AgEcon SEARCH
RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

The World's Largest Open Access Agricultural & Applied Economics Digital Library

This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.

Help ensure our sustainability.

Give to AgEcon Search

AgEcon Search
<http://ageconsearch.umn.edu>
aesearch@umn.edu

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

POWER INDICES AND PROBABILISTIC VOTING ASSUMPTIONS

by

Dennis Leech
Department of Economics
University of Warwick

No. 325

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

UNIVERSITY OF WARWICK
COVENTRY

POWER INDICES AND PROBABILISTIC VOTING ASSUMPTIONS

by

Dennis Leech
Department of Economics
University of Warwick

No. 325

August 1989

This paper is circulated for discussion purposes only and its contents should be considered preliminary.

POWER INDICES AND PROBABILISTIC VOTING ASSUMPTIONS

by
Dennis Leech

August 1989

Department of Economics
Warwick University
Coventry CV4 7AL
England

ABSTRACT:

This paper compares the theoretical bases of the Shapley-Shubik and Banzhaf indices of voting power for a legislature with weighted voting. Definitions based on probabilistic-voting assumptions, useful both as behavioral descriptions and for computation in empirical applications, are compared in terms of necessary and sufficient conditions on the choice of voting probabilities. It is shown that the Shapley-Shubik index requires stronger conditions than the Banzhaf index: the former that voting probabilities be chosen by all players from a common uniform distribution on the unit interval, the latter only that voting probabilities be selected independently from any set of distributions (on the unit interval) which have a common mean of $1/2$. This result has a bearing on the theoretical criteria by which one may choose between the two indices in a voting context.

1. INTRODUCTION

Power indices for simple games have an important role in the empirical analysis of the distribution of voting power among individual members of a voting body. The two traditional and widely used power indices are those of Shapley and Shubik (1954) and Banzhaf (1965). Both employ a definition of voting power as the relative ability of each member of the voting body to change coalitions from losing to winning; both also have simple interpretations in terms of probabilistic voting. The voting models underlying the two indices, however, whether coalitional or probabilistic, are quite different; moreover, although in some empirical applications the indices do not differ too dramatically (for example, compare the results in Owen (1975a) and (1975b); also see Leech(1988) for a stockholder application), in many important cases they give significantly different results (for examples see Straffin(1977, 1978). There is a need, therefore, to choose between them on the basis of the theoretical plausibility of their respective assumptions in the particular voting situation under analysis.

This paper is concerned with the interpretation of the two power indices in terms of probability models: it shows that the Shapley-Shubik index requires much stronger assumptions about the manner in which the relevant probabilities are generated than does the Banzhaf index.

2. POWER INDICES AND PROBABILITY MODELS

A voting body, conceived as a weighted majority game, is a special case of a simple game (Shapley, 1962) defined for a set of players $N = \{1, 2, \dots, n\}$; player i casts a number of votes (or has weight) w_i . Coalitions of players are represented by subsets of N and are dichotomized into winning coalitions and losing coalitions on the basis of a quota q (conventionally $q \geq 1/2 \sum w_i$, to rule out improper games). The

game is generally represented by the notation: $\{q; w_1, \dots, w_n\}$. A subset S corresponds to a winning coalition if it satisfies:

$$\sum_{i \in S} w_i \geq q ,$$

and is losing otherwise.

We can define a power index for each player as the relative number of times that that player, by adding his weight to a losing coalition, can change it into a winning coalition; that is, the power index is the relative number of swings for that player. A swing for player i occurs if a coalition S_i exists such that:

$$\sum_{j \in S_i} w_j < q \quad \text{and} \quad \sum_{j \in S_i} w_j + w_i \geq q .$$

The Shapley-Shubik index for player i is defined as the number of swings relative to the number of orderings of the players, $n!$; the coalition model assumed is one where players are added sequentially in the build-up of the grand coalition, N . For a given swing, S_i , the index makes allowance for all possible permutations of the players conditional on that swing, $s_i!(n-s_i-1)!$, where s_i is the number of members of the coalition S_i . Thus, the Shapley-Shubik index is a vector ϕ whose i^{th} element is:

$$(1) \quad \phi_i = \sum_{S_i} \frac{s_i! (n - s_i - 1)!}{n!} , \quad i = 1, 2, \dots, n .$$

The importance attached to each swing therefore depends on the number of members of the coalition S_i and its complement $N - S_i$.

