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A LAPLACE APPROXIMATION TO THE MOMENTS OF A RATIO OF QUADRATIC FORMS

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SUMMARY

The Laplace method for integrals approximating is applied to give a general approximation for the kth moment of a ratio of quadratic forms in random variables. The technique utilizes the existence of a dominating peak at the boundary point on the range of integration. As closed form and tractable formulae do not exist in general, this simple approximation, which only entails basic algebraic operations, has evident practical appeal. We exploit the approximation to provide an approximate mean-bias function for the least squares estimator of the coefficient of the lag dependent variable in a first order stochastic difference equation.

Some key words: Approximate mean-bias; Boundary point; Generalized cumulant; Invariant polynomial; Saddlepoint approximation.

* Some of the work towards the completion of this paper was carried out while the author was a PhD. student in the Department of Econometrics, Monash University.

1. INTRODUCTION

Convenient closed form expressions for moments of a ratio of quadratic forms in random variables exist only in some special cases. For example, under normality, the computation of the moments of the Durbin-Watson test statistic under the null hypothesis is straightforward. Under the alternative hypothesis, however, these moments are cumbersome functions of infinite sums of invariant polynomials with multiple matrix arguments. As only the top order of these polynomials is tabulated, the exact finite sample formulae presented by Smith (1989) are essentially uncomputable.

The practitioner or theorist who requires any moment of this ratio faces either a laborious Monte Carlo experiment, or the exploitation of formulas involving unsolved integrals given by Sawa (1972) and Magnus (1986), under normality. The former is almost standard practice for obtaining the bias of the least squares estimator of the lag dependent variable in the first order stochastic difference equation. In contrast, Andrews (1993) has suggested a median-, rather than a mean-unbiased estimator in the same context, which avoids the unavailability of the first moment in a manageable form. Hoque and Peters (1986) have applied the latter approach in their computation of the bias and mean squared error of certain estimators within the ARMAX framework. This approach entails a numerical integration of derivatives of the joint moment generating function of the two quadratic forms.

In addition to the examples discussed above, many test statistics for autocorrelation, heteroscedasticity and structural breaks are ratios of quadratic forms. The k-class estimator in a simultaneous system, the R^2 statistic in the linear regression model, Stein-rule estimators, test statistics in unit root models and forecasts in dynamic linear models are further examples.

To date, the literature on this ratio has been largely concerned with its distributional properties under normality, but even so has failed to deliver serviceable means for computing its moments. Some contributions in the area are Gurland (1953), Imhof (1961), Anderson (1971, Ch6), Davies (1971), Johnson and Kotz (1971, Ch 29), Kumar (1973), Evans and Savin (1984), Magnus (1986), Farebrother (1990), Shively, Ansley and Kohn (1990) and Lieberman (1994). Limited departures from normality have been entertained by Box and Watson (1962), Kariya (1977), Kariya and Eaton (1977), Knight (1985), McCabe (1989), Peters (1989), Evans (1992) and others. An attempt to obtain the moments of this ratio by numerical integration of an Edgeworth expansion of the joint moment generating function of two quadratric forms, resulted in considerable computational difficulties, see Peters (1989, Sec. 5.2). The need for closed form expressions for the moments of this important statistic is clearly manifested through Peters' (1989) report. In this article, the Laplace method for integrals approximating is used to derive a general expression for the kth moment of a ratio of quadratic forms. The approximation can be easily executed in any standard computer package, and is particularly compact under the normality assumption.

A number of authors have recently used the Laplace method in deriving various approximations, particularly in Bayesian contexts and in integrating out unwanted variables. See, among others, Tierney and Kadane (1986), Tierney, Kass and Kadane (1989 a,b), DiCiccio and Martin (1991) and Daniels and Young (1991). These approximations rely on the existence of a single mode at an interior point of the range of integration. Erdelyi (1956), De Bruijn (1958), Olver (1974), Bleistein and Handelsman (1986) and Erkanli (1994) provide analogous treatment to the case where the maximum is attained at the boundary point, which proves useful in the present study.

