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THE STOCHASTIC SPECIFICATION OF DEMAND SHARE EQUATIONS

RESTRICTING BUDGET SHARES TO THE UNIT SIMPLEX

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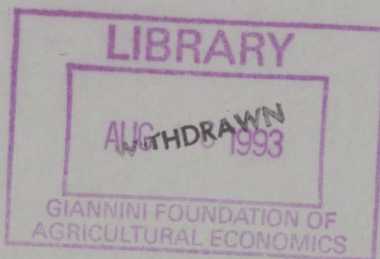## DEPARTMENT OF ECONOMETRICS

THE STOCHASTIC SPECIFICATION OF DEMAND SHARE EQUATIONS

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The Stochastic Specification of Demand Share Equations:

Restricting Budget Shares to the Unit Simplex.

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Abstract: The traditional approach to estimating systems of share equations is to append a multivariate normal stochastic component to the deterministic component. The aggregation restrictions are then exploited to delete one equation and estimation is then carried out by maximum likelihood. Such an approach does not exploit the aggregation restrictions to their full extent, as there remains a non-zero probability of shares being negative or greater than unity. In this paper we take an approach, compositional data analysis, and a distribution familiar to statisticians, the additive logistic normal, and apply them to the estimation of a system of demand share equations. The results of this new approach are then compared to those of the traditional approach.

Keywords: stochastic; demand share; simplex; additive logistic normal; Dirichlet distribution; multivariate normality.

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1. Introduction.

Economists are often interested in estimating a system of equations which relate to shares of a total. Examples of this are the shares of individual demands in total expenditure and the shares of inputs in the cost of production. In such modelling situations the shares should naturally be restricted to lie between zero and one and the sum of all the shares should equal one. Much attention is paid in economics to specifying the deterministic part of such share models. This paper, however, considers a new approach to the appropriate incorporation of the stochastic part in such models and the resultant estimation of the model.

Typically, share equations are estimated assuming that the disturbances (and thus the shares) follow a multivariate normal distribution. Obviously, since actual shares will necessarily fall in the zero-one interval, this choice of distribution does not impose the restriction that *each* share must lie between zero and unity. Even if we model shares using a 'regular' deterministic specification (such as Cooper and McLaren's (1992) Modified PIGLOG (MPIGLOG)), which restricts the deterministic component to lie between zero and unity (the monotonicity of the cost function, together with the budget constraint, gives this restriction), we are still confronted with the possibility of shares outside the unit interval, by virtue of the nature of the stochastic specification.

Extreme drawings from the multivariate normal distribution may force the share, which contains the deterministic component and the (additive) stochastic component, outside the zero-one interval. Therefore, use of the multivariate normal distribution will result in a

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non-zero probability of budget shares outside the zero-one interval. Although widely used, the multivariate normal distribution would thus seem to be an inappropriate choice when modelling shares.

Given this theoretical problem with the use of the multivariate normal distribution it is perhaps surprising that there have been few attempts to investigate the use of other multivariate distributions for the stochastic specification of systems of share equations. One paper which does consider this topic is Woodland (1979). In the context of estimating systems of demand share equations, Woodland recognises the problems associated with assuming that the disturbances (and thus the budget shares) follow a multivariate normal distribution. He argues that "the stochastic specification should respect the fact that shares cannot be negative, nor can they exceed unity" (p.362). Use of the normal distribution implies a positive probability that shares will not respect this constraint and that the covariance matrix for each observation is the same and the distribution of shares is symmetric about the mean. Since the mean will generally be different for each observation and since the shares must lie between zero and unity, it is highly unlikely that the true density functions for all observations are symmetric with a common covariance matrix. This reasoning suggests that we may argue that the normal distribution is invalid as a stochastic specification in the context of estimating share equations.

However, Woodland argues that if the lack of symmetry is foregone, the normal distribution may provide an adequate description of the true density function if the elements of the covariance matrix are small and the means are not near zero or unity. In such a case, the density outside the zero-one interval would be negligible. Although the normal distribution may be regarded as inappropriate from an economic theory

perspective, it may still be useful if it is deemed to be 'robust'. This was an issue which Woodland undertook to investigate.