The Banzhaf index is the number of swings normalized by the total number of swings in some sense; the model of coalition formation assumed has no regard for

orderings of the players, and therefore each swing receives equal importance. Let η_i be the number of swings for i and let $\bar{\eta} = \sum \eta_i$ be the total number of swings for all players. Two versions of the Banzhaf index employ two normalizations:

The Normalized Banzhaf index is the vector β whose i^{th} element is the number of swings for i relative to the total number of swings for all players:

$$(2) \quad \beta_i = \eta_i / \bar{\eta}, \quad i = 1, 2, \dots, n.$$

The Banzhaf Swing Probability is the vector β' whose i^{th} element is the number of swings for i relative to the total number of possible swings (the number of voting outcomes ignoring player i):

$$(3) \quad \beta'_i = \eta_i / 2^{n-1}, \quad i = 1, 2, \dots, n.$$

We can obtain the Normalized Banzhaf index from equation (3) by the relation:

$$(4) \quad \beta_i = \beta'_i / \sum \beta'_i.$$

The probabilistic nature of the normalization in equation (3) is obvious if we take all voting outcomes as being equally likely; likewise the Shapley-Shubik index has a direct probabilistic interpretation if we assume that all orderings are equiprobable.

Straffin (1977,1978) gives an alternative probabilistic basis for both the Shapley-Shubik index and the Banzhaf swing probability (hence the Normalized Banzhaf index from equation (4)), which does not depend on coalition models. This work is closely related to that of Owen (1972, 1982) on multilinear extensions of games as a means of simplifying the computation of the Shapley value. It is useful, in deriving the probabilistic representations of the indices, to begin by considering the general case before specializing to simple games.

A general n -person game has characteristic function v which associates a numerical value, $v(S)$, with each coalition $S \subseteq N$. The characteristic function satisfies conventional monotonicity conditions: $v(\emptyset) = 0$; and $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ if $i \notin S$. The multilinear extension of this game is a function $f(x)$ written:

$$(5) \quad f(x) = \sum_{S \subseteq N} \prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j) v(S) ,$$

where $x=(x_1, x_2, \dots, x_n)$ and $0 \leq x_i \leq 1$ for all i . If we regard the elements of the vector x heuristically as independent probabilities attaching to each player, indicating the likelihood of his joining a coalition, then equation (5) is the expected value of the characteristic function given x .

The marginal value for player i , given the probabilities x_j for $j \neq i$, is obtained on differentiating equation (5) with respect to x_i . Rewriting equation (5) as:

$$(6) \quad f(x) = \sum_{T; i \in T} \prod_{j \in T} x_j \prod_{j \notin T} (1 - x_j) v(T) + \sum_{S; i \notin S} \prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j) v(S) ,$$

where $S, T \subseteq N$, and differentiating equation (6) gives:

$$(7) \quad f_i(x) = \sum_{T; i \in T} \prod_{j \in T; j \neq i} x_j \prod_{j \notin T} (1 - x_j) v(T) - \sum_{S; i \notin S} \prod_{j \in S} x_j \prod_{j \notin S; j \neq i} (1 - x_j) v(S) .$$

Letting $T=S \cup \{i\}$, we can rewrite equation (7) as:

$$(8) \quad f_i(x) = \sum_{S; i \notin S} \prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j) [v(S \cup \{i\}) - v(S)] .$$

Expression (8) shows that $f_i(x)$ is the expected increment in the characteristic function by the addition of player i to a coalition, given the probability vector x .

For voting games the characteristic function is dichotomous, coalitions being either winning, with $v(S)=1$, or losing, with $v(S)=0$. In such cases the term in square brackets in expression (8) is zero except for a swing for player i , $S=S_i$.

Thus, we can write expression (8) as the probability (for given x) of a swing as:

$$(9) \quad f_i(x) = \sum_{S_i} \prod_{j \in S_i} x_j \prod_{j \notin S_i} (1 - x_j) .$$

This representation of the value of the game to player i assumes an arbitrary value for x corresponding to a particular voting issue (x_j being the probability of player j voting "aye" on a particular issue before the voting body). An index of power in an abstract sense must be independent of issues and additional assumptions made to enable the elimination of x from equation (9).

