MONASH
UNIVERSITY

AUSTRALIA

MARGINAL LIKELIHOOD BASED TESTS OF REGRESSION DISTURBANCES

Ismat Ara and Maxwell L. King

Working Paper No. 14/93

October 1993

DEPARTMENT OF ECONOMETRICS

ISSN 1032-3813

ISBN 0 7326 0382 X

MARGINAL LIKELIHOOD BASED TESTS OF REGRESSION DISTURBANCES

Ismat Ara and Maxwell L. King

Working Paper No. 14/93

October 1993

DEPARTMENT OF ECONOMETRICS

MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA.

MARGINAL LIKELIHOOD BASED TESTS OF REGRESSION DISTURBANCES

by

Ismat Ara and Maxwell L. King*

Department of Econometrics
Monash University
Clayton, Victoria 3168, Australia

May 1993 revised version

Abstract

This paper presents general formulae for the likelihood ratio (LR), Wald (W), Lagrange multiplier (LM) and asymptotic locally most mean powerful (ALMMP) tests of linear regression disturbances using marginal likelihood methods. These tests can be derived by treating the maximal invariant statistic for these testing problems as the observed data. By way of illustration, the marginal-likelihood-based LR, W, LM and ALMMP tests are constructed for the separate problems of testing for general AR(4) disturbances and testing for the presence of Hildreth-Houck random coefficients. Empirical size calculations reported here and elsewhere suggest that this approach results in tests whose true sizes are much closer to the nominal size than their conventional counterparts.

---

* This research was supported in part by an ARC grant. We are grateful for comments from seminar audiences at the Universities of California San Diego, Rochester, Iowa, and Canterbury as well as Michigan State, Texas A & M and Southern Methodist Universities. We are particularly grateful to Murray Smith for helping us translate his results to get the lemma given in section 2.1 and to Alan Morgan and Vladimir Rouderfer for research assistance.

1 Introduction

Because of the non-experimental nature of almost all economic data, there is an extensive literature on diagnostic testing of econometric models. Much of this literature is concerned with testing for spherical disturbances in the linear regression model; see for example Godfrey (1988), Judge *et al.* (1985), King (1987a, 1987b), Pagan and Hall (1983) and Pagan (1984). There is a general emphasis on three testing procedures based on the likelihood function, namely the likelihood ratio (LR), Wald and Lagrange multiplier (LM) tests. Unfortunately, the accuracy in small-samples of the asymptotic critical values of these tests is questionable. For example, King (1987a, p.59) concluded that as a test for autocorrelation in the linear regression model, "the LR test is a particularly unreliable test". Breusch and Pagan (1979), Godfrey (1978), Griffiths and Surekha (1986) and Honda (1988) all concluded that the true small-sample size of the LM test for heteroscedasticity is typically much lower than its nominal size. Also, Moulton and Randolph (1989) reported remarkable inaccuracies for the asymptotic critical values of the LM test for error components in regressions with 126 to 506 observations.

One possible explanation is that the presence of nuisance parameters (the regression coefficients and disturbance variance) causes biases in key estimates used in the tests. As Durbin and Watson (1971) first showed, this problem of nuisance parameters can be overcome by the use of invariance arguments. A maximal invariant statistic can be found whose distribution does not depend on the nuisance parameters. Because all invariant test statistics can be expressed as functions of the maximal invariant, optimal invariant testing procedures can be constructed by treating the maximal invariant as the observed data (see, e.g., King and Hillier (1985) and King (1987b)).

An alternative approach that has been attracting attention, particularly in the context of modelling linear regression disturbances, is the method of marginal likelihood which was first suggested by Kalbfleisch and Sprott (1970). There is growing evidence that its use can reduce estimation bias. A good example is that of estimating the error variance in the classical linear model. The maximum likelihood (ML) estimator is well known to be biased while that based on the marginal likelihood is the standard unbiased estimator. Also see Tunnicliffe Wilson (1989) and the references therein. A related estimator of covariance-matrix parameters of the linear regression model is the restricted (or residual) ML (REML) estimator, see e.g., Harville (1977) and Robinson (1987). This involves maximizing the marginal likelihood function of all parameters of the covariance matrix. The scale parameter, σ^2, is often a nuisance parameter so it is typically more appropriate to use the marginal likelihood function of all disturbance parameters except σ^2. This is the approach we favour.

There is some evidence that the use of marginal likelihood methods can produce more accurate tests of regression disturbances. In contrast to the performance of the conventional LR test for autocorrelation, Corduas (1986) found that the marginal-likelihood-based LR test has good small-sample properties. Moulton and Randolph (1989) reported that the LM test for error components in regression disturbances constructed from the restricted likelihood function, can be far more accurate than its conventional counterpart.

The LR, Wald and LM tests are two-sided test procedures, although they can be converted into one-sided tests for one-dimensional testing problems. Often, one-sided hypotheses arise naturally through economic theory and functional considerations such as variances always being positive. An alternative form of the LM test for one-sided tests of more than one parameter has recently been suggested by King and Wu (1990). Their test statistic is based on the sum of scores. In the absence of nuisance parameters, this test is locally most mean powerful (LMMP) as it maximizes the mean slope of the power hypersurface at the null hypothesis. The information matrix can be used to construct an asymptotic test based on the sum of the scores.

In section 2 of this paper, we derive general formulae for the LR, Wald, LM and asymptotic LMMP (ALMMP) tests of regression disturbances based on the maximal invariant statistic as the observed data. We show that this approach is equivalent to constructing these tests based on the marginal likelihood function. The application of these tests to the separate problems of testing for AR(p) disturbances and testing for Hildreth-Houck (1968) random regression coefficients is considered in section 3. Section 4 reports a Monte Carlo study designed to compare conventional and marginal likelihood based tests for these two problems. Some concluding remarks are made in the final section.

2. Theory

Consider the normal linear model with non-spherical disturbances

$$y = X\beta + u ; \qquad u \sim N(0,\sigma^2\Omega(\theta)) , \tag{1}$$

where y is $n\times1$, X is $n\times k$, nonstochastic and of rank $k < n$, and $\Omega(\theta)$ is a symmetric matrix that is positive definite for θ ($p\times1$) in a subset of R^p which is of interest. Without loss of generality, it is assumed that

$\Omega(0) = I_n$. We are interested in testing $H_0 : \theta = 0$ against either H_a : $\theta \neq 0$ or $H_a^+ : \theta > 0$ where, in this context, > denotes ≥ for each component with at least one strict inequality. It is well known that this testing problem is invariant to transformations of the form

$$y \longrightarrow \eta_0 y + X\eta \qquad (2)$$

where η_0 is a positive scalar and η is a k×1 vector.

2.1 Preliminaries

Let $m = n-k$, $M = I_n - X(X'X)^{-1}X'$, $z = My$ be the ordinary least squares (OLS) residual vector from (1) and P be an m×n matrix such that $PP' = I_m$ and $P'P = M$. Observe that the m×1 vector Pz is a LUS residual vector given that under H_0, $Pz \sim N(0,\sigma^2 I_m)$ (see King (1987a, section 5)). The vector

$$v = Pz / (z'P'Pz)^{1/2}$$

is a maximal invariant under the group of transformations given by (2) for our problem. The density of v under (1) can be shown to be

$$f(v;\theta)dv = \frac{1}{2}\,\Gamma(m/2)\,\pi^{-m/2}\,|P\Omega(\theta)P'|^{-1/2}\,s^{-m/2}\,dv \qquad (3)$$

where $s = v'(P\Omega(\theta)P')^{-1}v = \hat{u}'\Omega(\theta)^{-1}\hat{u} / z'z$, $\hat{u}$ is the generalized least squares (GLS) residual vector assuming covariance matrix $\sigma^2\Omega(\theta)$ and dv denotes the uniform measure on the surface of the unit m-sphere.

The principle of invariance implies that invariant tests can be constructed by treating v as the observed data and (3) as its density function. In our case, the restriction to invariant tests is well accepted (see for e.g. Durbin and Watson (1971) and King (1987b)). We can therefore treat (3) as a likelihood function for θ and derive standard tests such as the LR, Wald and LM tests.

This approach is equivalent to using the marginal likelihood for θ which, from Tunnicliffe Wilson (1989), is given by

$$f_m(\theta|y) = |\Omega(\theta)|^{-1/2}|X'\Omega(\theta)^{-1}X|^{-1/2}(\hat{u}'\Omega(\theta)^{-1}\hat{u})^{-m/2}. \quad (4)$$

As functions of θ, $f(v;\theta)$ and $f_m(\theta|y)$ are identical up to a multiplicative constant and so can be treated as equivalent likelihood functions. This is because from Verbyla (1990), we have

$$|P\Omega(\theta)P'| = |X'X|^{-1}|\Omega(\theta)||X'\Omega(\theta)^{-1}X| \quad (5)$$

so that

$$f(v;\theta) \,/\, f_m(\theta|y) = \frac{1}{2}\,\Gamma(m/2)\,\pi^{-m/2}\,|X'X|^{1/2}\,(z'z)^{m/2}$$

which is not a function of θ. When repeated evaluations of the likelihood function are required, it may be preferable to use (4). This is because often $|\Omega(\theta)|$ is known and only the determinant of the $k\times k$ matrix $X'\Omega(\theta)^{-1}X$ need be evaluated numerically. In deriving the score vector and information matrix we found it easier to work with (3).

