



AgEcon SEARCH
RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

The World's Largest Open Access Agricultural & Applied Economics Digital Library

This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.

Help ensure our sustainability.

Give to AgEcon Search

AgEcon Search
<http://ageconsearch.umn.edu>
aesearch@umn.edu

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

MONASH

WP 8/96

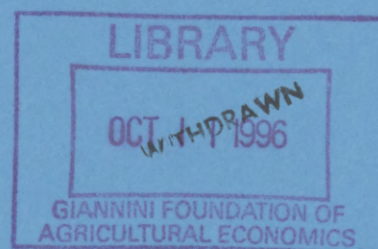
ISSN 1032-3813

ISBN 0 7326 0789 2

MONASH UNIVERSITY



AUSTRALIA



**A COMPARISON OF THE ACCURACY OF ASYMPTOTIC
APPROXIMATIONS IN THE DYNAMIC REGRESSION MODEL USING
KULLBACK-LEIBLER INFORMATION**

Ranjani Atukorala and Maxwell L. King

**Working Paper 8/96
August 1996**

DEPARTMENT OF ECONOMETRICS

**A COMPARISON OF THE ACCURACY OF ASYMPTOTIC
APPROXIMATIONS IN THE DYNAMIC REGRESSION MODEL USING
KULLBACK-LEIBLER INFORMATION**

by

Ranjani Atukorala and Maxwell L. King

**Department of Econometrics
Monash University
Clayton, Victoria 3168
Australia**

Key Words and Phrases: large sample asymptotics; Monte Carlo simulations;
small disturbance asymptotics; t test.

Abstract

This paper illustrates the use of the Kullback-Leibler Information (KLI) measure for assessing the relative quality of two approximations to an unknown distribution from which we can obtain simple random drawings. The illustration involves comparing the large-sample and small-disturbance asymptotic distributions under the null hypothesis of a t statistic from the dynamic linear regression model. We find very clear evidence in favour of the use of p-values and critical values from the small-disturbance Student's t distribution rather than from the large-sample standard normal distribution in this case.

1. Introduction

Because of the complexity of models econometricians are required to work with, they often use test statistics whose exact distributions under the null hypothesis are unknown. They typically use asymptotic approximations in the hope that these will provide reasonably accurate critical values or p-values. Sometimes this approach works well while in other circumstances it can be extremely poor. For an example in this latter category, see Table 1 of King and McAleer (1987).

There are also different kinds of asymptotic approximations. The most common is first-order large-sample asymptotics which involves finding the distribution of the statistic in question (suitably normalized) in the limit as the sample size goes to infinity. Then there are second-order or higher-order asymptotics which attempt to improve on the quality of first-order asymptotics. This typically involves finding a power series expansion (usually involving powers of $n^{-1/2}$ where n is the sample size) of either the distribution function or the density function of the statistic and then constructing an approximation using more than the first term of this expansion; see for example Rothenberg (1984). An alternative approach is small-disturbance asymptotics which involves using the distribution of the test statistic in the limit as the disturbance variance goes to zero. This has received considerably less attention in the literature than large sample asymptotics. Its use has been suggested and investigated by Kadane (1974) and Morimune and Tsukuda (1984) in the context of testing linear simultaneous equation models and by Inder (1986, 1990), Nankervis and Savin (1987), King and Wu (1991) and King and Harris (1995) with respect to tests of the dynamic linear regression model.

Typically different approaches yield different approximations and it is not clear which approach is best in any particular setting. While it might seem that second-order approximations are better than first-order approximations, this is not always the case. If the higher-order terms in the expansion involve unknown parameters, then approximating these with estimates can sometimes worsen the approximation error rather than improve it. There is also the added worry of stochastic adjustments to test statistics badly affecting the power of the resultant test. In view of these considerations, it is highly desirable to have a simple method for comparing the quality of two or more approximations to an unknown distribution.