Straffin (1977) shows that we can obtain both the Shapley-Shubik index and the Banzhaf swing probability from equation (9) by treating the x_j 's as random variables and making different assumptions about how they are generated. In particular, Straffin shows that we can obtain the respective indices by making these assumptions:

Independence Assumption: The x_j 's are selected independently from the uniform distribution on the interval $[0,1]$.

Homogeneity Assumption: A number π is selected from the uniform distribution on $[0,1]$ and x_j is set equal to π for all j .

The independence assumption (which leads to the Banzhaf index) is conventionally interpreted as meaning that each voter will vote "aye" with a probability selected independently of the other voters. But the assumption of a uniform distribution means that the range of attitudes allowed for each voter is the same; each voter has the same likelihood of being committed (a value of x_j close to 0 or 1) or uncommitted (x_j close to $1/2$) on any issue. On the other hand, the homogeneity assumption (which gives the Shapley-Shubik index) is equivalent to assuming all players to have a common set of values reflected in a common probability of voting "aye," π . The assumption of a uniform distribution means that all sets of values or attitudes receive equal importance. But we demonstrate here that it is not necessary to assume that attitudes follow a uniform distribution for the

Banzhaf index and that the independence assumption above is unduly strong. These findings affect the heuristic interpretation of the probability model assumed appropriate for the use of the index.

We can derive both indices from equation (9) by building in these assumptions respectively and taking its expectation. We may write the expectation in general as:

$$(10) \quad E(f_i(x)) = \int_0^1 \int_0^1 \dots \int_0^1 f_i(x) g(x) dx_1 dx_2 \dots dx_n ,$$

where $g(x)$ is the joint density of x .

The homogeneity assumption leads to the Shapley-Shubik index because in this case $g(x)=1$ for all x , and we can write equation (10) (as a path integral):

$$(11) \quad \phi_i = E(f_i(\pi)) = \int_0^1 f_i(\pi) d\pi ,$$

where:

$$(12) \quad f_i(\pi) = \sum_{S_i} \pi^{s_i} (1 - \pi)^{n - s_i - 1} .$$

Rewriting equation (11) as:

$$(13) \quad \phi_i = \sum_{S_i} \int_0^1 \pi^{s_i} (1 - \pi)^{n - s_i - 1} d\pi ,$$

and using the Beta integral,

$$B(s_i + 1, n - s_i) = \frac{s_i! (n - s_i - 1)!}{n!} = \int_0^1 \pi^{s_i} (1 - \pi)^{n - s_i - 1} d\pi ,$$

gives the Shapley-Shubik index, equation (1). Straffin's homogeneity assumption is clearly necessary and sufficient.

Now considering the derivation of the Banzhaf index from equation (10), we show that Straffin's independence assumption, as stated above, is unduly strong. The condition that the probabilities are drawn from the uniform distribution is not necessary and that, in fact, nothing need be assumed about the form of the distribution; all that is required is the much weaker condition that the voting probabilities are selected independently from distributions with mean 1/2.

Assuming the x_j 's to be selected independently from distributions (not necessarily uniform) on $[0,1]$ with $E(x_j)=1/2$, the joint density in equation (10) becomes:

$$(14) \quad g(x) = \prod_{j \neq i} g_j(x_j) ,$$

where $g_j(x_j)$ is the marginal density for x_j .

Substituting equations (9) and (14) into equation (10) gives:

$$\begin{aligned} E(f_i(x)) &= \sum_{S_i} \int_0^1 \dots \int_0^1 \prod_{j \in S_i} x_j \prod_{j \notin S_i} (1 - x_j) \prod_{j \neq i} g_j(x_j) dx_1 \dots dx_n \\ &= \sum_{S_i} \prod_{j \in S_i} \int_0^1 x_j g_j(x_j) dx_j \prod_{j \notin S_i} \int_0^1 (1 - x_j) g_j(x_j) dx_j \\ &= \sum_{S_i} \frac{1}{2^{n-1}} = f_i(1/2) = \frac{\eta_i}{2^{n-1}} = \beta_i , \end{aligned}$$

which is the Banzhaf swing probability, equation (3).