A comparison was made of the theoretical implications of assuming a normal and a Dirichlet distribution for shares in Woodland's paper. Both distributions assume that the expected shares are equal to the deterministic shares. However, differences arise in terms of symmetry, restrictions on the range of shares and the covariance matrix for the shares. The normal distribution implies that the distribution for a particular share is symmetric about the mean, the range of shares is *not* restricted to the zero-one interval and the covariance matrix for the shares is constant and does not depend on the expected shares. The Dirichlet distribution results in the distribution for a particular share being asymmetric, except in the case where the mean is 0.5. The distribution restricts the shares to lie between zero and unity with probability one and the covariance matrix for the shares depends on the expected shares. For these reasons, Woodland considered the Dirichlet distribution to be an attractive alternative to the normal distribution.

The two stochastic specifications were used in three applications: estimating a production function, modelling utility (work – leisure choice) and modelling consumer preferences for meat. The first application was to a time series containing little variability in the explanatory variables. The second and third applications were to cross section data containing rather large variations in the explanatory variables. Woodland found that, although the economic applications and degree of variability in the data were different, the models based on the normal and Dirichlet distributions have similar estimates for the parameters in each of the three applications. He considered that this

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provided "some empirical justification for the use of the normal (distribution) even though it may not be a strictly appropriate specification for a system of share equations" (p. 380).

In this paper we consider the use of a modelling approach, compositional data analysis - hereafter CODA (Aitchison (1986a)) - which takes account of all restrictions on shares. Using data on budget shares we compare it with the traditional economists' approach of using multivariate normality. We find that the CODA approach shows up an independence assumption in the use of the Dirichlet which goes some way to explaining the findings in Woodland (1979).

The plan of the rest of this paper is as follows: section two describes the CODA approach to specifying and estimating models for data on shares. Section three compares the 'traditional' and CODA approaches to estimating systems of equations for share data. Section four contains the results of applying the two approaches to two demand systems for budget shares (the Almost Ideal Demand System - AIDS - and the Indirect Addilog - IA) using Australian data. Finally, section five contains some concluding remarks.

2. Compositional Data Analysis.

The restriction that shares must lie between zero and unity and that they sum to unity applies not only to shares in economics. Statisticians in other disciplines (e.g. geology, medicine, biology) also use share data which must obey the same two restrictions. Such data is termed 'compositional data' and is analysed using methods of compositional data analysis. In this section we give a brief overview of the methods of compositional data analysis (a more detailed

exposition may be found in Aitchison (1986a)). We begin with some definitions.

A *composition* is a matrix of proportions (i.e. shares), denoted by W which is of dimension $T \times N$ where T denotes the number of rows (observational units) and N denotes the number of columns (number of parts or shares in the composition). Therefore, compositional data consists of a set of proportions. From this definition of a composition, two constraints are obvious. Firstly, each proportion must lie between zero and unity. Secondly, as the proportions are 'parts of a whole', they must sum to unity.

A *basis* X is a $T \times N$ matrix of positive components each of which is recorded on the same measurement scale. For example, in the consumer demand context $x_{t1}, \ldots, x_{tN}$ (= x_t, the tth row of X) represent expenditure in dollars on each of N goods by consumer t. This is usually the form of data gathered for use in estimation of demand equations. By rescaling the rows of X such that they sum to unity, we form the *composition*. In our example, we take the expenditure on good i (x_{ti}) by a consumer and divide by total expenditure on all goods (m_t) by a consumer to find the proportional expenditure on good i (w_{ti}) by the consumer. This "constraining operation" therefore takes us from N dimensional positive real space (the positive orthant of $\mathbb{R}^N$) to a restricted part of n (= N - 1) dimensional real space, $\mathscr{S}^n$, which is termed the 'n dimensional unit simplex'.

Since a simplex is simply a restricted part of real space, the distinction must be made in terms of the "length" of the simplex of interest. In compositional data analysis, the length of the simplex must be unity from the restriction that the components of a composition

sum to unity. We therefore refer to such (length restricted) real space as the unit simplex to distinguish it from a restricted (not necessarily unit length) part of real space. Thus the unit simplex corresponds to the zero-one interval to which economic theory says shares should be restricted.