The aim of this paper is to advance the case for using the Kullback-Leibler Information (KLI) measure for assessing the relative quality of two approximations. To the best of our knowledge, this idea was first suggested by White (1994) but has yet to be applied to compare two asymptotic approximations. We illustrate its use with a comparison of two approximations in the context of the t test of the lagged dependent variable coefficient in the dynamic linear regression model. The large-sample distribution under the null hypothesis is standard normal while its small-disturbance asymptotic distribution is Student's t , as shown by Nankervis and Savin (1987). Through the approach illustrated in this paper, we are able to conclude that the small-disturbance approximation is typically more accurate in this case although, as the sample size increases, the two approximations converge. This finding adds further to the growing evidence that suggests small-disturbance asymptotics deserve closer attention than they are currently getting from the econometrics profession.

The organisation of the remainder of the paper is as follows. The theory behind the use of the KLI measure for assessing the relative quality of two approximating distributions is outlined in section 2. The dynamic linear regression model and the t-test of the coefficient of the lagged dependent variable are introduced in section 3. Section 4 provides the details of the Monte Carlo simulations and the results of these experiments are reported in section 5. Some concluding remarks are made in section 6.

2. Theory

In order to evaluate the quality of an approximating distribution, we need a convenient measure of distance or difference or divergence between distributions. One such measure is the KLI measure introduced by Kullback and Leibler (1951). Let $g(x)$ be the true density function of a $q \times 1$ random vector x and $f(x)$ be an approximating density for x . The KLI measure is defined as

$$\begin{aligned} I(g; f) &= E[\log\{g(x)/f(x)\}] \\ &= \int_{R^q} \log\{g(x)/f(x)\} g(x) dx. \end{aligned}$$

Its usefulness as a measure of the quality of approximation comes from the following properties:

- (i) $I(g; f) \geq 0$ for all g and f .
- (ii) $I(g; f) = 0$ if and only if $g(x) = f(x)$ almost everywhere.

As observed by Renyi (1961, 1970), the KLI measure can be interpreted as the surprise experienced on average when we believe $f(x)$ is the true underlying distribution and we are told it is in fact $g(x)$. The smaller the value of $I(g;f)$, the less the surprise and the closer we consider the approximating distribution $f(x)$ to be to the true distribution $g(x)$. Also note that $I(g;f)$ is the expected value of the log of the likelihood ratio which, according to the Neyman-Pearson Lemma, provides the best test of $H_0: x \sim g(x)$ against $H_1: x \sim f(x)$. In fact if x_1, \dots, x_m was a simple random sample of size m from either H_0 or H_1 , then the most powerful test can be based on rejecting H_0 for small values of

$$\frac{1}{m} \sum_{i=1}^m \log\{g(x_i)/f(x_i)\}$$

which, when $g(x)$ is the true distribution, is the standard estimate of

$$I(g;f) = E\{\log(g(x)/f(x))\}$$

from a simple random sample of size m . In this sense we feel confident in using $I(g;f)$ as a measure of distance between $g(x)$ and $f(x)$. For further discussion of the KLI measure, see Kullback (1959), Renyi (1961, 1970), Maasoumi (1993) and White (1994). For more on other measures of distance or divergence between distributions see Maasoumi (1993).

As White (1994) observed, a comparison of the adequacy of two approximating densities of $g(x)$, namely $f_1(x)$ and $f_2(x)$, can be made using the difference in KLI measures. This difference is

$$D(f_1, f_2) = I(g; f_1) - I(g; f_2)$$

$$\begin{aligned} &= \int_{R^q} \log\{g(x)/f_1(x)\}g(x)dx - \int_{R^q} \log\{g(x)/f_2(x)\}g(x)dx \\ &= \int_{R^q} \log\{f_2(x)/f_1(x)\}g(x)dx. \end{aligned}$$

Obviously, if this difference is positive, f_2 is the better approximation, while if it is negative, f_1 is better. Unfortunately g is unknown but, typically through Monte Carlo methodology, we are able to obtain a simple random sample of m observations from g . If these observations are denoted by x_1, x_2, \dots, x_m then $D(f_1, f_2)$ can be estimated by

$$\begin{aligned} d(f_1, f_2) &= \frac{1}{m} \sum_{i=1}^m \log\{f_2(x_i)/f_1(x_i)\} \\ &= \frac{1}{m} \sum_{i=1}^m \log f_2(x_i) - \frac{1}{m} \sum_{i=1}^m \log f_1(x_i). \end{aligned} \quad (1)$$

