THE 'PURE THEORY OF THE TRUE COST-OF-LIVING INDEX' REVISITED

by

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This paper is circulated for discussion purposes only and its contents should be considered preliminary.

1. Introduction

The basic consumer theoretic approach to cost-of-living indices may be traced through the following landmark contributions: A.A. Konus ((8)), R. Frisch ((2)), H. Hotelling ((4)), H. Wold ((18)), F.M. Fisher and K. Shell ((1)). Fisher and Shell (F-S) are especially concerned with the problems posed by taste and quality change, which are discussed in the context of particular parameterizations of the utility function.

The purpose of this paper is to show that the whole analysis of F-S, with the exception of their brief discussion of corner solutions, can be derived from an alternative approach that is simpler and, it is hoped, more intuitively appealing. Section II of this paper sets out the basic theory of the true cost-of-living index, clarifying, it is hoped, an issue on which F-S are not explicit enough ([1]). Section III deals with the problem of taste change. Section IV deals with the quality change. The derivation of the main results of the underlying consumer theory is stt out in appendices A, B, C.

The labels 'duality' or 'indirect utility function/expenditure function' approach may be attached to this area of economic theory. Key references in the field, in chronological order, include the following: H.Hotelling ((4)), P. Samuelson ((14)), J. Ville ((17)), R.W. Shephard ((16)), L. McKenzie ((13)), S. Karlin ((7)), H.S. Houthakker ((5)), ((6)), P. Samuelson ((15)), L.J. Lau ((9)), ((10)).

In another working paper a similar approach is taken to F-S's recent M.I.T. Working Paper 59 on national output deflators.

II The Expenditure Function and Cost-of-Living Indices

F-S state on p.102: "When tastes change, Laspeyres and Paasche indices cease to become approximations to the same thing (our italics) and become approximations to different things". The criticism that can be made of this is that in the absence of the property of homotheticity of the utility function, even when there is no taste change, Laspeyres and Paasche indices are not in general approximations to the same thing in the sense of being bounds. The reason is that there are two true cost-of-living indices ([2]).

Assume that a consumer with a strictly quasi-concave utility function $u(x_1, x_2 \ldots x_n)$, maximizes utility subject to the given parameters, budget y and prices $p_1, p_2 \ldots p_n$. We can characterize the maximal levels of utility he can attain for different levels of the parameters by the indirect utility function

$$u = V(\frac{p_1}{y}, \frac{p_2}{y} \ldots \frac{p_n}{y}).$$

In addition, we can characterize the minimum expenditure of attaining a given level of utility, for different values of the prices, by the 'expenditure function' $y = m(p_1, p_2 \ldots p_u, u)$. The expenditure function can be obtained by inverting the indirect utility function with respect to y. This is discussed more formally in appendices A and B.

The first concept of the pure cost-of-living index is that from the Laspeyres point of view. Wold ([18])calls it the Laspeyres-Konus index. If $\hat{y}, \hat{p}_1, \hat{p}_2 \ldots \hat{p}_n$ are the base period parameter values and $p_1, p_2 \ldots p_n$ are the current period parameter values, the definition of the Laspeyres-Konus index given a particular utility function (with implied indirect utility and expenditure function) is

$$P_{LK} = \frac{m(p_1, p_2 \ldots p_n, \hat{u})}{m(\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n, \hat{u})}$$

$$= \frac{m(p_1, p_2 \ldots p_n, \hat{u})}{\hat{y}}$$

where $\hat{u}$ is the maximum utility level attainable under parameter values $\hat{y}, \hat{p}_1, \ldots \hat{p}_n$ (i.e.

$$\hat{u} = V(\frac{\hat{p}_1}{\hat{y}}, \frac{\hat{p}_2}{\hat{y}} \ldots \frac{\hat{p}_n}{\hat{y}}) \qquad)$$

The Laspeyres index

$$P_L = \frac{\sum_{i=1}^{n} \hat{x}_i p_i}{\sum_{i=1}^{n} \hat{x}_i \hat{p}_i} = \frac{\sum_{i=1}^{n} \hat{x}_i p_i}{\hat{y}}$$

where $(\hat{x}_1, \hat{x}_2 \ldots \hat{x}_n)$ is the bundle chosen under base period parameter values. Since $m(p_1, p_2 \ldots p_n, \hat{u})$ is the minimum expenditure necessary to reach utility level $\hat{u}$, $P_{LK} \leq P_L$.

The analogous Paasche-Konus index is defined as

$$P_{PK} = \frac{m(p_1, p_2 \ldots p_n, u^*)}{m(\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n, u^*)}$$

where u^* is the maximal level of utility attainable under parameter values $\hat{y}, p_1, p_2 \ldots p_n$

$$P_{PK} = \frac{\hat{y}}{m(\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n, u^*)}$$

The Paasche index

$$P_P = \frac{\sum_{i=1}^{n} x_i^* p_i}{\sum_{i=1}^{n} x_i^* \hat{p}_i} = \frac{\hat{y}}{\sum_{i=1}^{n} x_i^* \hat{p}_i}$$

where $(x_1^*, x_2^* \ldots x_n^*)$ is the bundle chosen under parameter values $\hat{y}, p_1, p_2 \ldots p_n$. However being the minimum expenditure required to reach utility level u^*,

$$m(\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n, u^*) \leq \sum_{i=1}^{n} x_i^* \hat{p}_i$$

Hence $P_{pk} \geq P_p$.

If the direct utility function is homothetic, it can be readily shown that the indirect utility function is homothetic. This implies that the expenditure functions can be written in the form $d(u)\,\bar{m}(p_1, p_2 \ldots p_n)$ which implies that P_{lk} and P_{pk} are independent of the reference level of utility and are identical. However, in general, when u^* and $\hat{u}$ are different, there are two distinct pure cost-of-living indices even in the absence of taste change.

After presenting verbal arguments to the effect that the Paasche index is more relevant to the true cost-of-living F-S give this formal interpretation of 'the true cost-of-living index (p.102.' "Given base period prices of goods $\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n$, base period income $\hat{y}$, current prices of goods $p_1, p_2 \ldots p_n$, the problem is to find that income y such that the representative consumer is <u>currently</u> (their italics) indifferent between facing current prices with income y and facing base period prices with base period income. The true cost-of-living index is then $\frac{y}{\hat{y}}$." This is somewhat curious:

$$\frac{y}{\hat{y}} = \frac{m(p_1, p_2 \ldots p_n, \hat{u})}{\hat{y}} = P_{lk}$$

But in general there is <u>no relationship</u> between the Laspeyres-Konus index P_{lk} and the Paasche index P_p. This apparently then is inconsistent with the case they have just made for the Paasche index as being more relevant. This argument which is simply that in the case of changing tastes, using the current bundle for weighting prices is a better reflection of current tastes, has an obvious appeal. However it should have been made in the context of the relevant concept of the pure cost-of-living index ([3]).

