L} \frac{\partial X}{\partial L} \right\} \Big/ D$$

Substitution for U_y, U_L, U_{yy}, U_{Ly} from (9) yields

(10)' $$\frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha} X\, y^{a-1} L^{b-2} \left\{ a(1 - \eta) - b \right\} \Big/ D.$$

and in the case of a = 1, b = 0,

$$\frac{dL}{d\alpha} = \left\{ \frac{\partial P}{\partial \alpha} X (1 - \eta) \right\} \Big/ \left(L^2 \frac{\partial^2 y}{\partial L^2} \right)$$

(ii) Vanek-Meade assumption; θ = constant.

The total differential of (6) can be rewritten

$$D\ dL = -\left\{ \frac{U_y}{L} \frac{\partial V}{\partial \alpha} \frac{\partial X}{\partial L} - \frac{\partial P}{\partial \alpha}\left(U_y \frac{X}{L^2} - U_{yy} \frac{\partial y}{\partial L} \frac{X}{L} - U_{Ly} \frac{X}{L}\right)\right\} d\alpha ,$$

where V is marginal revenue.

By the Vanek-Meade assumption $\frac{\partial V}{\partial \alpha} = \left(1 + \frac{1}{\theta}\right) \frac{\partial P}{\partial \alpha}$, hence

$$(11) \qquad \frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha} \left\{\left(\frac{U_y}{L} + U_{yy} \frac{U_L}{U_y} - U_{Ly}\right) \frac{X}{L} - \frac{U_y}{L} \left(1 + \frac{1}{\theta}\right) \frac{\partial X}{\partial L} \right\} \Big/ D.$$

Substitution for the case of (9) yields

$$(11)' \qquad \frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha}\ X\ y^{a-1}\ L^{b-2} \left[a \left\{ 1 - \eta\left(1 + \frac{1}{\theta}\right)\right\} - b \right] \Big/ D$$

when a = 1 and b = 0

$$\frac{dL}{d\alpha} = \frac{\partial P}{\partial \alpha}\ X \left\{ 1 - \frac{\partial X}{\partial L} \frac{L}{X} \left(1 + \frac{1}{\theta}\right) \right\} \Big/ \left(L^2 \frac{\partial^2 y}{\partial L^2} \right)$$

$$= \frac{\partial P}{\partial \alpha} \left\{ \frac{X}{L} - \frac{\partial X}{\partial L} \left(1 + \frac{1}{\theta}\right) \right\} \Big/ \left(L \frac{\partial^2 y}{\partial L^2} \right)$$

$$= \frac{\partial P}{\partial \alpha} \left\{ \frac{PX}{L} - \frac{P \partial X}{\partial L} \left(1 + \frac{1}{\theta}\right) \right\} \Big/ \left(L\ P \frac{\partial^2 y}{\partial L^2} \right)$$

$$= \frac{\partial P}{\partial \alpha} \left(\frac{PX}{L} - y \right) \Big/ \left(LP \frac{\partial^2 y}{\partial L^2} \right)$$