An estimate of the standard error of this estimator is given by $\hat{\sigma} / m$ where

$$\hat{\sigma}^2 = \frac{1}{m-1} \sum_{i=1}^m [\log\{f_2(x_i)/f_1(x_i)\} - d(f_1, f_2)]^2.$$

3. The Dynamic Linear Regression Model

Our interest is in the linear dynamic regression model

$$y_t = \alpha y_{t-1} + x_t' \beta + u_t, \quad t = 1, \dots, n, \quad (2)$$

where y_t is the dependent variable, x_t is a $k \times 1$ vector of exogenous variables, α and β are an unknown parameter scalar and $k \times 1$ vector respectively and u_t is the

is required to complete the model. Following Nankervis and Savin (1987) and others, we assume $y_0 = 0$ with no loss of generality in the fixed start-up model when an intercept is present. If there are n observations available on each variable then the parameters are estimated using the last $n-1$ observations. The model for these observations can be written as

$$y = \alpha y_{-1} + X\beta + u$$

where y , y_{-1} and u are $(n-1) \times 1$ vectors and X is an $(n-1) \times k$ matrix which we will assume has full-column rank.

The ordinary least squares estimators of α and β are

$$\hat{\alpha} = y'_{-1} My / y'_{-1} My_{-1}$$

and

$$\hat{\beta} = (X'X)^{-1} X'(y - \hat{\alpha}y_{-1}),$$

respectively, where $M = I_{n-1} - X(X'X)^{-1}X'$. The associated estimator of σ^2 , denoted s^2 , is

$$\begin{aligned} s^2 &= (y - \hat{\alpha}y_{-1} - X\hat{\beta})'(y - \hat{\alpha}y_{-1} - X\hat{\beta}) / (n - k - 2) \\ &= (y - \hat{\alpha}y_{-1})' M (y - \hat{\alpha}y_{-1}) / (n - k - 2). \end{aligned}$$

In the context of this model, we are interested in the problem of testing $H_0: \alpha = \alpha^*$ using the t test whose test statistic is given by

$$t_\alpha = (\hat{\alpha} - \alpha^*) (y'_{-1} My_{-1})^{1/2} / s.$$

It is well known that under H_0 , t_α has a standard normal large sample asymptotic distribution under the following regularity conditions (Schmidt (1976, p.97)):

(i) The elements of X are uniformly bounded in absolute value as $n \rightarrow \infty$;

(ii)
$$\lim_{n \rightarrow \infty} \frac{1}{n-i-1} \sum_{t=2}^{n-i} x_t' x_{t+i}$$

exists for any integer i and is nonsingular for $i = 0$;

(iii)
$$|\alpha^*| < 1.$$

Observe that these regularity conditions impose important restrictions on both the regressors and α^* . For example, trending regressors and $\alpha^* = 1$ are excluded. In the latter case, the large sample asymptotic distribution of t_α depends on the regressors and the unknown parameters. In some cases it is $N(0,1)$. On the other hand, Dickey and Fuller (1979) derived it for $\beta = 0$ when x_t is just the intercept term and when x_t comprises the intercept and linear trend and found in both cases it is not $N(0,1)$.

Alternatively, Nankervis and Savin (1987) (also see King and Wu, 1991) have shown that under H_0 , t_α has a Student's t distribution with $n-k-2$ degrees of freedom as σ^2 tends to zero. The only restrictions on X are that it be nonstochastic (or that the inference be conditional on X) and that $ML(I_{n-1} - \alpha^* L)^{-1} \neq 0$ where L is the $(n-1) \times (n-1)$ matrix

$$L = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & 0 & & & & & 0 & 0 \\ 0 & 1 & & & & & 0 & 0 \\ \cdot & & \cdot & & & & & \cdot \\ \cdot & & & \cdot & & & & \cdot \\ \cdot & & & & \cdot & & & \cdot \\ \cdot & & & & & \cdot & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 \end{bmatrix}.$$

There is no restriction on α^* . Notice that in our testing problem, σ^2 is a nuisance parameter. Because the small disturbance distribution is the limit of a sequence of null distributions for different values of this nuisance parameter, it will work well for testing problems with very little non-similarity. In our case, the test becomes more similar under Schmidt's regularity conditions as $n \rightarrow \infty$, but for some regressors we know it can be very non-similar when $\alpha^* = 1$.