Three operations used in CODA are of interest: subcomposition, amalgamation and partition. If we were interested in focussing on a subset of components of the composition, we could form a *subcomposition* by rescaling the components of interest such that they sum to unity in the subcomposition. Forming a subcomposition is useful both in reducing the dimension of the problem and in that it also preserves the ratio relationships. That is, the ratio of any two components of a subcomposition is the same as the ratio of the corresponding two components in the full composition. The subcomposition is actually formed by linear projection in the unit simplex.

Amalgamations are essentially compositions in which various components are aggregated to form new components. That is, an N part composition is separated into C ($\leq$ N) mutually exclusive and exhaustive subsets and the components within each subset are added together to form a C part composition (i.e. an amalgamation). The amalgamation operation transforms the problem from $\mathcal{S}^n$ to $\mathcal{S}^c$ (where c = C – 1) and is thus another useful dimension reducing operation. It would be particularly important in analysing, say, budget shares, as there is a potentially enormous number of goods among which the consumer's budget is allocated and this would result in the large dimensions of the problem precluding meaningful analysis. By amalgamating expenditure on certain goods we reduce the dimensions of the problem to a manageable size

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We may form a *partition* by combining the operations of amalgamation and forming subcompositions. This may be useful when we are interested in applying both operations, for example combining expenditure into groups and focussing on components of such groups and again has the added advantage of reducing the dimension of the problem.

Aitchison (1986a) points out that the major barrier to statistical analysis of compositional data is that the constraints such data must satisfy lead to the lack of an interpretable (covariance) structure. He therefore suggests the use of several transformations to overcome this problem. Thus underlying the CODA approach is the idea that by transforming w we may obtain a set of new variables which are amenable to analysis by traditional multivariate statistical methods.

Of particular interest to us is the use of the logratio transformation[1]. This leads to the *logratio covariance matrix* (Σ) which has typical element:

$$\sigma_{ij} = \text{cov}\{\log(w_i/w_N),\ \log(w_j/w_N)\} \qquad (i,\ j = 1,\ \ldots,\ n). \qquad (1)$$

This covariance matrix allows us to analyse the variability between any two components in the composition relative to a third component which is always the same (w_N). Σ is non-negative definite and is specified as a traditional variance-covariance matrix for $\log(w_i/w_N)$ where $i = 1,\ \ldots,\ n$. While the parts of the composition are treated asymmetrically, since in each ratio the denominator is the same (w_N), it is important to note that re-ordering the components and changing the component used as the denominator in the logratios makes no difference to statistical procedures involving the logratio covariance

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[1] Wherever possible in the remainder of this paper the observational subscript t (= 1, ..., T) will be suppressed.

matrix Σ. Thus all statistical procedures are invariant to the choice of component used as the denominator for the logratios (for details and proofs, see Aitchison (1986a, Ch. 5)). Note also that this invariance property is similar to that in the 'traditional' approach where statistical procedures are invariant to the choice of equation to delete.

By using the logratio transformation, we have defined a set of new variables for analysis (y) which will have mean μ and covariance matrix Σ. To complete the framework for statistical analysis it is necessary to make some distributional assumptions. An obvious candidate distribution for our new logratio variables is the multivariate normal distribution. To discover the implication of this assumption on the distribution of the composition w, we note that we move from $\mathbb{R}^n$ to $\mathscr{S}^n$ by applying a particular one-to-one transformation (the additive logistic transformation). This is defined by:

$$w_i = \exp(y_i)/\{\exp(y_1) + \ldots + \exp(y_n) + 1\} \qquad (i = 1, \ldots , n). \qquad (2)$$

$$w_N = 1 - w_1 - \ldots - w_n = 1/\{\exp(y_1) + \ldots + \exp(y_n) + 1\}. \qquad (3)$$

The inverse transformation ($\mathscr{S}^n$ to $\mathbb{R}^n$) is the logratio transformation and defines the y_i's as:

$$y_i = \log(w_i/w_N) \qquad (i = 1, \ldots , n) \qquad (4)$$

with Jacobian: $\qquad \mathrm{jac}(y|w^{(n)}) = (w_1 \ldots w_N)^{-1}. \qquad (5)$

