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Saddlepoint Approximation for the Least Squares Estimator in First-Order Autoregression

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SUMMARY

In an earlier article, Phillips (1978) extended Daniels' (1956) approximation to the density of a modified correlation coefficient to obtain a saddlepoint approximation to the density of the least squares estimator in the first order non-circular autoregression. It was demonstrated that the resulting approximation is undefined in a substantial part of the tails. In this note, we establish that a suitable deformation of the contour of integration leads to a different type of saddlepoint approximation which is defined everywhere on the support of the density. It is further shown that the relative error of this approximation is bounded in the extreme tails.

Some Key Words: Autoregression; Branch point; Quadratic forms; Saddlepoint approximation.

*Some of the work towards the completion of this paper was carried out while the author was a PhD. student in the Department of Econometrics, Monash University.

1. INTRODUCTION

The model

$$y_t = \alpha y_{t-1} + u_t, \quad t = -1, 0, 1, \ldots, \text{ with } u_t \sim N(0, \sigma^2),$$

has been the subject of numerous studies in time series regression. Phillips (1978) examined the performances of the Edgeworth and saddlepoint approximations to the density of the least squares estimator of α,

$$\hat{\alpha} = \sum_{t=1}^{T} y_t y_{t-1} \Big/ \sum_{t=1}^{T} y_{t-1}^2$$

$$= y' C_1 y / y' C_2 y,$$

where $y' = (y_0, \ldots, y_T)$ and C_1 and C_2 are given by Phillips (1978) equation (6). It was established, both analytically and empirically, that Daniels' (1956) saddlepoint approximation to the density of $\hat{\alpha}$ is unavailable in a substantial part of the tails. The failure of the approximation is attributed to the existence of a branch point of the integrand, such that it is not possible to deform the contour of integration to pass through a suitable saddlepoint. The problem apparently occurs because a term in the determinant appearing in the joint mgf of $y' C_1 y$ and $y' C_2 y$ has been neglected prior to inversion. Although the error incurred on the final approximation is exponentially small, the approximation breaks down in the tails. See Daniels (1956) and Phillips (1978). In Section 2 we use a result by Lieberman (1994) to provide an alternative saddlepoint approximation which is available over the entire range of $\hat{\alpha}$. The relative error of this approximation is bounded in the extreme tails. A numerical evaluation carried out in section 3 confirms that the modification results in good overall performance, especially in the extreme tails of the distribution. All proofs are contained in the appendix.

2. APPROXIMATE DENSITY AND TAIL BEHAVIOUR

To establish an approximation which is reliable in the tails, we write $\hat{\alpha}$ as

$$\hat{\alpha} = \frac{v' R_\alpha' C_1 R_\alpha v}{v' R_\alpha' C_2 R_\alpha v}, \quad v \sim N(0, \sigma^2),$$

where

$$R_\alpha = \begin{bmatrix} b & 0 \dots & & 0 \\ \alpha b & 1 & 0 \dots & 0 \\ \alpha^2 b & \alpha & 1 \; 0 \dots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha^T b & \alpha^{T-1} & \dots \alpha & 1 \end{bmatrix}, \quad b = \begin{array}{ll} (1-\alpha^2)^{-\frac{1}{2}} & \text{if } \alpha \in (-1,1) \\ 0 & \text{otherwise,} \end{array}$$

and note that $\hat{\alpha}$ is homogeneous of degree 0 in v. As $\hat{\alpha}$ is a ratio of quadratic forms in normal variables, it follows from the appendix that a saddlepoint approximation for the density of $\hat{\alpha}$ is

$$\hat{f}(\hat{\alpha}) = \frac{\text{tr}[(I-2\hat{\omega}D)^{-1} R_\alpha' C_2 R_\alpha] \exp\left\{-\frac{1}{2}\sum_{t=0}^{T} \log(1-2\hat{\omega}d_t)\right\}}{\left\{4\pi \sum_{t=0}^{T} [d_t^2/(1-2\hat{\omega}d_t)^2]\right\}^{\frac{1}{2}}}, \qquad (1)$$

with $d_0 \geq d_1 \geq \dots \geq d_T$, being the ordered eigenvalues of $D = D(\hat{\alpha}) = R_\alpha' (C_1 - \hat{\alpha} C_2) R_\alpha$ and $\hat{\omega}$ satisfying

$$\sum_{t=0}^{T} \frac{d_t}{1-2\hat{\omega}d_t} = 0. \qquad (2)$$

We see that $\text{pr}(\hat{\alpha} < x) = \text{pr}(v' D(x) v < 0)$ is non-trivial, if and only if $d_0 > 0$ and $d_T < 0$. A suitable solution for (2) for which (1) is always defined, is clearly a saddlepoint satisfying $\hat{\omega} \in (1/2d_T, 1/2d_0)$. It follows then that the approximation (1) must be defined for all $\hat{\alpha}$ on the support of $f(\hat{\alpha})$.