The question is, which of these two approximating asymptotic distributions should be used to calculate p-values and critical values for t_α ?

4. The Monte Carlo Experiment

In order to answer this question, we conducted a Monte Carlo experiment with the aim of estimating $D(f_1, f_2)$ using (1) for a range of situations based on (2) where f_1 is the standard normal density function and f_2 is the density of the Student's t distribution with $n - k - 2$ degrees of freedom. That is

$$f_1(x) = (2\pi)^{-1/2} \exp(-x^2/2)$$

and

$$f_2(x) = \Gamma\{(v+1)/2\} / \left[(\pi v)^{1/2} \Gamma(v/2) \{1 + (x^2/v)\}^{(v+1)/2} \right]$$

where $\Gamma(\cdot)$ is the gamma function and $v = n - k - 2$. The following sets of exogenous variables were used in the Monte Carlo experiment.

X1 : ($k = 3$). The three exogenous regressors are a constant dummy, Australian private capital movements and Australian Government capital movements commencing 1968(1).

X2 : ($k = 5$). The five exogenous regressors are a constant dummy, quarterly Australian household disposable income commencing 1959(4), quarterly final consumption expenditure commencing 1959(4) and these latter two variables lagged one quarter.

X3 : ($k = 5$). The five exogenous regressors are a constant dummy and four independent series each generated as independent samples from the uniform distribution with range $[0,1]$.

These three X matrices provide a range of different regressor types with the capital movement series in X1 showing high variability and strong seasonality and the X2 variables being more typical of macroeconomic data.

Because our interest is in how the approximations perform as σ and n vary, we calculated $d(f_1, f_2)$ using $m = 20,000$ for all combinations of $\sigma = 0.05, 0.1, 0.5, 1, 10, 50, 500$ and 5000 ; and $n = 10, 20, 30, 40, 50$ and 60 . For X2 and X3, we also included calculations for $n = 70, 80$ and 90 . All calculations were performed for $\alpha^* = 0, 0.25, 0.5, 0.75, 0.95$ and 1 . Throughout, all β values were set to one.

5. Results

Selected estimated values of $D(f_1, f_2)$ (denoted d) and their standard errors (s.e.) are presented in Tables 1 - 3. Note that significant positive values indicate that the Student's t distribution provides the better approximation to the true null distribution of the t statistic, while significant negative values suggest the standard normal distribution is better.

The results very clearly indicate that the Student's t distribution better approximates the true distribution than does the standard normal distribution. In the majority of cases, the positive $d(f_1, f_2)$ value is significantly different from zero. At worst, some $d(f_1, f_2)$ values suggest the two approximating distributions are equally as good, which might be expected because the two distributions converge in large samples. Of the total of 1152 $d(f_1, f_2)$ values calculated in our study, only 20 were negative. Of these 16 were within one standard error of zero, 3 were within two standard errors and the remaining negative value was 2.5 standard errors from zero. The majority of these negative values occurred for the artificially generated X3 data set when $\alpha^* = 0$.

As expected, because of the convergence of the two approximating distributions, the $d(f_1, f_2)$ values almost always decrease as n increases, *ceteris paribus*. We also see a tendency for $d(f_1, f_2)$ to increase as α^* increases, particularly for large σ values. One might expect $d(f_1, f_2)$ to decrease as σ increases, *ceteris paribus*. However, the actual pattern is more complicated than this. For small α^* values we do indeed see $d(f_1, f_2)$ decline as σ increases, although for

X2 and X3 it declines and then increases. As n increases, and particularly as α^* increases, there is a greater tendency for $d(f_1, f_2)$ to increase as σ increases. The largest differences in favour of the Student's t distribution occur at the largest σ values, the smallest n values and $\alpha^* = 1.0$. Thus, somewhat ironically, we observe that the small-disturbance asymptotic approximation typically improves its performance relative to the large-sample approximation as the disturbance variance increases.

6. Concluding Remarks

Our results show very clearly that the small-disturbance Student's t distribution does a better job of approximating the true null distribution than does the large-sample standard normal distribution. As expected, the largest differences are for small sample sizes. Somewhat unexpectedly we have also found some large differences for large σ values, particularly when α^* is around 1.0.