In Section 2 we derive our general approximation for the moments of a ratio of quadratic forms in random variables. Under certain restrictions, the true moments are available in tractable forms. It is demonstrated that, in this special case, the approximate mean collapses to the exact formula and the error of approximate higher order moments is $O(n^{-1})$. Sufficient conditions under which the approximation error is $O(n^{-1})$ in the general set-up are established and an expansion up to $O(n^{-3})$ is obtained. The usefulness of the approximation is demonstrated in Section 3, with the introduction of parsimonious representations for the approximate bias function and mean squared error of the least squares estimator of the coefficient of the lag dependent variable in the linear regression model. The accuracy of the technique is evaluated numerically in Section 4.

2.1 APPROXIMATE MOMENTS OF A RATIO OF QUADRATIC FORMS

Let x be $n \times 1$ random vector with density function $f(x)$. Let matrices F and G be non-stochastic $n \times n$, F symmetric and G positive definite. Denote the joint moment generating function of $x'Fx$ and $x'Gx$ by

$$M(\omega_1, \omega_2) = E \exp(\omega_1 x'Fx + \omega_2 x'Gx),$$

and assume that it converges in a strip containing in its interior the origin. The statistic under consideration is $r = x'Fx/x'Gx$. If $x \sim N(0, \sigma^2 I_n)$ and G is an identity matrix, then the moments of r are computable functions in the diagonal elements of F. See for example, Henshaw (1968). For the more general case under normality, the moments are complicated functions of infinite sums of invariant polynomials with multiple matrix arguments. Under non-normality, there do not exist any tractable formulae for these moments, hence the motivation for manageable and computable approximations. Assuming that $E(x'Gx)$ and $E\{(x'Fx)^k\}$ exist, $k \geq 1$, the following theorem is proved in the appendix.

THEOREM 1. *The Laplace approximation for the* k*th moment of* r *about the origin is*

$$E_L(r^k) = \frac{E\{(x'Fx)^k\}}{\{E(x'Gx)\}^k}. \tag{1}$$

While it is difficult to provide a general characterization of sequences of F and G matrices which uniquely determines the order the error of the approximation, it is possible to establish conditions on the cumulants of the numerator and denominator quadratic forms under which the boundedness of the error of $E_L(r^k)$ is ensured. Denote the pth cumulant of x'Gx by κ_p and the $\gamma = 1 + m$ order, $\delta = k + m$ degree generalized cumulant of the product of $(x'Fx)^k$ and x'Gx, by κ_{km}. For a comprehensive discussion on generalized cumulants, the reader is refered to McCullagh (1989, Ch3). It follows from the appendix that, in terms of these cumulants, the sufficient conditions for $E_L(r^k)$ to have an error of $O(n^{-1})$ are:

Condition 1. $\kappa_p = O(n)$, $(p = 1,2,\ldots)$.

Condition 2. $\kappa_{k0} = E\{(x'Fx)^k\} = O(n^k)$, $(k = 1,2,\ldots)$.

Condition 3. $\kappa_{km} = O(n^{\ell})$, with $\ell \le k$ $(k,\ m = 1,2,\ldots)$.

Conditions 1-3 are automatically satisfied if r is a ratio of two independent chi-square variates with degrees of freedom both of order $O(n)$. While conditions 1-3 are not necessary for a formal development of an expansion for $E(r^k)$, the error of the approximation is established under them. Given these conditions, the Laplace expansion for $E(r^k)$ is

$$E(r^k) = E_L(r^k) + b_{n1} + b_{n2} + O(n^{-3}), \tag{2}$$

where

$$b_{n1} = \frac{k(k+1)}{2}\left[\frac{E\{(x'Fx)^k\}\kappa_2}{\{E(x'Gx)\}^{k+2}}\right] - k\left[\frac{\kappa_{k1}}{\{E(x'Gx)\}^{k+1}}\right] = O(n^{-1}) \tag{3}$$

and

$$b_{n2} = \frac{k(k+1)}{2}\left[\frac{\kappa_{k2}}{\{E(x'Gx)\}^{k+2}}\right] - \frac{k(k+1)(k+2)}{2}\left[\frac{3E\{(x'Fx)^k\}\kappa_3 + \kappa_{k1}\kappa_2}{\{E(x'Gx)\}^{k+3}}\right]$$

$$+ \frac{k(k+1)(k+2)(k+3)}{8}\left[\frac{E\{(x'Fx)^k\}\kappa_2^2}{\{E(x'Gx)\}^{k+4}}\right] = O(n^{-2}). \quad (4)$$