Thus if the logratio composition y has an n dimensional normal distribution ($\mathscr{N}^n(\mu,\Sigma)$) then the composition w will have an additive logistic normal distribution ($\mathscr{L}^n(\mu,\Sigma)$) where μ is the the mean of the logratios and Σ denotes the logratio covariance matrix. It can be shown that if $y \sim \mathscr{N}^n(\mu,\Sigma)$ and $w \sim \mathscr{L}^n(\mu,\Sigma)$ then the basis x will follow a

multivariate lognormal distribution. Finally, any subcomposition, amalgamation or partition will also follow an additive logistic normal distribution.

In contrast to the additive logistic normal, the Dirichlet distribution for **w** can only arise from a basis **x** which follows a Gamma distribution where the components of the basis (the marginal distributions) are independent and equally scaled. The imposition of independence in the Dirichlet distribution is a strong assumption which will make the Dirichlet distribution of little use if we believe that a composition contains even weak forms of dependence among the components. This independence assumption will thus render the Dirichlet distribution inadequate for describing or modelling the observed patterns of variability in many compositional data sets.

Woodland's (1979) approach to modelling shares with a Dirichlet distribution is therefore particularly restrictive as it relies on the assumption that expenditure on each good is Gamma distributed and independent of expenditure on other goods. Given that consumers allocate expenditure on goods subject to a budget constraint and that various goods are substitutes for, or complements to, other goods, the assumption of independence would seem to be unrealistic in the context of budget share analysis.

The additive logistic normal distribution has several advantages over the Dirichlet distribution. It is better able to describe actual patterns of variability in a composition as it can accommodate dependent and independent covariance structures. Thus we are able to test for independence using parametric hypothesis tests and could fit a model to describe compositional dependence if any is found to exist.

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The additive logistic normal distribution also allows for relatively simple estimation, modelling and hypothesis testing of the parameters ($\underline{\mu}$ and Σ) and for validation tests regarding the distributional assumption.

Finally in this section we consider how we can apply one particular form of statistical analysis (regression modelling) to compositional data. Essentially, the task is to model the conditional distribution or density function for the composition w. Earlier we discovered that the 'trick' was to apply a transformation and work with the logratio composition y in $\mathbb{R}^n$ instead of using the composition w in $\mathscr{S}^n$. Thus the model is specified as: $Y = ZB + U$, where Y is a $T \times n$ matrix of logratios, Z is the $T \times k$ full rank covariate matrix, B is a $k \times n$ matrix of parameters and U is a $T \times n$ matrix of errors where the rows of U, $\underline{u}_t$, are distributed as $N^n(0, \Sigma)$ and are independent across t.

Maximum Likelihood estimation of the logratio linear model gives the estimator of B as: $\hat{B} = (Z'Z)^{-1}Z'Y$. Thus we can see that this is a standard multivariate regression and the usual techniques of estimation and hypothesis testing apply. Further, should we wish to test the validity of the assumption that w ~ additive logistic normal then as the mappings are one-to-one we may test the validity of the assumption that y ~ multivariate normal.

3. Application to Budget Share Modelling.

In this section we give an overview of the 'traditional' approach to specifying and estimating systems of demand share models. We then describe how the statistical, CODA, approach may be combined with the economic specification of the deterministic component of such models to

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yield specifications which constrain both observed and implied shares to the unit simplex.

Under the 'traditional' approach, we would specify a general model for the demand shares as:

$$w_i = W_i(Z, \underline{\beta}) + u_i \qquad (i = 1, \ldots , N) \qquad (6)$$

where w_i is the budget share for good i; $W_i(Z, \underline{\beta})$ is the deterministic component and is expressed as a function of exogenous variables (Z), which are typically the prices of all N goods and total expenditure, and the unknown parameters ($\underline{\beta}$); u_i is the additive stochastic component. To impose normality we assume that $u = (u_1, \ldots, u_n)$ follows an n-variate normal distribution with a mean of 0 and an n × n positive definite covariance matrix Ω. Since,

$$\sum_{i=1}^{N} w_i = 1 = \sum_{i=1}^{N} W_i(Z, \underline{\beta}) \qquad (7)$$

it follows that $u_N = - \sum_{i=1}^{n} u_i$ and u_N also follows a normal distribution.