Our method does not enable us to gauge how close each approximating distribution is to the true null distribution. In order to calculate the KLI measure for each of these approximations, we need the density function of the true distribution. We are currently experimenting with non-parametric density estimation methods to solve this problem and expect to report on this work in a future paper.

References

- Dickey, D.A. and W.A. Fuller, (1979), Distribution of the estimators for autoregressive time series with a unit root, *Journal of the American Statistical Association* 74, 427-431.
- Inder, B.A., (1986), An approximation to the null distribution of the Durbin-Watson statistic in models containing lagged dependent variables, *Econometric Theory* 2, 413-428.
- Inder, B.A., (1990), A new test for autocorrelation in the disturbances of the dynamic linear regression model, *International Economic Review* 34, 341-354.
- Kadane, J.B., (1974), Testing a subset of the overidentifying restrictions, *Econometrica* 42, 853-867.
- King, M.L. and D.C. Harris, (1995), The application of the Durbin-Watson test to the dynamic regression model under normal and non-normal errors, *Econometric Reviews* 14, 487-510.
- King, M.L. and M. McAleer, (1987), Further results on testing AR(1) against MA(1) disturbances in the linear regression model, *Review of Economic Studies* 54, 649-663.
- King, M.L. and P.X. Wu, (1991), Small-disturbance asymptotics and the Durbin-Watson and related tests in the dynamic regression model, *Journal of Econometrics* 47, 145-152.
- Kullback, S., (1959), *Information Theory and Statistics*. New York: John Wiley and Sons.

- Kullback, S. and R.A. Leibler, (1951), On information and sufficiency, *Annals of Mathematical Statistics* 22, 79-86.
- Maasoumi, E., (1993), A compendium to information theory in economics and econometrics, *Econometric Reviews* 12, 137-181.
- Morimune, K. and Y. Tsukuda, (1984), Testing a subset of coefficients in a structural equation, *Econometrica* 52, 427-448.
- Nankervis, J.C. and N.E. Savin, (1987), Finite sample distributions of t and F statistics in an AR(1) model with an exogenous variable, *Econometric Theory* 3, 387-408.
- Renyi, A., (1961), On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium in Mathematical Statistics*. Berkeley: University of California Press.
- Renyi, A. (1970), *Probability Theory*. Amsterdam: North-Holland.
- Rothenberg, T., (1984), Approximating the distributions of econometric estimators and test statistics. In Griliches, Z. and Intriligator, M.D. (eds.) *Handbook of Econometrics* 2. Amsterdam: North-Holland.
- Schmidt, P., (1976), *Econometrics*. New York: Marcel Dekker.
- White, H. (1994), *Estimation, Inference and Specification Analysis*. Cambridge: Cambridge University Press.

Table 1: Estimated differences in KLI between standard normal and Student's *t* approximations (*d*) and associated standard errors (s.e.) for X_1 .*