III <u>Taste Change</u>

Taste change is assumed to augment the first good. If b is the taste change parameter $u = u(b.x_1, x_2 \ldots x_n)$. We have shown in appendix C, that the corresponding expenditure function is in the form:

$$y = m(\frac{p_1}{b}, p_2 \ldots p_n, u).$$

Some useful properties of this function are derived in appendices B and C.

We follow the F-S discussion of the Laspeyres-Konus true cost-of-living index from the viewpoint of current tastes, even though we have argued that the Paasche-Konus is more appropriate to their viewpoint.

The Laspeyres-Konus true cost-of-living index, given tastes represented by $u = u(b.x_1, x_2 \ldots x_n)$, is defined by

$$P_{LK} = \frac{m(\frac{p_1}{b}, p_2 \ldots p_n, \hat{u})}{m(\frac{\hat{p}_1}{b}, \hat{p}_2 \ldots \hat{p}_n, \hat{u})} = \frac{y}{\hat{y}}$$

where $\hat{u}$ is the maximal level of utility attainable under parameter values $\hat{y}, \hat{p}_1, \hat{p}_2 \ldots \hat{p}_n, b$.

The F-S discussion can be separated into three parts. Part 1 looks at the way y is affected by a change in b (in the context of price changes). Part 2 examines the fact that if tastes have changed, the base period bundle will not be consistent with the above expenditure function . Part 3 examines second order effects, i.e. how the size of price changes influences the effect of a change on b on y.

Part 1: We examine the effect of a change in b on y and hence $y/\hat{y}$ since $\hat{y}$ is fixed. F-S Corollary 3.1, p.110, states that if prices are unchanged then $\frac{\partial y}{\partial b} = 0$.

This is obvious since if prices are unchanged, $y = \hat{y}$ for all values of b . However a simple check on the result follows from total differentiation w.r.t. b of

$$y = m(\frac{p_1}{b}, p_2 \ldots p_n, \hat{u})$$

$$\frac{\partial y}{\partial b} = \frac{\partial m}{\partial b}\bigg|_{u=\hat{u}=\text{const.}} + \frac{\partial m}{\partial \hat{u}} \cdot \frac{\partial \hat{u}}{\partial b} \qquad \text{(III.1)}$$

where $\hat{u} = V(\frac{\hat{p}_1}{b.\hat{y}}, \frac{\hat{p}_2}{\hat{y}}, \ldots \frac{\hat{p}_n}{\hat{y}})$

$$\frac{\partial m}{\partial \mathfrak{a}} = \frac{1}{\lambda} \qquad ([4]) \qquad \text{(III.2)}$$

N.B. $\frac{\partial m}{\partial \mathfrak{a}} \neq \frac{1}{\hat{\lambda}}$ since $\frac{\partial m}{\partial \mathfrak{a}}$ is being evaluated not at

$\frac{\hat{p}_1}{b} \ldots \hat{p}_n$ but rather at $\frac{p_1}{b}, p_2 \ldots p_n$, where

$\lambda = \frac{\bar{\lambda}}{y} = \frac{\partial v}{\partial y}$ in the notation of the appendices.

$$\frac{\partial \mathfrak{a}}{\partial b} = \frac{\partial}{\partial b}\left(v\left(\frac{\hat{p}_1}{b.\hat{y}}, \frac{\hat{p}_2}{\hat{y}} \ldots \frac{\hat{p}_n}{\hat{y}}\right)\right)$$

$$= \frac{\hat{\lambda}\, \hat{f}_1\, \hat{x}_1}{b} \qquad \text{by (C.6*)} \qquad \text{(III.3)}$$

According to result (C.7),

$$\frac{\partial m}{\partial(\frac{p_1}{b})} = b.x_1 = h_1(\frac{p_1}{b}, p_2 \ldots p_n, \mathfrak{a}) \qquad ([6])$$

$$\therefore \frac{\partial m}{\partial b} = \frac{-p_1}{b^2} \frac{\partial m}{\partial(\frac{p_1}{b})} = \frac{-p_1 x_1}{b} \qquad \text{(III.4)}$$

Hence

$$\frac{\partial y}{\partial b} = \frac{\hat{\lambda}}{\lambda} \frac{\hat{p}_1 x_1}{b} - \frac{p_1 x_1}{b} \qquad \text{(III.5)}$$

$$= \frac{\hat{\mathfrak{a}}_1\, \hat{x}_1 - u_1 x_1}{\lambda} \qquad \text{(III.6)}$$

This proves F-S's theorem 3.1, p.110.

F-S theorem 3.2(C) follows immediately: if all prices change in proportion so that $p_i = k\hat{p}_i$ $i=1\ldots n$, then $\frac{\partial y}{\partial b} = 0$.

Proof: $\lambda = k.\hat{\lambda}$, $p_1 = k\hat{p}_1$, $\hat{x}_1 = x_1$ from the homogeneity of degree zero of the demand function for x_1 in the prices.

It is also obvious from the fact that $\frac{y}{\hat{y}} = k$ which is independent of b.

We now prove F-S theorem 3.2(A):

if $p_i = \hat{p}_i$ $\quad i = 2 \ldots n$,

then sign $\left(\frac{\partial y}{\partial b}\right)$ = sign $(p_1 - p_1)$ if the Marshallian demand function for good one is elastic w.r.t. p_1, with opposite signs if the demand is inelastic and

$\frac{\partial y}{\partial b} = 0$ if the Marshallian own-price elasticity $= -1$

Proof:

$$\frac{\partial y}{\partial b} = \frac{\hat{u}_1 \hat{x}_1 - u_1 x_1}{\lambda} \quad \text{by} \qquad \text{(III.6)}$$

Hence sign $\left(\frac{\partial y}{\partial b}\right)$ = sign $(\hat{u}_1 \hat{x}_1 - u_1 x_1)$.

Thus we need to investigate what happens to $u_1 x_1$ as p_1 changes from $\hat{p}_1$. We note that constant

$$= \hat{u} = v\left(\frac{\hat{p}_1}{b.\hat{y}}, \frac{\hat{p}_2}{\hat{y}} \cdots \frac{\hat{p}_n}{\hat{y}}\right) = v\left(\frac{\hat{p}_1}{b.y}, \frac{\hat{p}_2}{y} \cdots \frac{\hat{p}_n}{y}\right)$$

(III.7)

Implicitly this defines the relationship between y and p_1.