The approximation (1) is substantially different from those provided by Phillips (1978) and Daniels (1956), since an exact joint mgf has been employed in its derivation. The drawback with the current methodology is in the difficulty in establishing the order of the error analytically over the entire range of $\hat{\alpha}$. It is shown in the appendix, though, that when $|\hat{\alpha}|$ is sufficiently large, $f(\hat{\alpha}) = \hat{f}(\hat{\alpha})\{1+O(T^{-1})\}$. The chief advantage is of course, having a reliable approximation which is guaranteed to deliver an answer, even in the extreme tails region.

For computational purposes, it may be more convenient to rewrite (1) and (2) equivalently as

$$\hat{f}(\hat{\alpha}) = \frac{[\mathrm{tr}(\hat{A}^{-1}R_\alpha{}' C_2 R_\alpha)]\left|\hat{A}\right|^{-\frac{1}{2}}}{\left\{4\pi\,\mathrm{tr}[(\hat{A}^{-1}D)^2]\right\}^{\frac{1}{2}}}, \tag{3}$$

where $\hat{A} = A(\hat{\omega}) = I - 2\hat{\omega}D$ and $\hat{\omega}$ satisfies

$$\mathrm{tr}(\hat{A}^{-1}D) = 0. \tag{4}$$

Although the eigenvalues of D are not explicitly involved in (3) and (4), it is necessary to constrain a line search in solving (4) so that $\hat{\omega}$ belongs to the interval $(1/2d_T, 1/2d_0)$. The expressions for $\hat{f}(\hat{\alpha})$ appearing in (1) and (3) are only the leading term of a saddlepoint approximation. For additional accuracy, the expansion is

$$f(\hat{\alpha}) = \hat{f}(\hat{\alpha})\left\{1 - \frac{2\mathrm{tr}[(\hat{A}^{-1}D)^2\hat{A}^{-1}R_\alpha{}' C_2 R_\alpha]}{\mathrm{tr}(\hat{A}^{-1}R_\alpha{}' C_2 R_\alpha)\mathrm{tr}[(\hat{A}^{-1}D)^2]} + \frac{3\mathrm{tr}[(\hat{A}^{-1}D)^4]}{2[\mathrm{tr}(\hat{A}^{-1}D)^2]^2}\right.$$

$$\left. + \frac{2\,\mathrm{tr}(\hat{A}^{-1}D\hat{A}^{-1}R_\alpha{}' C_2 R_\alpha)\,\mathrm{tr}[(\hat{A}^{-1}D)^3]}{\mathrm{tr}(\hat{A}^{-1}R_\alpha{}' C_2 R_\alpha)[\mathrm{tr}[(\hat{A}^{-1}D)^2]]^2} - \frac{10[\mathrm{tr}[(\hat{A}^{-1}D)^3]]^2}{3[\mathrm{tr}[(\hat{A}^{-1}D)^2]]^3} + \ldots\right\}. \tag{5}$$

We establish in the appendix that when $|\hat{\alpha}|$ is large, the error of neglecting further terms in the expansion (5) is $O(T^{-2})$.

