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A Variational Approach to Network Games

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Summary

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Keywords: Network Games, Variational Inequalities, Lowest Eigenvalue, Shock Propagation

JEL Classification: C72, D85, H41, C61, C62

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Department of Economics, Indiana University, Bloomington, IN 47408, USA. Email: emelo@iu.edu. I'm grateful to Rabah Amir, Bob Becker, Agostino Capponi, Juan Carlos Escanciano, Ben Golub, Armando Gomes, P. J. Healy, Ehud Kalai, Brian Rogers, Frank Page, Sudipta Sarangi, Yves Zenou, and Junjie Zhou for very useful comments. I also thank to seminar participants at Indiana University, the Midwest Economic Theory and International Trade Conference, and the Third Annual Conference of Network Science and Economics.

1. Introduction

The outcomes of a wide range of economic and social situations are determined by who interacts with whom. Examples include public good provision, trade, investment decisions, and information collection among many others. In all these examples, agents' decisions are determined and influenced by the choices of their friends and acquaintances. Formally, these interactions are modeled through a network game, where the nodes of a fixed network (graph) represent players and the links represent social or economic relationships amongst players. Linked players interact with other linked players implying that the outcomes of the game depend on the entire network structure. In this context we would like to understand how the network structure determines the actions that players take and the payoffs that they receive. Similarly, we would like to understand the comparative statics of network games and how local exogenous shocks may propagate to the rest of the network.

Networks are complex objects and even for simple cases the equilibrium analysis and comparative statics exercises turn out to be a challenging task. Recognizing this complexity, the theoretical and empirical literature has focused on games with quadratic payoffs and linear best responses. This approach was introduced by Ballester et al. [2006]. They show in the context of strategic complements, and exploiting the Perron-Frobenius' Theorem, that the largest eigenvalue of the network characterizes the properties of an interior equilibrium. Based on these findings, Bramoulle et al. [2014] provide a unified framework to analyze games of strategic complements and strategic substitutes. They show that the class of network games with linear best responses can be analyzed using tools from the theory of potential games (Monderer and Shapley [1996]) and convex optimization. Their main result is that the equilibria of the game and their properties are determined by the sparsity of the network structure, as measured by its lowest eigenvalue.

However, while tractable, this framework is very particular and little is known beyond the linear case and games without a potential game structure. In fact, it is easy to find relevant economic games where players' best responses are nonlinear, so that Ballester et al. [2006] and Bramoulle et al. [2014]'s results do not apply.[1] From an applied point of view, this lack of knowledge precludes us to understand the role of the network structure in determining the equilibria and corresponding comparative statics. Moreover, this limits empirical applications of network games models.

In this paper we consider an alternative approach to the study of games played on fixed networks, which does not require specific functional forms on players' payoffs nor the existence of a potential function. Formally, we tackle the problem of strategic interaction in networks borrowing tools from the mathematical theory of *Variational Inequalities*

[1] For example, Allouch [2015] shows that in the context of public goods in networks players' best responses are nonlinear.

(Facchinei and Pang [2003]). We employ these techniques to characterize the outcomes and the comparative statics of the game as a function of the network topology.[2]

We focus on network games of strategic substitutes and strategic complements with continuous actions sets. We make at least four general contributions. First, we establish the existence and uniqueness of a Nash equilibrium (NE) for a general class of network games (Theorem 1). We exploit the fact that finding a NE is equivalent to find a solution to a Variational Inequality problem (VI). Based on this equivalence, for games of strategic substitutes (complements) we show that the existence and uniqueness of a NE is determined by the lowest (largest) eigenvalue of the network, a parameter measuring players' payoff concavity, and a parameter capturing the strength of the strategic interaction among players.

Second, we characterize the comparative statics and the effect of local shocks.[3] We show that any *locally unique* NE is a continuous function of the parameters of the game (Theorem 2). We fully characterize the effect of a parametric change at the node level showing how a local shock propagates to the entire network as a function of its topology. In addition, we show that exogenous changes on the network topology (adding or deleting links) has a bounded effect on the equilibrium play (Proposition 4). The tightness of our bound is inversely related with the value of the lowest eigenvalue of the network.[4]

As a third contribution, we study the notion of approximate NE in networks. Intuitively, an approximate NE allows for situations in which players may choose an *approximate best response*. We show that the precision of an approximation depends on a precise relationship between a parameter chosen by the researcher and the lowest (or largest) eigenvalue of the network (Theorem 3). From a practical point of view, our approximation result may be useful as a stopping criteria when implementing algorithms to compute the equilibrium of the game.

In our fourth contribution, we apply our framework to the study of Aggregative Network games, games of mixed interactions, and Bayesian Network games. In economic terms, an aggregative network game is one in which each player' payoff is affected by its own action and the *aggregate* of the actions taken by his neighbors. A key ingredient in this class of games is the notion of an *aggregator function*, which captures how players "aggregate" their neighbors strategies. Examples in this class of games include models of competition

[2]We note that the book by Nagurney [1999] discusses the use of variational inequality techniques in the context of traffic networks, spatial pricing, and general equilibrium. However, Nagurney [1999] does not discuss games played in fixed networks as here. Thus her results cannot be applied to the problem studied in this paper.

[3]Consistent with previous literature, we define a local shock as exogenous parametric change at the node (player) level.

[4]In the case of strategic complements the tightness of the bound depends on the largest eigenvalue of the network.

(Cournot and Bertrand with and without product differentiation), patent races, models of contests and fighting, public good provision, congestion games, among many others. We show that the existence, uniqueness, and the comparative statics of an equilibrium are determined by the lowest eigenvalue of the network, a measure of players' payoffs concavity, and the magnitude of the derivative of the aggregator function. In terms of comparative statics, we show that the effect of a local shock is amplified by the value of the first derivative of the aggregator function.

A network game with mixed interactions is a game in which the strategic behavior allows for complementarity and subtitutibility among players. Formally, the network describing the connections among players, can be decomposed into two subnetworks; one capturing strategic subtitubility and another one capturing strategic complementarity. We show that the existence and uniqueness, the comparative statics, and the approximation of a NE follows from a simple adaptation of our results. In particular, we derive a precise relationship between the largest eigenvalue of the subnetwork capturing complementarity and the lowest eigenvalue of the subnetwork capturing substitubility respectively.

Finally, in applying our framework to network game of incomplete information, we show how a simple modification of our results allows us to establish the existence and uniqueness of a Bayesian Nash Equilibrium.

1.1. **Related Literature.** Our paper belongs to the growing literature of games played on fixed networks.[5] The main line of research has taken a traditional approach looking at the Nash equilibria of the game, given the network structure and the parameters describing players' payoffs.[6] In particular, this literature assumes that players' payoffs are quadratic, so that their best responses are linear functions. The main advantage of this class of games is that we may focus on how the direct effects, which are a function of the network topology, determine the nature and shape of Nash equilibria. This approach was introduced by Ballester et al. [2006] who study a network game of strategic complements. They show that for small direct effects, there exists a unique and interior Nash equilibrium.[7] The case of large direct effects has been studied in Bramoulle and Kranton [2007]. They show that under strategic substitubility, there are generally multiple equilibria.

The closest paper to ours is Bramoulle et al. [2014], which provides a unified treatment of network games with linear best responses. Their analysis cover the cases of small and

[5]For a general survey of this literature we refer the reader to Jackson and Zenou [2014] and Bramoullé and Kranton [2016].

[6]An alternative line of research has focused on network games under incomplete information. The main paper in this line is the work by Galeotti et al. [2010], which introduces the idea that players have incomplete information about about network links.

[7]Their analysis has been extended by Corbo et al. [2007] and Ballester and Calvo-Armengol [2010], which consider more general interaction patterns, and applications to the analysis of peer effects may be found in Calvo-Armengol et al. [2009] and Liu et al. [2014]

large direct effects, focusing on games of strategic substitutes. Combining the theory of potential games with spectral analysis, they establish the key result that the uniqueness of a Nash equilibrium depends on the lowest eigenvalue of the network. In addition, they provide some comparative statics results, which critically depend on the parametric structure of the problem. Our analysis differs from theirs in at least four fundamental aspects. First, Bramoulle et al. [2014]'s approach works only for the class of games with linear best responses, while our framework does not require any type of parametric assumption on players' payoffs. In fact, we show that their uniqueness result is a particular case of our approach. Second, we fully characterize the local uniqueness of a NE, the comparative statics, and the effect of local shocks. These two topics are not analyzed by Bramoulle et al. [2014].

A third important difference is our analysis of approximated equilibrium. Finally, from a technical point of view our analysis does not require the existence of a potential function, which is key in Bramoulle et al. [2014]'s framework. In particular, we show that by employing variational methods, we may disregard the assumption that the network game has the structure of a potential game. The main implication of this technical difference is that we are able to analyze a larger class of games than the one considered by Bramoulle et al. [2014].

A few recent papers have started to look at network games with nonlinear payoffs. Elliot and Golub [Forthcoming] study Pareto efficiency in the context of public goods. However, they do not discuss strategic behavior in networks. The work by Belhaj et al. [2014] studies network games of strategic complements, assuming a particular nonlinear functional form on players' best responses. Their analysis can be seen as a particular case of our results.[8] The papers by Baetz [2015] and Hiller [2017] study the uniqueness of a NE under the assumption of concave best responses. However, the focus of both papers is on the network formation process. The paper by Allouch [2015] studies a class of public goods in networks with nonlinear best responses. He establishes the existence and uniqueness of a NE in terms of the lowest eigenvalue of the network. We show that Allouch [2015]'s result is a particular case of our framework. In particular, it is shown that his analysis may be seen as a special case of the class of aggregative network games. Two final differences are our comparative statics and approximation analysis. None of these topics are studied by Allouch [2015]. Finally, it is worth mentioning that the recent work by Parise and Ozdaglar studies variational inequalities and network games. They focus on the notion of P-matrix and they consider other classes of games (e.g. congestion games). Their analysis and ours can be seen as complementaries.

The rest of the paper is organized as follows.§2 introduces the problem of strategic interaction in networks. §3 studies the existence and uniqueness of a Nash equilibrium.

[8]See section 6 for details.

§4 presents the analysis of comparative statics and shock propagation in networks. §6 studies the class of aggregative network games. §7and 8 study network games of mixed interactions and games of incomplete information respectively. §9 concludes. Technical details and proofs are gathered in appendix A.

2. The environment

There is a set of n players denoted by N. Each player i simultaneously chooses an action $x_i \in \mathcal{X}_i \subseteq \mathbb{R}_+$. We assume that $\mathcal{X}_i = [0, b_i]$ where $b_i < \infty$. According to this, let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$ denote the action profile of all players. Let $\mathbf{x}_{-i} \in \mathcal{X}_{-i} = \prod_{j \neq i} \mathcal{X}_j$ denote a profile of actions for players other than i. The players are embedded in a fixed network, represented by a n-by-n matrix $\mathbf{G}$. The entry (i,j)th is denoted by g_{ij}, where $g_{ij} = 1$ if player i and j are connected, and $g_{ij} = 0$ otherwise. The terms g_{ij}s represent links between two players, and $\mathbf{L} = \{(i,j) \in N \times N : g_{ij} = 1\}$ denotes the set of such links. For each player i, we set $g_{ii} = 0$. The network $\mathbf{G}$ is assumed to be *undirected*, which is equivalent to say that $g_{ij} = g_{ji}$ for all $(i,j) \in \mathbf{L}$. A player j who is linked to player i is called player i's neighbor. The set of player i's neighbors is defined by $N_i = \{j \in N : g_{ij} = 1\}$ and player i's degree is defined as $d_i = |N_i|$. Let $\bar{\mathbf{x}}_{-i} = \sum_{j=1}^n g_{ij} x_j$ for all $i \in N$.

An important element in our analysis are the eigenvalues of $\mathbf{G}$, which may be denoted as:

$$\lambda_{min}(\mathbf{G}) = \lambda_1(\mathbf{G}) \leq \lambda_2(\mathbf{G}) \leq \cdots \leq \lambda_n(\mathbf{G}) = \lambda_{max}(\mathbf{G}).$$

We remark two properties of the eigenvalues of $\mathbf{G}$. First, given that $\mathbf{G}$ is a nonnegative matrix, by the Perron-Frobenius Theorem (Horn and Johnson [1990, Thm. 8.3.1]) we know that $\lambda_{max}(\mathbf{G}) > 0$. This fact, impliest our second observation which is related to the fact that $\lambda_{min}(\mathbf{G}) < 0$. To see this, note that for any square matrix the sum of its eigenvalues is equal to its trace. From the definition of $\mathbf{G}$ it follows that its trace must be zero. Because $\lambda_{max}(\mathbf{G}) > 0$ we find that necessarily $\lambda_{min}(\mathbf{G})$ must be *strictly* negative.

We now describe players' payoffs. Formally, player i's payoff depend on his own action x_i, the actions of others $\mathbf{x}_{-i}$, and the network $\mathbf{G}$. According to this, we define player i's payoffs as

$$u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}),$$

where $\boldsymbol{\theta}_i \in \Theta_i \subseteq \mathbb{R}^{k_i}$ with Θ_i being compact and convex and $k_i \geq 1$ for $i = 1, \ldots, n$. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n) \in \Theta = \prod_{i=1}^n \Theta_i$ denote the parameter vector for the game.

Throughout the paper we assume the following two conditions on players' payoff functions:

Assumption 1. *For all player $i \in N$ and $\boldsymbol{\theta}_i \in \Theta_i$, we assume the following:*

(i) *The payoff function* $u_i(\cdot, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})$ *is continuously twice differentiable and strictly concave on* x_i *for all* $\mathbf{x}_{-i} \in \mathcal{X}_{-i}$.

(ii) *There exists a constant* $\bar{\kappa} > 0$ *such that*

$$\inf_{i \in N} \inf_{\mathbf{x} \in \mathcal{X}} \left| \frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_i^2} \right| \geq \bar{\kappa}$$

Condition (i) states that players' payoffs are strictly concave on their own strategy. Condition (ii) assumes that the terms $\left|\frac{\partial^2 u_i}{\partial x_i^2}\right|$ are uniformly bounded from below.

Assumption 2. *For all player* $i \in N$ *and* $\boldsymbol{\theta}_i \in \Theta_i$*, we assume the following:*

(i) *The cross partial derivative*

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i}$$

is well defined for all $\mathbf{x}$*, and* $(i, j) \in \mathbf{L}$.

(ii) *There exists a constant* $\kappa > 0$ *such that*

$$\sup_{(i,j) \in \mathbf{L}} \sup_{\mathbf{x} \in \mathcal{X}} \left| \frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} \right| \leq \kappa.$$

(iii) *The cross partial derivative*

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial \theta_{il} \partial x_i}$$

is well defined for all θ_{il} *with* $l = 1, \ldots, k_i$ *and* $\mathbf{x} \in \mathcal{X}$.

Previous assumption describes conditions on the the behavior of cross partial derivatives. Intuitively, condition (ii) imposes that for each player i the influence of changes on their neighbors actions has a bounded impact on his marginal utility.

Assumptions 1 and 2 are the only requirements we impose on players' payoffs. Departing from previous literature, we do not assume specific functional forms (See the discussion in examples 1 and 2 below).

In terms of strategic interaction, we focus on games of strategic substitutes and strategic complements (Bulow et al. [1985]) with an especial emphasis in the former class. Formally, this class of games is defined as follows.

Definition 1. *Given a parameter vector* $\boldsymbol{\theta}$ *and a network* $\mathbf{G}$*, we say that the game has the property of strategic substitutes if for each player* i *the following condition holds:*

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} < 0 \quad \textit{for all } (i, j) \in \mathbf{L},\ \mathbf{x} \in \mathcal{X}.$$

Similarly, we say that the game has the property of strategic complements if for all player $i \in N$ the following condition holds:

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} > 0 \quad \textit{for all } (i,j) \in \mathbf{L},\ \mathbf{x} \in \mathcal{X}.$$

Previous definition covers a large class of games. Important example of games covered by definition 1 are public goods, Cournot competition, and sharing information games.

Example 1. Linear-quadratic network games*: In this class of games players' payoffs are given by*

$$u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}) = x_i - \frac{1}{2}x_i^2 - \theta_i x_i \bar{\mathbf{x}}_{-i} \quad \textit{for all } i \in N,$$

with $\theta_i \geq 0$. The parameter θ_i measures how much i and j affect each others payoffs, given they are linked. Equivalently the parameter θ_i is a measure of the effect of $\mathbf{G}$ on player i's payoffs.
It is easy to see that

$$\frac{\partial^2 u_i(x_i, \boldsymbol{\theta}_i; , \theta_i, \mathbf{G})}{\partial x_i^2} = -1,$$

$$\frac{\partial^2 u_i(x_i, \boldsymbol{\theta}_i; \theta_i, \mathbf{G})}{\partial x_j \partial x_i} = -\theta_i g_{ij},$$

and

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial \theta_i \partial x_i} = -\bar{\mathbf{x}}_{-i}$$

implying that assumptions 1 and 2 are satisfied.