Explicitly $\quad y = m\left(\frac{p_1}{b}, \hat{p}_2 \cdots \hat{p}_n, \hat{u}\right)$ (III.8)

$$\text{Consider} \quad \frac{\partial (u_1 x_1)}{\partial p_1} = \frac{\partial}{\partial p_1}\left(\lambda \frac{p_1}{b} x_1\right) = \frac{1}{b}\left(p_1 x_1 \frac{\partial \lambda}{\partial p_1} + \lambda x_1 + \lambda p_1 \frac{\partial x_1}{\partial p_1}\right) \qquad \text{(III.9)}$$

$$\lambda = \frac{\partial v}{\partial y}\left(\frac{p_1}{b.y}, \frac{\hat{p}_2}{y} \cdots \frac{\hat{p}_n}{y}\right)$$

$$\frac{\partial\lambda}{\partial p_1} = \frac{\partial}{\partial p}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial p_1}\right) = \frac{\partial V}{\partial p_1}\left(\frac{\partial V}{\partial p_1}\Bigg|_{y \text{ constant}} + \frac{\partial V}{\partial y} \cdot \frac{\partial V}{\partial p_1}\right) \qquad \text{(III.10)}$$

using (C.8)

$$\frac{\partial\lambda}{\partial p_1} = \frac{\partial}{\partial y}\left\{-\lambda\,\frac{D_1}{b}\left(\frac{\hat{p}_1}{b.y}, \frac{\hat{p}_2}{y} \ldots \frac{\hat{p}_n}{y}\right) + \lambda h_1\left(\frac{p_1}{b}, \hat{p}_2 \ldots \hat{p}_n, \hat{u}\right)\right\} \quad ([6])$$

$$= -\frac{\partial\lambda}{\partial y} \cdot x_1 - \frac{\lambda\partial D_1}{b\partial y} + \frac{\partial\lambda}{\partial y} \cdot x_1$$

$$= -\frac{\lambda\partial D_1}{b\partial y} \qquad \text{(III.11)}$$

Substitute into (III.9)

$$\therefore \frac{\partial(u_1 x_1)}{\partial p_1} = \frac{\lambda}{b}\quad p_1 \frac{\partial x_1}{\partial p_1} - \frac{x_1}{b} \cdot \frac{\partial D_1}{\partial y} \quad + x_1 \qquad \text{(III.12)}$$

But

$$x_1 = \frac{1}{b} h_1 \quad \frac{p_1}{b}, \hat{p}_2 \ldots \hat{p}_n, \hat{u})$$

and by the Slutsky equation

$$\frac{\partial h_1}{\partial p_1} - x_1 \cdot \frac{\partial D_1}{\partial y} = \frac{\partial D_1}{\partial p_1}$$

$$\therefore \frac{\partial(u_1 x_1)}{\partial p_1} = \frac{\lambda x_1}{b}\left[\frac{p_1}{x_1} \cdot \frac{\partial D_1}{\partial p_1} + 1\right] = \frac{\lambda x_1}{b}\left[\eta_{11} + 1\right] \qquad \text{(III.13)}$$

where η_{11} is the Marshallian own price elasticity of good one. The theorem follows for the cases where, consistently, either

$\eta_{11} < -1$, or $\eta_{11} > -1$ or $\eta_{11} = -1$ over the interval $\hat{p}_1$ to p_1. ([5])

F-S theorem 3.2(B) states that if $p_i = \hat{p}_i$ i=1 ... n, i≠j then sign $(\frac{\partial y}{\partial b})$ = sign $(p_j - \hat{p}_j)$ if the jth good is a gross complement for the 1st good, with opposite signs if the jth good is a gross substitute for the first, and $\frac{\partial y}{\partial b} = 0$ if the Marshallian cross price elasticity = 0.

<u>Proof.</u> Now $y = m(\frac{\hat{p}_1}{b} \ldots \hat{p}_{j-1}, p_j, \hat{p}_{j+1} \ldots \hat{p}_n, \hat{u})$ (III.14)

$$\frac{\partial}{\partial p_j}(u_1 x_1) = \frac{1}{b}\left(p_1 x_1 . \frac{\partial \lambda}{p_j} + \lambda p_1 \frac{\partial x_1}{\partial p_j}\right)$$

$$\frac{\partial \lambda}{\partial p_j} = \frac{\partial}{\partial p_j}\left\{\frac{\partial V}{\partial y}\left(\frac{\hat{p}_1}{b.y}, \frac{\hat{p}_2}{y} \ldots \frac{\hat{p}_{j-1}}{y}, \frac{p_j}{y}, \frac{\hat{p}_{j+1}}{y} \ldots \frac{\hat{p}_n}{y}\right)\right\}$$

$$= -\lambda . \frac{\partial D_j}{\partial y} \tag{III.15}$$

$$\therefore \frac{\partial}{\partial p_j}(u_1 x_1) = \frac{\lambda p_1}{b}\left[-x_1 . \frac{\partial D_j}{\partial y} + \frac{\partial h_1}{b \partial p_j}\right] \tag{III.16}$$

where $x_1 = h_{\frac{1}{b}}(\frac{\hat{p}_1}{b} \ldots \hat{p}_{j-1}\ p_j, \hat{p}_{j+1} \ldots \hat{p}_n, \hat{u})$

By the Slutsky equation $\frac{\partial D_j}{\partial p_1} = -x_1 . \frac{\partial D_j}{\partial y} + \frac{\partial h_j}{\partial p_1}$

Since compensated cross price effects are symmetric,

$$\frac{1}{b}\frac{\partial h_1}{\partial p_j} = \frac{\partial h_j}{\partial p_1} \qquad ([6])$$

$$\therefore \frac{\partial (u_1 x_1)}{\partial p_j} = \frac{\lambda x_1}{b}\left(\frac{p_1}{x_j} . \frac{\partial D_j}{\partial p_1}\right) = \frac{\lambda x_1}{b}\eta_{j1} \tag{III.17}$$

Hence theorem 3.2(B) follows if consistently either $\eta_{j1} < 0$, or $\eta_{j1} > 0$ or $\eta_{j1} = 0$.

Part (2): We examine the consequences of the fact that, if tastes have changed since the base period, the weights in a Laspeyres index do not correspond to the optimal bundle chosen under the preference representation:

$$u = u(b_1.x_1, x_2 \ldots x_n) = V\left(\frac{\hat{p}_1}{b.\hat{y}}, \frac{\hat{p}_2}{b.\hat{y}} \ldots \frac{\hat{p}_n}{b.\hat{y}}\right)$$

Thus $P_L = \dfrac{\Sigma\, p_i\, \tilde{x}_i}{\Sigma\, \hat{p}_i\, \tilde{x}_i}$ and the bounding relationship between

$P_{LK} \equiv \dfrac{y}{\hat{y}}$ and P_L breaks down. The differences between $\hat{x}_i$ and $\tilde{x}_i$ caused by taste change w.r.t. the first good are easy to evaluate by varying b in the Marshallian demand functions:

$$\hat{x}_i = D_i\left(\frac{\hat{p}_1}{b.\hat{y}}, \frac{\hat{p}_2}{\hat{y}} \ldots \frac{\hat{p}_n}{\hat{y}}\right) \qquad i=2 \ldots n$$

$$b.\hat{x}_1 = D_1\left(\frac{\hat{p}_1}{b.\hat{y}}, \frac{\hat{p}_2}{\hat{y}} \ldots \frac{\hat{p}_n}{\hat{y}}\right)$$

$$\frac{\partial \hat{x}_1}{\partial b} = -\frac{\hat{x}_1}{b} + \frac{1}{b}\,\frac{\partial D_1}{\partial(\frac{p_1}{b})} \cdot -\left(\frac{p_1}{b^2}\right) = -\frac{\hat{x}_1}{b}(1 + \eta_{11})$$

$$\frac{\partial \hat{x}_i}{\partial b} = \frac{1}{b}\,\frac{\partial D_i}{\partial(\frac{p_1}{b})}\left(-\frac{p_1}{b^2}\right) = -\frac{\hat{x}_i}{b}\,\eta_{i1}$$

$$\text{where} \quad \eta_{i1} = \frac{p_1}{x_i}\,\frac{\partial D_i}{\partial p_1} = \frac{p_1}{bx_i}\,\frac{\partial D_i}{\partial(\frac{p_1}{b})} \quad \text{and} \quad \eta_{11} = \frac{p_1}{x_1} \cdot \frac{\partial(\frac{D_1}{b})}{\partial\, p_1} \qquad ([8])$$

Part (3): Part (1) above considered qualitatively how the adjustment in the Laspeyres-Konus pure cost of living index from the view point of current tastes, was related to how prices had changed since the base period. We now investigate how the size of the adjustment is related to the size of the price change.