The log likelihood function implied by (3) is

$$L(\theta) = \text{constant} - \frac{1}{2}\log|P\Omega(\theta)P'| - \frac{m}{2}\log\left[v'(P\Omega(\theta)P')^{-1}v\right]. \quad (6)$$

The scores are

$$\frac{\partial L(\theta)}{\partial\theta_i} = -\frac{1}{2}\,\mathrm{tr}\left[(P\Omega(\theta)P')^{-1}\,P\,\frac{\partial\Omega(\theta)}{\partial\theta_i}\,P'\right]$$

$$+\frac{m}{2}\left[v'(P\Omega(\theta)P')^{-1}v\right]^{-1}\left[v'(P\Omega(\theta)P')^{-1}\left(P\,\frac{\partial\Omega(\theta)}{\partial\theta_i}\,P'\right)\left(P\Omega(\theta)P'\right)^{-1}v\right]$$

$$= -\frac{1}{2}\,\mathrm{tr}\left[\Delta(\theta)\,\frac{\partial\Omega(\theta)}{\partial\theta_i}\right] - \frac{m}{2}\left[\hat{u}'\,\frac{\partial\Omega(\theta)^{-1}}{\partial\theta_i}\,\hat{u} \Big/ \hat{u}'\Omega(\theta)^{-1}\hat{u}\right] \qquad (7)$$

where from Rao (1973, p.77)

$$\Delta(\theta) = \Omega(\theta)^{-1} - \Omega(\theta)^{-1}X(X'\Omega(\theta)^{-1}X)^{-1}X'\Omega(\theta)^{-1}$$

$$= P'(P\Omega(\theta)P')^{-1}P\ .$$

$$\frac{\partial^2 L(\theta)}{\partial\theta_i\partial\theta_j} = \frac{1}{2}\,\mathrm{tr}\left[\Delta(\theta)\,\frac{\partial\Omega(\theta)}{\partial\theta_j}\,\Delta(\theta)\,\frac{\partial\Omega(\theta)}{\partial\theta_i}\right] - \frac{1}{2}\,\mathrm{tr}\left[\Delta(\theta)\,\frac{\partial^2\Omega(\theta)}{\partial\theta_i\partial\theta_j}\right]$$

$$+ \frac{m}{2}\left\{u'\Delta(\theta)\,\frac{\partial^2\Omega(\theta)}{\partial\theta_i\partial\theta_j}\,\Delta(\theta)u - 2u'\Delta(\theta)\,\frac{\partial\Omega(\theta)}{\partial\theta_j}\,\Delta(\theta)\,\frac{\partial\Omega(\theta)}{\partial\theta_i}\,\Delta(\theta)u\right\}$$

$$\left\{u'\Delta(\theta)u\right\}^{-1} + \frac{m}{2}\left\{u'\Delta(\theta)\,\frac{\partial\Omega(\theta)}{\partial\theta_i}\,\Delta(\theta)uu'\Delta(\theta)\,\frac{\partial\Omega(\theta)}{\partial\theta_j}\,\Delta(\theta)u\right\}$$

$$\left(u'\Delta(\theta)u\right)^{-2}.$$

In order to derive the information matrix we need the following result which is a special case of results given by Smith (1987, 1989).

LEMMA: Consider the linear regression model $y^* = X^*\beta + u^*$ where X^* is an $n\times k$ nonstochastic matrix and $u^* \sim N(0,\sigma^2 I_n)$. Let e be the OLS residual vector from this regression so that

$$e = (I_n - X^*(X^{*\prime}X^*)^{-1}X^{*\prime})u^* = M^*u^*\ .$$

(i) If A is any $n\times n$ symmetric matrix, then

$$E(e'Ae/e'e) = \mathrm{tr}(M^*A)/m\ .$$

(ii) If A_1 and A_2 are any two $n\times n$ symmetric matrices then

$$E\left[e'A_1ee'A_2e/(e'e)^2\right] = \left\{tr(M^*A_1)tr(M^*A_2) + 2tr(M^*A_1M^*A_2)\right\} \Big/ (m^2+2m).$$

We wish to find the expected value of $-\dfrac{\partial L(\theta)}{\partial\theta_i\partial\theta_j}$ assuming $u \sim N(0,\sigma^2\Omega(\theta))$. Now

$$\Delta(\theta)u = \left(\Omega(\theta)^{-1/2}\right)'M^*\Omega(\theta)^{-1/2}u = \left(\Omega(\theta)^{-1/2}\right)'e$$

where e is the OLS residual vector from the transformed regression

$$\Omega(\theta)^{-1/2}y = \Omega(\theta)^{-1/2}X\beta + \Omega(\theta)^{-1/2}u \tag{8}$$

in which $\Omega(\theta)^{-1/2}u \sim N(0,\sigma^2I_n)$ and

$$M^* = I_n - \Omega(\theta)^{-1/2}X\left(X'\Omega(\theta)^{-1}X\right)^{-1}X'\left(\Omega(\theta)^{-1/2}\right)'.$$

Also observe that $u'\Delta(\theta)u = e'e$. Therefore

$$E\left\{-\frac{\partial^2L(\theta)}{\partial\theta_i\partial\theta_j}\right\} = -\frac{1}{2}\,tr\left[\Delta(\theta)\frac{\partial\Omega(\theta)}{\partial\theta_j}\Delta(\theta)\frac{\partial\Omega(\theta)}{\partial\theta_i}\right]$$

$$+ tr\left[M^*\Omega(\theta)^{-1/2}\frac{\partial\Omega(\theta)}{\partial\theta_j}\Delta(\theta)\frac{\partial\Omega(\theta)}{\partial\theta_i}\left(\Omega(\theta)^{-1/2}\right)'\right]$$

$$-\frac{1}{2}\left\{tr\left[M^*\Omega(\theta)^{-1/2}\frac{\partial\Omega(\theta)}{\partial\theta_i}\left(\Omega(\theta)^{-1/2}\right)'\right]tr\left[M^*\Omega(\theta)^{-1/2}\frac{\partial\Omega(\theta)}{\partial\theta_j}\left(\Omega(\theta)^{-1/2}\right)'\right]\right.$$

$$\left.+ 2\,tr\left[M^*\Omega(\theta)^{-1/2}\frac{\partial\Omega(\theta)}{\partial\theta_i}\left(\Omega(\theta)^{-1/2}\right)'M^*\Omega(\theta)^{-1/2}\frac{\partial\Omega(\theta)}{\partial\theta_j}\left(\Omega(\theta)^{-1/2}\right)'\right]\right\}\Big/(m+2)$$

$$= \frac{m}{2(m+2)}\,tr\left[\Delta(\theta)\frac{\partial\Omega(\theta)}{\partial\theta_i}\Delta(\theta)\frac{\partial\Omega(\theta)}{\partial\theta_j}\right] - \frac{1}{2(m+2)}\left[\Delta(\theta)\frac{\partial\Omega(\theta)}{\partial\theta_i}\right]tr\left[\Delta(\theta)\frac{\partial\Omega(\theta)}{\partial\theta_j}\right]$$

(9)

In what follows, let

$$s(\theta) = \frac{\partial L(\theta)}{\partial \theta}$$

denote the score vector whose i^{th} component is given by (7) and let $\hat{\theta}$ denote the maximum marginal likelihood estimator of θ, i.e. that value of θ which maximizes (6). Also, let $I(\theta)$ denote the information matrix whose $(i,j)^{th}$ element is given by (9).

2.2 The Likelihood Ratio Test

Given $\hat{\theta}$, the marginal-likelihood-based LR test of H_0 against $H_a : \theta \neq 0$ rejects H_0 for large values of

$$- \log|P\Omega(\hat{\theta})P'| - m \log\left(\hat{u}'\Omega(\hat{\theta})^{-1}\hat{u} \;/\; z'z\right) \tag{10}$$

which, under H_0, has an asymptotic chi-squared distribution with p degrees of freedom. The first term in (10) can be evaluated using (5). The GLS residual vector $\hat{u}$ is that from (1) assuming u has covariance matrix $\sigma^2\Omega(\hat{\theta})$. Thus $\hat{u}'\Omega(\hat{\theta})^{-1}\hat{u}$ can be calculated as the sum of squared OLS residuals from (8) with $\theta = \hat{\theta}$.