From an applied perspective, the main advantage of this class of games is that players' best reponses are linear functions. In particular, for each player $i \in N$ we get:

$$\beta_i(\mathbf{x}_{-i}; \boldsymbol{\theta}, \mathbf{G}) = \max\{1 - \theta_i \bar{\mathbf{x}}_{-i}, 0\}.$$

We shall refer to this class of games as linear-quadratic games (Jackson and Zenou [2014]). ■

Example 2. Cournot networks*: We now present a networked version of the traditional Cournot model of competition. Nodes represent firms and links may be seen as the degree of substitution between them. Each firm faces an inverse demand function denoted by $P_i(x_i + \theta_i \bar{\mathbf{x}}_{-i})$, where the parameter $\theta_i > 0$ takes into account the network interaction. Assume that $P_i(\cdot)$ is strictly concave, and continuously twice differentiable on $\mathcal{X}$. In addition, assume that each firm has a cost function $c_i(x_i) = \theta_{i2}\frac{x_i^2}{2}$, which is strictly convex, and continuously twice differentiable on $\mathcal{X}$. Accordingly, firms' profits are*

$$u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}) = x_i P_i(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) - \theta_{i2}\frac{x_i^2}{2}, \forall i \in N.$$

It is straightforward to check that

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_i^2} = 2P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) - \theta_{i2} < 0,$$

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial \theta_i \partial x_i} = P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i})\bar{\mathbf{x}}_{-i} < 0,$$

and

$$\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} = \left[P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i})\right] \theta_{i1} g_{ij} < 0.$$

Last inequality states that the game has the property of strategic substitutes.

Let

$$\bar{\kappa} = \inf_{i \in N, \mathbf{x} \in \mathcal{X}} \{|2P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) - \theta_{i2}|\}$$

and

$$\kappa = \sup_{i \in N, \mathbf{x} \in \mathcal{X}} |\{|P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i})|\}.$$

From previous description, it follows that this networked version of Cournot model satisfies assumptions 1 and 2. ■

2.1. **Nash Equilibrium.** We focus on Nash Equilibrium (NE) outcomes. We begin noting that thanks to assumption 1, a NE may be characterized as a solution $\mathbf{x}^*$ to the system of first order conditions

$$\begin{aligned} \frac{\partial u_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_i} &\leq 0 \quad \forall i \in N, \\ x_i^* &\geq 0. \end{aligned} \tag{1}$$

In order to make progress, most of the literature has assumed specific functional forms on players' payoff functions. The seminal papers by Ballester et al. [2006] and Bramoulle et al. [2014] assume that players' payoffs are quadratic functions. Assuming this specific structure has two advantages. First, expression (1) boils down to a linear system in which the role of the network $\mathbf{G}$ is explicit in determining not only the uniqueness of a NE, but also properties such stability and local equilibria. Second, and from a technical point of view, this particular class of games is suitable to be analyzed using potential games and convex optimization techniques (e.g. Bramoulle et al. [2014]). The literature following this approach has coined the term of *linear-quadratic network games* (Jackson and Zenou [2014]).

However, while appealing, the assumption of linear best responses turn out to be a very restrictive condition, limiting applications of network games to important economic problems. For instance, this assumption rules out public good games where players may have concave (and nonlinear) payoffs, and Cournot games on networks with non linear market demand.

2.2. **Variational Inequalities.** In this paper we employ variational methods, which allows us to overcome the problem of assuming particular functional forms on players' payoffs. In particular, we introduce tools from the theory of Variational Inequality Problems.[9]

This theory enables us to analyze network games in an alternative way. Formally, by employing these variational methods we may exploit the information contained in players' first order conditions. To formalize this idea, define the map $f(\cdot;\boldsymbol{\theta},\mathbf{G}):\mathcal{X}\longrightarrow\mathbb{R}^n$ as follows:

$$f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})=\begin{pmatrix}-\frac{\partial u_1(x_1,\mathbf{x}_{-1};\boldsymbol{\theta}_1,\mathbf{G})}{\partial x_1}\\ \vdots \\ -\frac{\partial u_n(x_n,\mathbf{x}_{-n};\boldsymbol{\theta}_n,\mathbf{G})}{\partial x_n}\end{pmatrix}. \tag{2}$$

Previous expression is the payoff gradient of the game. With this definition in place, we establish the following well-known result.

Proposition 1. *Let assumption 1 hold. Then $\mathbf{x}^*$ is a NE of the network game if and only if $\mathbf{x}^*$ solves the following problem*

$$(\mathbf{y}-\mathbf{x}^*)^T f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G})\geq 0,\quad \forall\mathbf{y}\in\mathcal{X}. \tag{3}$$

The problem of finding a solution to (3) is known as the *Variational Inequality Problem* (VI) in the mathematical programming literature. Proposition 1 states that finding a NE is equivalent to solve a VI. This connection was first noticed by Gabay and Moulin [1980] and extended by Harker [1991]. In this paper we show that this characterization is key to understand how the outcomes of the game are determined by the network structure. For ease notation we denote problem (3) as $\mathbf{VI}(f,\mathcal{X})$.

The theory of existence and uniqueness of solutions to a VI is vast, and most of these results focus on structural properties of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$. We exploit these results to understand the role of $\mathbf{G}$ in shaping the equilibrium outcomes. In §3 we discuss this approach in detail. In addition, understanding the equilibria of game in terms of a VI, allows us to carry out comparative statics exercises. We shall exploit this feature in §4.

3. Equilibrium Analysis

In this section we establish the existence and uniqueness of a NE. To this end we exploit the variational characterization in proposition 1. Thanks to this result, we focus on the monotonicity of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ to understand the connection between the equilibria of the game and the topology of the network $\mathbf{G}$. Furthermore, the monotonicity of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ will play a role in understanding the comparative statics of network games.

[9] For a general treatment of the subject we refer the reader to Nagurney [1999] and Facchinei and Pang [2003].

Throughout the paper we shall consider the following notions of monotonicity.

Definition 2. *Let $g : B \longrightarrow \mathbb{R}^n$, with $B \subset \mathbb{R}^n$. The map g is said to be monotone on B if for every pair $\mathbf{x}$ and $\mathbf{y}$ in B we have*

$$(\mathbf{x}-\mathbf{y})^T(g(\mathbf{x})-g(\mathbf{y})) \geq 0.$$

It is said to be strictly monotone on B, if for every pair $\mathbf{x}$ and $\mathbf{y}$ in B with $\mathbf{x} \neq \mathbf{y}$, we have

$$(\mathbf{x}-\mathbf{y})^T(g(\mathbf{x})-g(\mathbf{y})) > 0.$$

Finally, the map g is said to be strongly monotone on B with modulus $\mu > 0$, if for every pair $\mathbf{x}$ and $\mathbf{y}$ in B with $\mathbf{x} \neq \mathbf{y}$, we have

$$(\mathbf{x}-\mathbf{y})^T(g(\mathbf{x})-g(\mathbf{y})) > \mu\|\mathbf{x}-\mathbf{y}\|^2 > 0.$$

The concept of monotone map may be seen as a generalization of the notion of positive definite matrices to nonlinear maps (More and Rheinboldt [1973]).[10] Thus in order to determine the monotonicity of a map g we can use the information contained on its Jacobian matrix. Next section exploits this fact.[11]

3.1. **The Jacobian of the game.** In this section we show how the monotonicity of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is determined by the eigenvalues of its Jacobian matrix. In order to introduce this matrix, define the terms $f_{ii}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ and $f_{ij}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ as

$$f_{ii}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) \equiv -\frac{\partial^2 u_i(x_i,\mathbf{x}_{-i};\boldsymbol{\theta}_i,\mathbf{G})}{\partial x_i^2} \quad \text{for all } i \in N,$$

and

$$f_{ij}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) \equiv -\frac{\partial^2 u_i(x_i,\mathbf{x}_{-i};\boldsymbol{\theta}_i,\mathbf{G})}{\partial x_j \partial x_i} \quad \text{for all } (i,j) \in \mathbf{L}.$$

Using previous definitions, the Jacobian of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is defined as the n-by-n matrix

$$\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) \quad = \quad [f_{ij}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})]_{i,j} \quad \forall \mathbf{x} \in \mathcal{X}.$$

Decomposing the Jacobian in terms of its *diagonal* and *off diagonal* elements we get

$$\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \mathbf{D}(\mathbf{x}) + \mathbf{N}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}). \tag{4}$$

[10] We point out that for the case of g being an affine map, the notions of strictly and strongly monotonicity are equivalent. To see this consider the affine map $g(\mathbf{x}) = \mathbf{M}\mathbf{x}+\mathbf{q}$ where $\mathbf{M} \in \mathbb{R}^{n\times n}$ is a constant matrix, and $\mathbf{q} \in \mathbb{R}^n$ is a parameter vector. Then for $\mathbf{x} \neq \mathbf{y}$ we find that $(\mathbf{x}-\mathbf{y})^T(g(\mathbf{x})-g(\mathbf{y})) = (\mathbf{x}-\mathbf{y})^T\mathbf{M}(\mathbf{x}-\mathbf{y})$. Thus the definitions of strictly and strongly monotone coincide.

[11] Elsewhere, the notion of monotone map has been implemented in the context of multidimensional mechanism design. In particular, the papers by Saks and Yu [2005], Lavi and Swamy [2009],Berger et al. [2009], Ashlagi et al. [2010], and Archer and Kleinberg [2014], relate the incentive compatibility (truthful implementation) of a mechanism to its monotonicity properties.

$\mathbf{D}(\mathbf{x})$ is a diagonal matrix whose entries are the terms $f_{ii}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$s, which by assumption 1 are strictly positive. Intuitively, this matrix captures players' payoffs concavity.

The key component in (4) is the matrix $\mathbf{N}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$, which captures the off diagonal elements of the Jacobian. The entries of this matrix are defined as

$$\mathbf{N}_{ij}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \begin{cases} f_{ij}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) & \text{if } (i,j)\in\mathbf{L}; \\ 0 & \text{otherwise }. \end{cases} \tag{5}$$

From previous definition, it follows that $\mathbf{N}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ may be interpreted as a *weighted connection matrix*. Formally, the matrix $\mathbf{N}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is a transformation of the original network $\mathbf{G}$, where for each link $(i,j)\in\mathbf{L}$ we replace the original entries g_{ij}s by the terms $f_{ij}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$s. In economic terms, these latter terms capture the *nature* and *intensity* of strategic interaction amongst players. For the case of strategic substitutes we have $f_{ij}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})>0$ for all $(i,j)\in\mathbf{L}$, so that $\mathbf{N}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is a nonnegative matrix.

Summarizing, the Jacobian matrix encapsulates all the needed information about players' payoff function concavity, the topology of the underlying network $\mathbf{G}$, and the *nature* and *strength* of players' strategic interaction.

From a mathematical point of view, the relevance of the Jacobian is that the monotonicity of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is determined by the positive definiteness of $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$. The following well-known result formalizes this relationship (Facchinei and Pang [2003, Prop. 2.3.2]).[12]

Proposition 2. *Let assumption 1 hold. Then*

(i) $f(\mathbf{x};\boldsymbol{\theta};\mathbf{G})$ *is monotone iff* $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ *is positive semidefinite* $\forall\mathbf{x}\in\mathcal{X}$.
(ii) $f(\mathbf{x};\boldsymbol{\theta};\mathbf{G})$ *is strictly monotone if* $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ *is positive definite* $\forall\mathbf{x}\in\mathcal{X}$.
(iii) $f(\mathbf{x};\boldsymbol{\theta};\mathbf{G})$ *is strongly monotone if* $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ *is strongly positive definite* $\forall\mathbf{x}\in\mathcal{X}$.

Proposition 2 implies that looking at the eigenvalues of the Jacobian of the game we can determine the monotonicity of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$. This fact will play a key role in our analysis.

It is worth noticing that in applying proposition 2, the Jacobian does not need to be symmetric. Thus in determining the positive definiteness properties of $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ we must look at the eigenvalues of its symmetric part, which is defined by

$$\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \mathbf{D}(\mathbf{x}) + \bar{\mathbf{N}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}), \tag{6}$$

where $\bar{\mathbf{N}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) \equiv \frac{1}{2}[\mathbf{N}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) + \mathbf{N}^T(\mathbf{x};\boldsymbol{\theta},\mathbf{G})]$.

[12]Let $\mathbf{M}$ be a n-by-n matrix and let $\bar{\mathbf{M}} = \frac{1}{2}[\mathbf{M}+\mathbf{M}^T]$ denote its symmetric part. $\mathbf{M}$ is said to be semi positive definite if for all $\mathbf{y}\neq\mathbf{0}$ we have $\mathbf{y}^T\bar{\mathbf{M}}\mathbf{y}\geq 0$. $\mathbf{M}$ is said to be positive definite if for all $\mathbf{y}\neq\mathbf{0}$ we have $\mathbf{y}^T\bar{\mathbf{M}}\mathbf{y}>0$. Finally, $\mathbf{M}$ is said to be strongly positive definite if there exists a scalar $\mu>0$ such that $\mathbf{y}^T\bar{\mathbf{M}}\mathbf{y}\geq\mu\|\mathbf{y}\|^2>0$ for all $\mathbf{y}\neq\mathbf{0}$.

From (6) we note that the positive definiteness of the Jacobian depends on the eigenvalues of $\mathbf{D}(\mathbf{x})$ and $\bar{\mathbf{N}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$. More precisely, note that by using lemma 1 in Appendix A we get

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \geq \lambda_{min}(\mathbf{D}(\mathbf{x})) + \lambda_{min}(\bar{\mathbf{N}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})). \tag{7}$$

Inequality (7) deserves some comments. First, We point out that thank to assumption 1 it follows that $\lambda_{min}(\mathbf{D}(\mathbf{x})) > 0$ for all $\mathbf{x} \in \mathcal{X}$. In addition, and assuming that players' strategies are strategic substitutes, we note that $\lambda_{min}(\bar{\mathbf{N}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) < 0$. Combinining these two observations allow us to conclude that the positive defininetess of the Jacobian depends on the relationship between $\lambda_{min}(\mathbf{D}(\mathbf{x}))$ and $\lambda_{min}(\bar{\mathbf{N}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}))$. The second observation is that given assumptions 1 and 2 inequality (7) can be expressed in a much simpler way. In particular, lemma 2 in Appendix A establishes that

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \geq \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}) \quad \forall \mathbf{x} \in \mathcal{X}. \tag{8}$$

Thus the information to determine the positive definiteness of $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is governed by the parameters $\kappa, \bar{\kappa}$ and $\lambda_{min}(\mathbf{G})$. As we shall see this simple observation is key in deriving our results.

Finally, we remark that for the case of strategic complements, Eq. (8) becomes:

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \geq \bar{\kappa} - \kappa\lambda_{max}(\mathbf{G}) \quad \forall \mathbf{x} \in \mathcal{X}. \tag{9}$$

3.2. **Strong Monotonicity: Existence and uniqueness.** We now show how the $\kappa, \bar{\kappa}$, and $\lambda_{min}(\mathbf{G})$ provide key information to determine the uniqueness of a NE. In doing so we begin establishing the following result.

Proposition 3. *Let assumptions 1 and 2 hold. Assume that the network game has the property of strategic substitutes. If the condition*

$$|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$$

holds, then $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone with modulus $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}) > 0$.

The intuition behind proposition 3 is simple. From inequality (8) it follows that the condition $|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$ ensures us that the eigenvalues of the $\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ are strictly positive and *uniformly bounded* from below, implying the strong positive definiteness of this matrix. As a result, by proposition 2(iii) we conclude that the map $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone with modulus $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G})$.

A direct implication of proposition 3 is the following result.

Theorem 1. *Let assumptions 1-2 hold. Assume that the network game has the property of strategic substitutes. If the condition*

$$|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$$

holds, then there exists a unique NE.

Four remarks are in order. Our first observation is about the existence of a NE, which follows from assumptions 1 and 2. In particular, the combination of these two assumptions allows us to apply More [1974, Thm. 3.1] ensuring that problem 3 has at least one solution. Second, the uniqueness is a direct consequence of proposition 3. This fact implies that the uniqueness of a NE depends on $\lambda_{min}(\mathbf{G})$, which captures how sustitubility among players' actions is amplified as a function of the sparsity of the network $\mathbf{G}$, $\bar{\kappa}$, which captures players' payoff functions concavity, and κ which captures the strength of the interaction among players. In other words, our equilibrium condition combines structural properties of players' payoffs functions with spectral properties of the network $\mathbf{G}$.