Consider

$$\frac{\partial}{\partial p_1}\left(\frac{\partial y}{\partial b}\right) = \frac{\partial}{\partial p_1}\left(\frac{\hat{\lambda}\hat{p}_1\hat{x}_1}{b\lambda} - \frac{p_1}{b}x_1\right) = \frac{1}{b}\left\{-\frac{1}{\lambda^2}\frac{\partial\lambda}{\partial p_1}(\hat{\lambda}\hat{p}_1\hat{x}_1) - \frac{p_1}{b}\frac{\partial h_1}{\partial p_1} - x_1\right\}$$

Since

$$\frac{\partial\lambda}{\partial p_1} = -\frac{\lambda}{b}\frac{\partial D_1}{\partial y}, \quad \frac{\partial}{\partial p_1}\left(\frac{\partial y}{\partial b}\right) = \frac{1}{b}\left\{\frac{\partial D_1}{\partial y}\frac{\hat{\lambda}}{b\lambda}\hat{p}_1\hat{x}_1 - \frac{p_1}{b1}\frac{\partial h_1}{\partial p_1} - x_1\right\} \qquad \text{(III.18)}$$

$$\left.\frac{\partial}{\partial p_1}\left(\frac{\partial y}{\partial b}\right)\right|_{\text{at } \hat{p}_1\ldots\hat{p}_n,\hat{y}} = \frac{\hat{p}_1}{b}\left(\frac{\hat{x}_1}{b}\frac{\partial D_1}{\partial y} - \frac{\partial h_1}{b\partial p_1}\right) - \frac{\hat{x}_1}{b} = -\frac{\hat{x}_1}{b}(\eta_{11}+1) \qquad \text{(III.19)}$$

using Slutsky's equation and $p_1=\hat{p}_1$, $x_1=\hat{x}_1$, $\lambda=\hat{\lambda}$

and

$$\frac{\partial}{\partial p_j}\left(\frac{\partial y}{\partial b}\right) = \frac{1}{b}\left\{-\frac{1}{\lambda^2}\frac{\partial\lambda}{\partial p_j}(\hat{\lambda}\hat{p}_1\hat{x}_1) - \frac{p_1}{b}\frac{\partial h_1}{\partial p_j}\right\} \qquad \text{(III.20)}$$

$$\left.\frac{\partial}{\partial p_j}\left(\frac{\partial y}{\partial b}\right)\right|_{\text{at } \hat{p}_1\ldots\hat{p}_n,\hat{y}} = -\frac{\hat{x}_1}{b}\cdot\eta_{1j} \qquad \text{(III.21)}$$

since $\frac{\partial\lambda}{\partial p_j} = -\lambda\frac{\partial D_j}{\partial y}$

and $\frac{1}{b}\frac{\partial h_1}{\partial p_j} = \frac{\partial h_j}{\partial p_1}$

This proves the first part of F-S theorems 3.4(A) and (B).

It is clear that these results on sign $\left(\frac{\partial}{\partial p_j}\left(\frac{\partial y}{\partial b}\right)\right)$ are local, i.e. for small displacements of p_1 from $\hat{p}_1$ and p_j from $\hat{p}_j$.

From (III.18)

$$\frac{\partial}{\partial p_1}\left(\frac{\partial y}{\partial b}\right) = \frac{\hat{\lambda}}{\lambda}\,\frac{\hat{p}_1\hat{x}_1}{p_1}\left(\eta_{11}^{fn} - \eta_{11}\right) - x_1\eta_{11}^{fn} - x_1 \qquad \text{(III.22)}$$

where $\eta_{11}^{fn} = \dfrac{p_1}{b\,x_1} \cdot \dfrac{\partial h_1}{\partial p_1}$

and $\eta_{11} = \dfrac{p_1}{b\,x_1} \cdot \dfrac{\partial D_1}{\partial p_1}$

But

$$\frac{\hat{\lambda}}{\lambda}\,\frac{\hat{p}_1\hat{x}_1}{p_1} - x_1 = \frac{1}{p_1}\,\frac{\partial y}{\partial b}$$

$$\therefore \frac{\partial}{\partial p_1}\left(\frac{\partial y}{\partial b}\right) = \frac{1}{p_1}\,\frac{\partial y}{\partial b} \cdot \eta_{11}^{fn} - \frac{\hat{\lambda}}{\lambda}\,\frac{\hat{p}_1\hat{x}_1}{p_1}\,\eta_{11} - \frac{\hat{\lambda}}{\lambda}\,\frac{\hat{p}_1\hat{x}_1}{p_1} + \frac{1}{p_1}\,\frac{\partial y}{\partial b}$$

$$= \frac{1}{p_1}\left(\frac{\partial y}{\partial b}\right)\left(\eta_{11}^{fn} + 1\right) - \frac{\hat{\lambda}}{\lambda}\,\frac{\hat{p}_1\hat{x}_1}{p_1}\left(\eta_{11} + 1\right) \qquad \text{(III.23)}$$

This proves F-S lemma 3.7 and shows that we get the <u>global</u> result:

if either <u>both</u> $\eta_{11} < -1$ <u>and</u> $\eta_{11}^{fn} \leq -1$ or <u>both</u> $\eta_{11} > -1$ <u>and</u> $\eta_{11}^{fn} \geq -1$, then $\dfrac{\partial}{\partial p_1}\left(\dfrac{\partial y}{\partial b}\right) > 0$ and $\dfrac{\partial}{\partial p_1}\left(\dfrac{\partial y}{\partial b}\right) < 0$ respectively.