2.3 The Wald Test

The Wald test based on (6) rejects H_0 for large values of

$$\hat{\theta}' I(\hat{\theta})\hat{\theta} = \frac{1}{2(m+2)} \sum_{i=1}^{p} \sum_{j=1}^{p} \hat{\theta}_i \hat{\theta}_j \left\{ m \operatorname{tr}\left[\Delta(\hat{\theta}) \frac{\partial\Omega(\hat{\theta})}{\partial\theta_i} \Delta(\hat{\theta}) \frac{\partial\Omega(\hat{\theta})}{\partial\theta_j}\right] - \operatorname{tr}\left[\Delta(\hat{\theta}) \frac{\partial\Omega(\hat{\theta})}{\partial\theta_i}\right] \operatorname{tr}\left[\Delta(\hat{\theta}) \frac{\partial\Omega(\hat{\theta})}{\partial\theta_j}\right]\right\}. \tag{11}$$

This statistic has an asymptotic $\chi^2(p)$ distribution under H_0. In the special case of $p = 1$, (11) becomes

$$\hat{\theta}^2\left\{m\ \mathrm{tr}\left[\left(\Delta(\hat{\theta})\frac{\partial\Omega(\hat{\theta})}{\partial\theta}\right)^2\right] - \left(\mathrm{tr}\left[\Delta(\hat{\theta})\frac{\partial\Omega(\hat{\theta})}{\partial\theta}\right]\right)^2\right\} \Big/ (2m+4)$$

and the formula for the standard error of $\hat{\theta}$ is

$$\mathrm{se}(\hat{\theta}) = \left[\left\{m\ \mathrm{tr}\left[\left(\Delta(\hat{\theta})\ \frac{\partial\Omega(\hat{\theta})}{\partial\theta}\right)^2\right] - \left(\mathrm{tr}\left[\Delta(\hat{\theta})\ \frac{\partial\Omega(\hat{\theta})}{\partial\theta}\right]\right)^2\right\} \Big/ (2m+4)\right]^{-1/2}.$$

Therefore, when $p = 1$, H_0 can be tested using $\hat{\theta}$ / $\mathrm{se}(\hat{\theta})$ which has a standard normal asymptotic distribution under H_0.

2.4 The Lagrange Multiplier Test

Often it is inconvenient to obtain $\hat{\theta}$. The LM test has the advantage that it does not involve $\hat{\theta}$. Let $A_i = \frac{\partial\Omega(0)}{\partial\theta_i} = -\frac{\partial\Omega(0)^{-1}}{\partial\theta_i}$. Then the i^{th} element of $s(0)$ is

$$s_i(0) = \frac{\partial L(0)}{\partial\theta_i} = \frac{m}{2}\ z'A_iz \ /\ z'z - \mathrm{tr}[MA_i] \ /\ 2$$

and the $(i,j)^{th}$ element of $I(0)$ is given by

$$m\ \mathrm{tr}[MA_iMA_j] \ /\ (2m+4) - \mathrm{tr}[MA_i]\ \mathrm{tr}[MA_j] \ /\ (2m+4)\ .$$

The LM test of H_0 against $H_a : \theta \neq 0$ rejects H_0 for large values of

$$s(0)'\ I(0)^{-1}\ s(0) \qquad (12)$$

assuming an asymptotic $\chi^2(p)$ distribution under H_0.

In the special case of $p = 1$, (12) becomes

$$\frac{\left(\frac{m}{2}\, z'Az \,/\, z'z - \mathrm{tr}[MA] \,/\, 2\right)^2}{\left\{m\, \mathrm{tr}[(MA)^2] - (\mathrm{tr}[MA])^2\right\} \,/\, (2m+4)} .$$

Observe that the square root of this statistic can be written as

$$\frac{z'Az \,/\, z'z - \mathrm{tr}[MA] \,/\, m}{\left(2\left\{m\, \mathrm{tr}[(MA)^2] - (\mathrm{tr}[MA])^2\right\} \Big/ \left\{(m^2(m+2)\right\}\right)^{1/2}} . \qquad (13)$$

It has an asymptotic standard normal distribution under H_0.

King and Hillier (1985) have shown that for $p = 1$, rejecting H_0 for large values of $z'Az/z'z$ is the LBI test of H_0 against H_a^+. It results in the same class of critical regions as the above one-sided LM test. Furthermore, rejecting H_0 for large values of (13) assuming a N(0,1) distribution is identical to applying the LBI test of H_0 using the two moment normal approximation to obtain critical values. Evans and King (1985) have found this approximation gives reasonably accurate critical values for tests of autocorrelation and heteroscedasticity. This is in contrast to the literature on the accuracy of the asymptotic critical values for the standard LM test.

2.5 The LMMP Test

From King and Wu (1990), the marginal-likelihood-based LMMP test of H_0 against $H_a^+ : \theta > 0$, rejects H_0 for large values of

$$\sum_{i=1}^{p} s_i(0) \Big/ \left\{\sum_{i=1}^{p} \sum_{j=1}^{p} I(0)_{ij}\right\}^{1/2}$$

$$= \frac{1}{2}\left\{m\, z'Az \,/\, z'z - \mathrm{tr}[MA]\right\} \Big/ \left\{\sum_{i=1}^{p} \sum_{j=1}^{p} I(0)_{ij}\right\}^{1/2} \qquad (14)$$

which has a N(0,1) asymptotic distribution under H_0, where $A = \sum_{i=1}^{p} A_i$. Our lemma implies that the numerator of (14) has mean zero and variance

$$\left\{m \; tr[(MA)^2] - (tr[MA])^2\right\} \Big/ \left\{2(m+2)\right\} .$$

This suggests that the LMMP test can be based on (13) in which $A = \sum_{i=1}^{p} A_i$. When an asymptotic critical value is used to apply a LMMP test, we call it an asymptotic LMMP (ALMMP) test.

3. Applications to Testing for AR(p) Disturbances and Hildreth-Houck Random Coefficients

This section is concerned with the application of the above theory to the problems of testing for general AR(p) disturbances and testing for Hildreth-Houck (1968) random coefficients in (1).

3.1 Testing for General AR(p) Disturbances

Consider the AR(p) disturbance process for (1),

$$u_t = \theta_1 u_{t-1} + \theta_2 u_{t-2} + \dots + \theta_p u_{t-p} + \varepsilon_t \qquad (15)$$

where $\varepsilon_t \sim IN(0,\sigma^2)$, $t = 1,\dots,n$. We will assume a stationary disturbance process which requires the autoregressive parameters θ_i, $i = 1,\dots,p$, to be such that the roots of the characteristic equation

$$1 - \theta_1\nu - \theta_2\nu^2 \dots - \theta_p\nu^p = 0$$

lie outside the unit circle. Our interest is in testing

$$H_0 : \theta_1 = \theta_2 = \dots = \theta_p = 0$$

against the alternative that at least one θ_i is nonzero, $i = 1,\dots,p$.

Under (15), $u \sim N(0,\sigma^2\Omega(\theta))$ where (see van der Leeuw (1992))

$$\Omega(\theta) = [L'L - NN']^{-1}, \tag{16}$$

in which L is the n×n matrix

$$L = \begin{bmatrix} 1 & 0 & & & & & 0 \\ -\theta_1 & 1 & . & & & & 0 \\ . & . & & . & & & . \\ . & . & & & & & . \\ -\theta_p & -\theta_{p-1} & & \cdot\!\cdot & & & . \\ 0 & -\theta_p & & & . & & . \\ . & & . & & & . & . \\ . & & & . & & 1 & 0 \\ 0 & 0 & \cdots & -\theta_p & \cdot\;\cdot & -\theta_1 & 1 \end{bmatrix}$$

and N is the n×p matrix of zeros but with the top p×p block being

$$\begin{bmatrix} -\theta_p & -\theta_{p-1} & -\theta_{p-2} & \cdot\;\cdot\;\cdot & -\theta_1 \\ 0 & -\theta_p & -\theta_{p-1} & & -\theta_2 \\ 0 & 0 & -\theta_p & & . \\ . & & & \ddots & . \\ 0 & 0 & 0 & \cdot\;\cdot\;\cdot & -\theta_p \end{bmatrix}.$$

From (16), it is possible to deduce that the inverse of the Cholesky decomposition of $\Omega(\theta)$, which we denote by $H(\theta) = \Omega(\theta)^{-1/2}$, is equal to L but with the top left p×p block replaced by the lower triangular matrix

$$\begin{bmatrix} h_{11} & 0 & 0 & \cdot\;\cdot & 0 \\ h_{21} & h_{22} & 0 & & \\ . & . & . & & \\ . & . & & . & \\ . & . & & & \\ h_{p1} & h_{p2} & & & h_{pp} \end{bmatrix}.$$

The h_{ij} values can be calculated recursively in the order indicated:

$$h_{p,p} = \left(1 - \theta_p^2\right)^{1/2}$$

$$h_{p,p-i} = \left(-\theta_i - \theta_{p-i}\theta_p\right) \Big/ h_{pp}, \qquad i = 1,\ldots,p-1 .$$

For $q = p-1, p-2, \ldots, 2$ and $m = p-q$,

$$h_{q,q} = \left(1 + \sum_{i=1}^{q-1} \theta_i^2 - \sum_{i=m+1}^{p} \theta_i^2 - \sum_{i=1}^{m} h_{q+i,q}^2\right)^{1/2}$$

$$h_{q,q-k} = \left(-\theta_k - \sum_{i=1}^{q-k-1} \theta_i\theta_{i+k} - \sum_{i=m+1}^{p-k} \theta_i\theta_{i+k} - \sum_{i=q+1}^{p} h_{i,q}h_{i,q-k}\right) \Big/ h_{qq},$$

for $k = 1,\ldots,q-2$, and

$$h_{q,1} = \left(-\theta_{q-1} - \theta_{m+1}\theta_p - \sum_{i=q+1}^{p} h_{i,q}h_{i,1}\right) \Big/ h_{qq} .$$

Finally,

$$h_{11} = \left(1 - \theta_p^2 - \sum_{i=1}^{p-1} h_{i+1,1}^2\right)^{1/2} .$$

The LR and Wald tests of H_0 require maximum marginal likelihood estimates of θ. Based on (4) and $H(\theta)$, this involves maximizing

$$\log f_m(\theta|y) = \log\left(\prod_{i=1}^{p} h_{ii}\right) - \tfrac{1}{2}\log|X^{*\prime}X^*| - \tfrac{m}{2}\log(e'e), \qquad (17)$$

where $X^* = H(\theta)X$ and e is the OLS residual vector from the regression

$$H(\theta)y = H(\theta)X + H(\theta)u . \qquad (18)$$

Let $\hat{\theta}$ denote the value of θ which maximizes (17) and let $\hat{h}_{ij}$ and $\hat{X}^*$ denote h_{ij} and X^*, respectively, evaluated at $\theta = \hat{\theta}$. Furthermore, let $\hat{e}$ denote the OLS residuals from (18) with $\theta = \hat{\theta}$.