If $x'Fx$ and $x'Gx$ are independent, then all terms involving κ_{km}, $m \geq 1$, vanish, resulting in a simplification of (3) and (4). It is convenient to use McCullagh's (1989 p. 60) formula for the conversion of generalized cumulants into ordinary moments in expressing the relevant terms appearing in (3) and (4) as

$$\kappa_{k1} = E\{(x'Fx)^k(x'Gx)\} - E\{(x'Fx)^k\}E(x'Gx)$$

and

$$\kappa_{k2} = E\{(x'Fx)^k(x'Gx)^2\} - 2E(x'Gx)E\{(x'Fx)^k(x'Gx)\} - E\{(x'Gx)^2\}E\{(x'Fx)^k\} + 2\{E(x'Gx)\}^2E\{(x'Fx)^k\}.$$

These moments can be easily recovered from the Edgeworth expansion of the joint moment generating function of two quadratic forms, given by Peters (1989); see also Knight (1985).

Under the normality assumption, the approximation is even more appealling, as the expressions involved simplify greatly. Specifically, if $x \sim N(\mu, \Omega)$, we can write the Laplace approximations for the mean and second raw moment explicitly as

$$E_L(r) = \frac{\mu'F\mu + tr(\Omega F)}{\mu'G\mu + tr(\Omega G)} \quad (5)$$

and

$$E_L(r^2) = \frac{\{\mu'F\mu + tr(\Omega F)\}^2 + 2\{tr(\Omega F)^2 + 2\mu'F\Omega F\mu\}}{\{\mu'G\mu + tr(\Omega G)\}^2}. \quad (6)$$

2.2 An example

Let y be $n \times 1$, X $n \times s$ of rank s and non-stochastic, β $s \times 1$ and fixed, and u an $n \times 1$ disturbance vector. The general linear model is $y = X\beta + u$.

Under the null hypothesis of no serial correlation in the disturbances, $u \sim N(0, \sigma^2 I_n)$. The Durbin-Watson test statistic is then

$$r = \frac{u'PA_1Pu}{u'Pu},$$

where A_1 is the first differencing matrix and $P = I_n - X(X'X)^{-1}X$ is idempotent. Then r is distributed independently of its own denominator, so that

$$E(r^k) = \frac{E\{(u'PA_1Pu)^k\}}{E\{(u'Pu)^k\}}, \tag{7}$$

see Durbin and Watson (1950, p. 419). As seen from (1) and (7), the approximation for the mean is exact. From Smith (1986), $E\{(u'Pu)^k\} = (2\sigma^2)^k(\frac{1}{2}n - \frac{1}{2}s)_k$, where $(a)_k = a(a + 1)\ldots(a + k - 1)$. Therefore, the error of the approximation for $k \geq 2$ is

$$\frac{E_L(r^k)}{E(r^k)} = \frac{2^k(\frac{1}{2}n - \frac{1}{2}s)_k}{(n-s)^k}$$

$$= 1 + O(n^{-1}).$$

We note that the error of $O(n^{-1})$ holds for $k \geq 2$ regardless of the design matrix X.