3. NUMERICAL COMPARISONS

While we can readily obtain the missing values in Phillips' (1978) Table 1 by numerically integrating (3) or (5), it is much easier to do so by an application of the Lugannani-Rice (1980) formula. From Lieberman (1994), we know that

$$\Pr(\hat{\alpha} > x) \cong 1 - \Phi(\hat{\xi}) + \phi(\hat{\xi})\left(\frac{1}{\hat{z}} - \frac{1}{\hat{\xi}}\right), \tag{6}$$

where

$$\hat{z} = \hat{\omega}[2\,\mathrm{tr}\{(\hat{A}^{-1}D)^2\}]^{\frac{1}{2}},$$

$$\hat{\xi} = (\log|\hat{A}|)^{\frac{1}{2}}\,\mathrm{sgn}(\hat{\omega}),$$

$D = D(x)$, Φ and ϕ are the standard normal cdf and pdf respectively and $\hat{\omega}$ is as defined by (2) or (4). $\text{Sgn}(\hat{\omega})$ is negative, 0, positive when $v'Dv - \text{tr}(D)$, is negative, 0, positive, respectively. For $v'Dv = \text{tr}(D)$, the Lugannani-Rice formula reduces to

$$\text{pr}(\hat{\alpha} > x) = \tfrac{1}{2} - \text{tr}(D^3) / 3\{\text{tr}(D^2)\}^{\frac{3}{2}} \pi^{\frac{1}{2}}.$$

As with the density approximation, the saddlepoint used in (6) is real and unique, giving a non-trivial tail area approximation over the entire range of $\hat{\alpha}$.

As an illustration, we use (6) to compute

$$\Pr\left\{\left|T^{\frac{1}{2}}(\hat{\alpha} - \alpha) / \left(1 - \alpha^2\right)^{\frac{1}{2}}\right| > x\right\}$$

in the interval $2 \leq x \leq 3$, where the breakdown in Phillips' (1978) saddlepoint approximation mainly occurs. Additional values corresponding to x = 4,5 are incorporated in order to demonstrate the robustness of the approximation in the extreme tails region. All computations were carried out in the SHAZAM (White et al. (1990)) package. The program incorporates the EIGENVAL and DAVIES options in calculating the eigenvalues of a square matrix and the distribution of a ratio of quadratic forms, respectively. The latter is the benchmark for all comparisons. It agrees with Phillips' Table 1 exact values to a third decimal place. The results are reported in Table 1.

Clearly, the Lugannani-Rice approximation is excellent over the reported interval of $\hat{\alpha}$. Accuracy increases towards the far tails of the distribution for both T=10 and T=30 and for all α values. We observe in particular the approximation's exceptional behaviour for x = 4, 5. It should also be stressed that the approximation is superior to Phillips' (1978) EDGE A and EDGE B approximations which are available in the relevant range, but do not seem to be satisfactory.

APPENDIX

Let $A = A(\omega) = I - 2\omega D$,

$$B(\omega) = \mathrm{tr}(A^{-1}R_\alpha' C_2 R_\alpha) \tag{A1}$$

and

$$h(\omega) = -\tfrac{1}{2}\log|A|. \tag{A2}$$

Instead of adopting Daniels' (1956) approach, Lieberman (1994) applied Geary's (1944) formula in inverting the exact joint mgf of $v' R_\alpha' C_1 R_\alpha v$ and $v' R_\alpha' C_2 R_\alpha v$. The result can be readily deduced to be

$$f(\hat{\alpha}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B(\omega) e^{h(\omega)} d\omega \qquad c > 0. \tag{A3}$$

A saddlepoint approximation for $f(\hat{\alpha})$ is $\hat{f}(\hat{\alpha}) = [2\pi h''(\hat{\omega})]^{-\frac{1}{2}} B(\hat{\omega}) e^{h(\hat{\omega})}$, with $h'(\hat{\omega}) = 0$, giving (1), or equivalently (3). This is a somewhat non-standard saddlepoint approximation, since both $B(\hat{\omega})$ and $h(\hat{\omega})$ evolve with T. To examine the order of the error, we expand (A3) fully as

$$f(\hat{\alpha}) \equiv \frac{B(\hat{\omega})e^{h(\hat{\omega})}}{[2\pi h''(\hat{\omega})]^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \left\{1 + \frac{iB'(\hat{\omega})z}{B(\hat{\omega})[h''(\hat{\omega})]^{\frac{1}{2}}} - \frac{B''(\hat{\omega})z^2}{2B(\hat{\omega})h''(\hat{\omega})} + \ldots\right\}$$

$$\left\{1 - \frac{ih^{(3)}(\hat{\omega})z^3}{6[h''(\hat{\omega})]^{\frac{3}{2}}} + \frac{h^{(4)}(\hat{\omega})z^4}{24[h''(\hat{\omega})]^2} + \frac{[h^{(3)}(\hat{\omega})]^2 z^6}{72[h''(\hat{\omega})]^3} + \ldots\right\} dz, \tag{A4}$$