A convenient form of the marginal-likelihood-based LR test is to reject H_0 for large values of

$$\log|X'X| + 2\log\left(\prod_{i=1}^{p}\hat{h}_{ii}\right) - \log|\hat{X}^{*\prime}\hat{X}^{*}| - m\log(\hat{e}'\hat{e}/z'z) \qquad (19)$$

which, under H_0, has an asymptotic $\chi^2(p)$ distribution.

In order to construct the Wald test, first observe that

$$\frac{\partial\Omega(\theta)}{\partial\theta_i} = -\Omega(\theta)\,\frac{\partial\Omega(\theta)^{-1}}{\partial\theta_i}\,\Omega(\theta)$$

and from (16)

$$\frac{\partial\Omega(\theta)^{-1}}{\partial\theta_i} = (B_i(\theta) + B_i'(\theta)) - (C_i(\theta) + C_i'(\theta)) \qquad (20)$$

where $B_i(\theta)$ is the n×n matrix

$$B_i(\theta) = \begin{bmatrix}
\theta_i & \theta_{i+1} & \cdots & \theta_p & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\theta_{i-1} & \theta_i & \cdots & \theta_{p-1} & \theta_p & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \ddots & & & & & & \vdots \\
\theta_1 & \theta_2 & & \theta_{p-i+1} & \cdots & \theta_p & & & & & 0 \\
-1 & \theta_1 & & & & & \ddots & & & & \vdots \\
0 & -1 & & & & & & \theta_p & 0 & & 0 \\
\vdots & & \ddots & & & & & \vdots & & & \vdots \\
\vdots & & & & & & -1 & \theta_1 & 0 & & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0
\end{bmatrix}$$

and $C_i(\theta)$ is the n×n matrix of zeros whose top left i×p block is

$$\begin{bmatrix} \theta_i & \theta_{i+1} & . & . & \theta_p & 0 & . & . & . & 0 \\ \theta_{i-1} & \theta_i & & & \theta_{p-1} & \theta_p & & & & . \\ . & . & & & & & . & & & . \\ . & . & & & & & & . & & . \\ . & . & & & & & & & . & . \\ \theta_2 & \theta_3 & & & & & & & & 0 \\ \theta_1 & \theta_2 & . & . & . & . & . & . & . & \theta_p \end{bmatrix} .$$

Define $D_i(\theta) = \frac{\partial \Omega(\theta)^{-1}}{\partial \theta_i} \Omega(\theta)$ so that

$$\Delta(\theta) \frac{\partial \Omega(\theta)}{\partial \theta_i} = -M(\theta)^* D_i(\theta)$$

where $M(\theta)^* = I_n - \Omega(\theta)^{-1}X(X'\Omega(\theta)^{-1}X)^{-1}X$. The Wald test statistic given by (11) simplifies to

$$\hat{\theta}' I(\hat{\theta})\hat{\theta} = \frac{1}{2(m+2)} \sum_{i=1}^{p} \sum_{j=1}^{p} \hat{\theta}_i \hat{\theta}_j \Big\{ m \, tr\Big[M(\hat{\theta})^* D_i(\hat{\theta}) M(\hat{\theta})^* D_j(\hat{\theta})\Big]$$

$$- \, tr\Big[M(\hat{\theta})^* D_i(\hat{\theta})\Big] tr\Big[M(\hat{\theta})^* D_j(\hat{\theta})\Big]\Big\} . \tag{21}$$

The LM test requires

$$A_i = \frac{-\partial \Omega(0)^{-1}}{\partial \theta_i}$$

which from (20) implies A_i is a matrix of zeros with two off-diagonals of ones which begin at the $(i+1,1)^{th}$ and $(1,i+1)^{th}$ elements. Thus

$$tr[MA_i] = 2 \sum_{\ell=1}^{n-i} m_{\ell,\ell+i}$$

and

$$tr[MA_i MA_j] = 4 \sum_{q=1}^{n-j} \sum_{\ell=1}^{n-i} m_{q,\ell} \, m_{q+j,\ell+i}$$

where m_{ij} is the $(i,j)^{th}$ element of M. Furthermore,

$z'A_i z = 2 \sum_{\ell=1}^{n-i} z_\ell z_{\ell+i}$ so that the LM test statistic is of the form of (12) where

$$s_i(0) = m \sum_{\ell=1}^{n-i} z_\ell z_{\ell+i} \Bigg/ \sum_{\ell=1}^{n} z_\ell^2 - \sum_{\ell=1}^{n-i} m_{\ell,\ell+i} \tag{22}$$

and the $(i,j)^{th}$ element of I(0) is

$$2m \sum_{q=1}^{n-j} \sum_{\ell=1}^{n-i} m_{q,\ell} m_{q+j,\ell+i} \Big/ (m+2) - 2 \left(\sum_{\ell=1}^{n-i} m_{\ell,\ell+i} \right) \left(\sum_{\ell=1}^{n-j} m_{\ell,\ell+j} \right) \Big/ (m+2). \tag{23}$$

3.2 Testing for Hildreth-Houck Random Coefficients

The Hildreth-Houck model assumes the regression coefficients at time t are generated as $\beta_t = \beta + \varepsilon_t$ where $\varepsilon_{it} \sim IN(0,\tau_i^2)$, $i = 1, \ldots, k$, and ε_t is independent of ε_s, $t \neq s$. Also, assuming $x_{1t} = 1$, the error term u_t is now part of ε_{1t}. Thus

$$y_t = x_t'\beta + x_t'\varepsilon_t = x_t'\beta + w_t$$

where x_t is the k×1 vector of regressors at time t. Therefore we have a regression model whose error term is w_t with $E(w_t) = 0$ and

$$\begin{aligned} var(w_t) &= \sum_{i=1}^{k} \tau_i^2 x_{it}^2 \\ &= \sigma^2(1 + r_t'\theta) \end{aligned}$$

where $\tau_1^2 = \sigma^2$, $\theta_{i-1} = \tau_i^2/\sigma^2$, $r_t = \left(x_{2t}^2, x_{3t}^2, \ldots, x_{kt}^2\right)'$ and $\theta = \left(\theta_1, \theta_2, \ldots, \theta_{k-1}\right)'$. The testing problem becomes one of testing $H_0 : \theta = 0$ against $H_a^+ : \theta > 0$ in the context of (1) where

$$\Omega(\theta) = \text{diag}(1 + r_1'\theta, \ldots, 1 + r_n'\theta) . \tag{24}$$

The alternative hypothesis is H_a^+ because θ_i, $i = 1, \ldots, k-1$, are ratios of variances which must be nonnegative.

One could use the LR or Wald test based on ML estimates of θ which take account of the constraint $\theta \geq 0$. There is also Gouriéroux, Holly and Monfort's (1980) Kuhn-Tucker test. Unfortunately, the asymptotic distributions of these test statistics under H_0 are probability mixtures of chi-squared distributions, making them extremely difficult to use if k is large. We therefore shall not consider these tests.

Breusch and Pagan's (1979) LM test ignores the one-sided nature of the problem and is based on rejecting H_0 for large values of

$$\frac{1}{2} f'\bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}'f \tag{25}$$

in which $\bar{X}$ is X with all elements replaced by their squares, and $f_t = z_t^2 / \hat{\sigma}^2 - 1$ where $\hat{\sigma}^2 = z'z/n$. An asymptotic test can be based on (25) having a $\chi^2(k-1)$ distribution under H_0, although a number of independent Monte Carlo studies report that this test rejects less frequently than it should under H_0. There are two directions in which the small-sample performance of this asymptotic test might be improved. The first is to construct an LM test using the marginal likelihood. The Evans and King (1985) study suggests that, at least for $k = 2$, this asymptotic test should be reasonably accurate in terms of its size. We can also take account of the one-sided nature of the testing problem, and apply King and Wu's (1990) ALMMP test. A third possibility involves combining both suggestions and deriving an ALMMP test from the marginal likelihood.