A third observation is that our result is *nonparametric* in the sense that we do not impose any specific functional forms to players' payoffs. Formally, by studying the equilibria of the game as the solutions to $\mathbf{VI}(f, \mathcal{X})$, we are able to exploit players' first order conditions. This fact represents a fundamental difference from previous contributions which have focused on ad-hoc specifications of players' best response functions. In fact, the class of games with linear best responses is a particular case of our framework. In §3.3 below we show how theorem 1 generalizes previous results in the literature.[13] Fourth, our result does not require the existence of a potential function, which has been the main tool in studying network games with linear best responses (See, e.g.,Bramoulle et al. [2014] and Bramoullé and Kranton [2016]). Thus theorem 1 applies to a larger class of games, being potential games a particular case.

It is worth pointing out that theorem 1 is different from Rosen [1965]'s uniqueness result. In particular, his uniqueness result applies under the assumption that $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ satisfies the property of strict diagonal concavity, which in terms of our framework is equivalent to assume strict monotonicity. There are two important differences between our result and Rosen's result. First, his result is stated in terms of what he calls "pseudo gradients", which are just the original gradients weighted for positive scalars. These scalars are unknown and in applications they need to be determined. Our result does not need to introduce auxiliary parameters, implying that theorem 1 is a *direct* way of determining the uniqueness of an equilibrium point. Second, we provide a precise condition in terms of $\bar{\kappa}$, κ, and $\lambda_{min}(\mathbf{G})$. This type of spectral characterization is not given by Rosen [1965].

Next result is a direct corollary of Theorem 1

Corollary 1. *Let assumptions 1-2 hold. Assume that the network game has the property of strategic complements. If the condition*

$$\lambda_{max}(\mathbf{G}) < \bar{\kappa}/\kappa$$

holds, then there exist a unique NE.

[13]Furthermore, in §6 we discuss how to specialize theorem 1 to the class of aggregative network games.

3.3. **Applications.** In this section we apply our existence and uniqueness result to three different classes of network games.

3.3.1. *Linear-quadratic network games.* In this class of games, players' payoffs are given by:

$$u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}) = x_i - \frac{1}{2}x_i^2 - \theta_i x_i \bar{\mathbf{x}}_{-i} \quad \text{for all } i \in N.$$

From this payoff specification, it is easy to see that the condition determining the strict monotonicity of $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ may be written as

$$(\mathbf{y} - \mathbf{x})^T[\mathbf{I} + \mathbf{V}(\boldsymbol{\theta}) \cdot \mathbf{G}](\mathbf{y} - \mathbf{x}) > 0,$$

where the term $\mathbf{I} + \mathbf{V}(\boldsymbol{\theta}) \cdot \mathbf{G}$ is the Jacobian of the game with $\mathbf{V}(\boldsymbol{\theta}) = Diag(\theta_i)_{i \in N}$.

In Appendix A we show that strong monotonicity of $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ is determined by the condition

$$|\lambda_{min}(\mathbf{G})| < \frac{1}{\bar{\theta}},$$

where $\bar{\theta} = \max_{i \in N}\{\theta_i\}$. This latter inequality is the condition established in theorem 1.

Previous analysis shows how the result for an homogeneous θ in Bramoulle et al. [2014] is a direct corollary of Theorem 1.

3.3.2. *Cournot Networks.* We consider the networked version of the traditional model of Cournot competition described in example 2. For this game, the Jacobian can be written as

$$\mathbf{J}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G}) = \mathbf{D}(\mathbf{x}) + \mathbf{V}(\mathbf{x}, \boldsymbol{\theta}) \cdot \mathbf{G}$$

where $\mathbf{D}(\mathbf{x}) = Diag(-(\theta_{i2} + 2P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i})))_{i \in N}$ and $\mathbf{V}(\mathbf{x}, \boldsymbol{\theta}) = Diag\left(-(P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i})\theta_{i1})\right)_{i \in N}$. Recalling that

$$\bar{\kappa} = \inf_{i \in N, \mathbf{x} \in \mathcal{X}} \{|2P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) - \theta_{i2}|\}$$

and

$$\kappa = \sup_{i \in N, \mathbf{x} \in \mathcal{X}} |\{|P_i'(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i}) + x_i P_i''(x_i + \theta_{i1}\bar{\mathbf{x}}_{-i})|\} \times \bar{\theta}_{i1},$$

with $\bar{\theta}_1 = \max_{i \in N}\{\theta_{i1}\}$.

Applying theorem 1 it follows that this game has a unique NE when $|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$. From the definitions of $\bar{\kappa}$ and κ we conclude that the uniqueness of an equilibrium depends on properties of the demand function.

3.3.3. *Public goods in networks.* In our second application we consider the public good game studied by Allouch [2015]. In order to see how our framework applies to this class of games, define players' payoffs as:

$$u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}) = -\frac{1}{2}(x_i + \bar{\mathbf{x}}_{-i} - \gamma_i(\theta_i + \bar{\mathbf{x}}_{-i}))^2, \quad \text{for } i = 1, \dots, n.$$

The function $\gamma_i(\cdot)$ is the demand function of the public good. The parameter θ_i may be interpreted as player i's income. Assume that for all i, $\gamma_i(\cdot)$ is continuously differentiable on $\mathcal{X}_i$. In addition, assume that the public good is normal, which mathematically is equivalent to say that for all player i the following inequality holds:

$$0 < \gamma_i'(\theta_i + \bar{\mathbf{x}}_{-i}) < 1 \quad \forall \mathbf{x} \in \mathcal{X}.$$

The Jacobian of this game is

$$\mathbf{J}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G}) = \mathbf{I} + \mathbf{V}(\mathbf{x}, \boldsymbol{\theta})\mathbf{G} \quad \forall \mathbf{x} \in \mathcal{X},$$

where $\mathbf{V}(\mathbf{x}, \boldsymbol{\theta}) = Diag(1 - \gamma_i'(\theta_i + \bar{\mathbf{x}}_{-i}))_{i \in N}$.

In order to incorporate the role of $\mathbf{G}$, Allouch [2015] introduces the notion of *network normality*, which formally can be written as:

$$1 + (1 - \gamma_i'(\theta_i + \bar{\mathbf{x}}_{-i}))\lambda_{min}(\mathbf{G}) > 0, \quad \forall i \in N, \mathbf{x} \in \mathcal{X}.$$

Previous condition, combined with the fact that players' best responses are given by $\beta_i(\mathbf{x}_{-i}; \boldsymbol{\theta}, \mathbf{G}) = \max\{\gamma_i(\theta_i + \bar{\mathbf{x}}_{-i}) - \bar{\mathbf{x}}_{-i}, 0\}$ for all $i \in N$, allows Allouch [2015, Thm. 1] to establish that network normality determines the uniqueness of a NE.

In appendix A we show that for this class of games, the condition in theorem 1 may be expressed as

$$|\lambda_{min}(\mathbf{G})| < \frac{1}{1 - {\gamma'}^*} \quad \forall \mathbf{x} \in \mathcal{X}.$$

where ${\gamma'}^* = \inf_{i \in N, \mathbf{x} \in \mathcal{X}}\{\gamma_i'(\theta_i + \bar{\mathbf{x}}_{-i})\}$.

It is worth noticing that, contrary to Allouch' analysis, the best response $\beta_i(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ does not play any role in our derivation. We only need to focus on the strong monotonicity of $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$. In §4 and 5, we retake this game to show how our approach extends Allouch's analysis in terms of comparative statics and ϵ-approximate equilibrium.

4. Comparative statics and shock propagation

In this section we study the comparative statics of network games. Our main goal is to characterize how a local exogenous shock (parametric change) at the node level propagates to the rest of the network. Formally, in understanding how a local shock affects the equilibrium of the game, we need to find conditions under which an equilibrium $\mathbf{x}^*$ may be written as a function of the parameter vector $\boldsymbol{\theta}$. The main tool for doing this is the

Implicit Function Theorem, which requires that the equilibrium $\mathbf{x}^*$ must be interior, i.e., the condition

$$f(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) = \mathbf{0}.$$

must hold.

Recalling the definition of $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$, it follows that previous requirement is equivalent to assume that

$$\frac{\partial u_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_i} = 0 \quad \forall i \in N,$$

which means that all players choose an interior strategy (a strictly positive number).

However, depeding on the type of strategic interaction, the structure of equilibria of network games may be very complex. For instance trategic substitutability may be very complex. In fact, it is easy to find examples where this *interiority* condition fails. For instance, in the context of public goods, Bramoulle et al. [2014] show that the set of equilibria of the game consist of a set "free riders", i.e., a set of players contributing 0, and a set of "contributors" which consists of players choosing an interior strategy.[14]

This technical caveat precludes us of applying directly the implicit function theorem to characterize how the equilibrium of the game varies in response to exogenous shocks.

In this section we borrow results the theory of sensitivity analysis in VI to overcome these technical problems. In particular, we show that for a small exogenous shock, the NE of the game is a continous function of $\boldsymbol{\theta}$. A direct consequence of the continuity of the NE, is the fact that for small exogenous shocks, the set of contributors and free riders does not change.

To keep our analysis as simple as possible, we restrict our attention to games with non degenerate Nash equilibria.

Definition 3. *For a given $\boldsymbol{\theta} \in \Theta$ and a network $\mathbf{G}$, we say that the equilibrium $\mathbf{x}^*$ is non degenerate if*

$$x_i^* + f_i(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) > 0 \quad \text{for all } i \in N.$$

The notion of non degenerate equilibria is useful for two reasons. First, assuming that the equilibrium is non degenerate, we can rule out situations where for a subset of players we have $f_i(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) = 0$ and $x_i^* = 0$. Consistent with this, for a non degenerate equilibrium $\mathbf{x}^*$ we define the set of active players as:

$$\mathcal{A} = \left\{ i \in N : f_i(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) = 0, x_i^* > 0 \right\},$$

while the set of inactive players is defined as

$$\mathcal{A}^c = \left\{ i \in N : f_i(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) > 0, x_i^* = 0 \right\}.$$

[14]Similarly, Belhaj et al. [2014] provide examples of network games of strategic complements, where at equilibrium some players are inactive.

A second advantage of assuming a non degenerate equilibrium, is that in economic terms the set $\mathcal{A}$ may be seen as the set of contributors, whereas the set $\mathcal{A}^c$ may interpreted as the set of non-contributors or free riders.

Notation: In stating our results, we use the following notation. For a NE $\mathbf{x}^*$ let $f^* = f(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G})$. A sub vector of f^* with elements $f_i(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G})$ for all $i \in \mathcal{A}$, will be denoted by $f^*_{\mathcal{A}}$. In a similar way we define $f^*_{\mathcal{A}^c}$. Let $\mathbf{J}^* = \mathbf{J}(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G})$. A submatrix of $\mathbf{J}^*$ with elements f^*_{ij}, with $i, j \in \mathcal{A}$ is denoted by $\mathbf{J}^*_{\mathcal{A}}$. The matrices $\mathbf{N}^*_{\mathcal{A}}$ and $\mathbf{G}_{\mathcal{A}}$ denote the subnetworks (weighted and unweighted) of active players. In both networks we do not include active players who are isolated (disconnected). In a similar way we define the matrix $\mathbf{D}^*_{\mathcal{A}}$. Finally, denote $u^*_i = u_i(\mathbf{x}^*; \boldsymbol{\theta}_i, \mathbf{G})$ for all $i \in N$.

In characterizing the comparative statics of network games, we introduce the cross-effect matrix of the game, which captures the strategic interaction amongst active players.

Definition 4. *For a Nash equilibrium* $\mathbf{x}^*$*, the cross effect matrix of the game is defined by*

$$\boldsymbol{W}^* = [\mathbf{I} + \mathbf{M}^*]^{-1},$$

where $\mathbf{M}^* = \mathbf{D}^{*-1}_{\mathcal{A}} \mathbf{N}^*_{\mathcal{A}}$.

From the definition of $\boldsymbol{W}^*$, it follows that its entries, denoted by w^*_{ij}, capture the interaction of players in the subnetwork $\mathbf{G}_{\mathcal{A}}$.[15]

Next definition is useful in establishing our results.

Definition 5. *For a Nash equilibrium* $\mathbf{x}^*$*, define the* $|\mathcal{A}|$*-by-*k *matrix* $\Gamma(\mathbf{x}^*, \boldsymbol{\theta})$ *whose entries are given by*

$$\tau_{is} = \frac{\partial^2 u^*_i}{\partial \theta_{il} \partial x_i},$$

for $i = 1, \ldots, n$ *and* $l = 1, \ldots, k_i$.

We now are ready to characterize the existence, uniqueness, and differentiability of local equilibria.

Theorem 2. *Let assumptions 1-2 hold and let* $\mathbf{x}^*$ *be a non degenerate NE. Assume that the network games has the property of strategic substitutes. Finally assume that the following condition holds:*

$$|\lambda_{min}(\mathbf{G}_{\mathcal{A}})| < \bar{\kappa}/\kappa. \tag{10}$$

Then for $\tilde{\boldsymbol{\theta}}$ *in a neighborhood of* $\boldsymbol{\theta}$*, there exists a locally unique once continuously differentiable equilibrium* $\mathbf{x}(\tilde{\boldsymbol{\theta}})$ *such that* $\mathbf{x}^* = \mathbf{x}(\boldsymbol{\theta})$ *and*

[15] It worth remarking that $\mathbf{W}^*$ is a $|\mathcal{A}|$-by-$|\mathcal{A}|$ nonnegative matrix.

$$\frac{\partial \mathbf{x}^*_{\mathcal{A}}}{\partial \theta_{jl}} = \mathbf{W}^* \hat{\mathbf{D}}^{*-1}_{\mathcal{A}} \tag{11}$$

where $\hat{\mathbf{D}}^{*-1}_{\mathcal{A}} = \mathbf{D}^{*-1}_{\mathcal{A}} \mathbf{e}_j \tau_{jl}$ *for* $j = 1, \ldots, n$ *and* $l = 1, \ldots, k_j$.

Theorem 2 establishes that for small perturbations of $\boldsymbol{\theta}$, the equilibrium point $\mathbf{x}^*$ varies continuously. A very important feature of this result is that the continuity of $\mathbf{x}^*$ is established *regardless* the structure of the equilibrium, i.e., continuity applies not only to active players but also to inactive players.

Second, from Eq. (11) it follows that $\mathbf{W}^*$ may be seen as a sufficient statistic to understand the role of the network $\mathbf{G}$ in propagating a shock. Concretely, the entries w^*_{ij} capture the *direct effect* on player i's strategy of a parametric change affecting player j. To see this, we note that for a particular player i formula (11) can be written

$$\frac{\partial x^*_i}{\partial \theta_{jl}} = w^*_{ij} \frac{\partial^2 u^*_j}{\partial \theta_{jl} \partial x_j} \cdot \left[-\frac{\partial^2 u^*_j}{\partial x^2_j} \right]^{-1} \quad \text{for } i = 1, \ldots, n.$$

Previous expression makes explicit the property that the effect of an exogenous change on θ_{jl} is captured by the terms w^*_{ij}'s. An important feature of this result is that player i and j do not need to have a link between them. Thus $\mathbf{W}^*$ provides all the needed information to understand how a shock propagates to the network of active players.

A third observation is about the fact that theorem 2 provides an explicit formula to compute the effect of exogenous shocks. In particular, this result allows us to handle situations where some players may be inactive. This represent a difference from previous comparative statics results, which assume that the unique equilibrium is interior (See, e.g., Acemoglu et al. [2016]).

Finally, from a technical point of view the proof of theorem 2 relies on arguments from the theory of sensitivity analysis in VI. In particular, in proving this result we use arguments from Kyparisis [1986, 1987] and Tobin [1986].

A direct corollary of previous result, is a first order approximation of the local equilibrium $\mathbf{x}^*$ around a neighborhood of $\boldsymbol{\theta}$.

Corollary 2. *Assume the conditions in theorem 2 hold. A first order approximation of* $\mathbf{x}^*$ *in a neighborhood of* $\boldsymbol{\theta}$ *is given by:*

$$\mathbf{x}(\tilde{\boldsymbol{\theta}}) = \mathbf{x}^* + \mathbf{W}^* \mathbf{D}^*_{\mathcal{A}} \Gamma^* \Delta \boldsymbol{\theta}, \tag{12}$$

where $\Delta \boldsymbol{\theta} = \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}$ *and* $\Gamma^* = \Gamma(\mathbf{x}^*, \boldsymbol{\theta}^*)$.

The effect of adding links to G. In general, and without additional structure, carrying out comparative statics in terms of $\mathbf{G}$ is a challenging task. Fortunately, the strong monotonicity of $f(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G})$ allows us to bound the effect of adding links on the equilibrium play. In doing so, we note that $\mathbf{G}$ is a subnetwork of $\mathbf{G}'$ if $\mathbf{G}$ has less links than $\mathbf{G}'$. We refer to this as $\mathbf{G} \subset \mathbf{G}'$. Accordingly we denote the entries of $\mathbf{G}'$ and $\mathbf{G}$ by g'_{ij} and g_{ij} respectively.