Thus we can write the log-likelihood for a sample of T observations as:

$$\ell = - \frac{T(\log(2\pi) + 1)}{2} - \frac{T\log|\hat{\Omega}|}{2} , \qquad (8)$$

where $\hat{\Omega}$ is the sample covariance matrix with i,j^{th} element:

$$\frac{\sum_{t=1}^{T} u_{ti} u_{tj}}{T} .$$

and u_{ti} is the tth observation on u_i. Finally, it should be noted that the estimates obtained by maximising this function are invariant to the equation which is deleted (Barten (1969)).

We have seen in section two that the CODA approach involves modelling the logratios of the shares. We note that in formulating a logratio model we may 'merge' the statistical, CODA, approach with the

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traditional approach to modelling systems of budget share equations. The CODA approach restricts the linear combination of deterministic and stochastic components for each share to lie between zero and unity. That this corresponds to the aggregation restrictions from economic theory is coincidental, as the statistical approach does not impose/recognise the restrictions arising from economic theory (for example, homogeneity and symmetry). We therefore refer to the 'combined' approach as producing models in which the deterministic specification is derived from economic theory and the stochastic specification is derived from the statistical analysis of compositional data.

The combined approach involves the use of the additive logratio transformation of the observed shares

$$y_i = \log (w_i/w_N).\tag{9}$$

Application of this transformation allows us to model the logratio shares as multivariate normal, with a conditional mean of $\mu_i(Z,\underline{\beta})$. The requirement that the μ_i be consistent with economic theory naturally suggests the specification:

$$\mu_i(Z,\underline{\beta}) = \log \left(\frac{W_i(Z,\underline{\beta})}{W_N(Z,\underline{\beta})}\right).\tag{10}$$

This gives the following general functional form for estimation:

$$y_i = \log \left(\frac{w_i}{w_N}\right) = \log \left(\frac{W_i(Z,\underline{\beta})}{W_N(Z,\underline{\beta})}\right) + u_i \qquad (i = 1, \dots, n)\tag{11}$$

where w_i is the observed share, $W_i(Z, \underline{\beta})$ is the functional form for the shares derived from economic theory and u_i is an additive stochastic component which is distributed as multivariate normal with mean 0 and variance–covariance matrix Σ. This implies that the observed shares, w_i, are distributed as additive logistic normal.

Thus, by applying the additive logratio transformation to the deterministic equations as specified under the traditional approach, adding a multivariate normal disturbance term and subsequently applying traditional multivariate regression techniques, we are able to model systems of demand equations whilst ensuring that individual shares implied by the model *cannot* lie outside the unit simplex. If necessary, the implied shares (composition) can be recovered from the logratio composition by applying the additive logistic transformation discussed earlier.

In summary, we have outlined a method of reformulating the traditional approach to modelling systems of share equations such that we constrain the budget shares to lie within the unit simplex. By applying the additive logratio transformation to the model and assuming multivariate normality of the disturbances, we can apply conventional multivariate regression techniques. Further, it should be noted that within the 'combined' approach, we are not constrained to specifying the deterministic component in the model to be linear in its logarithms. Just as the estimation and hypothesis testing procedures relating to a linear regression model may be extended to cover non-linear regression models, so too can the 'combined' approach to estimating and testing in the context of systems of demand share equations be extended to cover a non-linear deterministic specification in the logratio model.

Before proceeding to an empirical comparison of the two approaches, we now consider one class of demand systems (Addilog systems) which, although not originally formulated as such, represent an application of the 'combined' approach in economics. The particular example we consider is that of Bewley (1982b) who, in effect, models expenditure on commodities using the Australian Household Expenditure Survey data

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by specifying budget shares as having an additive logistic normal distribution. In adopting such a specification, he was working with what he terms the Generalised Addilog Demand System (GADS). The GADS specification has its foundations in both the Addilog model (Houthakker (1960)) and the Multinomial Logit model (Theil (1969)), but maintains the essential characteristics of the Addilog model. For further details of the derivation of this specification, see *inter alia* Bewley (1982a, 1982b, 1986).