For our problem

$$A_i = \operatorname{diag}\left(x_{i+1,1}^2, \ldots, x_{i+1,n}^2\right),$$

$$\operatorname{tr}[MA_i] = \sum_{t=1}^{n} m_{tt}\, x_{i+1,t}^2$$

and

$$\operatorname{tr}[MA_iMA_j] = \sum_{t=1}^{n} x_{j+1,t}^2 \sum_{q=1}^{n} x_{i+1,q}^2\, m_{tq}^2\,, \quad i,j = 1,..,k-1.$$

Thus the marginal-likelihood-based LM test rejects H_0 for large values of (12), where

$$s_i(0) = \frac{1}{2} \sum_{t=1}^{n} x_{i+1,t}^2 \left(z_t^2 / \tilde{\sigma}^2 - m_{tt}\right) \tag{26}$$

and the $(i,j)^{th}$ element of I(0) is

$$m \left\{\sum_{t=1}^{n} x_{j+1,t}^2 \sum_{q=1}^{n} x_{i+1,q}^2\, m_{tq}^2\right\} \Big/ (2m+4)$$

$$- \left\{\sum_{t=1}^{n} m_{tt}\, x_{i+1,t}^2\right\} \left\{\sum_{t=1}^{n} m_{tt}\, x_{j+1,t}^2\right\} \Big/ (2m+4) \tag{27}$$

in which $\tilde{\sigma}^2 = z'z/m$.

The ALMMP test derived from the standard likelihood function is based on rejecting H_0 for large values of

$$f'\bar{X}^*\ell \Big/ \left\{2 \sum_{t=1}^{n} \left[x_t^* - \frac{1}{n} \sum_{t=1}^{n} x_t^*\right]\right\}^{1/2} \tag{28}$$

assuming an N(0,1) asymptotic distribution under H_0, where $\bar{X}^*$ is $\bar{X}$ with the first column of ones deleted, ℓ is a $(k-1)\times 1$ vector of ones and

$$x_t^* = \sum_{i=2}^{k} x_{it}^2\,.$$

From (14), the corresponding ALMMP test derived from the marginal likelihood rejects H_0 for large values of

$$\frac{\sum_{t=1}^{n} x_t^* \left(z_t^2 / \tilde{\sigma}^2 - m_{tt}\right)}{\left(2\left\{m\left[\sum_{t=1}^{n} x_t^* \sum_{q=1}^{n} x_q^* m_{tq}^2\right] - \left[\sum_{t=1}^{n} m_{tt} x_t^*\right]^2\right\} \Big/ (m+2)\right)^{1/2}} \tag{29}$$

against an asymptotic $N(0,1)$ distribution under H_0.

4. Monte Carlo Size and Power Comparisons

Monte Carlo simulations were conducted to compare the small-sample size and power properties of classical and marginal-likelihood-based tests for AR(4) disturbances and for Hildreth-Houck random coefficients in the context of (1).

4.1 Experimental Design

For each testing problem, the first part of the study involved a comparison of estimated sizes using asymptotic critical values. The second part involved the use of the Monte Carlo method to estimate appropriate five percent critical values of each of the tests which were then used to compare powers at approximately the same significance level. In the case of testing for AR(4) disturbances, the tests involved were the classical LR, Wald (W) and LM tests as well as the marginal-likelihood-based LR (MLR), Wald (MW) and LM (MLM) tests. The test statistics of the latter three tests are respectively (19), (21) and (12) with $s_i(0)$ given by (22) and $I(0)$ by (23). The tests of H_0 against Hildreth-Houck random regression coefficients are the Breusch-Pagan LM (BPLM), MLM, ALMMP and the marginal likelihood based (MALMMP) tests.

The respective test statistics are (25), (12) with $s_i(0)$ given by (26) and I(0) by (27), (28) and (29).

The following n×k X matrices, chosen to represent a range of non-seasonal, seasonal and cross-section data, were used in the comparison:

X1 : (n×2). A constant and a linear trend.

X2 : (n×4). A constant and three quarterly seasonal dummy variables.

X3 : (n×3). A constant, the quarterly seasonally adjusted Australian household disposable income and private consumption expenditure series, commencing 1959(4).

X4 : (n×3). A constant, quarterly Australian private capital movements and Government capital movements commencing 1968(1). For k = 5, the additional regressors are these two variables lagged one quarter.

X5 : (n×3). Australian cross-sectional data classified according to eight categories of sex/marital status and eight categories of age. The variables are a constant, population, and number of households whose head belongs to the given population category for 1961. See Williams and Sams (1981) for further details of this data set.

X6 : (n×3). A constant and two regressors of independent drawings from the (0,20) uniform distribution and the log-normal distribution with a coefficient of variation of one, respectively.

For testing against AR(4) disturbances, sizes and powers were estimated for X1, X2, X3 and X4 (with k = 5) for n = 30 and 60. 1000 replications and the following parameter combinations in (15) were used: θ' = (0.3,0,0,0), (0.5,0,0,0), (0,0.3,0,0), (0,0,0,0.3), (0.3,0.2,0,0), (0.3,0.2,0.2,0), (0.3,0.2,0.2,0.1), (-0.3,0,0,0), (0,0,0,-0.3), (-0.3,0.2,-0.2,0.1), (0.3,-0.2,0.2,-0.2).

For testing against Hildreth-Houck coefficients, the design matrices X3, X4, X5 and X6 all with k = 3 and n = 20 and 60 were employed. 2000 replications and the following parameter combinations in (24) were used:

θ_1 = 0.0, 0.3, 0.7, 1.0, 3.0, 5.0,

θ_2 = 0.0, 0.3, 0.7, 1.0, 3.0, 9.0.

All tests are invariant to the values of β and σ^2. This follows from Breusch (1980) in the case of the LR and Wald tests and because the LM and LMMP tests are based on ratios of quadratic forms in OLS residuals. Consequently, without loss of generality, β_i, i = 1,...,k, and σ^2 were set equal to one in the simulations.

4.2 The Size Results

Table 1 reports the estimated sizes of the six tests against AR(4) disturbances when asymptotic critical values at the five percent nominal level are used. The corresponding estimated sizes of the four tests for Hildreth-Houck random coefficients are presented in table 2. In both tables, a star superscript denotes an estimated size significantly different from 0.05 at the one percent level.

Table 1 reveals that all estimated sizes of the classical LR and W tests for AR(4) disturbances are significantly below 0.05 and are almost zero. Also, there is no clear sign of improvement as n increases from 30 to 60. In contrast, the classical LM test has acceptable sizes, at least for the design matrices used in this study. Only in one case, for n = 30, is an estimated size significantly greater than 0.05. With respect to the marginal-likelihood-based tests for AR(4) disturbances, the estimated sizes of the MLR and MLM tests are most acceptable. In the case of the LR test, basing inferences on the marginal likelihood provides a vast improvement in asymptotic accuracy. On the other hand, the estimated sizes of the MW test are generally significantly greater than the nominal level, although there is a clear tendency for these sizes to converge to 0.05 as n increases.

For testing against Hildreth-Houck coefficients, all but one of the estimated sizes of the BPLM test are significantly below 0.05. This is consistent with results reported by Breusch and Pagan (1979), Godfrey (1978), Griffiths and Surekha (1986) and Honda (1988). The ALMMP critical values seem more accurate although there is a noticeable trend for estimated sizes to be significantly below 0.05 when n = 60. The estimated sizes of the marginal-likelihood-based tests are generally much closer to 0.05 than their respective conventional counterparts. This is particularly true for the LM test. Although there appears to be a tendency for the marginal-likelihood-based tests to have true sizes slightly above 0.05, overall their sizes are reasonably acceptable, ranging from 0.050 to 0.078.

Overall, it seems clear that the use of the marginal likelihood improves the accuracy of standard asymptotic critical values. The improvement is stunning in the case of the LR test, less obvious for the LM test and rather debatable for the ALMMP test. Only for the Wald test does the use of the marginal likelihood result in unacceptable sizes. A number of studies have found the Wald test can have rather inaccurate asymptotic critical values (see for example Lafontaine and White (1986) and Breusch and Schmidt (1988)). Our results may reflect yet a further problem with the Wald test rather than with the marginal likelihood.

4.3 Power Results

Estimated powers of the six tests for AR(4) disturbances using simulated critical values at the five percent level are presented in tables 3 and 4. A striking feature of these results is the erratic behaviour of the classical LR and Wald tests. Their range of powers is much greater than those of the other tests going from near zero to almost 0.95. For every X matrix, some of the estimated powers of these two tests are below the significance level, thus showing they are biased tests and have blind spots. This occurs at $\theta = (-0.3,0,0,0)'$, $(0,0,0,-0.3)'$ and $(-0.3,0.2,-0.2,0.1)'$ for most X matrices. Generally the powers of all tests increase as the sample size increases, *ceteris paribus*. The classical LR and Wald tests again provide the only exceptions which occur at the above three points for some X matrices.