Proposition 4. *Let assumption 1 and 2 hold and consider a network game of strategic substitutes. For the networks $\mathbf{G}$ and $\mathbf{G}'$ with $\mathbf{G} \subset \mathbf{G}'$ assume that the condition*

$$|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$$

holds. Let $\mathbf{x}^*$ be the unique NE for the network $\mathbf{G}$, and let $\mathbf{y}$ be a NE for the network $\mathbf{G}'$. Then

$$\|\mathbf{y} - \mathbf{x}^*\| \leq \frac{1}{\mu} \|f(\mathbf{y}; \boldsymbol{\theta}, \mathbf{G}) - f(\mathbf{y}; \boldsymbol{\theta}, \mathbf{G}')\|, \tag{13}$$

where $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G})$.

Proposition 4 provides a bound to quantify the impact on equilibrium play of adding (or severing) links to $\mathbf{G}$. A striking feature of bound (13) is the fact that its tightness depends on μ, which captures the role of the network topology. Intuitively, this result establishes that the equilibrium $\mathbf{x}^*$ varies "*continuously*" as a function of $\mathbf{G}$. A second observation is that the bound in (13) does not need to assume that under the network $\mathbf{G}'$ the game has a unique equilibrium. The only requirement is that under $\mathbf{G}'$ a NE exists. We remark that the strong monotonicity of $f(\mathbf{x}; \boldsymbol{\theta}; \mathbf{G})$ is key in proving and establishing the bound (13). Finally, it is worth mentioning that for the case of strategic complements the proposition 6 holds with $\mu = \bar{\kappa} - \kappa\lambda_{max}(\mathbf{G})$.

Example 3. *We apply our results to the model of public goods in networks discussed in §3. Assuming network normality, by the interlacing eigenvalue theorem Horn and Johnson [1990, p. 185], it follows that for the subnetwork $\mathbf{G}_{\mathcal{A}}$ and the equilibrium $\mathbf{x}^*$ the following inequality holds:*

$$|\lambda_{min}(\mathbf{G}_{\mathcal{A}})| < \frac{1}{\max_{i \in \mathcal{A}}\{1 - \gamma'_i(\theta_i + \bar{\mathbf{x}}^*_{-i})\}}.$$

A direct consequence of previous condition is that theorem 2 applies, which implies that the $\mathbf{x}^*$ is a continuous function of the vector of incomes $\boldsymbol{\theta}$. Exploiting this feature, we may analyze how a small change on player j's income propagates to the rest of the network.

We begin noting that network normality implies that $\boldsymbol{W}^*$ is well defined.[16] *In particular, this matrix takes the form:*

$$\boldsymbol{W}^* = [\mathbf{I} + \mathbf{V}(\mathbf{x}^*, \boldsymbol{\theta}) \cdot \mathbf{G}_{\mathcal{A}}]^{-1},$$

[16]This follows from the fact that network normality implies that $\mathbf{J}^*_{\mathcal{A}}$ is invertible.

where $\mathbf{V}(\mathbf{x}^*, \boldsymbol{\theta}) = Diag(1 - \gamma_i'(\theta_i + \bar{\mathbf{x}}^*_{-i}))_{i \in \mathcal{A}}$.

Then applying formula (11), we find

$$\frac{\partial x_i^*}{\partial \theta_j} = w_{ij}^* \cdot \gamma_j'(\theta_j + \bar{\mathbf{x}}^*_{-j}) \quad \forall i \in \mathcal{A}.$$

5. Approximate equilibrium in Networks

In previous sections our analysis has been focused on the notion of NE, which assumes that players do not make mistakes. In this section we introduce a weaker equilibrium notion. In particular, we introduce the notion of ϵ-*Approximate NE*. Formally this concept is defined as follows.

Definition 6. *Let* $\epsilon > 0$ *and* $\boldsymbol{\theta} \in \Theta$. *We say that* $\hat{\mathbf{x}} \in \mathcal{X}$ *is a* ϵ*-approximate NE for the network game if it solves*

$$(\mathbf{y} - \hat{\mathbf{x}})^T f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}) \geq -\epsilon, \quad \forall \mathbf{y} \in \mathcal{X}. \tag{14}$$

Definition 6 deserves some discussion. First, we note that an approximate NE is just a solution to a modification of $\mathbf{VI}(f, \mathcal{X})$. Intuitively an approximate NE may be interpreted as a situation in which each player does not necessarily play his best response, given what his opponents are playing, but chooses a strategy which is no worse than ϵ from his best response.

As a second observation, we note that definition 6 resembles the notion of ϵ-NE employed in the context of large games. This literature mainly focuses in binary/discrete actions games without considering network structures.[17]

Next we introduce the notion of δ-near NE, which is closely related to definition 6.

Definition 7. *Let* $\mathbf{x}^*$ *be a NE for network game. We will say that* $\hat{\mathbf{x}} \in \mathcal{X}$ *is a* δ*-near NE if*

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\| \leq \delta.$$

We now are ready to establish the main result of this section

Theorem 3. *Let assumptions 1 and 2 hold. Let* $\mathbf{x}^*$ *be the unique NE of the game. Assume that the network game has the property of strategic substitutes. Then every* ϵ*-approximate NE* $\hat{\mathbf{x}}$ *is a* $\sqrt{\frac{\epsilon}{\mu}}$*-near NE of the network game. In particular*

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\| \leq \sqrt{\frac{\epsilon}{\mu}},$$

where $\mu = \bar{\kappa} + \kappa \lambda_{min}(\mathbf{G})$.

[17] See for instance Kalai [2004] and Azrieli and Shmaya [2013].

Three remarks are in order. Theorem 3 states that a solution to problem (14) allows us to find an approximation to the unique NE. The striking feature is that the precision of this approximation is a function of ϵ, a parameter chosen by the researcher, and the monotonicity modulus μ, which captures the role of the network topology. Thus in games where μ is very small, we may counterbalance this effect by choosing a smaller ϵ. Second, theorem 3 states that $\lambda_{min}(\mathbf{G})$ not only is key in determining the uniqueness and comparataive statics of a NE, but also in ϵ-approximate NE. From an applied point of view, theorem 3 is useful to generate stoping criteria when implementing algorithms to compute equilibrium points. Finally, we mention that for the case of strategic complements, theorem 3 holds with $\mu = \bar{\kappa} - \kappa\lambda_{max}(\mathbf{G})$.

6. Aggregative network games

An aggregative game is a game in which each player's payoff is affected by its own action and the *aggregate* of the actions taken by all the players. Many games studied in the literature can be cast as aggregative games, including models of competition, patent races, models of contests and fighting, congestion games, and public good provision games.

While aggregative games is a well studied subject in game theory, most of the literature has focused on situations without a network structure.[18] In this section we introduce the class of *Aggregative Network Games*. The key feature of this class of games is that the payoff for each player is affected by its own action and the aggregate of the actions taken by his *neighbors*. Our aim is to understand how the network topology and the structure of an aggregative game jointly determine the outcomes of the game and their properties. In doing so, we exploit the results developed in §2-5.

In this section we show that the existence, uniqueness, and comparative statics of an equilibrium are determined by a specific interaction between the aggregator function, which captures how players aggregates his neighbors' actions, and the network structure.

6.1. **Environment.** We specialize the environment described in §2 to introduce the class of aggregative network games with a special focus on the case of strategic substitutes. According to this, players' payoffs are defined by:

$$u_i(x_i, \varphi_i(\bar{\mathbf{x}}_{-i}); \boldsymbol{\theta}_i, \mathbf{G}) \quad \forall i \in N, \tag{15}$$

where $\bar{\mathbf{x}}_{-i} = \sum_{j=1}^{n} g_{ij}x_j$ and $\varphi_i(\bar{\mathbf{x}}_{-i}) = \varphi(\bar{\mathbf{x}}_{-i} + \theta_{i1})$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an "*aggregator*" summarizing the behavior of player i's neighbors.

Formally, the function φ may be interpreted as sufficient statistics capturing player i's neighbors behavior and the value of the parameter θ_{i1}. Two widely used aggregators are the identity function, i.e., $\varphi(\bar{\mathbf{x}}_{-i} + \theta_{i1}) = \bar{\mathbf{x}}_{-i} + \theta_{i1}$, and the average function, which in the

[18] For a recent treatment of this class of games we refer the reader to Acemoglu and Jensen [2013].

case of a network game is expressed by $\varphi(\bar{\mathbf{x}}_{-i} + \theta_{i1}) = d_i^{-1}\bar{\mathbf{x}}_{-i} + \theta_{i1}$ with $d_i = \sum_{j=1}^{n} g_{ij}$. For ease exposition, throughout this section we use the notation $\varphi_i = \varphi(\bar{\mathbf{x}}_{-i} + \theta_{i1})$ and $\boldsymbol{\varphi} = (\varphi_i)_{i\in N}$ respectively.

We consider a class of games satisfying two general and mild properties in terms of players' payoffs.

Assumption 3. *For all player $i \in N$, we assume:*

(i) *The payoff function $u_i(\cdot, \varphi_i; \boldsymbol{\theta}_i, \mathbf{G})$ is continuously twice differentiable and strictly concave on x_i for all $\mathbf{x}_{-i} \in \mathcal{X}_{-i}$ and $\boldsymbol{\theta}_i \in \Theta_i$.*

(ii) *There exists a constant $\kappa_1 > 0$ such that:*

$$\inf_{i\in N}\inf_{\mathbf{x}\in\mathcal{X}}\left\{\left|\frac{\partial^2 u_i(x_i, \varphi_i; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_i^2}\right|\right\} \geq \kappa_1.$$

(iii) *The cross partial derivative*

$$\frac{\partial u_i(x_i, \varphi_i; \boldsymbol{\theta}_i, \mathbf{G})}{\partial \varphi_i \partial x_i},$$

is strictly negative and well defined for all $\mathbf{x} \in \mathcal{X}$, $\boldsymbol{\theta}_i \in \Theta_i$, and $(i, j) \in \mathbf{L}$.

(iv) *There exists a constant $\kappa_2 > 0$ such that:*

$$\sup_{i\in N}\sup_{\mathbf{x}\in\mathcal{X}}\left\{\left|\frac{\partial^2 u_i(x_i, \varphi_i; \boldsymbol{\theta}_i, \mathbf{G})}{\partial \varphi_i \partial x_i}\right|\right\} \leq \kappa_2.$$

Previous conditions are just an adaptation of assumptions 1 and 2. Condition (i) establishes that players' payoffs are strictly concave on their own strategies. Condition (ii) adapt the uniformity condition in assumption 1. Condition (iii) formalizes the fact that the game is one of strategic substitutes. Finally, condition (iv) establishes that the impact of a change on the value of φ_i has a bounded effect on players' marginal utility.

Assumption 4. *For all $i \in N$ and θ_{i1} assume:*

(i) *The function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing and once continuously differentiable with $\varphi_i' > 0$ for all $\mathbf{x}_{-i} \in \mathcal{X}_{-i}$.*

(ii) *There exists a constant $\kappa_3 > 0$ such that*

$$\sup_{i\in N}\sup_{\mathbf{x}_{-i}\in\mathcal{X}_{-i}} \varphi_i' \leq \kappa_3.$$

Part (i) makes explicit the class of aggregator functions under consideration. Concretely, it establishes that the function φ is strictly increasing and differentiable on $\mathbb{R}_+$. It is worth emphasizing that neither concavity or convexity of φ is assumed. Part (ii) establishes that the first derivative of φ is bounded. This condition is satisfied by many aggregators functions commonly used in economic models. For instance, when $\varphi_i = d_i^{-1}\bar{\mathbf{x}}_{-i} + \theta_{i1}$, assumption 4 is automatically satisfied with $\varphi_i' = d_i^{-1}$ for all i and the parameter κ_2 being defined as $\kappa_3 = \max_{j=1,\ldots,N}\{d_j^{-1}\}$.

Combining Eq. (15) with assumption 3 and 4, the Jacobian of the game may be expressed as:

$$\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \mathbf{D}(\mathbf{x}) + \mathbf{V}(\mathbf{x};\boldsymbol{\theta},\boldsymbol{\varphi}) \cdot \mathbf{G}, \tag{16}$$

where $\mathbf{D}(\mathbf{x}) = Diag\left(-\frac{\partial^2 u_i(x_i,\varphi_i;\boldsymbol{\theta}_i,\mathbf{G})}{\partial x_i^2}\right)$ and $\mathbf{V}(\mathbf{x};\boldsymbol{\theta},\boldsymbol{\varphi}) = Diag\left(-\frac{\partial^2 u_i(x_i,\varphi_i;\boldsymbol{\theta}_i,\mathbf{G})}{\partial \varphi_i \partial x_i} \cdot \varphi_i'\right)$.

Similar to the analysis in §3, the Jacobian in Eq. (16) provides us a simple way to understand the role of $\boldsymbol{\varphi}$, the concavity of players' payoff functions, and the network $\mathbf{G}$ in determining the structure and comparative statics of the equilibria of the game.

Theorem 4. *Let assumptions 3 and 4 hold. Assume that the game has the property of strategic substitutes. If the condition*

$$|\lambda_{min}(\mathbf{G})| < \frac{\kappa_1}{\kappa_2\kappa_3}$$

is satisfied then:

(i) *The map $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone with modulus $\mu = \kappa_1 + \kappa_2\kappa_3\lambda_{min}(\mathbf{G})$.*
(ii) *There exists a unique NE.*

Three remarks are in order. First, theorem 4 is a specialization of our general result in theorem 1. In particular, part (i) states the strong monotonicity of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ in terms of $\lambda_{min}(\mathbf{G})$ and the parameters κ_1, κ_2, and κ_3. Second, and as a direct consequence of part (i), the existence, uniqueness, and stability of a NE is established.. Thus in this class of games the study of Nash equilibria boils down to understand the relationship between the network topology and the conditions stated in assumptions 3 and 4.

Finally, we point out that theorem 4 contributes not only to the literature on network games, but also to the literature on aggregative games in general. As we said before, this latter literature focuses on situations in which all players are connected, which in terms of our framework is equivalent to assume that the underlying network is complete. In fact, for the case of the complete network, we know that $|\lambda_{min}(\mathbf{G})| = 1$ so that the condition in theorem 4 reduces to $\kappa_2\kappa_3 < \kappa_1$.

Modifying assumption 3(iii) to the case of strictly positive cross partial derivatives, we get the following direct corollary of theorem 4.

Corollary 3. *Let assumptions 3 and 4 hold. Assume that the game has the property of strategic complements. If the condition*

$$\lambda_{max}(\mathbf{G}) < \frac{\kappa_1}{\kappa_2\kappa_3}$$

is satisfied then:

(i) *The map* $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ *is strongly monotone with modulus* $\mu = \kappa_1 - \kappa_2\kappa_3\lambda_{min}(\mathbf{G})$.
(ii) *There exists a unique NE.*

Example 4. *In order to gain some intuition about how theorem 4 may be applied to concrete economic models, we reconsider Allouch [2015]'s public goods model. We begin recalling in Allouch's model, players' payoffs are defined as:*

$$u_i(x_i, \varphi_i; \mathbf{G}) = -\frac{1}{2}(x_i + \varphi_i)^2, \quad \textit{for } i = 1, \ldots, n,$$

where $\varphi_i = \bar{\mathbf{x}}_{-i} - \gamma_i(\theta_{i1} + \bar{\mathbf{x}}_{-i})$.

First, note that $\left|\frac{\partial^2 u_i}{\partial x_i^2}\right| = 1$ *for all* $i \in N, \mathbf{x} \in \mathcal{X}$ *which implies that assumption 3*(ii) *is satisfied. Finally, in order to satisfy assumption 4 we introduce a simple modification to the model. Define the constant* $\gamma^{*\prime} = \inf_{i,\mathbf{x}_{-i}}\{\gamma_i'(\theta_{i1} + \mathbf{x}_{-i})\}$. *Using this definition we find that the derivative of the aggregator* $\varphi_i = \bar{\mathbf{x}}_{-i} - \gamma_i(\theta_{i1} + \bar{\mathbf{x}}_{-i})$ *is bounded by* $\kappa_2 \equiv (1 - \underline{\gamma})$.[19] *As a conclusion, assumptions 3 and 4 are satisfied and theorem 4 can be applied.* ■

6.2. **Comparative statics.** We begin analyzing the effect of an exogenous change on θ_{js} applying the results from §4. We recall that $\mathcal{A}$ denotes the set of active players, which for sake of exposition is assumed to be connected.

For this class of games, the matrix $\mathbf{W}^*$ is written as:

$$\mathbf{W}^* = [\mathbf{I} + \hat{\mathbf{V}}_{\mathcal{A}}(\mathbf{x};\boldsymbol{\theta},\boldsymbol{\varphi})\mathbf{G}_{\mathcal{A}}]^{-1}$$

with $\hat{\mathbf{V}}_{\mathcal{A}}(\mathbf{x};\boldsymbol{\theta},\boldsymbol{\varphi}) = Diag\left(\frac{\partial u_i^*}{\partial\varphi_i\partial x_i}\varphi_i'/\frac{\partial^2 u_i^*}{\partial x_i^2}\right)_{i\in\mathcal{A}}$.