3. APPROXIMATELY UNBIASED ESTIMATORS

Consider the model $y_t = \alpha y_{t-1} + x_t'\beta + u_t$, where $u_t \sim N(0, \sigma^2)$ for all t, $(t = 1,\ldots,n)$. The ordinary least squares estimator of α is

$$\hat{\alpha} = \frac{y'_{t-1}Py_t}{y'_{t-1}Py_{t-1}} = \frac{z'Sz}{z'Bz},$$

where $z' = (y_1\ldots y_n)$, $S = \frac{1}{2}(D_1'PD_2 + D_2'PD_1)$, $B = D_1'PD_1$, $D_1 = (I_{n-1} \vdots 0)$ and $D_2 = (0 \vdots I_{n-1})$. Under the assumptions that the process has been stabilized at $t = 1$, so that $E(y_1) = E(y_0)$ and $\text{var}(y_t) = \text{var}(y)$, for all t, $z \sim N(\mu, \sigma^2\Omega)$, with

$$\mu = \begin{pmatrix} b & 0 \ldots & & & 0 \\ \alpha b & 1 & 0 \ldots & & 0 \\ \alpha^2 b & \alpha & 1 & 0 \ldots & 0 \\ & & \cdot & & \\ \alpha^{n-1} b & \alpha^{n-2} & \ldots & \alpha & 1 \end{pmatrix} \begin{pmatrix} x'_1\beta \\ x'_2\beta \\ \vdots \\ \\ x'_n\beta \end{pmatrix} = HX'\beta, \text{ say,}$$

where $b = 1/(1-\alpha)$. $\Omega = LL'$, with L being the same matrix as H, except that b is replaced by $(1 - \alpha^2)^{-1/2}$ in the first column. $\hat{\alpha}$ is a ratio of quadratic forms in normal variables. The mean-bias of $\hat{\alpha}$ can be usefully approximated using (5). The Laplace approximations for the bias and mean squared error functions of this estimator are

$$\text{Bias}_L(\hat{\alpha}) = E_L(\hat{\alpha}) - \alpha$$

$$= \frac{\mu'(S-\alpha B)\mu + \text{tr}\{\Omega(S-\alpha B)\}}{\mu' B\mu + \text{tr}(\Omega B)} \tag{8}$$

and

$$\begin{aligned} \text{Mse}_L(\hat{\alpha}) &= E_L(\hat{\alpha}^2) - \{E_L(\hat{\alpha})\}^2 + \{\text{Bias}_L(\hat{\alpha})\}^2 \\ &= \frac{\{\mu' S\mu + \text{tr}(\Omega S)\}^2 + 2\{\text{tr}(\Omega S)^2 + 2\mu' S\Omega S\mu\}}{\{\mu' B\mu + \text{tr}(\Omega B)\}^2} \\ &\quad + \alpha\left\{\alpha - 2\,\frac{\mu' S\mu + \text{tr}(\Omega S)}{\mu' B\mu + \text{tr}(\Omega B)}\right\}. \end{aligned} \tag{9}$$

Notice that $\mu = \mu(\alpha)$ and $\Omega = \Omega(\alpha)$, so that for any given values of α, β, σ^2, X and n, the computation of (8) and (9) is instant.

4. ACCURACY OF THE APPROXIMATION.

Table 1 compares the Laplace approximations to the bias and mean squared error of $\hat{\alpha}$ with a Monte Carlo simulation consisting of 2000 replications. The SHAZAM package (White et al. 1990) is used with the random number generator, ranseed = 123, throughout. The data have been used in previous studies of the same nature, specifically, Inder (1986) and Hoque and Peters (1986). The various experimental designs are;

Design 1; X consists of a constant and Maddala and Rao's (1973) GNP data, $\beta' = (0, 1)$, $\sigma = 1, 8$ and $n = 32, 76$.

Design 2; X consists of a constant and a linear time trend, $\beta' = (0, 1)$, $\sigma = 1$ and $n = 20, 40, 60, 80$.

Design 3; X consists of a constant and the Durbin and Watson (1951) consumption of spirit data in the U.K., $\beta' = (0, 1, 1)$, $\sigma = 1$ and $n = 30, 60$.

The approximations for the bias generally compare favourably with the simulated figures, particularly in designs 1 and 2 and with the larger sample sizes. They are almost always superior to the corresponding mean squared error approximations. Given that the latter is an outcome of an approximation to a double integral, this feature is not surprising. We emphasize though, as already noted by Tierney and Kadane (1986, Sec. 6) in a similar context, that the approximations should be regarded as simple and tractable devices which should by no means replace exact calculations, should extremely accurate results be necessary.