In terms of power, no one test dominates the others. The minimum power of each of the classical tests is significantly lower than that of its marginal-likelihood-based counterpart. For $n = 30$, the average power of the LR test for each X matrix is slightly higher than that for the MLR test. A similar pattern holds for the Wald test while the reverse is the case for the LM test. For $n = 60$, however, the average power of each of the marginal-likelihood-based tests is higher than that of the respective classical test. The differences in average power are very large for the LR and Wald tests, ranging between 0.111 and 0.164 for the LR test and 0.083 and 0.142 for the Wald test.

The question of which test has the best overall power is a difficult one because different power curves dominate at different points. Marginal-likelihood-based tests seem more reliable in that they have higher minimum powers, at least for the points we considered. The MLM test always has the highest average power of the six tests for n = 30 while the MLR (X1, X2, X3) and MLM (X4) tests share that distinction when n = 60.

We now discuss the estimated powers of the tests for Hildreth-Houck coefficients. A feature of these results is that the ALMMP and MALMMP tests are identical when appropriate critical values are found. This is because the two test statistics can be written as monotonic functions of

$$\sum_{t=1}^{n} x_t^* z_t^2 / z'z$$

and it is from this statistic that both tests derive their power. Selected estimated powers for X3, X4, X5 and X6 are presented in tables 5, 6, 7 and 8, respectively. The high degree of variation with the choice of design matrix is not surprising because the degree of heteroscedasticity induced by the random coefficients depends on the variation in the squared regressors. With a few minor exceptions, for a given X matrix, there is very little variation in power as θ_1 and/or θ_2 change. There is a tendency for powers to increase as θ_1 or θ_2 increases, *ceteris paribus*, although there are many exceptions.

A comparison of the BPLM and MLM tests shows that the MLM test has better power for half the regressions, namely X3, X4, and X5 when n = 60 and X3 when n = 20. An explanation is that the MLM test has its power curve more correctly centred at H_0. At any point under H_a, the slopes of the two power curves will differ. When appropriate critical values are used for both tests, this may mean the BPLM test has higher power than the MLM test over about half the unrestricted parameter space. Of course, we are only interested in the positive quadrant of this space.

For design matrices X3 and X4, the one-sided (M)ALMMP test is always more powerful than the two-sided BPLM and MLM tests. For X6 and particularly for X5, the dominance of the (M)ALMMP test is less clear-cut. In the case of X6, the (M)ALMMP test is always most powerful except on the boundary $\theta_1 = 0.0$ and also occasionally when $\theta_2 = 3.0, 9.0$ and θ_1 is small. On the boundary $\theta_1 = 0.0$, the (M)ALMMP test loses power as n increases in contrast to the two LM tests. For X5, particularly when $n = 20$, the LM tests are more powerful than the (M)ALMMP test for $\theta_2 = 0$ and as θ_1 increases for a wider range of θ_2 values.

Recently Wu (1991) compared the powers of the exact LMMP invariant (LMMPI) test with the power envelope for a range of testing problems involving the disturbances of (1). He noted that the LMMPI test can be viewed as a sum of locally best invariant tests for each of the parameters being tested. He found the test works well when these individual statistics, in our case monotonic functions of

$$\sum_{t=1}^{n} x_{t+1,t}^2 \; z_t^2/z'z,$$

are positively correlated and poorly when they are negatively correlated. We note that the ALMMP and MALMMP tests for Hildreth-Houck regression coefficients work best when the squared regressors corresponding to the coefficients under test are positively correlated.

5. Concluding Remarks

This paper presents general formulae for the LR, Wald, LM and ALMMP tests of covariance matrices of regression disturbances based on the marginal likelihood function. These tests can also be derived using standard methods by treating the maximal invariant statistic as the observed data. Monte Carlo results reported by Corduas (1986), Evans and King (1985) and in this paper suggest that this approach results in tests whose true sizes are closer to the nominal size than their conventional counterparts. The Wald test in very small samples may be a possible exception to this general observation. The traditional LM, LR and Wald tests have often been found to have inaccurate critical values for many different econometric testing problems. The approach used in this paper can be used with the expectation of improving accuracy in a range of other testing situations.

Our results indicate that the use of marginal-likelihood-based tests does not necessarily result in increased power everywhere under the alternative hypothesis. It does appear, however, that these tests have better centered power curves in the sense that they are less likely to have points under the alternative hypothesis with power below the size and near zero. This is particularly evident for the LR and Wald tests for AR(4) disturbances. Thus in terms of both size and power, we conclude that a marginal-likelihood based test is likely to be more reliable than its classical counterpart.

References

Breusch, T.S., 1980, Useful invariance results for generalized regression models, Journal of Econometrics 13, 327-340.

Breusch, T.S. and A.R. Pagan, 1979, A simple test for heteroscedasticity and random coefficient variation, Econometrica 47, 1287-1294.

Breusch, T.S. and P. Schmidt, 1988, Alternative forms of the Wald test: How long is a piece of string?, Communications in Statistics A 17, 2789-2795.

Corduas, M., 1986, The use of marginal likelihood in testing for correlation in time series regression, Unpublished M.Phil. thesis (University of Lancaster).

Durbin, J. and G.S. Watson, 1971, Testing for serial correlation in least squares regression III, Biometrika 58, 1-19.

Evans, M.A. and M.L. King, 1985, Critical value approximations for tests of linear regression disturbances, Australian Journal of Statistics 27, 68-83.

Godfrey, L.G., 1978, Testing for multiplicative heteroscedasticity, Journal of Econometrics 8, 227-236.

Godfrey, L.G., 1988, Misspecification tests in econometrics: The Lagrange multiplier principle and other approaches, (Cambridge University Press, Cambridge).

Gourieroux, C., A. Holly and A. Monfort, 1980, Kuhn-Tucker, likelihood ratio and Wald tests for nonlinear models with inequality constraints on the parameters, Discussion Paper No. 770, Harvard Institute of Economic Research (Harvard University).

Griffiths, W.E. and K. Surekha, 1986, A Monte Carlo evaluation of the power of some tests for heteroscedasticity, Journal of Econometrics 31, 219-231.

Harville, D.A., 1977, Maximum likelihood approaches to variance component estimation and to related problems, Journal of the American Statistical Association 72, 320-338.

Hildreth, C. and J.P. Houck, 1968, Some estimators for a linear model with random coefficients, Journal of the American Statistical Association 63, 584-595.

Honda, Y., 1988, A size correction to the Lagrange multiplier test for heteroscedasticity, Journal of Econometrics 38, 375-386.

Judge, G.G., W.E. Griffiths, R.C. Hill, H. Lutkepohl and T.C. Lee, 1985, The theory and practice of econometrics, 2nd ed., (John Wiley and Sons, New York).

Kalbfleisch, J.D. and D.A. Sprott, 1970, Application of likelihood methods to models involving large numbers of parameters, Journal of the Royal Statistical Society B 32, 175-194.

King, M.L., 1987a, Testing for autocorrelation in linear regression models: A survey, in Specification analysis in the linear model, eds. M.L. King and D.E.A. Giles, (Routledge and Kegan Paul, London), 19-73.

King, M.L., 1987b, Towards a theory of point-optimal testing, Econometric Reviews 6, 169-218.

King, M.L. and G.H. Hillier, 1985, Locally best invariant tests of the error covariance matrix of the linear regression model, Journal of the Royal Statistical Society B 47, 98-102.

King, M.L. and P.X. Wu, 1990, Locally optimal one-sided tests for multi-parameter hypotheses, presented at the Sixth World Congress of the Econometric Society, Barcelona, Working Paper 2/90 (Department of Econometrics, Monash University, Melbourne).

Lafontaine, F. and K.J. White, 1986, Obtaining any Wald statistic you want, Economics Letters 21, 35-40.

Moulton, B.R. and W.C. Randolph, 1989, Alternative tests of the error components model, Econometrica 57, 685-693.

Pagan, A.R., 1984, Model evaluation by variable addition, in Econometrics and quantitative economics, eds. D.F. Hendry and K.F. Wallis, (Basil Blackwell, Oxford), 103-133.

Pagan, A.R. and A.D. Hall, 1983, Diagonostic tests as residual analysis, Econometric Reviews 2, 159-218.

Rao, C.R., 1973, Linear statistical inference and its application, 2nd ed., (Wiley, New York).

Robinson, D.L., 1987, Estimation and use of variance components, The Statistician 36, 3-14.

Smith, M.D., 1987, Quadratic forms in normally distributed variables: Some econometric applications, Unpublished Ph.D. thesis (Monash University, Melbourne).

Smith, M.D., 1989, On the expectation of quadratic forms in normal variables, Journal of Multivariate Analysis 31, 244-257.

Tunnicliffe Wilson, G., 1989, On the use of marginal likelihood in time series model estimation, Journal of the Royal Statistical Society B 51, 15-27.

van der Leeuw, J, 1992, The covariance matrix of ARMA-errors in closed form, Mimeo. (Tilburg University).

Verbyla, A.P., 1990, A conditional derivation of residual maximum likelihood, Australian Journal of Statistics 32, 227-230.