Proposition 5. *Let assumptions 3 and 4 hold and assume that the game has the property of strategic substitutes. In addition, assume that*

$$|\lambda_{min}(\mathbf{G}_{\mathcal{A}})| < \frac{\kappa_1}{\kappa_2\kappa_3}$$

holds and let $\mathbf{x}^*$ *be a non degenerate NE. Then for* $\tilde{\boldsymbol{\theta}}$ *in a neighborhood of* $\boldsymbol{\theta}$, *there exists a locally unique once continuously differentiable equilibrium* $\mathbf{x}(\tilde{\boldsymbol{\theta}})$ *such that* $\mathbf{x}^* = \mathbf{x}(\boldsymbol{\theta})$ *with:*

$$\frac{\partial \mathbf{x}_i^*}{\partial\theta_j} = \mathbf{W}^*\hat{\mathbf{D}}_{\mathcal{A}}^{*-1} \tag{17}$$

where $\hat{\mathbf{D}}_{\mathcal{A}}^{*-1} = \mathbf{D}_{\mathcal{A}}^{*-1}\mathbf{e}_j\tau_{js}$ *with* $\tau_{j1} = \frac{\partial^2 u_j^*}{\partial\varphi_j\partial x_j}\varphi_j'^*\left[-\frac{\partial^2 u_j^*}{\partial x_j^2}\right]^{-1}$, $\varphi_j'^* = \varphi'(\bar{\mathbf{x}}_{-j}^* + \theta_{j1})$, *and* $\tau_{js} = \frac{\partial^2 u_j^*}{\partial\theta_{js}\partial x_j}\cdot\left[-\frac{\partial^2 u_j^*}{\partial x_j^2}\right]^{-1}$ *for* $s = 2, \ldots, k_j$.

[19] It is worth noticing that the class of public goods with Gorman polar payoffs automatically satisfies this condition.

Previous proposition is a direct consequence of the results in §4. The new insight is the role played by $\varphi_i'^*$, which amplifies the effect of an exogenous change on θ_{j1}. Thus the aggregator function not only plays a role in determining the uniqueness and stability of an equilibrium, but also plays a role in determining the magnitude of exogenous shocks.

Next result reestablishes the bound in proposition 4 highlighting the role of the function φ.

Proposition 6. *Let assumptions 3 and 4 hold and consider a network game of strategic substitutes. For the networks* $\mathbf{G}$ *and* $\mathbf{G}'$ *with* $\mathbf{G} \subset \mathbf{G}'$ *assume that the condition* $\mathbf{G} \subset \mathbf{G}'$*, and assume that the condition*

$$|\lambda_{min}(\mathbf{G})| < \frac{\kappa_1}{\kappa_2\kappa_3}$$

holds. Let $\mathbf{x}^*$ *be the unique NE for the network* $\mathbf{G}$*, and let* $\mathbf{y}$ *be a NE for the network* $\mathbf{G}'$*. Then*

$$\|\mathbf{y} - \mathbf{x}^*\| \leq \frac{\kappa_2\kappa_3}{\mu}\|\Delta\mathbf{y}\| \tag{18}$$

where $\mu = \kappa_1 + \kappa_2\kappa_3\lambda_{min}(\mathbf{G})$ *and* $\Delta\mathbf{y} = (\sum_{j=1}^{n}(g_{ij} - g'_{ij})y_j)_{j\in N}$*.*

6.3. **Applications.** In this section we show how our framework applies to two specific classes of aggregative network games.[20] We begin by analyzing the case where the aggregator takes the form of a simple sum. In our second application we study the case where the aggregator enters additively on players' payoffs. In the context of strategic complements, this type of network games has recently been studied by Belhaj et al. [2014] and Acemoglu et al. [2016]. We show how our framework complements in an important way their findings.

6.3.1. *Application 1.* In our first application, we consider that for each player i the aggregator φ_i takes the form

$$\varphi_i = \bar{\mathbf{x}}_{-i} + \theta_{i1} \quad \forall i \in N. \tag{19}$$

From (19) it is easy to check that this aggregator function automatically satisfies assumption 4 with $\kappa_3 = 1$.

Under assumptions 3 and 4 the condition in theorem 4 boils down to check

$$|\lambda_{min}(\mathbf{G})| < \frac{\kappa_1}{\kappa_2}.$$

In terms of comparative statics, proposition 5 can easily be applied using the fact $\varphi_i'^* = 1$.

[20]The details are provided in Appendix A.

In bounding the effect of changing $\mathbf{G}$, bound (18) takes a simple form. In Appendix A we show that

$$\|f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}')\| \leq \kappa_2 \frac{\|\Delta\mathbf{y}\|}{\mu},$$

where $\mu = \kappa_1 + \kappa_2\lambda_{min}(\mathbf{G})$, $\Delta\mathbf{y} = \left(\sum_{j=1}^n (g_{ij} - g'_{ij})y_j\right)_{i\in N}$, and $\varphi_i(\bar{\mathbf{y}}_\lambda) = \lambda\sum_{j=1}^n g'_{ij}y_j + (1-\lambda)\sum_{j=1}^n g_{ij}y_j + \theta_{i1}$ for $0 < \lambda < 1$.

Hence, Eq. (18) can be expressed as:

$$\|\mathbf{x}^* - \mathbf{y}\| \leq \kappa_2 \frac{\|\Delta\mathbf{y}\|}{\mu}.$$

Finally, applying theorem 3 the gap between $\hat{\mathbf{x}}$, an ϵ-approximate NE, and the equilibrium of the game $\mathbf{x}^*$ may be written as

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\| \leq \sqrt{\frac{\epsilon}{\kappa_1 + \kappa_2\lambda_{min}(\mathbf{G})}}.$$

6.3.2. *Application 2.* We now analyze the case of aggregative games with payoffs:

$$u_i(x_i,\varphi_i;\mathbf{G}) = \theta_{i2}x_i - \frac{1}{2}x_i^2 - x_i\varphi_i \quad \forall i \in N. \tag{20}$$

The main feature in specification (20) is that the aggregator φ_i enters additively. This class of games has been studied by Belhaj et al. [2014] and Acemoglu et al. [2016] in the context of strategic complements. However, none of these papers provide the results discussed in this section.

From Eq. (20) is easy to check assumption 3 is satisfied with

$$\inf_{i\in N,\mathbf{x}\in\mathcal{X}} \left\{ \left| \frac{\partial^2 u_i(x_i,\varphi_i;\mathbf{G})}{\partial x_i^2} \right| \right\} = \min_{i\in N}\{\theta_{i2}\}$$

and

$$\sup_{i\in N,\mathbf{x}\in\mathcal{X}} \left\{ \left| \frac{\partial^2 u_i(x_i,\varphi_i;\mathbf{G})}{\partial\varphi_i\partial x_i} \right| \right\} = 1.$$

Previous expressions yield $\kappa_1 = \min_{i\in N}\{\theta_{i2}\}$ and $\kappa_2 = 1$. Assuming that assumption 4 holds, the condition in theorem 4 may be written as

$$|\lambda_{min}(\mathbf{G})| < \frac{\min_{i\in N}\{\theta_{i2}\}}{\kappa_3}.$$

Thus in applying theorem 4, we only need to focus on the parameters $\min_{i\in N}\{\theta_{i2}\}$, κ_3, and $\lambda_{min}(\mathbf{G})$, without assumptions about convexity or concavity of the functions φ_is.

In bounding the effect of modifying the network $\mathbf{G}$, simple algebra shows that bound (18) may be expressed as:

$$\|\mathbf{x}^* - \mathbf{y}\| \leq \frac{\kappa_3 \|\Delta \mathbf{y}\|}{\mu},$$

with $\mu = \min_{i \in N}\{\theta_{i2}\} + \kappa_3 \lambda_{min}(\mathbf{G})$ and $\Delta \mathbf{y} = \left(\sum_{j=1}^{n}(g_{ij} - g'_{ij})y_j\right)_{i \in N}$.

Finally, the approximation result in theorem 3 implies that for all $\epsilon > 0$, every ϵ-approximate NE $\hat{\mathbf{x}}$ satisfies:

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\| \leq \sqrt{\frac{\epsilon}{\min_{i \in N}\{\theta_{i2}\} + \kappa_3 \lambda_{min}(\mathbf{G})}}.$$

7. Network games of mixed interactions

In this section we show how our approach can be used to study games of mixed interactions. Intuitively, a game of mixed interaction is one in which players' strategies may be strategic complements and strategic substitutes. In the context of linear-quadratic network games, Ballester et al. [2006] allow for mixed interactions among players. Most recently, the paper by Konig et al. [2017] exploits this idea to study network of conflicts.

Our goal is to show how the results derived in §2-5 can be extended in a natural way to environments with mixed strategic interactions. An important difference with Ballester et al. [2006] and Konig et al. [2017] is that we do not impose specific functional forms in deriving our results. A key insight from our results is that the role of complementarity and sustitubility is fundamental in determining the existence, uniqueness, and the comparative of statics of a NE.

7.1. **Environment.** We specialize the environment described in §2 to allow for mixed interactions. Players' payoffs are defined by a general function

$$u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}) \quad \forall i \in N.$$

It is assumed that the functions u_is satisfy assumption 1.

The main difference with games analyzed in previous sections is that weFormally we assume that the network $\mathbf{G}$ may be written as

$$g_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ complements.} \\ -1 & \text{if } i \text{ and } j \text{ are substitutes .} \\ 0 & \text{Otherwise.} \end{cases} \tag{21}$$

Let $\mathbf{L}^+$ and $\mathbf{L}^-$ be the set of links between complements and substitutes respectively. Similarly define $g^+_{ij} = \max\{g_{ij}, 0\}$ and $g^-_{ij} = \min\{g_{ij}, 0\}$ and $\mathbf{G}^+ = (g^+_{ij})_{(i,j) \in \mathbf{L}^+}$ and $\mathbf{G}^- = (g^-_{ij})_{(i,j) \in \mathbf{L}^-}$. Previous definitions implies that the network $\mathbf{G}$ can be expressed as:

$$\mathbf{G} = \mathbf{G}^+ - \mathbf{G}^-.$$

Throught this section we shall assume $\mathbf{G}^+$ and $\mathbf{G}^-$ are connected.

Formally, a game with mixed interactions is defined as follows:

Definition 8. *Given a parameter vector $\boldsymbol{\theta}$ and a (mixed) network $\mathbf{G} = \mathbf{G}^+ - \mathbf{G}^-$, we say that the game has the property of mixed interactions for each player $i \in N$ the following conditions hold:*

$$\begin{aligned} \frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} &< 0 \quad \textit{for all } (i,j) \in \mathbf{L}^-,\ \mathbf{x} \in \mathcal{X}, \\ \frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} &> 0 \quad \textit{for all } (i,j) \in \mathbf{L}^+,\ \mathbf{x} \in \mathcal{X}. \end{aligned}$$

The Jacobian of the game. Let $f(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G})$ be the map defined in Eq. (2). Based on this definition, we get:

$$\mathbf{J}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G}) = \mathbf{D}(\mathbf{x}) + \mathbf{N}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G}^+) + \mathbf{N}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G}^-) \tag{22}$$

The matrix $\mathbf{D}(\mathbf{x})$ is a diagonal matrix capturing players' payoff concavity. The matrices $\mathbf{N}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G}^+)$ and $\mathbf{N}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G}^-)$ captures the role of $\mathbf{G}^+$ and $\mathbf{G}^-$ respectively. Formally, these matrices are defined as:

$$\mathbf{N}_{ij}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G}^-) = \begin{cases} f_{ij}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G}) & \text{if } (i,j) \in \mathbf{L}^-; \\ 0 & \text{otherwise .} \end{cases}$$

and

$$\mathbf{N}_{ij}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G}^+) = \begin{cases} f_{ij}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G}) & \text{if } (i,j) \in \mathbf{L}^+; \\ 0 & \text{otherwise .} \end{cases}$$

It is worth noticing that $f_{ij}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G}) > 0$ for$(i,j) \in \mathbf{L}^-$ and $f_{ij}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G}) < 0$ for$(i,j) \in \mathbf{L}^+$.

With these definitions in hand we are ready to establish the following result

Proposition 7. *Let assumptions 1 and 2 hold. If the following condition*

$$|\lambda_{min}(\mathbf{G}^-)| < \frac{\bar{\kappa} - \kappa \lambda_{max}(\mathbf{G}^+)}{\kappa}, \tag{23}$$

is satisfied, then:

(i) *The map $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ is strongly monotone with modulus μ given by*

$$\mu = \bar{\kappa} + \kappa \lambda_{min}(\mathbf{G}^-) - \kappa \lambda_{max}(\mathbf{G}^+).$$

(ii) *There exists a unique NE.*

Two comments are in order. First, Eq. (23) highlights the fact that a specific relationship between $\lambda_{min}(\mathbf{G}^-)$ and $\lambda_{max}(\mathbf{G}^+)$ determines the existence and uniqueness of a NE. Intuitively this condition means that the topology of both network must "interact" in a very particular way in order to obtain a uniquen NE. A second observation is related to the fact that for the case of $\mathbf{G}^+$ being empty, Eq. (7) boils down to $|\lambda_{min}(\mathbf{G}^-)| < \bar{\kappa}/\kappa$, which is exactly the condition in theorem 1. Similarly, for the case of $\mathbf{G}^-$ being empty, Eq. (23) becomes $\lambda_{max}(\mathbf{G}^+) < \bar{\kappa}/\kappa$ which boils down to the uniqueness result for games with strategic complementarities. Nexy corollary summarizes this observation.

Corollary 4. *Let assumptions 1 and 2 hold.*

(i) *Let $\mathbf{G}^+$ be the empty network. If the following condition is satisfied*

$$|\lambda_{min}(\mathbf{G}^-)| < \frac{\bar{\kappa}}{\kappa},$$

there exists a unique NE.

(ii) *Let $\mathbf{G}^-$ be the empty network. If the following condition is satisfied*

$$\lambda_{min}(\mathbf{G}^+) < \frac{\bar{\kappa}}{\kappa},$$

there exists a unique NE.

In order to caractherize the comparative statics of this class of games, we define $\mathbf{G}^+_{\mathcal{A}}$ and $\mathbf{G}^-_{\mathcal{A}}$ as the subnetworks of $\mathbf{G}^+$ and $\mathbf{G}^-$ respectively.

Proposition 8. *Let assumptions 1 and 2 hold. Let $\mathbf{x}^*$ be a non degenerate NE, and assume that the following condition holds:*

$$|\lambda_{min}(\mathbf{G}^-_{\mathcal{A}})| < \frac{\bar{\kappa} - \kappa\lambda_{max}(\mathbf{G}^+_{\mathcal{A}})}{\kappa}. \tag{24}$$

Then for $\tilde{\boldsymbol{\theta}}$ in a neighborhood of $\boldsymbol{\theta}$, there exists a locally unique once continuously differentiable equilibrium $\mathbf{x}(\tilde{\boldsymbol{\theta}})$ such that $\mathbf{x}^* = \mathbf{x}(\boldsymbol{\theta})$ and

$$\frac{\partial \mathbf{x}^*_{\mathcal{A}}}{\partial \theta_{js}} = \mathbf{W}^* \hat{\mathbf{D}}^{*-1}_{\mathcal{A}}$$

where $\hat{\mathbf{D}}^{*-1}_{\mathcal{A}} = \mathbf{D}^{*-1}_{\mathcal{A}} \mathbf{e}_j \tau_{js}$ for $j = 1, \ldots, n$ and $s = 1, \ldots, k_j$.

There is an important difference between proposition 8 and theorem 2. In particular, an exogeneous change affecting player j will affect player i in a different way depending on the they type of relationship; complements or substitutes. This information is captured in the construction of the $\mathbf{W}^*$ whose entries depends on the nature of strategic interaction (complememts or substitutes).

Finally we can adapt our approximation result (theorem 3) in order to show how the unique equilibrium of this game can be approximated.

Proposition 9. *Let assumptions 1 and 2 hold. Let* $\mathbf{x}^*$ *be the unique NE of the game. Then every* ϵ*-approximate NE* $\hat{\mathbf{x}}$ *is a* $\sqrt{\frac{\epsilon}{\mu}}$*-near NE of the network game. In particular*

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\| \leq \sqrt{\frac{\epsilon}{\mu}},$$

where $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}^-) - \kappa\lambda_{max}(\mathbf{G}^+)$.

Proposition 9 states that the quality of the approximation depends on the nature of conflict among players which is captured by $\lambda_{max}(\mathbf{G}^+)$ and $\lambda_{min}(\mathbf{G}^-)$. Recalling that $\lambda_{max}(\mathbf{G}^+) > 0$ and $\lambda_{min}(\mathbf{G}^-) < 0$, it follows that the quality of the approximation will depend on the difference between these two measures. Thus in general we may expect that approximating the equilibrium $\mathbf{x}^*$ may be harder that the case analyzed in §4.