where we have set $-z^2 = h''(\hat{\omega})(\omega - \hat{\omega})^2$. Apart from constants, the typical term in the expansion (A4) involves the product

$$\left[\frac{B^{(j)}(\hat{\omega})}{B(\hat{\omega})[h''(\hat{\omega})]^{\frac{j}{2}}}\right]\left[\sum_{k=3}^{\infty} \frac{h^{(k)}(\hat{\omega})}{[h''(\hat{\omega})]^{\frac{k}{2}}}\right]^m \qquad \begin{matrix} j = 0, 1, 2, \ldots \\ k = 3, 4, \ldots \\ m = 0, 1, 2, \ldots. \end{matrix} \tag{A5}$$

In view of (A1) and (A2), the relevant derivatives in (A5) are

$$B^{(j)}(\hat{\omega}) = 2^j j!\ \mathrm{tr}[(\hat{A}^{-1}D)^j \hat{A}^{-1} R_\alpha' C_2 R_\alpha] \qquad j = 0, 1, 2, \ldots \tag{A6}$$

and

$$h^{(j)}(\hat{\omega}) = 2^{j-1}(j-1)!\ \mathrm{tr}[(\hat{A}^{-1}D)^j], \qquad j = 2, 3, \ldots. \tag{A7}$$

When $|\hat{\alpha}| \to \infty$, $|\hat{\omega}| \to \infty$ and so, we may rewrite (A6) and (A7) fully as

$$B^{(j)}(\hat{\omega}) = 2^j j!\,\hat{\omega}^{-(j+1)}\hat{\alpha}^{-1}\mathrm{tr}\{\{[(\hat{\omega}\hat{\alpha})^{-1}I - 2(\hat{\alpha}^{-1}R_\alpha' C_1 R_\alpha - R_\alpha' C_2 R_\alpha)]^{-1}$$

$$(\hat{\alpha}^{-1} R_\alpha' C_1 R_\alpha - R_\alpha' C_2 R_\alpha)\}^j [(\hat{\omega}\hat{\alpha})^{-1} I - 2(\hat{\alpha}^{-1}R_\alpha' C_1 R_\alpha - R_\alpha' C_2 R_\alpha)]^{-1} R_\alpha' C_2 R_\alpha\},$$

$$j = 0,1,2,\ldots \qquad \text{(A8)}$$

and

$$h^{(j)}(\hat{\omega}) = 2^{j-1}(j-1)!\,\hat{\omega}^{-j}\,\mathrm{tr}\{[[(\hat{\omega}\hat{\alpha})^{-1} I - 2(\hat{\alpha}^{-1} R_\alpha' C_1 R_\alpha - R_\alpha' C_2 R_\alpha)]^{-1}$$

$$(\hat{\alpha}^{-1} R_\alpha' C_1 R_\alpha - R_\alpha' C_2 R_\alpha)]^j\}, \qquad j = 2,3,\ldots. \qquad \text{(A9)}$$

Substituting (A8) and (A9) into (A5), it follows that

$$\lim_{|\hat{\alpha}|\to\infty}\left[\frac{B^{(j)}(\hat{\omega})}{B(\hat{\omega})[h''(\hat{\omega})]^{\frac{j}{2}}}\right]\left[\sum_{k=3}^{\infty}\frac{h^{(k)}(\hat{\omega})}{[h''(\hat{\omega})]^{\frac{k}{2}}}\right]^m = \frac{2^{\frac{j}{2}}\,j!}{(T+1)^{\frac{j}{2}}}\left[\sum_{k=3}^{\infty}\frac{2^{k-1}(k-1)!}{(T+1)^{\frac{k}{2}-1}}\right]^m, \qquad \text{(A10)}$$

with $j = 0,1,2,\ldots$, $k = 3,4,5,\ldots$ and $m = 0,1,2,\ldots$. Since all the odd terms in (A4) vanish on integration, the only relevant factors involving T in (A10) are such that $j + (\sum_{k=3} k)^m \geq 2$ and even. Thus, the highest order term in (A10) is $O(T^{-1})$. We provide the expansion to this order fully in (5). The error of neglecting further terms is $O(T^{-2})$.