Williams, P. and D. Sams, 1981, Household headship in Australia: Further developments to the IMPACT Project's Econometric Model of Household Headship, IMPACT Project Research Centre Working Paper BP-26, (Melbourne University).

Wu, P.X., 1991, One-sided and partially one-sided multiparameter hypothesis testing in econometrics, Unpublished Ph.D. thesis (Monash University, Melbourne).

Table 1: Estimated sizes of the LR, W, LM, MLR, MW and MLM tests based on asymptotic critical values at the five percent nominal level.

Design Matrix	n	Classical likelihood based tests			Marginal likelihood based tests		
		LR	W	LM	MLR	MW	MLM
X1	30	.000*	.003*	.059	.056	.091*	.059
	60	.003*	.004*	.052	.053	.074*	.058
X2	30	.006*	.018*	.076*	.061	.112*	.063
	60	.004*	.005*	.063	.042	.058	.051
X3	30	.000*	.003*	.056	.053	.110*	.058
	60	.002*	.003*	.059	.061	.080*	.062
X4	30	.002*	.013*	.053	.067	.193*	.067
	60	.002*	.003*	.062	.060	.091*	.062

* denotes significantly different from 0.05 at the one percent level.

Table 2: Estimated sizes of the BPLM, MLM, ALMMP and MALMMP tests based on asymptotic critical values at the five percent nominal level.

Design Matrix	n	Classical likelihood based tests		Marginal likelihood based tests	
		BPLM	ALMMP	MLM	MALMMP
X3	20	.028*	.038	.054	.056
	60	.030*	.036*	.056	.064*
X4	20	.036*	.052	.056	.058
	60	.006*	.006*	.064*	.062
X5	20	.016*	.041	.078*	.076*
	60	.030*	.030*	.067*	.070*
X6	20	.022*	.044	.050	.058
	60	.040	.062	.054	.066*

* denotes significantly different from 0.05 at the one percent level.

Table 3: Estimated powers of the six tests for AR(4) disturbances using simulated critical values at the five percent level and design matrices X1 and X2.

n	Test												
		θ_1	0.3	0.5	0.0	0.0	0.3	0.3	0.3	-0.3	0.0	-0.3	0.3
		θ_2	0.0	0.0	0.3	0.0	0.2	0.2	0.2	0.0	0.0	0.2	-0.2
		θ_3	0.0	0.0	0.0	0.0	0.0	0.2	0.2	0.0	0.0	-0.2	0.2
		θ_4	0.0	0.0	0.0	0.3	0.0	0.0	0.1	0.0	-0.3	0.1	-0.2
								X1					
30	LR		.361	.718	.102	.084	.453	.451	.419	.005	.073	.006	.311
	MLR		.158	.450	.166	.147	.248	.269	.246	.172	.159	.691	.251
	W		.351	.710	.098	.088	.441	.453	.421	.006	.061	.006	.314
	MW		.112	.284	.134	.144	.133	.086	.051	.199	.196	.724	.228
	LM		.102	.313	.171	.205	.173	.171	.151	.221	.173	.758	.136
	MLM		.180	.500	.240	.210	.363	.402	.369	.171	.132	.735	.162
60	LR		.560	.932	.168	.136	.717	.776	.786	.000	.042	.001	.421
	MLR		.375	.849	.401	.381	.594	.735	.751	.408	.371	.922	.560
	W		.545	.927	.169	.136	.698	.768	.780	.000	.038	.001	.417
	MW		.353	.816	.370	.341	.529	.572	.453	.420	.386	.929	.548
	LM		.277	.780	.350	.331	.531	.692	.691	.442	.405	.959	.296
	MLM		.382	.850	.439	.367	.676	.800	.795	.366	.333	.953	.302
								X2					
30	LR		.286	.678	.128	.077	.448	.522	.535	.002	.030	.009	.188
	MLR		.154	.440	.138	.119	.267	.373	.415	.151	.121	.412	.180
	W		.294	.675	.137	.083	.452	.532	.556	.002	.029	.009	.202
	MW		.149	.405	.115	.092	.232	.327	.322	.148	.170	.310	.199
	LM		.190	.488	.131	.071	.342	.432	.500	.171	.227	.474	.154
	MLM		.154	.465	.182	.175	.344	.486	.553	.140	.087	.530	.100
60	LR		.580	.950	.239	.149	.772	.877	.898	.002	.038	.001	.424
	MLR		.404	.885	.399	.372	.661	.832	.874	.440	.375	.856	.556
	W		.551	.942	.224	.145	.749	.861	.894	.002	.031	.001	.419
	MW		.388	.861	.360	.310	.633	.807	.852	.408	.378	.839	.539
	LM		.404	.864	.352	.207	.691	.853	.864	.386	.461	.865	.337
	MLM		.404	.862	.415	.357	.708	.865	.886	.384	.316	.891	.290

Table 4: Estimated powers of the six tests for AR(4) disturbances using simulated critical values at the five percent level and design matrices X3 and X4.

n	Test	θ_1 / θ_2 / θ_3 / θ_4										
		θ_1: 0.3	0.5	0.0	0.0	0.3	0.3	0.3	-0.3	0.0	-0.3	0.3
		θ_2: 0.0	0.0	0.3	0.0	0.2	0.2	0.2	0.0	0.0	0.2	-0.2
		θ_3: 0.0	0.0	0.0	0.0	0.0	0.2	0.2	0.0	0.0	-0.2	0.2
		θ_4: 0.0	0.0	0.0	0.3	0.0	0.0	0.1	0.0	-0.3	0.1	-0.2
		X3										
30	LR	.346	.689	.074	.066	.390	.381	.342	.006	.070	.002	.285
	MLR	.146	.374	.149	.131	.214	.231	.200	.151	.145	.672	.219
	W	.356	.703	.077	.078	.414	.386	.356	.006	.069	.002	.300
	MW	.079	.222	.109	.125	.103	.076	.048	.160	.157	.680	.171
	LM	.105	.256	.178	.205	.127	.107	.088	.222	.176	.753	.143
	MLM	.202	.479	.238	.189	.354	.367	.327	.156	.132	.718	.170
60	LR	.555	.931	.138	.118	.694	.711	.712	.001	.049	.000	.444
	MLR	.347	.815	.384	.359	.544	.678	.681	.387	.355	.921	.562
	W	.545	.925	.129	.118	.687	.707	.710	.001	.047	.000	.456
	MW	.277	.750	.309	.306	.442	.544	.512	.388	.341	.920	.486
	LM	.219	.713	.313	.329	.406	.575	.556	.455	.408	.959	.304
	MLM	.353	.825	.406	.353	.616	.749	.722	.335	.294	.954	.284
		X4										
30	LR	.320	.671	.093	.055	.413	.471	.486	.006	.057	.004	.253
	MLR	.145	.387	.137	.118	.239	.346	.375	.124	.133	.572	.175
	W	.311	.674	.091	.053	.394	.466	.482	.005	.055	.004	.254
	MW	.098	.248	.098	.105	.145	.189	.186	.117	.115	.582	.142
	LM	.094	.296	.129	.115	.168	.292	.350	.178	.138	.599	.110
	MLM	.138	.406	.167	.132	.289	.444	.493	.113	.079	.568	.096
60	LR	.604	.945	.146	.092	.720	.750	.780	.000	.058	.001	.472
	MLR	.320	.800	.330	.329	.532	.691	.747	.342	.301	.905	.485
	W	.623	.949	.152	.092	.727	.760	.795	.000	.056	.001	.499
	MW	.291	.760	.313	.305	.506	.670	.733	.334	.304	.905	.444
	LM	.264	.742	.282	.271	.467	.671	.699	.395	.422	.941	.281
	MLM	.344	.811	.374	.334	.613	.795	.816	.329	.276	.938	.257

Table 5: Estimated powers for X3 and the BPLM, MLM and ALMMP/MALMMP tests using simulated critical values at the five percent significance level.