7.1.1. *Application.* In order to gain some intuition we consider a particular class of payoff functions. Concretely we assume that players' payoffs are given by the following specification:

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}) = x_i - \frac{1}{2}x_i + x_i\varphi(\bar{\mathbf{x}}^+_{-i} + \theta_{i1}) - x_i\varphi(\bar{\mathbf{x}}^-_{-i} + \theta_{i2}), \tag{25}$$

where $\bar{\mathbf{x}}^+_{-i} = \sum_{j=1}^n g^+_{ij}x_j$ and $\bar{\mathbf{x}}^-_{-i} = \sum_{j=1}^n g^-_{ij}x_j$ with $\theta_{i1} > 0$ and $\theta_{i2} > 0$. Similar to §6, the terms φ_i represent player i's aggregator function. We assume that this function satisfies assumption 4.

Thus the class of games with payoffs (25) can be seen as an aggregative network game which allows for mixed interaction. In fact, it is easy to check:

$$\begin{aligned}
\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} &= \varphi(\bar{\mathbf{x}}^+_{-i} + \theta_{i1})g^+_{ij} > 0 \quad \text{for all } (i,j) \in \mathbf{L}^+ \\
\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_j \partial x_i} &= -\varphi(\bar{\mathbf{x}}^-_{-i} + \theta_{i2})g^-_{ij} < 0 \quad \text{for all } (i,j) \in \mathbf{L}^-
\end{aligned}$$

The following result is a direct adaptation of proposition 7.

Proposition 10. *Let assumptions 3 and 4 hold. If the condition*

$$|\lambda_{min}(\mathbf{G}^-)| < \frac{\kappa_1}{\kappa_2\kappa_3} - \lambda_{max}(\mathbf{G}^+)$$

is satisfied then:

(i) *The map* $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ *is strongly monotone with modulus*

$$\mu = \kappa_1 + \kappa_2\kappa_3(\lambda_{min}(\mathbf{G}^-) - \lambda_{max}(\mathbf{G}^+)).$$

(ii) *There exists a unique NE.*

Similarly, the comparative statics and approximation results in propositions 8 and 9 can be adapted for this class of games.

8. Bayesian Network games

In previous sections we have focused on network games of complete information. The goal of this section is to show how our framework can be adapted to analyze games of incomplete information. In particular, we relax the assumption that players knows the parameter vector $\boldsymbol{\theta}$. Instead, we treat $\boldsymbol{\theta}$ as a random vector whose values depends on the realizations of the underlaying state of the nature. We show how a simple adaptation of the results in §3 allows us to establish the existence and uniqueness of a Bayesian Nash Equilibrium (BNE).

Some previous papers have studied network games of incomplete information. The paper by Galeotti et al. [2010] studies Bayesian network games with discrete actions, showing how incomplete information can be used to reduce the number of equilibria. In the context of of linear-quadratic games with complementarities, de Marti and Zenou [2015] state the uniqueness of a BNE. The closest paper to ours, is the recent contribution by Ui [2016] who also applies VI to Bayesian games. He shows that when the payoff gradient of the game is strongly monotone at *every* state, then there exists a unique BNE.

The findings in this section differs from Ui [2016]'s in at least three important aspects. First, Ui [2016] we provide an explicit condition to guarantee that the payoff gradient is strongly monotone. In addition, our characterization requires that strong monotonicity must be satisfied only on a subset of states. Second, we show how our result can be applied to games of mixed interactions, a class of games not covered by Ui [2016]'s results. Finally, our uniqueness result applies to network games beyond the linear-quadratic case.

8.1. **Players, networks, and strategies.** Consider a Bayesian game with a set of players $N = \{1, \ldots, n\}$. As before, player $i \in N$ has a set of actions $\mathcal{X}_i \subseteq \mathbb{R}_+$,which is assumed to be a convex and compact set. We write $\mathcal{X} = \prod_{i\in N} \mathcal{X}_i$ and $\mathcal{X}_{-j} = \prod_{j\neq i} \mathcal{X}_j$. Players are embedded in a network, which is represented by $\mathbf{G}$. Similar to §2 we assume that $\mathbf{G}$ is connected. For a given network $\mathbf{G}$, player i's payoff function is a measurable function $u_i(\cdot;\cdot,\mathbf{G}) : \mathcal{X} \times \Omega \longrightarrow \mathbb{R}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. For a particular realization of ω, we define the parameter vector $\boldsymbol{\theta}(\omega) = (\boldsymbol{\theta}_1(\omega), \ldots, \boldsymbol{\theta}_n(\omega)) \in \Theta$.[21]

Player i's information is given by a measurable mapping $\eta_i : \Omega \longrightarrow \mathcal{Y}_i$, where $(Y_i, \mathcal{Y}_i)$ is a measurable space. Player i's strategy is a measurable mapping $s_i : Y_i \longrightarrow \mathcal{X}_i$ with $\mathbb{E}(\eta_i) < \infty$. We consider two strategies s_i^1 and s_i^2 as the same strategy if $s_i^1(\eta_i(\omega)) = s_i^2(\eta_i(\omega))$ almost surely. Let $\mathbf{S}_i$ denote player i's set of strategies and define $\mathbf{S} = \prod_{i\in N} \mathbf{S}_i$ and $\mathbf{S}_{-i} = \prod_{j\neq N} \mathbf{S}_j$. We assume that $\mathbb{E}(u_i(\mathbf{s}; \boldsymbol{\theta}_i(\omega), \mathbf{G}))$ exists for all $\mathbf{s} \in \mathbf{S}$.

Throught this section we assume the following

Assumption 5. *For all player $i \in N$ we assume the following:*

[21] We recall that $\Theta_i \subseteq \mathbb{R}^{k_i}$ with $k_i \geq 1$ and $\Theta = \prod_{i=1}^{n} \Theta_i$.

(i) $u_i(\cdot, \mathbf{s}_{-i}; \boldsymbol{\theta}_i(\omega), \mathbf{G}) : \mathcal{X}_i \longrightarrow \mathbb{R}$ *is continuously twice differentiable and strictly concave on* x_i *for each* $\mathbf{x}_{-i}$ *and a.s. for* $\omega \in \Omega$

$$\mathbb{E}\left(\left[\frac{\partial u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i(\omega), \mathbf{G})}{\partial x_i}\right]^2\right) < \infty.$$

(ii) *There exists a constant* $\bar{\kappa}(\omega) > 0$

$$\inf_{i \in N} \inf_{\mathbf{x} \in \mathcal{X}} \left|\frac{\partial^2 u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i(\omega), \mathbf{G})}{\partial x_i^2}\right| \geq \bar{\kappa}(\omega) \quad \textit{a.s. for } \omega \in \Omega.$$

Assumption 6. *For all player* $i \in N$ *we assume the following:*

(i) *The cross partial derivative*

$$\frac{\partial^2 u_i(\mathbf{x}; \boldsymbol{\theta}_i(\omega), \mathbf{G})}{\partial x_j \partial x_i}$$

is well defined for all $\mathbf{x} \in \mathcal{X}$ *and* $(i,j) \in \mathbf{L}$ *a.s for* $\omega \in \Omega$.

(ii) *There exists a constant* $\kappa(\omega) > 0$

$$\sup_{(i,j) \in \mathbf{L}} \sup_{\mathbf{x} \in \mathcal{X}} \left|\frac{\partial^2 u_i(\mathbf{x}; \boldsymbol{\theta}_i(\omega), \mathbf{G})}{\partial x_j \partial x_i}\right| \leq \kappa(\omega) \quad \textit{a.s for } \omega \in \Omega.$$

Throughout this section we fix N, $\mathcal{X}$, and $(\Omega, \mathcal{F}, \mathbb{P})$ and denote $\mathbf{u} = (u_i)_{i \in N}$ and $\boldsymbol{\eta} = (\eta_i)_{i \in N}$. According to this, a strategy profile $\mathbf{s} \in \mathbf{S}$ is is a Bayesian Nash equilibrium iff

$$\mathbb{E}(u_i(\mathbf{s}(\boldsymbol{\eta}); \boldsymbol{\theta}_i(\omega), \mathbf{G})|\eta_i) \geq \mathbb{E}(u_i(x_i, \mathbf{s}_{-i}(\boldsymbol{\eta}_{-i}); \boldsymbol{\theta}_i(\omega), \mathbf{G})|\eta_i) \quad \text{a.s. for } \omega \in \Omega.$$

The gradient payoff of the game is defined as $f(\mathbf{x}; \boldsymbol{\theta}(\omega), \mathbf{G}) = \left(-\frac{\partial u_i(\mathbf{x}; \boldsymbol{\theta}_i(\omega), \mathbf{G})}{\partial x_i}\right)_{i \in N}$.

Next result establishes the connection between VI and BNE.

Proposition 11. *Let assumption 5 hold. Then* $\mathbf{s}^* \in \mathbf{S}$ *is BNE of the network game iff* $\mathbf{s}^*$

$$\mathbb{E}((\mathbf{s} - \mathbf{s}^*(\boldsymbol{\eta}))^T f(\mathbf{s}^*(\boldsymbol{\eta}); \boldsymbol{\theta}(\omega), \mathbf{G})) \geq 0 \quad \textit{for all } \mathbf{s} \in \mathbf{S}. \tag{26}$$

It is easy to see that previous result is a direct extension of proposition 1 to the case of the incomplete information games.

8.2. **The Jacobian the of the game.** Similar to §3 we can write down the Jacobian of the game. Using assumptions 5(i) and 6(i) it follows that

$$\mathbf{J}(\mathbf{x}; \boldsymbol{\theta}(\omega), \mathbf{G}) = \mathbf{D}(\mathbf{x}; \boldsymbol{\theta}(\omega)) + \mathbf{N}(\mathbf{x}; \boldsymbol{\theta}(\omega), \mathbf{G}) \quad \text{a.s. for } \omega \in \Omega. \tag{27}$$

It is easy to see that expression (27) is a stochastic version of the Jacobian defined in Eq. (4).

Applying lemma 1 in Appendix A for all $\mathbf{x} \in \mathcal{X}$ we get

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x}; \boldsymbol{\theta}(\omega), \mathbf{G})) \geq \lambda_{min}(\mathbf{D}(\mathbf{x}; \boldsymbol{\theta}(\omega))) + \lambda_{min}(\bar{\mathbf{N}}(\mathbf{x}; \boldsymbol{\theta}(\omega), \mathbf{G})) \quad \text{a.s. for } \omega \in \Omega. \tag{28}$$

Using assumptions 5(ii) and 6(ii), we can apply lemma 2 in Appendix A (in an a.s. sense) to establish that for all $\mathbf{x} \in \mathcal{X}$ the following lower bound holds:

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta}(\omega),\mathbf{G})) \geq \bar{\kappa}(\omega) + \kappa(\omega)\lambda_{min}(\mathbf{G}) \quad \text{a.s. for } \omega \in \Omega. \tag{29}$$

Thus similar to the case of complete information, to determine the positive definiteness of $\mathbf{J}(\mathbf{x};\boldsymbol{\theta}(\omega),\mathbf{G})$ the parameters $\kappa(\omega), \bar{\kappa}(\omega)$ and $\lambda_{min}(\mathbf{G})$ are key.

With this observation in place, we are ready to establish the main result of this section.

Theorem 5. *Let assumption 5 and 6 hold. Assume that the game has the property of strategic substitutes. Suppose that the following inequality hold:*

$$|\lambda_{min}(\mathbf{G})| \leq \bar{\kappa}(\omega)/\kappa(\omega) \quad \textit{a.s. for } \omega \in \Omega. \tag{30}$$

In addition assume that inequality (30) is strict a.s. for $\omega \in \mathcal{M} \subseteq \Omega$ where $\mathbb{P}(\mathcal{M}) > 0$. Then there exists a unique BNE.

We remark two features of theorem 5. First, the uniqueness condition is weaker than the one assumed in theorem 1. In particular, we only need to assume that inequality (30) is strict only on a subset $\mathcal{M}$ of Ω. The implication of this requirement is that we only need to verify that $f(\mathbf{x};\boldsymbol{\theta}(\omega),\mathbf{G})$ is strongly monotone a.s. for $\omega \in \mathcal{M}$, whereas for $\omega \in \Omega \setminus \mathcal{M}$ the map $f(\mathbf{x};\boldsymbol{\theta}(\omega),\mathbf{G})$ is monotone. The second observation is related to the fact that theorem 5 generalizes the existence and uniqueness result in Ui [2016]. In particular, our result provides a specific condition in terms of the network topology without assuming specific functional forms on players' payoffs. Similarly, Ui [2016]'s uniqueness result assumes that the payoff gradient of the game must be strongly monotone a.s. for all $\omega \in \Omega$, while we only requires that such a condition must be satisfied on a subset $\mathcal{M} \subseteq \Omega$ with $\mathbb{P}(\mathcal{M}) > 0$.

The following corollary establishes the uniqueness of a BNE for the case of strategic complements.

Corollary 5. *Let assumption 5 and 6 hold. Assume that the game has the property of strategic complements. Suppose that the following inequality hold:*

$$\lambda_{max}(\mathbf{G}) \leq \bar{\kappa}(\omega)/\kappa(\omega) \quad \textit{a.s. for } \omega \in \Omega. \tag{31}$$

In addition assume that inequality (31) is strict a.s. for $\omega \in \mathcal{M} \subseteq \Omega$ where $\mathbb{P}(\mathcal{M}) > 0$. Then there exists a unique BNE.

Similarly, theorem 5 can be extended to the case of network games with mixed interactions.

Corollary 6. *Consider a Bayesian network game of mixed interactions and let assumption 5 and 6 hold. Suppose that the following inequality hold:*

$$|\lambda_{min}(\mathbf{G}^{-})| \leq \frac{\bar{\kappa}(\omega)}{\kappa(\omega)} - \lambda_{max}(\mathbf{G}^{+}) \quad \textit{a.s. for } \omega \in \Omega. \tag{32}$$

In addition assume that inequality (32) is strict a.s. for $\omega \in \mathcal{M} \subseteq \Omega$ where $\mathbb{P}(\mathcal{M}) > 0$. Then there exists a unique BNE.

8.3. **Applications.** In order to see how previous results apply, we analyze the linear-quadratic case with incomplete information. Concretely, we consider a network game where players' payoffs are given by

$$u_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\theta}_i(\omega), \mathbf{G}) = x_i - \frac{1}{2}x_i^2 - \theta_i(\omega)x_i\bar{\mathbf{x}}_{-i} \quad \text{a.s. for } \omega \in \Omega,\ i \in N. \tag{33}$$

Previous specification makes clear that the parameter $\theta_i(\omega)$ is a random variable. According to this the random vector $\boldsymbol{\theta} = (\theta_i(\omega))_{i\in N}$ a.s. for $\omega \in \Omega$.

The Jacobian of this game can be written as:

$$\mathbf{J}(\mathbf{x}; \omega, \mathbf{G}) = \mathbf{I} + \mathbf{V}(\boldsymbol{\theta}(\omega))\mathbf{G} \quad \text{a.s. for } \omega \in \Omega.$$

Define $\bar{\theta}(\omega) = \max_{i\in N}\{\theta_i(\omega)\}$ a.s. for $\omega \in \Omega$. Then the condition in theorem 5 can be expressed as $1 + \bar{\theta}(\omega)\lambda_{min}(\mathbf{G}) \geq 0$ a.s. for $\omega \in \Omega$ and for a subset $\mathcal{M} \subseteq \Omega$ the inequality is strict. Thus the conditions for the uniqueness of a BNE can be written as:

$$\begin{aligned} |\lambda_{min}(\mathbf{G})| &\leq \frac{1}{\bar{\theta}(\omega)} \quad \text{a.s. for } \omega \in \Omega, \\ |\lambda_{min}(\mathbf{G})| &< \frac{1}{\bar{\theta}(\omega)} \quad \text{a.s. for } \omega \in \mathcal{M}. \end{aligned}$$

As a second application, consider the class of aggregative games network games analyzed in §6.3. Accordingly players' payoffs are given by:

$$u_i(x_i, \varphi_i; \boldsymbol{\theta}_i(\omega), \mathbf{G}) = \theta_{i2}(\omega)x_i - \frac{1}{2}x_i^2 - x_i\varphi(\bar{\mathbf{x}}_{-i} + \theta_{i1}(\omega)) \quad \forall i \in N. \tag{34}$$

Following similar notation used in §5, we can define

$$\kappa_3(\omega) = \sup_{i\in N} \sup_{\mathbf{x}_{-i}\in\mathcal{X}_{-i}} \varphi'(\bar{\mathbf{x}}_{-i} + \theta_{i1}(\omega))$$

and

$$\underline{\theta}(\omega) = \min_{i\in N}\{\theta_{i2}(\omega)\} \quad \text{a.s. for } \omega \in \Omega.$$

Then applying theorem 5 to the class of games with payoffs (34), we find that the equilibrium is unique when the following inequalities are satisfied:

$$\begin{aligned} |\lambda_{min}(\mathbf{G})| &\leq \frac{\underline{\theta}(\omega)}{\kappa_3(\omega)} \quad \text{a.s. for } \omega \in \Omega, \\ |\lambda_{min}(\mathbf{G})| &< \frac{\underline{\theta}(\omega)}{\kappa_3(\omega)} \quad \text{a.s. for } \omega \in \mathcal{M} \subseteq \Omega. \end{aligned}$$

Thus applying theorem 5 we conclude that the class of aggregative network games with payoffs (34) has a unique BNE.