5. CONCLUSION

We have applied the Laplace method for integrals in establishing a general approximation for the kth moment of a very large class of statistics. The technique utilizes the existence of a dominating peak at a boundary point on the range of integration. For a subclass of statistics that includes the Durbin - Watson test statistic under the null, the approximate mean is exact. The relative error of approximate higher order moments is $O(n^{-1})$. An expansion up to $O(n^{-3})$ was obtained under general conditions on the cumulants of the quadratic forms involved in the ratio.

Of particular importance is the simplicity of the approximations. Only basic algebraic operations are required. Unlike what is often the case for saddlepoint approximations, not even a line search is necessary. There does not at present seem to exist any closed form convenient alternative for the calculation of these moments.

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APPENDIX

Proof of Theorem 1

We first show that (1) constitutes the Laplace approximation to $E(r^k)$, then justify conditions 1-3 and the order of the terms in the expansion (2). Let $\omega_2 = \sum_{j=1}^{k} \omega_{2j}$. The kth moment of r about the origin can be readily shown to be

$$E(r^k) = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \left. \frac{\partial^k M(\omega_1, \sum_{j=1}^{k} \omega_{2j})}{\partial \omega_1^k} \right|_{\omega_1 = 0} d\omega_{21} \ldots d\omega_{2k}. \qquad \text{(A1)}$$

This multiple integral has not been previously solved. To express (A1) in a form suitable for the application of the method of Laplace, we first write the integrand as

$$\left. \frac{\partial^k M(\omega_1, \sum_{j=1}^{k} \omega_{2j})}{\partial \omega_1^k} \right|_{\omega_1 = 0} = g_k(0, \sum_{j=1}^{k} \omega_{2j}) \exp\{h(0, \sum_{j=1}^{k} \omega_{2j})\},$$

where

$$g_k(0, \sum_{j=1}^{k} \omega_{2j}) = \left\{ \left. \frac{\partial^k M(\omega_1, \sum_{j=1}^{k} \omega_{2j})}{\partial \omega_1^k} \right|_{\omega_1 = 0} / M(0, \sum_{j=1}^{k} \omega_{2j}) \right\} \qquad \text{(A2)}$$

and

$$h(0, \sum_{j=1}^{k}\omega_{2j}) = \log M(0, \sum_{j=1}^{k}\omega_{2j}).$$

As G is positive definite,

$$h_{\omega_{2j}}(0, \sum_{j=1}^{k}\omega_{2j}) = \left[\int_{\mathbb{R}^n} \exp\{(x'Gx)\sum_{j=1}^{k}\omega_{2j}\}f(x)dx\right]^{-1}$$

$$\int_{\mathbb{R}^n} (x'Gx)\exp\{(x'Gx)\sum_{j=1}^{k}\omega_{2j}\}f(x)dx > 0,$$

for all $x \in \mathbb{R}^n$ and $\omega_{2j} \in \mathbb{R}$, $(j = 1,\ldots,k)$. On the range of integration, the monotonicity of $h(0, \sum_{j=1}^{k}\omega_{2j})$ implies that its maximum is attained at the boundary points $\tilde{\omega}_{21}=\ldots=\tilde{\omega}_{2k} = 0$. Applying the method of Laplace, we have

$$E(r^k) = \int_{-\infty}^{0}\cdots\int_{-\infty}^{0} g_k(0, \sum_{j=1}^{k}\omega_{2j}) \exp\{h(0, \sum_{j=1}^{k}\omega_{2j})\}d\omega_{21}\ldots d\omega_{2k}$$

$$\simeq \frac{g_k(0, 0) \exp\{h(0, 0)}{\prod_{j=1}^{k} h_{\omega_{2j}}(0, 0)}.$$