Similarly,

$$\frac{\partial}{\partial p_j}\left(\frac{\partial y}{\partial b}\right) = \frac{1}{b}\left\{\frac{\hat{\lambda}}{\lambda} \cdot \frac{\partial D_j}{\partial y}\,\hat{p}_1\hat{x}_1 - \frac{p_1}{b}\,\frac{\partial h_1}{\partial p_j}\right\} \qquad \text{(III.24)}$$

But

$$x_1\,\frac{\partial D_j}{\partial y} = \frac{\partial h_j}{\partial p_1} - \frac{\partial D_j}{\partial p_1}\,, \quad \therefore \quad \frac{\partial D_j}{\partial y} = \frac{x_j}{p_1 x_1}\left(\eta_{j1}^{fn} - \eta_{j1}\right)$$

and $\frac{\hat{\lambda}}{\lambda} \frac{\hat{p}_1 \hat{x}_1}{x_1} = x_1 + \frac{1}{p_1} \frac{\partial y}{\partial b}$

$$\frac{\partial}{\partial p_j} \left(\frac{\partial y}{\partial b}\right) = \frac{1}{b} \left\{ \left(x_1 + \frac{1}{p_1} \frac{\partial y}{\partial b}\right) \frac{x_j}{x_1} \left(\eta^{[n}_{j1} - \eta_{j1}\right) - x_j \, \eta^{[n}_{j1} \right\}$$

$$= \frac{1}{b} \left\{ \frac{1}{p_1} \frac{\partial y}{\partial b} \frac{x_j}{x_1} \eta^{[n}_{j1} - \frac{\hat{\lambda}}{\lambda} \frac{\hat{p}_1 \hat{x}_1}{p_1} \frac{x_j}{x_1} \eta_{j1} \right\}$$

$$= \frac{1}{b} \left\{ \frac{1}{p_j} \frac{\partial y}{\partial b} \eta^{[n}_{1j} - \frac{\hat{\lambda}}{\lambda} \frac{\hat{p}_1 \hat{x}_1}{p_1} \cdot \frac{x_j}{x_1} \eta_{j1} \right\} \qquad \text{(III.25)}$$

where $\eta_{j1} \equiv \frac{p_1}{x_j} \frac{\partial D_j}{\partial p_1}$ and $\eta^{[n}_{1j} \equiv \frac{p_j}{x_1} \frac{\partial h_1}{\partial p_j}$

which proves the second part of F-S lemma 3.7 ([9]) and theorem 3.4 (B).

IV Quality Change

All of F-S's analysis of quality change consists of finding the parameterization of quality change in the direct utility function implied by various more or less simple adjustments (for quality change) in the Laspeyres-Konus index. If b is the quality parameter w.r.t. good one ($b=1$ for the base period), then

$$P_{LK} = \frac{m(p_1, p_2 \ldots p_n, b, u)}{m(\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n, 1, \hat{u})} = \frac{y}{\hat{y}}$$

We abstract from market price changes and focus on the effect of changes in b .

F-S theorem 5.1 (B) states that an adjustment in p_1, which is independent of $(x_1, x_2 \ldots x_n)$, will correct for quality change in good one if and only if $u = u(h(b).x_1, x_2 \ldots x_n)$.

Formally if:

$$m(p_1^*, p_2 \ldots p_n, 1, \mathfrak{a}) = m(p_1, p_2 \ldots p_n, b, \mathfrak{a}) \qquad \text{(IV.1)}$$

we want $\frac{\partial p_1^*}{\partial b}$ to be independent of $(x_1, \ldots, x_n)$.

Differentiating (IV.1) totally w.r.t. b :

$$\frac{\partial m}{\partial p_1^*} \cdot \frac{\partial p_1^*}{\partial b} + \frac{\partial m}{\partial V} \cdot \frac{\partial \mathfrak{a}}{\partial b} = \frac{\partial m}{\partial b} + \frac{\partial m}{\partial V} \cdot \frac{\partial \mathfrak{a}}{\partial b} \qquad \text{(IV.2)}$$

$$\therefore \quad \frac{\partial p_1^*}{\partial b} = \frac{\partial m}{\partial b} \Big/ \frac{\partial m}{\partial p_1^*}$$

Totally differentiating

$$\mathfrak{a} = v\left(\frac{p_1}{y} \ldots \frac{p_n}{y}, b\right)$$

where $\quad y = m(p_1, \ldots p_n, b, \mathfrak{a})$

we obtain $0 = \frac{\partial v}{\partial y} \cdot \frac{\partial m}{\partial b} + \frac{\partial v}{\partial b}$

$$\therefore \frac{\partial m}{\partial b} = \frac{-1}{\lambda} \frac{\partial v}{\partial b} = \frac{-u_b}{\lambda}$$

We showed in appendix B that $\frac{\partial m}{\partial p_1} = x_1 = h_1(p_1 \ldots p_n, b, \mathfrak{a})$

Hence

$$\left.\frac{\partial p_1^*}{\partial b}\right|_{\text{evaluated at } p_1 \ldots p_n} = \frac{-u_b}{\lambda x_1} = \frac{-p_1 u_b}{x_1 u_1} \qquad \text{(IV.3)}$$

since $u_1 = \frac{\partial u}{\partial x_1} = \lambda \frac{p_1}{b}$.

We require $\frac{\partial p_1^*}{\partial b}$ and hence $\frac{u_b}{x_1 u_1}$ to be independent of $x_1, x_2 \ldots x_n$.

By the Leontief separation theorem ((11)), ((12)), ((3)), pp.390-391), this is true if and only if $\frac{u_b}{u_1} = H(b)$ which is true if and only if $u = u(h(b).x_1, x_2 \dots x_n)$.

Now in appendix C and our discussion of taste change, we saw that this implied an expenditure function of the form

$$y = m(h(b).p_1, p_2 \dots p_n, u).$$

<u>Theorem:</u> The form $u(h(b).x_1, x_2 \dots x_n)$ implies the form:

$$y = m(f(b).p_1, p_2 \dots p_n, u) \text{ and conversely.}$$

<u>Proof:</u>

$$\frac{m_b}{m_1} = \frac{-p_1}{x_1} \frac{u_b}{u_1}$$

where $m_b = \frac{\partial m}{\partial b}$, $m_1 = \frac{\partial m}{\partial p_1}$.

Hence if $\frac{u_b}{u_1} = x_1.H(b)$ then $\frac{m_b}{m_1} = -p_1 H(b)$

which by the Leontief theorem is true if and only if the expenditure function has the form $m(f(b).p_1, p_2 \dots p_n, u)$.

Conversely,

$$\frac{m_b}{m_1} = p_1 F(b) \Rightarrow \frac{u_b}{x_1 u_1} = -F(b) \Rightarrow u = u(h(b).x_1, x_2 \dots x_n).$$

This is an important result ([10]), since the quality correction can then be carried out independently <u>both</u> of $p_2 \dots p_n$ and $x_1 \dots x_n$. This is a very special case as we shall show.

F-S theorem 5.1 (A) states that $\frac{\partial p_1^*}{\partial b}$ is independent of $(x_2 \ldots x_n)$ if and only if

$$u = u(h(b,x_1), x_2 \ldots x_n)$$

Proof.

$$\frac{\partial}{\partial x_j}\left(\frac{\partial p_1^*}{\partial b}\right) = 0 \qquad j=2\ldots n$$

implies $\frac{u_b}{x_1 u_1} = H(b, x_1)$ which is true if and only if

$u = u(h(b,x_1), x_2 \ldots x_n)$ by the Leontief separation theorem.

The natural question arises: does this form of the utility function correspond with the form of the expenditure function

$$m(f(b,p_1), p_2 \ldots p_n, u) \qquad ?$$

The answer must be in the negative in general.