9. Conclusions

In this paper we have introduced variational methods to the study of network games. Using these techniques, we established the existence, uniqueness, and the comparative statics of a NE for a general class of network games. An important feature of our results is that we do not have to assume specific functional forms on players' payoff functions. Our analysis shows how the interaction between the lowest (largest) eigenvalue of the network, a parameter measuring players' payoff concavity, and a parameter capturing the strength of the strategic interaction among players contain all the relevant information to understand the outcomes of the game.

Our second contribution is the comparative statics analysis of network games. We fully characterized the effect of exogenous shocks at the node level. Our characterization sheds light about the role of the network topology in propagating shocks. From an economic point of view, our analysis shows that the notion of *interdependence* applies far beyond the class of games with linear best responses (Bramoullé and Kranton [2016]). From a technical point of view, our comparative statics analysis exploits the theory of sensitivity analysis in VI Kyparisis [1986, 1987] and Tobin [1986].

As a third contribution we show how the quality of approximation of a NE is determined by the network topology. Our approximation result is useful when numerical methods are implemented.

Finally, we applied our results to the study of aggregative network games, games of mixed interactions, and Bayesian network games. For all of these games, we derive simple conditions highlighting the role of the network topology.

Appendix A. Proofs

A.1. **Preliminaries.** We begin this section defining the eigenvalues of a symmetric matrix. Let $\mathbf{A}$ be a n-by-n symmetric matrix. The i-th eigenvalue of $\mathbf{A}$ is denoted by $\lambda_i(\mathbf{A})$ for $i = 1, \ldots, n$. The lowest and largest eigenvalue of $\mathbf{A}$ are denoted by $\lambda_{min}(\mathbf{A}) = \lambda_1(\mathbf{A})$ and $\lambda_{max}(\mathbf{A}) = \lambda_n(\mathbf{A})$, where

$$\lambda_{min}(\mathbf{A}) \leq \lambda_2(\mathbf{A}) \leq, \cdots, \leq \lambda_{n-1}(\mathbf{A}) \leq \lambda_{max}(\mathbf{A}).$$

Next lemma is a simplified version of Weyl's inequality (Horn and Johnson [1990, Thm. 4.3.1]). For completeness we provide its proof.

Lemma 1 (Weyl's inequality)**.** *Let $\mathbf{J} = \mathbf{D} + \mathbf{N}$ with $\mathbf{D}$ and $\mathbf{N}$ n-by-n symmetric matrices. Then the following inequality holds*

$$\lambda_{min}(\mathbf{J}) \geq \lambda_{min}(\mathbf{D}) + \lambda_{min}(\mathbf{N}).$$

Proof. In order to proof this inequality we use the well knwon fact that the the $\lambda_{min}(\cdot)$ is a concave function.[22] In particular for $\alpha \in [0, 1]$ and for symmetric matrices $\mathbf{D}$ and $\mathbf{N}$ we get

$$\lambda_{min}(\alpha\mathbf{D} + (1 - \alpha)\mathbf{N}) \geq \alpha\lambda_{min}(\mathbf{D}) + (1 - \alpha)\lambda_{min}(\mathbf{N}).$$

In particular, for $\alpha = 1/2$, we get

$$\lambda_{min}(\mathbf{J}) \geq \lambda_{min}(\mathbf{D}) + \lambda_{min}(\mathbf{N}).$$

□

Definition 9. *A matrix $\mathbf{M}(\mathbf{x})$ whose elements $m_{ij}(\mathbf{x})$ are functions defined on $B \subset \mathbb{R}^n_+$ is said to be positive semidefinite (p.s.d) on B, if for every $\mathbf{y} \in B$, we have*

$$\mathbf{y}^T \cdot \mathbf{M}(\mathbf{x}) \cdot \mathbf{y} \geq 0.$$

It is said to be positive definite (p.d) on B, if for every $\mathbf{y} \in B$, we have

$$\mathbf{y}^T \cdot \mathbf{M}(\mathbf{x}) \cdot \mathbf{y} > 0.$$

It is said to be strongly positive definite (s.p.d.) on B if there exists a scalar $\mu > 0$ such that for every $\mathbf{x} \in B$, we have

$$\mathbf{y}^T\mathbf{M}(\mathbf{x})\mathbf{y} \geq \mu\|\mathbf{y}\|^2, \quad \forall \mathbf{y} \in B.$$

[22]To see this, we note that for a symmetric matrix $\mathbf{J}$ its lowest eigenvalue can be obtained as $\lambda_{min}(\mathbf{J}) = \min_{\mathbf{z} \neq \mathbf{0}} \frac{\mathbf{z}^T\mathbf{J}\mathbf{z}}{\mathbf{z}^T\mathbf{z}}$. Because the the operator min is concave, it follows that $\lambda_{min}(\cdot)$ is concave.

A.2. **Proof of Proposition 1.** ($\Longrightarrow$) Let $\mathbf{x}^*$ be a NE and fix player $i \in N$. By assumption $-u_i(x_i^*, \mathbf{x}_{-i}^*; \theta_i, \mathbf{G})$ is strictly convex with respect to x_i. By the minimum principle, and defining $f_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}) = -\frac{\partial u_i(x_i^*, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_i}$, we obtain

$$(y_i - x_i^*) f_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}) \geq 0, \quad \forall y_i \in \mathcal{X}_i.$$

Thus, if $\mathbf{x}^*$ is a NE, and by concatenating previous expression, it follows that $\mathbf{x}^*$ must solve $\mathbf{VI}(f, \mathcal{X})$.

($\Longleftarrow$) Suppose that $\mathbf{x}^*$ solves $\mathbf{VI}(f, \mathcal{X})$. Defining $f_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}) = -\frac{\partial u_i(x_i^*, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G})}{\partial x_i}$, previous condition implies that for each player $i \in N$ we get

$$(y_i - x_i^*) f_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}) \geq 0, \quad \forall y_i \in \mathcal{X}_i.$$

By the convexity of $-u_i$ we get

$$-u_i(y_i, \mathbf{x}_{-i}^*, \boldsymbol{\theta}_i; \mathbf{G}) \geq -u_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}) + (y_i - x_i^*) f_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}).$$

Using the fact that $(y_i - x_i^*) f_i(x_i^*, \mathbf{x}_{-i}; \boldsymbol{\theta}_i, \mathbf{G}) \geq 0$ we obtain

$$u_i(x_i^*, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}) \geq u_i(y_i, \mathbf{x}_{-i}^*; \boldsymbol{\theta}_i, \mathbf{G}), \quad \forall y_i \in \mathcal{X}_i.$$

Because last expression holds for all $i \in N$ the conclusion follows. $\square$

A.3. **Proof of proposition 2.** We only proof part *ii*). The argument for *i*) and *iii*) is identical. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two arbitrary vectors in $\mathcal{X}$. Let

$$\phi(\lambda) = (\mathbf{x}_1 - \mathbf{x}_2)^T f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2; \boldsymbol{\theta}, \mathbf{G}), \quad 0 \leq \lambda \leq 1.$$

Since, by assumption $\mathcal{X}$ is convex, $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in \mathcal{X}$ for all $0 \leq \lambda \leq 1$. From the definition of $\phi(\lambda)$, we get

$$\phi(1) - \varphi(0) = (\mathbf{x}_1 - \mathbf{x}_2)^T (f(\mathbf{x}_1; \boldsymbol{\theta}, \mathbf{G}) - f(\mathbf{x}_2; \boldsymbol{\theta}, \mathbf{G})).$$

By the mean value theorem we know that $\phi(1) - \phi(0) = \phi'(\bar{\lambda})$ for some $0 < \bar{\lambda} < 1$. Noting that $\phi'(\bar{\lambda}) = (\mathbf{x}_1 - \mathbf{x}_2)^T \cdot \mathbf{J}(\mathbf{x}_{\bar{\lambda}}; \boldsymbol{\theta}, \mathbf{G}) \cdot (\mathbf{x}_1 - \mathbf{x}_2)$ with $\mathbf{x}_{\bar{\lambda}} = \bar{\lambda}\mathbf{x}_1 + (1-\bar{\lambda})\mathbf{x}_2$, we find that

$$(\mathbf{x}_1 - \mathbf{x}_2)^T (f(\mathbf{x}_1; \boldsymbol{\theta}, \mathbf{G}) - f(\mathbf{x}_2; \boldsymbol{\theta}, \mathbf{G})) = (\mathbf{x}_1 - \mathbf{x}_2)^T \mathbf{J}(\mathbf{x}_{\bar{\lambda}}; \boldsymbol{\theta}, \mathbf{G})(\mathbf{x}_1 - \mathbf{x}_2). \tag{35}$$

From equation (35) we get that if $\mathbf{J}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ is positive definite, then

$$(\mathbf{x}_1 - \mathbf{x}_2)^T (f(\mathbf{x}_1; \boldsymbol{\theta}, \mathbf{G}) - f(\mathbf{x}_2; \boldsymbol{\theta}; \mathbf{G})) \geq \lambda_{min}(\mathbf{J}(\mathbf{x}_{\bar{\lambda}}; \boldsymbol{\theta}, \mathbf{G})) \|\mathbf{x}_1 - \mathbf{x}_2\|^2 > 0 \quad \forall \mathbf{x} \in \mathcal{X},$$

so that we conclude $f(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ is strictly monotone. $\square$

In order to prove proposition 3 we need the following technical result.

Lemma 2. *Let assumptions 1 and 2 hold. In a network game of strategic substitutes, if the condition*

$$|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$$

holds, then $\mathbf{J}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})$ *is strongly positive definite with modulus* $\mu = \bar{\kappa} + \kappa \lambda_{min}(\mathbf{G})$.

Proof. To analyze the positive definiteness of $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ we consider its symmetric part. Applying lemma 1 combined with the definition of $\lambda_{min}(\bar{\mathbf{N}})$ we get

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \quad \geq \quad \lambda_{min}(\mathbf{D}(\mathbf{x})) + \min_{\mathbf{z}\neq\mathbf{0}} \frac{\mathbf{z}^T\bar{\mathbf{N}}\mathbf{z}}{\mathbf{z}^T\mathbf{z}}.$$

By assumption 2 it follows that $\min_{\mathbf{z}\neq\mathbf{0}} \frac{\mathbf{z}^T\bar{\mathbf{N}}\mathbf{z}}{\mathbf{z}^T\mathbf{z}} = \min_{\mathbf{z}\neq\mathbf{0}} \frac{\mathbf{z}^T[\mathbf{N}+\mathbf{N}^T]\mathbf{z}}{2\mathbf{z}^T\mathbf{z}} \geq \kappa \min_{\mathbf{z}\neq\mathbf{0}} \frac{\mathbf{z}^T\mathbf{G}\mathbf{z}}{\mathbf{z}^T\mathbf{z}} = \kappa\lambda_{min}(\mathbf{G})$. This latter fact implies combined with assumption 1 implies

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \quad \geq \quad \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}).$$

Finally the condition $|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$ implies that $\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \geq \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}) > 0$, which allows us to conclude the strong monotonicity of the Jacobian where the monotonicity modulus μ can be defined as $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G})$. ☐

A.4. **Proof of Proposition 3.** By Lemma 2, we know the condition $|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$ implies that $\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly for all $\mathbf{x} \in \mathcal{X}$. Combining this fact with proposition 2(iii), we conclude $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone mapping with monotonicity modulus $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G})$. ☐

Lemma 3. *[More (1974)] Let $f(\mathbf{x},\boldsymbol{\theta},\mathbf{G})$ be the map defined in Eq.(2). Assume that for some fixed $\mathbf{y} \in \mathcal{X}$ the condition:*

$$\frac{(\mathbf{x}-\mathbf{y})^T(f(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}))}{\|\mathbf{x}-\mathbf{y}\|} \longrightarrow \infty$$

holds as $\|\mathbf{x}\| \longrightarrow \infty$, $\mathbf{x} \in \mathcal{X}$. Then there exists a NE.

Proof. This result follows from Facchinei and Pang [2003, Thm. 2.3.3(b)]. ☐

A.5. **Proof of Theorem 1.**

(i) Existence. By proposition 1 By proposition 3 we know that $f(\mathbf{x},\boldsymbol{\theta},\mathbf{G})$ is strongly monotone. For a fixed $\mathbf{y} \in \mathcal{X}$ strong monotonocity implies that

$$\frac{(\mathbf{x}-\mathbf{y})^T(f(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}))}{\|\mathbf{x}-\mathbf{y}\|} \geq \mu\|\mathbf{x}-\mathbf{y}\|.$$

For $\|\mathbf{x}\| \longrightarrow \infty$ it follows that

$$\frac{(\mathbf{x}-\mathbf{y})^T(f(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}))}{\|\mathbf{x}-\mathbf{y}\|} \longrightarrow \infty.$$

Then by lemma 3 the existence of a NE follows.

(ii) Uniqueness. By mean of contradiction, assume that the network game has two NE denoted by $\mathbf{x}^*$ and $\mathbf{y}^*$ respectively. From Proposition 3 we know that the condition $|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$ implies that $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone. Then we must have

$$\begin{aligned} 0 &< (\mathbf{x}^* - \mathbf{y}^*)^T(f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y}^*;\boldsymbol{\theta},\mathbf{G})), \\ &= \mathbf{x}^{*T}f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G}) - \mathbf{x}^{*T}f(\mathbf{y}^*;\boldsymbol{\theta},\mathbf{G}) - \mathbf{y}^{*T}f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G}) + \mathbf{y}^{*T}f(\mathbf{y}^*;\boldsymbol{\theta},\mathbf{G}), \\ &= -\mathbf{x}^{*T}f(\mathbf{y}^*;\boldsymbol{\theta},\mathbf{G}) - \mathbf{y}^{*T}f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G}) \leq 0, \end{aligned}$$

which is a contradiction. Thus we conclude that the NE is unique. □

A.6. **Proofs of section 3.4.** Here we provide the details for the applications in §3.4.

A.6.1. *Linear quadratic games.* For this class of games the strict monotonicity of $f(\mathbf{x},\boldsymbol{\theta},\mathbf{G})$ is determined by the positive definiteness of $\mathbf{J}(\mathbf{x},\boldsymbol{\theta},\mathbf{G}) = \mathbf{I}+\mathbf{V}(\boldsymbol{\theta})\cdot\mathbf{G}$. Because this Jacobian is asymmetric, we look at its symmetric part, which is given by:

$$\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \mathbf{I} + \frac{1}{2}\left[\mathbf{V}(\boldsymbol{\theta})\cdot\mathbf{G} + \mathbf{G}\cdot\mathbf{V}(\boldsymbol{\theta})\right].$$

Applying lemma 2, it follows that the lowest eigenvalue of $\frac{1}{2}[\mathbf{V}(\boldsymbol{\theta})\cdot\mathbf{G}+\mathbf{G}\cdot\mathbf{V}(\boldsymbol{\theta})]$ is bounded below by $\lambda_{min}(\mathbf{G})\cdot\bar{\theta}$, where $\bar{\theta} = \max_{i\in N}\{\theta_i\}$.

It follows then that $\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is positive definiteness when $1+\lambda_{min}(\mathbf{G})\cdot\bar{\theta} > 0$. This latter condition implies, by proposition 3, that $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone when $|\lambda_{min}(\mathbf{G})| < \frac{1}{\bar{\theta}}$. Then by theorem 1 the existence and uniqueness of a NE follows at once. □

A.7. **Proof of corollary 1.**

A.7.1. *Public goods in networks.* For this class of games the Jacobian is given by:

$$\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \mathbf{I} + \mathbf{V}(\mathbf{x},\boldsymbol{\theta})\cdot\mathbf{G}.$$

Because this matrix is asymmetric, we look at its symmetric part given by:

$$\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \mathbf{I} + \frac{1}{2}\left[\mathbf{V}(\mathbf{x};\boldsymbol{\theta})\cdot\mathbf{G} + \mathbf{G}\cdot\mathbf{V}(\mathbf{x};\boldsymbol{\theta})\right].$$

For a given $\mathbf{x}$, define $\gamma' = \min_{i\in N}\{\gamma_i'(\theta_i + \bar{x}_{-i})\}$. Combining this definition with lemma 1 we find $\frac{1}{2}[\mathbf{V}(\mathbf{x},\boldsymbol{\theta})\cdot\mathbf{G}+\mathbf{G}\cdot\mathbf{V}(\mathbf{x},\boldsymbol{\theta})]$, we get

$$\lambda_{min}\left(\frac{1}{2}[\mathbf{V}(\mathbf{x},\boldsymbol{\theta})\cdot\mathbf{G}+\mathbf{G}\cdot\mathbf{V}(\mathbf{x},\boldsymbol{\theta})]\right) \geq (1-\gamma')\lambda_{min}(\mathbf{G}).$$

By proposition 3, we know that $|\lambda_{min}(\mathbf{G})| < \frac{1}{1-\gamma'}$ implies that $f(\mathbf{x},\boldsymbol{\theta},\mathbf{G})$ is strongly monotone with modulus $\mu = 1 + (1-\gamma')\lambda_{min}(\mathbf{G})$. Applying theorem 1 we conclude that the network game has a unique NE. □

A.7.2. *Cournot in networks.* In this model, the symmetric part of the Jacobian is given by

$$\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G}) = \mathbf{D}(\mathbf{x}) + \frac{1}{2}\left[\mathbf{V}(\mathbf{x},\boldsymbol{\theta})\cdot\mathbf{G} + \mathbf{G}\cdot\mathbf{V}(\mathbf{x},\boldsymbol{\theta})\right],$$

where $\mathbf{D}(\mathbf{x}) = Diag(c_i'' - 2P_i' - x_i P_i'')_{i\in N}$ and $\mathbf{V}(\mathbf{x},\boldsymbol{\theta}) = Diag\left(-(P_i' + P_i'' x_i)\theta_i\right)_{i\in N}$.