For convenience in estimation, the budget shares in the GADS model are converted to *centred* logratio form by applying a logratio transformation of the form $y_i = \log(w_i / \tilde{w})$ where $\tilde{w}$ is the geometric mean of the shares. This transformation results in multivariate normality of the disturbance term in the model for the *centred* logratios. However, since the dependent variable $\log(w_{ti} / \tilde{w}_t)$ uses the geometric mean of the shares, the covariance matrix for the disturbance terms will not be Σ but Γ – the *centred* logratio covariance matrix (see Aitchison (1986a)). One complication involved in using this specification is that Γ is singular. Statisticians using a CODA approach would solve·this by using the generalised inverse of Γ. Bewley models $\log(w_i / \tilde{w})$ which may also be written as $(\log w_i - ((1/N) \sum_{j=1}^{N} \log w_j))$, or $\Sigma(w_i^* - \bar{w}^*)$ where $w_i^* = \log w_i$. Since $\Sigma(w_i^* - \bar{w}^*) = 0$, the centred logratios add to zero, illustrating the singularity of Γ. Bewley (1982b) adopts the usual procedure of deleting one of the equations and estimating the remaining n equations. The matrix Γ is then n × n instead of N × N and no longer singular. Once again, the results are invariant to the choice of equation to delete.

Thus Bewley (1982b) has, in effect, adopted a 'combined' approach to modelling budget shares which will impose the restrictions that the

shares sum to unity *and* are each constrained to lie between zero and unity. This particular deterministic specification does not automatically impose any other restrictions arising from economic theory (for example, homogeneity and symmetry).

The assumption of additive logistic normality and the use of the *centred* logratio form of transformation in Bewley's work arise naturally from the specific functional form used for the budget share equations (GADS). We would argue that such a specification in fact arises naturally from an appropriate stochastic specification.

4. Empirical Comparison.

Thus far, we have a theoretical difference between the two approaches: the 'combined' approach restricts the stochastic shares to the unit simplex, the 'traditional' approach does not. Whether the imposition of the restriction that shares lie between zero and unity makes any difference to the results is an empirical question, the answer to which is not obvious *a priori*.

In the empirical comparison of the two approaches quarterly Australian National Accounts data for the period 1969:3 to 1992:3 is used. Australian total expenditure is divided into four categories: food (F), cigarettes and tobacco (T), alcoholic drinks (A) and 'other expenditure' (O). The last category is a residual, containing expenditure on all other goods and services. The budget shares for each category (w_F, w_T, w_A, w_O) are constructed as expenditure for the category as a proportion of total expenditure. The price series for each category (p_F, p_T, p_A, p_O) is derived as an 'implicit price deflator' by taking the ratio of current to constant ($m1984/85)

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expenditure. Total per capita expenditure (m) is measured in (current) millions of dollars per thousand persons.

In this application we have tried to follow what appears to be a standard classification system of expenditure (food, tobacco/alcohol, clothing/footwear, other) fairly closely. The main difference is in *amalgamating* the clothing/footwear and other categories and dividing the tobacco/alcohol categories into two separate categories. The amalgamation is useful in reducing the dimension of the problem, while the division of tobacco/alcohol into two separate categories is carried out primarily in search of two categories with mean budget shares "close" to zero. In the light of Woodland's (1979) findings it is thought that if any major differences are to arise in terms of implied budget shares outside the unit simplex, they will do so in categories with mean shares "close" to zero (or unity).

In terms of the empirical comparison, two models are used: the Almost Ideal Demand System (A.I.D.S.) and the Indirect Addilog (IA) Demand System. Under both the 'traditional' and 'combined' approaches the estimating equations have an additive error which is distributed as multivariate normal. Note, however, that although both of these multivariate normal distributions have mean 0 they have different covariance matrices; Ω for the 'traditional' and Σ for the 'combined'.