Proof: The expenditure function is linear homogeneous in the prices

$$\therefore\ m(f(b,p_1), p_2 \ldots p_n, u) \equiv p_1.m(f(b,1), \frac{p_2}{p_1}, \ldots \frac{p_n}{p_1}, u)$$

$$\equiv m(p_1.f(b,1), p_2 \ldots p_n, u).$$

According to our theorem,(p.15), this implies a direct utility function of the form $u(h(b).x_1, x_2 \ldots x_n)$ which is a more special case than $u(h(b,x_1), x_2 \ldots x_n)$.

Another way of seeing this is to consider:

$$\frac{u_b}{x_1 u_1} = H(b, x_1) = H(b, m_1) = -\frac{1}{p_1}\frac{m_b}{m_1}$$

But since m_1 is not in general independent of $p_2 \ldots p_n$, the expenditure function is not in general separable w.r.t. b and p_1.

The remaining results such as theorem 5 (B), which states that if $m(p_1, p_2^*, p_3 \ldots p_n, 1, \bar{u}) = m(p_1, p_2 \ldots p_n, b, \bar{u})$ then $\frac{\partial p_2^*}{\partial b}$ is independent of $x_3 \ldots x_n$ if and only if $u = u(g_1(x_1, b), g_2(x_1,x_2,b), x_3 \ldots x_n)$, are easily proved.

Theorem 3(B) follows immediately from the observation that

$$\frac{\partial p_2^*}{\partial b} = - \frac{p_2 u_b}{x_2 u_2}$$

and the use of the Leontief theorem.

Similarly, in the case where quality change in the first good augments each other good separately, i.e.

$$u = u(g_1(x_1,b), g_2(x_1,x_2,b), \ldots g_n(x_1,x_2,b)),$$

each price p_j can be adjusted independently of all x_i $i \neq 1$, $i \neq j$, since

$$\frac{\partial}{\partial x_i} \left(\frac{\partial p_j^*}{\partial b} \right) = 0 \quad \text{all} \quad i \neq 1, \quad i \neq j \quad \text{is equivalent to}$$

$$\frac{\partial}{\partial x_i} \left(\frac{u_b}{u_i} \right) = 0 \quad \text{all} \quad i \neq 1, \quad i \neq j \quad \text{which by the Leontief}$$

theorem is implied by the above utility function. This is F-S's theorem 5.4.

The implied expenditure functions for such utility functions are unfortunately not simple unless homotheticity of the utility function can be assumed.

Footnotes

[1] The point is dealt with in F-S footnote 7, p.134.

[2] This is, of course, well known: see Wold ((18)).

[3] One may say in defence of their analysing only P_{kk} that at one point they do specifically want to discuss the effect of taste change on a Laspeyres index: here, of course, P_{lk} is relevant. Further, it is not difficult to translate the results into the Paasche-Konus index case.

[4] λ is the Lagrangian multiplier in the problem:

$$\text{max. w.r.t. } x_1 \ldots\ldots x_n \quad L = u(b.x_1, x_2 \ldots x_n) - \lambda\left(\sum_{i=1}^{n} p_i x_i - y\right)$$

[5] There is some intuitive discussion of these results in footnote [7] below.

[6] It should be noted that the Hicksian demand function for good one is $\frac{1}{b} \cdot h_1\left(\frac{p_1}{b}, p_2, \; p_n, u\right)$ and the Marshallian demand function is $\frac{1}{b} \cdot D_1\left(\frac{p_1}{b.y}, \frac{p_2}{y}, \ldots \frac{p_n}{y}\right)$. For the other goods, these functions are simply $h_j\left(\frac{p_1}{b}, p_2, \cdots p_n, u\right)$ and $D_j\left(\frac{p_1}{b.y}, \frac{p_2}{y}, \cdot\cdot \frac{p_n}{y}\right)$ respectively, $j = 2 \ldots\ldots n$.

[7] F-S argue in footnote 15 that it is not at all intuitively obvious why these results should be in terms of Marshallian (gross) rather than Hicksian (net) elasticities. To this one could reply that there are strong intuitive reasons why the results should <u>not</u> be in terms of simple Hicksian elasticities. Take the case of a change in the first price only. The argument runs as follows: instead of looking at how the effect on y of a change in b is influenced by a change in p_1, we can just as well examine how the effect on y of a change in p_1 will be influenced by a change in b. But the effect on y of a small (say one unit) change in p_1 with utility constant is simply x_1. If the reference utility level were independent of b, then the effect of an increase in b on x_1 (works analogously to a reduction in p_1) would just depend on x_1 and the Hicksian own price elasticity. But a change in b also increases the reference level of utility: this is analogous to an income effect.

Since one man's intuition may be another's poison, a brief formal demonstration is in order. We consider

$$\frac{\partial}{\partial p_1}\left(\frac{\partial y}{\partial b}\right) \quad \text{and evaluate at } \hat{p}_1, \hat{p}_2, \ldots \hat{p}_n, b, \hat{y}.$$

$$'' = \frac{\partial}{\partial b}\left(\frac{\partial y}{\partial p_1}\right) = \frac{\partial}{\partial b}\left\{\frac{1}{b} \cdot h_1\left(\frac{p_1}{b}, \hat{p}_2 \cdot \; \hat{p}_n, \hat{u}\right)\right\}\Big|_{u = \text{const.}} + \frac{1}{b} \cdot \frac{\partial h_1}{\partial \hat{u}} \cdot \frac{\partial \hat{u}}{\partial b}$$

From (III.18) we see that the first term on the R.H.S. is equal to

Footnotes cont'd

$-\frac{1}{b}\left(x_1 + p_1 \cdot \frac{\partial x_1}{\partial p_1}\bigg|_{u\ \text{const.}}\right) = -\frac{x_1}{b}\left(\eta_{11}^{[n} + 1\right)$ and the second term is equal to $\frac{1}{b} \cdot \frac{\hat{\lambda}}{\lambda}\, \hat{p}_1 \left(\frac{\hat{x}_1}{b} \cdot \frac{\partial D_1}{\partial y}\right)$. Evaluating at $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_n, \hat{y}$, b we see that the second term equals $\frac{\hat{p}_1}{b} \cdot$ (income effect in the Slutsky equation) and hence we get the (local) result (III.19) in terms of the Marshallian elasticity.

[8] This establishes F-S theorem 3.3.

[9] Note that F-S define their $\eta_{j1}^{[n} = \frac{p_j}{x_1} \cdot \frac{\partial x_1}{\partial p_j}\bigg|_{u = \hat{u} = \text{const.}}$ on p.117 and $\eta_{ii} = \frac{p_1}{x_i} \cdot \frac{\partial x_i}{\partial p_1}\bigg|_{y\ \text{const.}}$ on p. 113. We remain with the latter notation.

[10] Important also in the sense that one requires this rather stringent assumption to obtain efficiency corrected price indices using the methods developed for durable goods by Cagan in "Measuring quality changes and the purchasing power of money: an exploratory study of automobiles", National Banking Review 1965, p.217-236 and in a more complete and sophisticated way by Robert E. Hall in "The measurement of quality change from vintage price data" in Griliches ed. The price statistics of the Federal Government, forthcoming 1971. The assumption plays an important role in my own forthcoming work on the U.S. used tractor market. Indeed some attempts are made there to subject the proposition to empirical testing.