Applying lemma 1 to $\frac{1}{2}[\mathbf{V}(\mathbf{x},\boldsymbol{\theta})\cdot\mathbf{G} + \mathbf{G}\cdot\mathbf{V}(\mathbf{x},\boldsymbol{\theta})]$ combined with the definition of κ we get:

$$\lambda_{min}\left(\frac{1}{2}[\mathbf{V}(\mathbf{x},\boldsymbol{\theta})\cdot\mathbf{G} + \mathbf{G}\cdot\mathbf{V}(\mathbf{x},\boldsymbol{\theta})]\right) \geq \kappa\lambda_{min}(\mathbf{G}).$$

By the definition of $\bar{\kappa}$ get

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \geq \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}).$$

Then $|\lambda_{min}(\mathbf{G})| < \bar{\kappa}/\kappa$ implies the existence and uniqueness of a NE. □

A.8. **Proof of Theorem 2.** By assumptions 1 and 2, it follows that the $\mathbf{J}^*_{\mathcal{A}}$ is well defined. Moreover, by lemma 2 we get:

$$\lambda_{min}(\mathbf{J}^*_{\mathcal{A}}) \geq \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}_{\mathcal{A}}).$$

From previous inequality, it follows that the condition $|\lambda_{min}(\mathbf{G}_{\mathcal{A}})| < \bar{\kappa}/\kappa$ implies that $\mathbf{J}^*_{\mathcal{A}}$ is strongly positive definite. This latter fact allows us to apply Tobin [1986, Cor. 2.1], impliying the existence of a locally unique NE. Finally, by Tobin [1986, Thm. 3.1] the locally unique equilibrium $\mathbf{x}^* = \mathbf{x}(\boldsymbol{\theta})$ is differentiable. Thus implicit differentiation yields formula (11). □

A.9. **Proof of Proposition 4.** . Let $\mathbf{x}^*$ be the unique equilibrium under $\mathbf{G}$. Similarly, let $\mathbf{y}$ be an equilibrium under $\mathbf{G}'$. From proposition 1 we know that $\mathbf{x}^*$ and $\mathbf{y}$ must satisfy:

$$\begin{aligned}(\mathbf{y}-\mathbf{x}^*)^T f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G}) &\geq 0,\\ (\mathbf{x}^*-\mathbf{y})^T f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}') &\geq 0.\end{aligned}$$

Adding up these expressions we get:

$$(\mathbf{y}-\mathbf{x}^*)^T(f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}') - f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G})) \leq 0.$$

This condition can be rewritten as:

$$(\mathbf{y}-\mathbf{x}^*)^T((f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}') - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G})) + f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G})) \leq 0.$$

It follows then

$$(\mathbf{y}-\mathbf{x}^*)^T(f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{x}^*;\boldsymbol{\theta},\mathbf{G})) \leq (\mathbf{y}-\mathbf{x}^*)^T(f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}')).$$

Combining the strong monotonicity of $f(\mathbf{x}, \boldsymbol{\theta}, \mathbf{G})$ with the Cauchy-Schwartz's inequality we get:

$$\|\mathbf{y} - \mathbf{x}^*\|^2 \mu \leq \|\mathbf{y} - \mathbf{x}^*\| \|f(\mathbf{y}; \boldsymbol{\theta}, \mathbf{G}) - f(\mathbf{y}; \boldsymbol{\theta}, \mathbf{G}')\|$$

Thus we conclude

$$\|\mathbf{y} - \mathbf{x}^*\| \leq \frac{1}{\mu} \|f(\mathbf{y}; \boldsymbol{\theta}, \mathbf{G}) - f(\mathbf{y}; \boldsymbol{\theta}, \mathbf{G}')\|.$$

□

A.10. **Proof of Theorem 3:** Let $\mathbf{x}^*$ be the unique NE. Let $\hat{\mathbf{x}}$ be a ϵ-approximate solution. Because $\hat{\mathbf{x}} \in \mathcal{X}$, it follows that

$$\begin{aligned} 0 &\leq (\hat{\mathbf{x}} - \mathbf{x}^*)^T f(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) = [f(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) - f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}) + f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G})]. \\ 0 &\leq (\hat{\mathbf{x}} - \mathbf{x}^*)^T (f(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G}) - f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G})) + (\hat{\mathbf{x}} - \mathbf{x}^*)^T f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}). \end{aligned}$$

Rearranging the last inequality we get the following:

$$(\hat{\mathbf{x}} - \mathbf{x}^*)^T (f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}) - f(\mathbf{x}^*; \boldsymbol{\theta}, \mathbf{G})) \leq (\hat{\mathbf{x}} - \mathbf{x}^*)^T f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}).$$

Using the fact that under assumptions 1 and 2 the map f is strongly monotone with modulus $\mu = \bar{\kappa} + \kappa \lambda_{min}(\mathbf{G})$, we get

$$\mu \|\hat{\mathbf{x}} - \mathbf{x}^*\|^2 \leq (\hat{\mathbf{x}} - \mathbf{x}^*)^T f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}).$$

Noting that $\hat{\mathbf{x}}$ is a ϵ-approximate NE, and using the fact that $(\mathbf{x}^* - \hat{\mathbf{x}})^T f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}) \geq -\epsilon$ may be written as $(\hat{\mathbf{x}} - \mathbf{x}^*)^T f(\hat{\mathbf{x}}; \boldsymbol{\theta}, \mathbf{G}) \leq \epsilon$, we obtain:

$$\mu \|\hat{\mathbf{x}} - \mathbf{x}^*\|^2 \leq \epsilon.$$

Finally, from previous expression we conclude that:

$$\|\hat{\mathbf{x}} - \mathbf{x}^*\| \leq \sqrt{\frac{\epsilon}{\mu}}.$$

A.11. **Proof of Theorem 4.**

(i) Strong Monotonicity: The Jacobian for this game is given by Eq. (16). Its symmetric part may be expressed as

$$\bar{\mathbf{J}}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G}) = \mathbf{D}(\mathbf{x}) + \frac{1}{2} \left[\mathbf{V}(\mathbf{x}; \boldsymbol{\theta}, \boldsymbol{\varphi}) \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{V}(\mathbf{x}; \boldsymbol{\theta}, \boldsymbol{\varphi}) \right].$$

Applying lemma 1, we get

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})) \geq \lambda_{min}(\mathbf{D}(\mathbf{x})) + \lambda_{min} \left(\frac{1}{2} \left[\mathbf{V}(\mathbf{x}; \boldsymbol{\theta}, \boldsymbol{\varphi}) \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{V}(\mathbf{x}; \boldsymbol{\theta}, \boldsymbol{\varphi}) \right] \right)$$

Using assumptions 3 and 4 we obtain:

$$\lambda_{min}(\bar{\mathbf{J}}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{G})) \geq \kappa_1 + \kappa_2 \kappa_3 \lambda_{min}(\mathbf{G}).$$

Thus when $|\lambda_{min}(\mathbf{G})| < \frac{\kappa_1}{\kappa_2\kappa_3}$ the map $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone with modulus $\mu = \kappa_1 + \kappa_2\kappa_3\lambda_{min}(\mathbf{G}) > 0$.

(ii) The existence and uniqueness follows from applying the same arguments used in proving theorem 1 combined with part(i). □

A.12. **Proof of proposition 5.** It follows from applying the same logic used in proving theorem 2. □

A.13. **Proof of Proposition 6.** By proposition 4 we know:

$$\|\mathbf{y} - \mathbf{x}^*\| \leq \frac{1}{\mu}\|f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}')\|.$$

By a mean value argument is easy to see

$$\|f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}')\| \leq \kappa_2\kappa_3\|\Delta\mathbf{y}\|.$$

We get

$$\|\mathbf{y} - \mathbf{x}^*\| \leq \frac{\kappa_2\kappa_3}{\mu}\|\Delta\mathbf{y}\|$$

□

A.14. **Proofs of sections 6.3.1 and 6.3.2.**

A.14.1. *Details section 6.3.1.* Here we provide the details needed to obtain the bound in section 6.3.1. First, for each $i \in N$, a mean value arguments allows us to write down the following:

$$f_i(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f_i(\mathbf{y};\boldsymbol{\theta},\mathbf{G}') = \frac{\partial^2 u_i(y_i,\varphi_i(\bar{\mathbf{y}}_\lambda);\boldsymbol{\theta}_i,\mathbf{G})}{\partial\varphi_i\partial x_i}\varphi_i'(\bar{\mathbf{y}}_\lambda)\Delta_i\mathbf{y},$$

where $\Delta_i\mathbf{y} = \sum_{j=1}^n(g_{ij} - g'_{ij})y_j$ and $\varphi_i(\bar{\mathbf{y}}_\lambda) = \lambda\sum_{j=1}^n g_{ij}y_j + (1-\lambda)\sum_{j=1}^n g'_{ij}y_j$ with $0 < \lambda < 1$.

It follows that

$$\|f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}'\| \leq \kappa_2\kappa_3\|\Delta\mathbf{y}\|.$$

Then

$$\|\mathbf{x}^* - \mathbf{y}\| \leq \frac{\kappa_2\kappa_3}{\mu}\|\Delta\mathbf{y}\|.$$

□

A.14.2. *Details section 6.3.2:* Here we provide the details needed to obtain the bound in section 6.3.2. For each $i \in N$, the mean value theorem allows us to write down the following inequality:

$$f_i(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f_i(\mathbf{y};\boldsymbol{\theta},\mathbf{G}') = \varphi_i'(\bar{\mathbf{y}}_\lambda)\Delta_i\mathbf{y},$$

where $\Delta_i\mathbf{y} = \sum_{j=1}^n (g_{ij} - g'_{ij})y_j$ and $\bar{\mathbf{y}}_\lambda = \lambda\sum_{j=1}^n g_{ij}y_j + (1-\lambda)\sum_{j=1}^n g'_{ij}y_j + \theta_i$ with $0 < \lambda < 1$.

By assumption 4, it follows that

$$\|f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{y};\boldsymbol{\theta},\mathbf{G}')\| \leq \kappa_3\|\Delta\mathbf{y}\|.$$

Hence we conclude

$$\|\mathbf{x}^* - \mathbf{y}\| \leq \frac{\kappa_3\|\Delta\mathbf{y}\|}{\mu}.$$

□

A.15. **Proof of proposition 7.** (i) We begin noting the a direct application of lemma 1, combined with assumptions 1 and 2, yields:

$$\lambda_{min}(\mathbf{J}(\mathbf{x};\boldsymbol{\theta},\mathbf{G})) \geq \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}^-) - \kappa\lambda_{max}(\mathbf{G}^+).$$

Under condition Eq. (23) previous expression is strictly positive. By proposition 2(iii) implies that $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ is strongly monotone with modulus $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}^-) - \kappa\lambda_{max}(\mathbf{G}^+)$.
(ii) The strong monotonicity of $f(\mathbf{x};\boldsymbol{\theta},\mathbf{G})$ combined with proposition 3 the existence and uniqueness of a NE follows at once. □

A.16. **Proof of proposition 8.** This result is a direct application of theorem 2. □

A.17. **Proof of proposition 9.** It follows from theorem 3 with $\mu = \bar{\kappa} + \kappa\lambda_{min}(\mathbf{G}^-) - \kappa\lambda_{max}(\mathbf{G}^+)$. □

Lemma 4. *Let assumption 5 and 6 hold. Suppose that the following inequality hold:*

$$|\lambda_{min}(\mathbf{G})| \leq \bar{\kappa}(\omega)/\kappa(\omega) \quad a.s.\ \omega \in \Omega. \tag{36}$$

In addition assume that inequality (36) is strict a.s. for $\omega \in \mathcal{M} \subseteq \Omega$ with $\mathbb{P}(\mathcal{M}) > 0$. Then

(i) *The Jacobian $\mathbf{J}(\mathbf{x};\boldsymbol{\theta}(\omega),\mathbf{G})$ is positive semidefinite a.s. for $\omega \in \Omega$.*
(ii) *The Jacobian $\mathbf{J}(\mathbf{x};\boldsymbol{\theta}(\omega),\mathbf{G})$ is strongly positive definite a.s. for $\omega \in \mathcal{M}$.*
(iii) *The map $f(\mathbf{x};\boldsymbol{\theta}(\omega),\mathbf{G})$ is monotone a.s. for $\omega \in \Omega$ and strongly monotone with modulus $\mu(\omega) = \bar{\kappa}(\omega) + \kappa(\omega)\lambda_{min}(\mathbf{G})$ a.s. for $\omega \in \mathcal{M}$.*

Proof. The proof of parts (i) and (ii) follow from definition 9 combined with Eq. (29). The proof of part (iii) is a direct application of proposition 2. □

A.18. **Proof of proposiition 5.** *Proof.* EXISTENCE: The existence follows from the compactness of $\mathcal{X}$ and the continuity of $f(\mathbf{s};\boldsymbol{\theta}(\omega),\mathbf{G})$ combined with proposition 2 in Ui [2016]. UNIQUENESS: Let $\mathbf{s}^1$ and $\mathbf{s}^2$ be two BNE. Then the following conditions must hold:

$$\mathbb{E}((\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta}(\omega),\mathbf{G})) \geq 0, \tag{37}$$

$$\mathbb{E}((\mathbf{s}^1(\boldsymbol{\eta})-\mathbf{s}^2(\boldsymbol{\eta}))^T f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta}(\omega),\mathbf{G})) \geq 0. \tag{38}$$

Adding up (37) and (38) we get:

$$\mathbb{E}((\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T (f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta}(\omega),\mathbf{G}) - f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta}(\omega),\mathbf{G})) \leq 0. \tag{39}$$

Last expression can be written as[23]

$$\begin{aligned}\mathbb{E}((\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T (f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}))) &= \int_{\Omega\setminus\mathcal{M}} (\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T (f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) d\mathbb{P}. \\ &+ \int_{\mathcal{M}} (\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T (f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) d\mathbb{P}\end{aligned}$$

By assumption inequality (32) holds a.e. for $\omega \in \Omega$. This condition implies that $f(\mathbf{s}(\boldsymbol{\eta}),\omega,\mathbf{G})$ is monotone. In particular, for the subset $\Omega \setminus \omega$ we can write down

$$\int_{\Omega\setminus\mathcal{M}} (\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T (f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) d\mathbb{P} \geq 0. \tag{40}$$

Similarly, for $\mathcal{M}$ we get

$$\int_{\mathcal{M}} (\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T (f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) d\mathbb{P} > \int_{\mathcal{M}} \mu(\omega)\|\mathbf{s}^1(\boldsymbol{\eta})-\mathbf{s}^2(\boldsymbol{\eta})\|^2 d\mathbb{P} > 0 \tag{41}$$

where $\mu(\omega) = \bar{\kappa}(\omega) + \kappa(\omega)\lambda_{min}(\mathbf{G})$.

From Eqs. (40) and (41) we find

$$\mathbb{E}((\mathbf{s}^2(\boldsymbol{\eta})-\mathbf{s}^1(\boldsymbol{\eta}))^T (f(\mathbf{s}^2(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}) - f(\mathbf{s}^1(\boldsymbol{\eta});\boldsymbol{\theta},\mathbf{G}))) \geq \int_{\mathcal{M}} \mu(\omega)\|\mathbf{s}^1(\boldsymbol{\eta})-\mathbf{s}^2(\boldsymbol{\eta})\|^2 d\mathbb{P} > 0,$$

which contradicts Eq. (39). Therefore we must have $\mathbf{s}^1(\boldsymbol{\eta}(\omega)) = \mathbf{s}^2(\boldsymbol{\eta}(\omega))$ a.s. for $\omega \in \Omega$. □

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