Under the 'traditional' approach the A.I.D.S. specification would result in the following demand share equation for the ith share:

$$w_i = (\alpha_i + \gamma_{iF} \log p_F + \gamma_{iT} \log p_T + \gamma_{iA} \log p_A + \gamma_{i0} \log p_0$$

$$+ \beta_i (\log m - (w_F \log p_F + w_T \log p_T + w_A \log p_A + w_0 \log p_0)))$$

$$+ \delta_{i2} D_2 + \delta_{i3} D_3 + \delta_{i4} D_4 + u_i \tag{12}$$

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where

$$\sum_i \alpha_i = 1 \qquad \sum_i \beta_i = 0 \qquad \sum_i \delta_{ir} = 0$$

$$\sum_i \gamma_{ij} = 0 \qquad \sum_j \gamma_{ij} = 0 \qquad \gamma_{ij} = \gamma_{ji}$$

$$(i, j = F, T, A, O; \; r = 2, 3, 4).$$

For simplicity, we have adopted the usual approach of using the Stone's price index as a deflator. It should be noted that D_2, D_3 and D_4 represent the seasonal dummy variables for the second, third and fourth quarters in each year. As a general rule, seasonal dummies have been appended to the deterministic component of the deterministic equation for the share. In these estimating equations the error terms u_i are assumed to follow a multivariate normal distribution and estimation is carried out by maximising the log-likelihood given earlier in section three of this paper.

The 'combined' approach to estimating the share equations with an A.I.D.S. specification would yield estimating forms of the demand share equations as:

$$y_i = \log\left(\frac{w_i}{w_0}\right) =$$

$$\log\left[\frac{\begin{pmatrix}\alpha_i + \gamma_{iF}\log p_F + \gamma_{iT}\log p_T + \gamma_{iA}\log p_A + \gamma_{i0}\log p_0 \\ + \beta_i(\log m - (w_F\log p_F + w_T\log p_T + w_A\log p_A + w_0\log p_0)) \\ + \delta_{i2}D_2 + \delta_{i3}D_3 + \delta_{i4}D_4\end{pmatrix}}{\begin{pmatrix}\alpha_0 + \gamma_{0F}\log p_F + \gamma_{0T}\log p_T + \gamma_{0A}\log p_A + \gamma_{00}\log p_0 \\ + \beta_0(\log m - (w_F\log p_F + w_T\log p_T + w_A\log p_A + w_0\log p_0)) \\ + \delta_{02}D_2 + \delta_{03}D_3 + \delta_{04}D_4\end{pmatrix}}\right] + u_i$$

$$(13)$$

where
$$\sum_j \alpha_j = 1 \qquad \sum_j \beta_j = 0$$
$$\sum_j \gamma_{jk} = 0 \qquad \sum_k \gamma_{jk} = 0$$
$$\gamma_{jk} = \gamma_{kj} \qquad \sum_j \delta_{jr} = 0$$
$$(i = F, T, A; \; j,k = F, T, A, O; \; r = 2, 3, 4).$$

In the 'combined' approach the shares have been assumed to be distributed as additive logistic normal and thus the logratios (and the errors) are distributed as multivariate normal. The model estimated here is therefore a three equation non-linear multivariate regression model and estimation is carried out in the usual manner.

The Indirect Addilog model is unusual in that in the economics literature it is traditionally estimated using what we refer to as the combined approach. This is usually done purely for convenience, as the resulting equations are log-linear. However, for completeness, we have included the estimation of the Indirect Addilog model as it would be estimated under the 'traditional' approach. For the ith share, we have the following equation specification:

$$w_i = \left[\frac{(\alpha_i \gamma_i (p_i/m)^{\gamma_i})}{\left[(\alpha_F \gamma_F (p_F/m)^{\gamma_F}) + (\alpha_T \gamma_T (p_T/m)^{\gamma_T}) + (\alpha_A \gamma_A (p_A/m)^{\gamma_A}) + (\alpha_O \gamma_O (p_O/m)^{\gamma_O}) \right]} \right]$$

$$+ \; \delta_{i1} D_1 + \delta_{i2} D_2 + \delta_{i3} D_3 + \delta_{i4} D_4 + u_i \qquad (14)$$

where $\sum_i \delta_{ir} = 0$, $\sum_r \delta_{ir} = 0$ $(i = F, T, A, O; \; r = 1, 2, 3, 4)$. We should note that D_1, D_2, D_3 and D_4 represent the seasonal dummy variables for each of the four quarters in each year. In the indirect addilog model, however, we require *four* such dummy variables as we do not have an intercept term. Again the error terms are assumed to follow

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a multivariate normal distribution and estimation is by the maximum likelihood method outlined in section three.