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Appendix A

In this appendix we set out the derivation, using Legendre's dual transformation, of the basic duality results in consumer theory. We follow an exposition similar to that of Lau ((9)).

We are given a twice differentiable function, which has the property of strict quasi-concavity w.r.t. $x_1 \dots x_n$.

PRIMAL: $$F = F(x_1 \dots x_n \; ; \; p_1 \dots p_m) \qquad \text{A. 1}$$

This property implies that the n definitional relations

TRANSFORMATION:

$$\nu_i = \frac{\partial F}{\partial x_i} \qquad \text{A. 2}$$

can be solved for

$$x_i = f_i \, (\nu_1 \dots \nu_n \; ; \; p_1 \dots p_m) \qquad \text{A. 3}$$

Define a new function

DUAL: $$G = \sum_{i=1}^{n} x_i \nu_i - F$$

$$= \sum_{i=1}^{n} f_i \nu_i - F(f_1 \dots f_n \; ; \; p_1 \dots p_m) \qquad \text{A. 4}$$

G is called Legendre's dual transformation of the primal function F.

The transformation works in reverse also:

INVERSE TRANSFORMATION:

$$\frac{\partial G}{\partial \nu_i} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial \nu_i} \cdot \nu_j + f_i - \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \cdot \frac{\partial f_j}{\partial \nu_i}$$

By A. 2

$$\frac{\partial G}{\partial \nu_i} = f_i = x_i \qquad \text{A. 5}$$

A most important aspect of the transformation is the following:

TRANSFORMATION W.R.T.
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$$\frac{\partial G}{\partial p_i} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial p_i} \cdot \nu_j - \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} \cdot \frac{\partial f_j}{\partial p_i} - \frac{\partial F}{\partial p_i}$$

By A.2

$$\frac{\partial G}{\partial p_i} = - \frac{\partial F}{\partial p_i} \qquad \text{A. 6}$$

We now set out this framework for the case first where F is the Lagrangian function of consumer theory and second where F is thisLagrangian function with the optimum conditions holding. We are given a twice differentiable, increasing, strictly quasi-concave utility function $u = u(x_1, \ldots x_n, b)$ where

changes in the parameter b are alternately interpreted as taste or quality changes. The Lagrangian will also be strictly quasi-concave.

PRIMAL [1]: $$F = L = u(x_1, \ldots x_n, b) - \bar{\lambda} \left(\sum_{i=1}^{n} \left(\frac{p_i}{y} \right) x_i - 1 \right) \qquad \text{A.1}^1$$

If the optimality conditions hold

PRIMAL*: $$F = L^* = u(x_1 \ldots x_n, b) \qquad \text{A.1*}$$

TRANSFORMATION[1]:

$$\nu_i = \frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \bar{\lambda} \left(\frac{p_i}{y} \right) \qquad i = 1 \ldots n \qquad \text{A.2}^1$$

which can be solved for

$$x_i = f_i \left(\nu_1 \ldots \nu_n ; \frac{p_1}{y}, \ldots \frac{p_n}{y}, b \right) \qquad \text{A.3}^1$$

TRANSFORMATION*: $\nu_i = \frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \bar{\lambda}(\frac{p_i}{y}) = 0$ A.2*

which can be solved for

$$x_i = f_i(0 \ldots 0; \frac{p_1}{y} \ldots \frac{p_n}{y}, b) = D_i(\frac{p_1}{y} \ldots \frac{p_n}{y}, b) \qquad \text{A.3*}$$

DUAL':

$$G = \sum_{i=1}^{n} x_i \nu_i - u(x_1 \ldots x_n, b) + \bar{\lambda}(\sum_{i=1}^{n} (\frac{p_i}{y}) x_i - 1) \qquad \text{A.4'}$$

DUAL*:

by A.3* $G^* = -u(x_1 \ldots x_n, b)$

$$-V(\frac{p_1}{y}, \ldots \frac{p_n}{y}, b) \qquad \text{A.4*}$$

INVERSE TRANSFORMATION': $\frac{\partial G}{\partial \nu_i} = x_i = f_i$ A.5'

Under the optimality conditions $\nu_i = 0$ and hence this inverse transformation is not defined.

TRANSFORMATION

W.R.T. PASSIVE VARIABLES': $\frac{\partial G}{\partial(\frac{p_i}{y})} = - \frac{\partial L}{\partial(\frac{p_i}{y})} = \lambda x_i$

$$\frac{\partial G}{\partial \lambda} = - \frac{\partial L}{\partial \lambda}, \frac{\partial G}{\partial b} = - \frac{\partial L}{\partial b} \qquad \text{A.6'}$$

TRANSFORMATION W.R.T. PASSIVE VARIABLES*:

$$\frac{\partial V}{\partial(\frac{p_i}{y})} = - \bar{\lambda} x_i, \quad \frac{\partial V}{\partial b} = \frac{\partial u}{\partial b} \qquad \text{A.6*}$$

Thus we have established a duality between the utility function in the quantities, x_i and the utility function in the real prices $\frac{p_i}{y}$:

$$u = u(x_1 \ldots x_n, b) \qquad V = V(\frac{p_1}{y}, \ldots \frac{p_n}{y}, b)$$

$$\frac{1}{\bar{\lambda}} \frac{\partial u}{\partial x_i} = (\frac{p_i}{y}) \quad \text{A.7} \quad , \quad \frac{1}{\bar{\lambda}} \frac{\partial V}{\partial(\frac{p_i}{y})} = -x_i \qquad \text{A.8}$$

$$\frac{\partial u}{\partial b} = \frac{\partial V}{\partial b} \qquad \text{A.9}$$

Since $\sum_{i=1}^{n} \frac{p_i}{y} x_i = 1$, A.7 implies that $\bar{\lambda} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot x_i$

A.7 and A.8 imply that $\bar{\lambda} = - \sum_{i=1}^{n} \frac{\partial V}{\partial(\frac{p_i}{y})} \cdot \frac{p_i}{y}$ A.10

APPENDIX B

We now sketch the derivation of the properties we require, of the indirect utility function and the expenditure function.

1. $V(\frac{p_1}{y} \ldots \frac{p_n}{y}, b)$ is monotonic increasing in y.

This is implied by the assumption that the direct utility function is continuous and increasing in its arguments (at least within the consumption set which is implicitly assumed to exist) which rules out "fat" indifference curves.

2. $V(\frac{p_1}{y}, \ldots \frac{p_n}{y}, b)$ can therefore be inverted to get the expenditure function $y = m(p_1, \ldots p_n, b, u)$

3. $m(p_1, \ldots p_n, b, u)$ is linear homogeneous in the prices. This is implied by the form of

$$u = V(\frac{p_1}{y} \ldots \frac{p_n}{y}, b) = V(\frac{kp_1}{ky} \ldots \frac{kp_n}{ky}, b)$$ where k is

any positive constant.