α^*	<i>n</i>		σ					
			.05	.5	10	50	500	5000
0	10	<i>d</i>	122.37	126.65	112.57	119.88	78.16	61.69
		s.e.	9.96	10.52	7.69	8.47	12.01	5.95
	20	<i>d</i>	8.88	8.81	10.59	8.21	1.28	.33
		s.e.	1.24	1.16	1.31	1.13	.88	.82
	30	<i>d</i>	2.26	3.93	2.59	2.49	1.67	.50
		s.e.	.60	.66	.59	.56	.58	.55
	40	<i>d</i>	1.20	1.71	1.04	1.85	1.07	1.05
		s.e.	.44	.46	.39	.44	.40	.42
	60	<i>d</i>	.35	.60	.76	.27	.36	.90
		s.e.	.25	.27	.30	.24	.25	.25
.5	10	<i>d</i>	110.89	108.70	130.31	133.40	116.68	118.25
		s.e.	7.45	8.89	11.01	10.54	7.35	11.81
	20	<i>d</i>	7.45	8.15	10.22	11.94	13.80	11.60
		s.e.	1.02	1.12	1.20	1.28	1.25	1.12
	30	<i>d</i>	3.17	2.37	3.27	3.36	5.92	6.76
		s.e.	.61	.56	.65	.68	.65	.71
	40	<i>d</i>	1.76	2.01	1.69	1.90	4.62	5.88
		s.e.	.45	.45	.38	.46	.49	.53
	60	<i>d</i>	.65	.52	.70	.86	1.11	4.14
		s.e.	.24	.26	.26	.26	.29	.35
.95	10	<i>d</i>	115.18	120.54	119.67	131.81	275.79	326.58
		s.e.	7.96	7.91	7.78	10.57	12.63	12.51
	20	<i>d</i>	8.09	7.40	7.41	9.78	38.77	93.13
		s.e.	1.11	1.03	1.03	1.26	1.79	2.35
	30	<i>d</i>	4.24	2.61	2.62	4.18	28.61	55.19
		s.e.	.68	.64	.64	.79	1.32	1.47
	40	<i>d</i>	1.17	1.48	1.50	2.10	20.09	40.64
		s.e.	.39	.46	.47	.59	.85	1.05
	60	<i>d</i>	.40	1.17	1.17	.76	2.83	27.36
		s.e.	.25	.29	.29	.26	.34	.68
1.0	10	<i>d</i>	124.14	119.44	128.65	115.22	263.17	404.26
		s.e.	9.13	8.81	8.79	7.73	11.09	20.88
	20	<i>d</i>	9.74	8.83	7.22	9.31	26.86	127.21
		s.e.	1.15	1.10	1.03	1.29	1.60	2.89
	30	<i>d</i>	3.05	2.62	3.59	2.91	19.50	86.70
		s.e.	.60	.56	.65	.73	1.02	1.87
	40	<i>d</i>	1.40	1.49	1.39	1.41	13.36	74.72
		s.e.	.41	.42	.41	.45	.68	1.43
	60	<i>d</i>	.90	.44	.35	.93	1.23	48.62
		s.e.	.26	.25	.25	.27	.25	.90

* All values have been multiplied by 10^3 .

Table 2: Estimated differences in KLI between standard normal and Student's *t* approximations (*d*) and associated standard errors (s.e.) for X^2_* .

α^*	<i>n</i>		σ					
			.05	.5	10	50	500	5000
0	10	<i>d</i>	627.54	433.56	332.84	339.99	398.25	396.06
		s.e.	69.67	56.38	26.26	35.29	49.28	47.67
	20	<i>d</i>	13.47	17.37	15.29	15.69	15.59	16.98
		s.e.	1.25	1.57	1.33	1.37	1.39	1.49
	40	<i>d</i>	3.43	4.57	3.76	6.06	3.95	4.46
		s.e.	.46	.47	.46	.65	.49	.47
	60	<i>d</i>	.92	1.48	1.75	1.51	1.42	1.49
		s.e.	.27	.30	.28	.29	.28	.30
	90	<i>d</i>	.50	.25	.47	.03	.36	.24
		s.e.	.18	.15	.17	.15	.16	.15
.5	10	<i>d</i>	503.59	539.75	768.02	763.95	707.04	675.52
		s.e.	33.76	44.98	61.08	57.90	52.41	45.21
	20	<i>d</i>	11.74	11.91	48.14	47.33	48.65	48.18
		s.e.	1.5	1.14	2.19	1.99	2.13	2.09
	40	<i>d</i>	1.95	3.10	16.45	17.42	15.76	18.55
		s.e.	.41	.46	.74	.79	.69	.88
	60	<i>d</i>	.49	1.86	7.41	7.80	8.04	7.32
		s.e.	.26	.29	.42	.44	.45	.42
	90	<i>d</i>	.61	.57	3.22	3.22	3.00	2.97
		s.e.	.19	.17	.24	.24	.24	.22
.95	10	<i>d</i>	549.17	513.50	872.21	1631.50	1675.92	1604.86
		s.e.	57.40	42.54	53.42	87.64	79.86	75.97
	20	<i>d</i>	12.97	16.12	124.04	231.17	238.23	241.28
		s.e.	1.25	1.49	3.38	4.47	4.75	4.74
	40	<i>d</i>	1.37	1.59	47.71	106.10	108.97	110.38
		s.e.	.41	.39	1.29	1.88	1.95	1.88
	60	<i>d</i>	1.11	.72	26.39	76.26	83.21	82.75
		s.e.	.28	.27	.77	1.23	1.27	1.27
	90	<i>d</i>	.54	.08	12.03	43.26	45.63	46.01
		s.e.	.17	.15	.40	.74	.75	.76
1.0	10	<i>d</i>	539.00	549.79	884.61	1695.82	1694.37	1801.58
		s.e.	39.10	45.28	60.50	79.03	78.44	94.65
	20	<i>d</i>	10.43	13.36	141.74	298.83	314.11	320.50
		s.e.	1.14	1.47	3.80	5.28	5.59	5.92
	40	<i>d</i>	1.23	1.73	85.93	172.26	179.37	174.44
		s.e.	.40	.43	1.70	2.34	2.47	2.39
	60	<i>d</i>	.77	.97	19.68	126.62	151.64	153.57
		s.e.	.27	.29	.59	1.51	1.74	1.74
	90	<i>d</i>	.13	-.07	4.26	72.19	101.36	103.84
		s.e.	.15	.15	.24	1.02	1.20	1.20