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Table 1 Exact values and saddlepoint approximations for

$$\Pr\left(\left|T^{\frac{1}{2}}(\hat{\alpha}-\alpha)/(1-\alpha^2)^{\frac{1}{2}}\right| > x\right)$$

	T=10 α=0.2		T=10 α=0.4	
X	Exact	Saddlepoint	Exact	Saddlepoint
2.00	0.0325	0.0272	0.0494	0.0474
2.10	0.0247	0.0212	0.0409	0.0396
2.20	0.0187	0.0164	0.0337	0.0329
2.30	0.0141	0.0125	0.0276	0.0271
2.40	0.0105	0.0094	0.0225	0.0221
2.50	0.0077	0.0069	0.0182	0.0180
2.60	0.0056	0.0050	0.0146	0.0144
2.70	0.0041	0.0036	0.0116	0.0115
2.80	0.0029	0.0026	0.0092	0.0091
2.90	0.0021	0.0018	0.0072	0.0071
3.00	0.0015	0.0013	0.0056	0.0055
4.00	6.7×10^{-5}	6×10^{-5}	2.9×10^{-4}	3×10^{-4}
5.00	7×10^{-6}	5.7×10^{-6}	2.2×10^{-5}	2.2×10^{-5}

	T=10 α=0.6		T=10 α=0.8	
X	Exact	Saddlepoint	Exact	Saddlepoint
2.00	0.0784	0.0781	0.1299	0.1322
2.10	0.0684	0.0686	0.1185	0.1211
2.20	0.0596	0.0600	0.1080	0.1107
2.30	0.0518	0.0522	0.0984	0.1011
2.40	0.0449	0.0454	0.0896	0.0923
2.50	0.0388	0.0393	0.0816	0.0841
2.60	0.0334	0.0339	0.0742	0.0766
2.70	0.0287	0.0292	0.0674	0.0697
2.80	0.0246	0.0250	0.0612	0.0634
2.90	0.0209	0.0213	0.0556	0.0576
3.00	0.0178	0.0181	0.0503	0.0522
4.00	2.7×10^{-3}	2.7×10^{-3}	0.0177	0.0184
5.00	2.4×10^{-4}	2.4×10^{-4}	0.0053	0.0056

Table 1 Exact values and saddlepoint approximations for

$$\Pr\left(\left|T^{\frac{1}{2}}(\hat{\alpha}-\alpha)/(1-\alpha^2)^{\frac{1}{2}}\right|>x\right)$$

(continued)

	T=30 α=0.2		T=30 α=0.4	
X	Exact	Saddlepoint	Exact	Saddlepoint
2.00	0.0392	0.0284	0.0456	0.0411
2.10	0.0302	0.0226	0.0369	0.0342
2.20	0.0230	0.0179	0.0299	0.0283
2.30	0.0174	0.0140	0.0241	0.0233
2.40	0.0131	0.0109	0.0195	0.0191
2.50	0.0098	0.0084	0.0157	0.0155
2.60	0.0072	0.0064	0.0126	0.0126
2.70	0.0053	0.0048	0.0101	0.0101
2.80	0.0039	0.0036	0.0081	0.0081
2.90	0.0028	0.0026	0.0064	0.0064
3.00	0.0020	0.0019	0.0051	0.0051
4.00	3.8×10^{-5}	3.8×10^{-5}	3.3×10^{-4}	3.2×10^{-4}
5.00	4.1×10^{-8}	1.3×10^{-7}	9.1×10^{-6}	9.0×10^{-6}

	T=30 α=0.6		T=30 α=0.8	
X	Exact	Saddlepoint	Exact	Saddlepoint
2.00	0.0605	0.0600	0.0952	0.0964
2.10	0.0519	0.0519	0.0854	0.0866
2.20	0.0445	0.0447	0.0766	0.0778
2.30	0.0382	0.0383	0.0686	0.0697
2.40	0.0326	0.0328	0.0614	0.0624
2.50	0.0278	0.0280	0.0549	0.0558
2.60	0.0236	0.0238	0.0490	0.0499
2.70	0.0200	0.0202	0.0437	0.0445
2.80	0.0169	0.0171	0.0390	0.0397
2.90	0.0143	0.0144	0.0347	0.0353
3.00	0.0120	0.0121	0.0308	0.0314
4.00	0.0018	0.0018	0.0090	0.0091
5.00	1.8×10^{-4}	1.8×10^{-4}	0.0023	0.0024