4. $\frac{\partial V}{\partial p_j} / \frac{\partial V}{\partial y} = -D_j(\frac{p_1}{y}, \ldots \frac{p_n}{y}, b) = -x_j$ which is Marshallian demand function. B.1

Proof: From A.8

$$\frac{1}{\bar{\lambda}} \cdot \frac{\partial V}{\partial(\frac{p_j}{y})} = -x_j$$

From A.10

$$\bar{\lambda} = -\sum_{i=1}^{n} \frac{\partial V}{\partial(\frac{p_i}{y})} \cdot \frac{p_i}{y}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial(\frac{p_i}{y})} \cdot \left(-\frac{p_i}{y^2}\right)$$

$$\therefore \quad \bar{\lambda} = y \cdot \frac{\partial V}{\partial y}$$ B.2

$$\frac{\partial V}{\partial p_j} = \frac{\partial V}{\partial (\frac{p_i}{y})} \cdot \frac{1}{y} = - \frac{\bar{\lambda}}{y} \cdot x_j \qquad \text{B.3}$$

$$\therefore \frac{\partial V}{\partial p_j} / \frac{\partial V}{\partial y} = -x_j = D_j(\frac{p_1}{y} \ldots \frac{p_n}{y}, b) \qquad \text{B.4}$$

5. $\frac{\partial m}{\partial p_j} = h_j(p_1, \ldots p_n, b, u) = x_j$ which is the Hicksian demand function

<u>Proof</u>: If we substitute $y = m(p_1 \ldots p_n, b, u)$ into

$$u = V(\frac{p_1}{y}, \ldots \frac{p_n}{y}, b) \text{ we obtain}$$

an identity in u. Holding u constant, we differentiate totally w.r.t. p_j

$$\therefore 0 = \left.\frac{\partial V}{\partial p_j}\right|_{y \text{ const}} + \frac{\partial V}{\partial y} \cdot \frac{\partial m}{\partial p_j}$$

where $y = m(p_1 \ldots p_n, b, u)$

Hence $\frac{\partial m}{\partial p_j} = - \frac{\partial V}{\partial p_j} / \frac{\partial V}{\partial y}$ where $y = m(p_1 \ldots p_n, b, u)$

$$= D_j(\frac{p_1}{y} \ldots \frac{p_n}{y}, b) \text{ where } y = m(p_1 \ldots p_n, b, u)$$

$$= x_j$$

But then x_j is a function of $p_1 \ldots p_n$, b and u

$$\frac{\partial m}{\partial p_j} = h_j(p_1 \ldots p_n, b, u) = x_j$$

APPENDIX C

In this we apply the analysis of appendix A to the special form of the utility function $u = u(b, x_1, x_2 \ldots x_n)$. We use similar labelling to make the analogy clear.

$$\text{PRIMAL}': \quad L = u(b.x_1, x_2 \ldots x_n) - \bar{\lambda}\left(\sum_{i=1}^{n} \left(\frac{p_i}{y}\right) x_i - 1\right) \qquad \text{C.1}'$$

$$\text{PRIMAL}^*: \quad L^* = u(b.x_1, x_2 \ldots x_n) \qquad \text{C.1}^*$$

$$\text{TRANSFORMATION}^*: \quad \nu_i = \frac{\partial u}{\partial x_i} - \bar{\lambda}\frac{p_i}{y} = 0 \qquad i=2\ldots n$$

$$\nu_1 = \frac{\partial u}{\partial x_1} - \bar{\lambda}\frac{p_1}{y} = 0 \qquad \text{C.2}^*$$

$$\text{i.e. } \frac{\nu_1}{b} = \frac{\partial u}{\partial (b.x_1)} - \bar{\lambda}\frac{p_1}{b.y} = 0$$

which can be solved for $x_i = D_i\left(\frac{p_1}{b.y}, \frac{p_2}{y} \ldots \frac{p_n}{y}\right) \quad i=2\ldots n$

$$b.x_1 = D_1\left(\frac{p_1}{b.y}, \frac{p_2}{y} \ldots \frac{p_n}{y}\right) \qquad \text{C.3}^*$$

$$\text{DUAL}^*: \quad G = -V\left(\frac{p_1}{b.y}, \frac{p_2}{y} \ldots \frac{p_n}{y}\right) \qquad \text{C.4}^*$$

substituting C.3* into the direct utility function.

TRANSFORMATION W.R.T. PASSIVE VARIABLES:

$$\frac{\partial V}{\partial\left(\frac{p_i}{y}\right)} = -\bar{\lambda}\, x_i \qquad i=1\ldots n$$

$$\text{i.e. } \frac{\partial V}{\partial\left(\frac{p_1}{y}\right)} = \frac{\partial V}{\partial\left(\frac{p_1}{by}\right)} \cdot \frac{1}{b} = -\bar{\lambda}\, x_1$$

$$\frac{\partial V}{\partial\left(\frac{p_1}{b.y}\right)} = -\bar{\lambda}\, bx_1 \qquad \text{C.6}^*$$

$$\text{and } \frac{\partial V}{\partial b} = -\bar{\lambda}\, bx_1\left(\frac{-p_1}{y.b^2}\right) = \frac{\bar{\lambda}}{y}\frac{p_1 x_1}{b} = \frac{\partial u}{\partial b}$$

We note that $\frac{\partial m}{\partial p_j} = h_j(\frac{p_1}{b}, p_2 \dots p_n, u) = x_j \qquad j=2\dots n$

and $\frac{\partial m}{\partial p_1} = x_1$ which implies $\frac{\partial m}{\partial (\frac{p_1}{b})} = b.x_1$ C.7

Similarly

$$- \frac{\partial V}{\partial p_j} / \frac{\partial V}{\partial y} = D_j(\frac{p_1}{b.y}, \frac{p_2}{y} \dots \frac{p_n}{y}) \qquad j=2\dots n$$

and $- \frac{\partial V}{\partial p_1} / \frac{\partial V}{\partial y} = x_1$ which implies $- \frac{\partial V}{\partial (\frac{p_1}{b})} / \frac{\partial V}{\partial y} = b.x_1$ C.8

These results have a straightforward intuitive interpretation: if x_1 is replaced by an "efficiency measure" $b.x_1 = x_1^*$ and p_1 is replaced by an "efficiency corrected price" $\frac{p_1}{b} = p_1^*$, all the results which can be derived for the problem,

$$\max u = u(x_1^*, x_2 \dots x_n) \text{ subject to } p_1^* x_1^* + \sum_{i=2}^{n} p_i x_i = y,$$

hold for $x_1^* = b.x_1$ and $p_1^* = \frac{p_1}{b}$.

N.B. The Hicksian demand function for good one has the form

$$x_1 = \frac{1}{b} . h_1(\frac{p_1}{b}, p_2 \dots p_n, u)$$

and the Marshallian demand function the form

$$x_1 = \frac{1}{b} . D_1(\frac{p_1}{b.y}, \frac{p_2}{y} \dots \frac{p_n}{y}).$$