Thus we can interpret the Banzhaf index as assuming that each player votes randomly and independently with a probability of 1/2. We have shown that this is equivalent to assuming that players select their voting probabilities randomly and

independently from distributions with mean $1/2$ without regard for the forms of those distributions. The fact that the index requires only stochastic independence and does not rely on a uniform distribution means its behavioral interpretation in terms of a probability model is much less restrictive

3. CONCLUSION

We have shown that the probabilistic-voting assumptions that Straffin proposes as the basis of the Banzhaf index are unduly strong. One need not assume both that voting probabilities be selected from a uniform distribution and that they be independent; it is sufficient that they be chosen independently from distributions that have a mean of $1/2$.

This result implies that the distributional assumption underpinning the Shapley-Shubik index is much stronger than that behind the Banzhaf index. The Shapley-Shubik index requires the specification of the whole distribution from which voting probabilities are selected -- that it be uniform -- while for the Banzhaf index only the mean need be specified.

This result has a bearing on the heuristic criterion by which one may choose between the two power indices in a particular voting situation. The Shapley-Shubik index is appropriate in situations where voting reflects a common set of values, but all possible sets of values are of equal weight (the uniform distribution of the common voting probability). The Banzhaf index is appropriate if voters act independently of each other. But it does not require that they share the same range of values; only that, on an average issue, they be as likely to vote one way as the other. For example the Banzhaf index would be the appropriate power index for a voting body whose members vote independently and which contains some persons who tend to be committed (high chance of choosing a voting probability close to zero or unity)

and some who were relatively uncommitted on most issues (voting probabilities clustered around 1/2).

REFERENCES

- Banzhaf, J. F. (1965). Weighted Voting Doesn't Work: A Mathematical Analysis. Rutgers Law Review, 19, 317-343.
- Leech, D. (1988). The Relationship between Shareholding Concentration and Shareholder Voting Power in British Companies: A Study of the Application of Power Indices for Simple Games. Management Science, 34, No.4 (April), 509-527.
- Owen, G. (1972). Multilinear Extensions of Games. Management Science, 18, No.5, Part 2, P64-P79.
- Owen, G. (1975a). Evaluation of a Presidential Election Game. American Political Science Review, 69, 947-953.
- Owen, G. (1975b). Multilinear Extensions and the Banzhaf Value. Naval Research Logistics Quarterly, 22, 741-750.
- Owen, G. (1982). Game Theory. (Second Ed.) New York: Academic Press.
- Shapley, L.S. (1962). Simple Games: An Outline of the Descriptive Theory. Behavioral Science, 7, 56-66.
- Shapley, L. S. and M. Shubik (1954). A Method for Evaluating the Distribution of Power in a Committee System. American Political Science Review, 48, 787-792.
- Straffin, P.D. (1977). Homogeneity, Independence and Power Indices. Public Choice, 30, 107-118.
- Straffin, P.D. (1978). Probability Models for Power Indices. In P.C. Ordeshook (Ed.). Game Theory and Political Science. New York: New York University Press, pp 477-510.

RECENT ISSUES

311	Progress and Providence in early Nineteenth Century Political Economy. (January 1989)	M. Berg
312	The Volume of Soviet Munitions Output, 1937-1944: a Re-evaluation. (February 1989)	M. Harrison
313	A Volume Index of the Total Munitions Output of the United Kingdom, 1939-1944. (February 1989)	M. Harrison
314	Workfare vs. Welfare: Incentive Arguments for Work Requirements in Poverty Alleviation Programs. (Revised: January 1989)	T. Besley
S. Coate
315	Imperfect Competition and the Taxation of Intermediate Goods (February, 1989)	G. D. Myles
316	Accounting for Time Use. (February 1989)	G. Pyatt
317	Exchange Rate Bands and Realignment in a Stationary Stochastic Setting. (Revised: November 1988)	M. Miller
P. Weller
318	The Stabilizing Properties of Target Zones. (May 1989)	M. Miller
P. Weller
J. Williamson
319	Total Output and the Productivity of Labour in Soviet Industry, 1940-1945.	M. Harrison
320	Soviet National Income and the Burden of Defence, 1937 and 1940-1944.	M. Harrison
321	Training and Contracts: Some Dynamic Aspects of Human Capital Formation.	G. D. Myles
322	An Inter-Establishment Study of Union Membership in Great Britain.	R. Naylor
P. Gregg
323	Union Wage Differentials, Product Market Influences and the Division of Rents.	M. Stewart
324	A Microeconomic Model of Intertemporal Substitution and Consumer Demand.	R. Blundell
M. Browning
C. Meghir
325	Power Indices and Probabilistic Voting Assumptions.	D. Leech