TEST	θ_1	$\theta_2 = 0$	$\theta_2 = .3$	$\theta_2 = .7$	$\theta_2 = 1$	$\theta_2 = 3$	$\theta_2 = 9$
				n = 20			
BPLM	0.0	.050	.070	.080	.080	.084	.084
MLM		.050	.080	.087	.088	.090	.090
(M)ALMMP		.050	.096	.110	.114	.122	.125
BPLM	.3	.072	.081	.084	.084	.085	.084
MLM		.080	.086	.090	.090	.090	.090
(M)ALMMP		.099	.109	.115	.116	.122	.125
BPLM	.7	.084	.084	.084	.084	.086	.086
MLM		.088	.088	.090	.090	.090	.090
(M)ALMMP		.113	.115	.118	.121	.123	.124
BPLM	1.0	.085	.085	.086	.085	.086	.086
MLM		.088	.088	.090	.090	.090	.090
(M)ALMMP		.115	.118	.120	.122	.123	.124
BPLM	3.0	.087	.088	.087	.086	.086	.086
MLM		.090	.091	.092	.092	.090	.090
(M)ALMMP		.124	.125	.125	.126	.125	.125
BPLM	5.0	.088	.088	.088	.088	.088	.088
MLM		.091	.092	.092	.092	.091	.090
(M)ALMMP		.126	.126	.126	.126	.126	.125
				n = 60			
BPLM	0.0	.050	.821	.869	.880	.894	.899
MLM		.050	.831	.876	.890	.898	.901
(M)ALMMP		.050	.874	.912	.922	.934	.938
BPLM	.3	.844	.873	.885	.887	.896	.900
MLM		.854	.882	.892	.896	.898	.901
(M)ALMMP		.894	.917	.927	.930	.936	.938
BPLM	.7	.886	.890	.895	.897	.900	.900
MLM		.892	.896	.899	.900	.900	.901
(M)ALMMP		.929	.932	.935	.936	.938	.939
BPLM	1.0	.894	.898	.899	.900	.901	.901
MLM		.900	.900	.900	.900	.900	.902
(M)ALMMP		.938	.938	.938	.938	.940	.940
BPLM	3.0	.906	.904	.904	.903	.904	.902
MLM		.908	.906	.906	.906	.904	.903
(M)ALMMP		.944	.944	.944	.944	.944	.942
BPLM	5.0	.907	.907	.907	.906	.904	.904
MLM		.910	.910	.908	.908	.906	.904
(M)ALMMP		.947	.946	.944	.944	.944	.944

Table 6: **Estimated powers for X4 and the BPLM, MLM and ALMMP/MALMMP tests using simulated critical values at the five percent significance level.**

TEST	θ_1	$\theta_2 = 0$	$\theta_2 = .3$	$\theta_2 = .7$	$\theta_2 = 1$	$\theta_2 = 3$	$\theta_2 = 9$
				n = 20			
BPLM	0.0	.050	.437	.437	.437	.437	.437
MLM		.050	.416	.416	.416	.416	.416
(M)ALMMP		.050	.502	.502	.502	.502	.502
BPLM	.3	.372	.347	.370	.379	.400	.424
MLM		.356	.322	.344	.354	.387	.404
(M)ALMMP		.437	.458	.474	.484	.496	.502
BPLM	.7	.372	.339	.347	.354	.385	.408
MLM		.356	.315	.322	.330	.364	.393
(M)ALMMP		.437	.450	.458	.465	.487	.499
BPLM	1.0	.372	.339	.344	.347	.377	.398
MLM		.356	.319	.318	.322	.350	.386
(M)ALMMP		.437	.446	.455	.458	.480	.494
BPLM	3.0	.372	.353	.340	.339	.347	.377
MLM		.356	.334	.322	.316	.322	.350
(M)ALMMP		.437	.440	.446	.447	.458	.480
BPLM	5.0	.372	.361	.348	.342	.344	.363
MLM		.356	.341	.328	.324	.318	.332
(M)ALMMP		.437	.438	.440	.443	.454	.472
				n = 60			
BPLM	0.0	.050	.982	.982	.982	.982	.982
MLM		.050	.990	.990	.990	.990	.990
(M)ALMMP		.050	.994	.994	.994	.994	.994
BPLM	.3	.994	.990	.988	.986	.983	.982
MLM		.992	.994	.993	.993	.990	.989
(M)ALMMP		.998	.998	.997	.996	.994	.994
BPLM	.7	.994	.994	.990	.990	.986	.984
MLM		.992	.996	.994	.994	.993	.990
(M)ALMMP		.998	.998	.998	.998	.995	.994
BPLM	1.0	.994	.993	.992	.990	.988	.984
MLM		.992	.996	.994	.994	.993	.990
(M)ALMMP		.998	.999	.998	.998	.996	.994
BPLM	3.0	.994	.993	.994	.993	.990	.988
MLM		.992	.996	.996	.997	.993	.993
(M)ALMMP		.998	.999	.999	.999	.998	.996
BPLM	5.0	.994	.993	.993	.993	.992	.988
MLM		.992	.994	.996	.996	.995	.994
(M)ALMMP		.998	.999	.999	.000	.998	.998

Table 7: Estimated powers for X5 and the BPLM, MLM and ALMMP/MALMMP tests using simulated critical values at the five percent significance level.

TEST	θ_1	$\theta_2 = 0$	$\theta_2 = .3$	$\theta_2 = .7$	$\theta_2 = 1$	$\theta_2 = 3$	$\theta_2 = 9$
				n = 20			
BPLM	0.0	.050	.596	.752	.800	.882	.907
MLM		.050	.524	.690	.745	.839	.880
(M)ALMMP		.050	.630	.774	.819	.892	.916
BPLM	.3	.627	.724	.786	.809	.866	.902
MLM		.606	.691	.750	.774	.839	.876
(M)ALMMP		.620	.738	.800	.828	.886	.910
BPLM	.7	.761	.792	.817	.830	.868	.897
MLM		.752	.774	.797	.809	.844	.872
(M)ALMMP		.742	.786	.819	.837	.882	.906
BPLM	1.0	.801	.818	.832	.842	.869	.897
MLM		.790	.802	.818	.824	.846	.872
(M)ALMMP		.776	.800	.827	.840	.880	.906
BPLM	3.0	.861	.884	.873	.874	.880	.892
MLM		.856	.856	.857	.858	.867	.873
(M)ALMMP		.827	.833	.838	.848	.870	.898
BPLM	5.0	.877	.879	.880	.880	.884	.892
MLM		.870	.869	.871	.871	.874	.880
(M)ALMMP		.835	.840	.845	.850	.869	.891
				n = 60			
BPLM	0.0	.050	.619	.800	.852	.956	.986
MLM		.050	.642	.810	.864	.957	.987
(M)ALMMP		.050	.657	.813	.872	.956	.982
BPLM	.3	.903	.929	.948	.959	.980	.984
MLM		.912	.936	.954	.964	.984	.987
(M)ALMMP		.897	.934	.960	.968	.986	.992
BPLM	.7	.975	.977	.982	.985	.990	.993
MLM		.976	.981	.984	.988	.992	.994
(M)ALMMP		.975	.982	.986	.988	.992	.996
BPLM	1.0	.985	.987	.988	.990	.993	.994
MLM		.988	.990	.993	.993	.995	.996
(M)ALMMP		.988	.990	.992	.992	.996	.997
BPLM	3.0	.998	.998	.999	.999	1.000	.998
MLM		.999	1.000	1.000	1.000	1.000	.999
(M)ALMMP		.998	.999	.999	1.000	1.000	1.000
BPLM	5.0	1.000	1.000	1.000	1.000	1.000	1.000
MLM		1.000	1.000	1.000	1.000	1.000	1.000
(M)ALMMP		1.000	1.000	1.000	1.000	1.000	1.000

Table 8: Estimated powers for X6 and the BPLM, MLM and ALMMP/MALMMP tests using simulated critical values at the five percent significance level.

TEST	θ_1	$\theta_2 = 0$	$\theta_2 = .3$	$\theta_2 = .7$	$\theta_2 = 1$	$\theta_2 = 3$	$\theta_2 = 9$
				n = 20			
BPLM	0.0	.050	.218	.326	.366	.462	.516
MLM		.050	.268	.385	.437	.536	.579
(M)ALMMP		.050	.078	.102	.110	.140	.163
BPLM	.3	.716	.710	.695	.690	.651	.596
MLM		.704	.694	.684	.674	.649	.622
(M)ALMMP		.816	.804	.788	.778	.706	.576
BPLM	.7	.733	.730	.726	.724	.698	.651
MLM		.720	.718	.714	.712	.692	.657
(M)ALMMP		.831	.824	.818	.814	.782	.694
BPLM	1.0	.736	.734	.733	.730	.714	.671
MLM		.721	.721	.716	.717	.704	.652
(M)ALMMP		.836	.830	.825	.822	.800	.734
BPLM	3.0	.739	.738	.736	.738	.736	.724
MLM		.728	.728	.728	.728	.722	.711
(M)ALMMP		.842	.840	.838	.837	.828	.805
BPLM	5.0	.740	.740	.740	.738	.739	.731
MLM		.729	.728	.728	.728	.726	.720
(M)ALMMP		.842	.842	.840	.840	.836	.816
				n = 60			
BPLM	0.0	.050	.590	.812	.871	.956	.977
MLM		.050	.618	.830	.885	.960	.984
(M)ALMMP		.050	.034	.034	.036	.041	.044
BPLM	.3	.992	.968	.988	.983	.970	.946
MLM		.990	.987	.984	.982	.970	.950
(M)ALMMP		.996	.995	.993	.990	.968	.810
BPLM	.7	.992	.992	.991	.990	.982	.968
MLM		.992	.991	.899	.989	.982	.968
(M)ALMMP		.996	.996	.996	.995	.990	.959
BPLM	1.0	.992	.992	.992	.991	.988	.974
MLM		.992	.991	.990	.990	.984	.973
(M)ALMMP		.997	.996	.996	.996	.993	.976
BPLM	3.0	.994	.994	.992	.992	.991	.988
MLM		.992	.992	.991	.991	.991	.985
(M)ALMMP		.997	.997	.997	.997	.996	.994
BPLM	5.0	.994	.994	.994	.992	.992	.990
MLM		.992	.992	.992	.992	.991	.990
(M)ALMMP		.997	.997	.997	.997	.996	.995