In view of (A2), $g_k(0, 0) = E\{(x'Fx)^k\}$ and since $h(0, \sum_{j=1}^{k}\omega_{2j})$ is the cumulant generating function of $x'Gx$, $h(0, 0) = 0$ and

$$h_{\omega_{2j}}(0, 0) = E(x'Gx).$$

Thus

$$E(r^k) \simeq \frac{E\{(x'Fx)^k\}}{\{E(x'Gx)\}^k} = E_L(r^k). \qquad \blacksquare$$

The order of the error of the approximation

In order to establish that conditions 1-3 ensure that the error of the approximation is $O(n^{-1})$, we first note that for any $j \in [1, k]$,

$$\left.\frac{\partial^p h(0, \sum_{j=1}^{k}\omega_{2j})}{\partial\omega_{2j}\ldots\partial\omega_{2j}}\right|_{\sum_{j=1}^{k}\omega_{2j}=0} = \kappa_p \quad (p = 1,2,\ldots)$$

and

$$\left.\frac{\partial^m g_k(0, \sum_{j=1}^{k}\omega_{2j})}{\partial\omega_{2j}\cdots\partial\omega_{2j}}\right|_{\sum_{j=1}^{k}\omega_{2j}=0} = \kappa_{km} \quad (m = 1,2,\ldots).$$

To see the latter, we put

$$K(u, \sum_{j=1}^{k}\omega_{2j}) = \log E \exp\{u(x'Fx)^k + (x'Gx)\sum_{j=1}^{k}\omega_{2j}\}.$$

Then

$$K_u(0, \sum_{j=1}^{k}\omega_{2j}) = g_k(0, \sum_{j=1}^{k}\omega_{2j})$$

and

$$K_{u\omega_{2j}\ldots\omega_{2j}}(0, 0) = \left.\frac{\partial^m g_k(0, \sum_{j=1}^{k}\omega_{2j})}{\partial\omega_{2j}\cdots\partial\omega_{2j}}\right|_{\sum_{j=1}^{k}\omega_{2j}=0} = \kappa_{km}, \text{ for any } j \in [1, k].$$

On detailed expansion of $E(r^k)$ about the origin, it is seen that

$$E(r^k) = \int_{-\infty}^{0}\cdots\int_{-\infty}^{0}\left[E\{(x'Fx)^k\} + \kappa_{k1}\sum_{j=1}^{k}\omega_{2j} + \kappa_{k2}\left(\sum_{j=1}^{k}\omega_{2j}\right)^2/2 + \ldots\right]$$
$$\left\{1 + \kappa_2\left(\sum_{j=1}^{k}\omega_{2j}\right)^2/2 + \kappa_3\left(\sum_{j=1}^{k}\omega_{2j}\right)^3/6 + \ldots\right\}$$
$$\exp\left\{E(x'Gx)\sum_{j=1}^{k}\omega_{2j}\right\} d\omega_{21}\ldots d\omega_{2k}$$

$$= \int_{0}^{\infty}\cdots\int_{0}^{\infty}\left[E\{(x'Fx)^k\} - \kappa_{k1}\sum_{j=1}^{k}t_j + \kappa_{k2}\left(\sum_{j=1}^{k}t_j\right)^2/2 + \ldots\right]$$
$$\left\{1 + \kappa_2\left(\sum_{j=1}^{k}t_j\right)^2/2 - \kappa_3\left(\sum_{j=1}^{k}t_j\right)^3/6 + \ldots\right\}$$
$$\exp\left\{-E(x'Gx)\sum_{j=1}^{k}t_j\right\} dt_1\ldots dt_k. \qquad \text{(A3)}$$

By the positive definiteness of G, $E(x'Gx) > 0$ and for any nonnegative integer s, we may use the result

$$\int_{0}^{\infty}\cdots\int_{0}^{\infty}\left(\sum_{j=1}^{k}t_j\right)^s \exp\left\{-E(x'Gx)\sum_{j=1}^{k}t_j\right\} dt_1\ldots dt_k = \frac{\Gamma(k+s)}{\Gamma(k)\{E(x'Gx)\}^{k+s}}. \qquad \text{(A4)}$$