* All values have been multiplied by 10^3 .

Table 3: Estimated differences in KLI between standard normal and Student's *t* approximations (*d*) and associated standard errors (s.e.) for X3.*

α^*	n		σ					
			.05	.5	10	50	500	5000
0	10	d	506.23	373.89	457.71	407.70	453.74	420.58
		s.e.	52.53	34.62	46.91	33.51	49.45	40.18
	20	d	11.14	8.63	4.74	1.14	2.81	1.91
		s.e.	1.28	1.40	1.03	.90	.95	.96
	40	d	2.20	.68	-.15	.84	.26	.72
		s.e.	.49	.41	.36	.42	.36	.43
	60	d	.58	.37	.13	-.14	-.23	.21
		s.e.	.26	.25	.24	.23	.23	.25
	90	d	.45	.06	-.01	.04	-.08	.11
		s.e.	.16	.16	.16	.15	.15	.15
.5	10	d	471.17	436.58	334.10	322.65	294.46	680.90
		s.e.	31.94	46.33	21.44	32.72	28.73	41.00
	20	d	14.40	16.81	23.20	19.19	.73	17.34
		s.e.	1.73	1.48	1.77	1.34	.87	1.58
	40	d	2.05	2.30	1.76	3.10	1.52	3.21
		s.e.	.47	.47	.44	.44	.49	.46
	60	d	.72	.46	.97	1.46	.65	.69
		s.e.	.25	.26	.26	.28	.27	.25
	90	d	.50	.42	.37	.91	.39	.63
		s.e.	.17	.18	.17	.18	.17	.18
.95	10	d	508.87	537.47	1635.58	624.99	692.65	415.17
		s.e.	42.22	56.78	85.85	51.96	42.68	41.98
	20	d	14.33	12.92	54.27	82.42	128.98	81.90
		s.e.	1.43	1.51	2.30	2.40	3.74	2.61
	40	d	1.77	1.98	26.87	33.01	29.05	23.76
		s.e.	.41	.41	1.05	1.02	.91	.85
	60	d	.72	.64	11.14	14.03	14.95	14.20
		s.e.	.28	.25	.45	.49	.51	.52
	90	d	.22	.45	5.58	8.05	7.18	6.84
		s.e.	.17	.16	.26	.31	.30	.30
1.0	10	d	437.44	501.11	1225.66	759.49	535.78	1525.24
		s.e.	28.63	36.80	72.19	45.80	38.00	86.35
	20	d	12.34	12.30	59.48	163.26	118.03	105.42
		s.e.	1.26	1.54	2.25	4.04	3.22	2.66
	40	d	1.12	1.04	24.46	53.19	49.14	62.11
		s.e.	.40	.39	.89	1.17	1.10	1.34
	60	d	.47	.45	11.52	35.43	35.78	39.78
		s.e.	.28	.24	.47	.75	.75	.82
	90	d	.09	.20	5.78	23.03	24.10	22.86
		s.e.	.14	.15	.33	.49	.47	.46

* All values have been multiplied by 10^3 .