An application of (A4) in (A3) yields an expansion of the form

$$E(r^k) = \frac{E\{(x'Fx)^k\}}{\{E(x'Gx)\}^k} + \frac{a_1}{\{E(x'Gx)\}^{k+1}} + \frac{a_2}{\{E(x'Gx)\}^{k+2}}\ldots, \qquad (A5)$$

where the a_i's involve $E\{(x'Fx)^k\}$, the κ_p's and κ_{km}'s, (i, k, m = 1,2,...; p = 2,3,...). The rate of decay of the series (A5) is therefore directly determined by the orders of magnitude of E(x'Gx) and the a_i's (i = 1,2,...). In view of (A3)-(A5), conditions 1-3 are seen to be sufficient for the approximation to have an error of $O(n^{-1})$. Direct if tedious derivation of the $O(n^{-1})$ and $O(n^{-2})$ terms under conditions 1-3 produces (3) and (4). The error of neglecting further terms is $O(n^{-3})$.

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Table 1. *Comparison of approximate (LAP) and simulated (MC) mean and mse*

Design 1

		$\sigma = 1$				$\sigma = 8$			
		bias		mse		bias		mse	
n	α	MC	LAP	MC	LAP	MC	LAP	MC	LAP
76	0.2	-0.000	-0.000	0.000	0.000	-0.021	-0.017	0.008	0.010
	0.4	-0.000	-0.000	0.000	0.000	-0.018	-0.014	0.005	0.013
	0.6	-0.000	-0.000	0.000	0.000	-0.011	-0.008	0.002	0.013
	0.8	-0.000	-0.000	0.000	0.000	-0.004	-0.003	0.000	0.011
	0.9	-0.000	-0.000	0.000	0.000	-0.001	-0.001	0.000	0.007
32	0.2	-0.002	-0.001	0.000	0.000	-0.057	-0.042	0.021	0.026
	0.4	-0.001	-0.001	0.000	0.000	-0.048	-0.033	0.014	0.032
	0.6	-0.001	-0.000	0.000	0.000	-0.028	-0.019	0.005	0.033
	0.8	-0.000	-0.000	0.000	0.000	-0.008	-0.006	0.001	0.022
	0.9	-0.000	-0.000	0.000	0.000	-0.003	-0.002	0.000	0.011

Design 2

	bias		mse		bias		mse	
α	MC	LAP	MC	LAP	MC	LAP	MC	LAP
	n = 80				n = 60			
0.2	-0.037	-0.031	0.013	0.016	-0.050	-0.041	0.019	0.022
0.4	-0.046	-0.036	0.013	0.025	-0.063	-0.048	0.019	0.033
0.6	-0.049	-0.036	0.011	0.042	-0.068	-0.047	0.016	0.056
0.8	-0.014	-0.010	0.001	0.040	-0.019	-0.012	0.002	0.047
0.9	-0.001	-0.001	0.000	0.009	-0.002	-0.001	0.000	0.011
	n = 40				n = 20			
0.2	-0.071	-0.062	0.029	0.033	-0.144	-0.129	0.071	0.074
0.4	-0.090	-0.072	0.030	0.050	-0.182	-0.149	0.082	0.105
0.6	-0.093	-0.068	0.025	0.082	-0.177	-0.132	0.070	0.156
0.8	-0.023	-0.016	0.003	0.062	-0.049	-0.037	0.011	0.109
0.9	-0.003	-0.003	0.000	0.017	-0.011	-0.011	0.002	0.043

Table 1. *(Continued)*

Design 3

	bias		mse		bias		mse	
α	MC	LAP	MC	LAP	MC	LAP	MC	LAP
	n = 60				n = 30			
0.2	-0.069	-0.061	0.021	0.023	-0.134	-0.121	0.052	0.054
0.4	-0.086	-0.072	0.023	0.035	-0.169	-0.142	0.063	0.078
0.6	-0.105	-0.084	0.025	0.062	-0.207	-0.166	0.076	0.126
0.8	-0.129	-0.099	0.028	0.131	-0.255	-0.203	0.096	0.210
0.9	-0.142	-0.107	0.031	0.226	-0.287	-0.232	0.112	0.269