When we use the 'combined' approach to specifying and estimating the Indirect Addilog model we find that the denominators cancel out and we are left with the following specification for the estimating equation:

$$y_i = \log(w_i/w_0)$$

$$= \log\left[\frac{(\alpha_i \gamma_i (p_i/m)^{\gamma_i}) + \delta_{11}D_1 + \delta_{12}D_2 + \delta_{13}D_3 + \delta_{14}D_4}{(\alpha_0 \gamma_0 (p_0/m)^{\gamma_0}) + \delta_{01}D_1 + \delta_{02}D_2 + \delta_{03}D_3 + \delta_{04}D_4}\right] + u_i \quad (15)$$

where

$$\sum_j \delta_{jr} = 0, \qquad \sum_r \delta_{jr} = 0$$

$$(i = F, T, A; \ j = F, T, A, O; \ r = 1, 2, 3, 4).$$

Note that for identification, we require the product $\alpha_0 \gamma_0$ to be normalised, and a normalisation of unity is chosen. This normalisation is imposed under both the 'traditional' and 'combined' approaches, as "the (α's) are only determined up to a scale factor" (Varian (1984, p.184)).

Before proceeding, we should note that the problem of shares outside the unit simplex in the 'traditional' approach is exacerbated when we have a system of demand share equations in which the deterministic component is not 'regular'. In such cases, this will increase the probability of shares that violate the unit interval. In terms of the specifications we have chosen to use, A.I.D.S. would pose the greatest problem, since it is well known that the deterministic component of this specification is not *globally* regular. However, it can be shown (see *inter alia* Bewley (1982b)) that under certain conditions the Indirect Addilog will be globally regular.

Furthermore, this 'irregularity' of A.I.D.S. poses a similar problem for the 'combined' approach to estimation. Under this approach, the shares implied by the model are constrained to lie between zero and unity. If the deterministic component of the model does not restrict the shares to the unit simplex then the consequences of using the 'combined' approach to estimation are not clear. Indeed, should the irregularity lead to a negative share, the model specification under the combined approach would not even be defined. Thus, use of the 'combined' approach to estimating systems of demand share equations that are not globally regular may be viewed either to cause further problems or solve this weakness of the demand system specification.

Aitchison (1986b) has developed a software package (called CODA) specifically for applying the compositional data analysis approach to estimating and testing share equations. However, this package has the major disadvantage that it is unable to cope with logratio equations that are complex and non-linear. As a result, each of the models is estimated under both the 'traditional' and 'combined' approaches by Full Information Maximum Likelihood (F.I.M.L.) using the LSQ option in TSP (see Hall *et al* (1991)). Under the 'combined' approach, the deterministic component is specified as the logarithm of the ratio of the relevant deterministic components in the 'traditional' approach. This means that we are able to maintain the same interpretation for the parameters under either approach and directly compare them.

Tables 1 through 4 contain the results of our estimation of the models under both approaches for the A.I.D.S. (symmetry and homogeneity constrained) and Indirect Addilog models, respectively. The four tables taken together show little if any difference between parameter estimates in the 'traditional' and 'combined' approaches. It should

also be noted that in these models there is no systematic pattern in the parameter estimates as between the two approaches. That is, given the small differences between the estimates under the two approaches, the 'combined' approach does not tend to systematically over- or under estimate parameter values, compared with the 'traditional' approach. The R^2 values for the 'traditional' and 'combined' approaches are not comparable since under the 'traditional' approach they relate to variations in the shares, whereas under the 'combined' approach they relate to variations in the logratio shares. A similar caveat applies to the Durbin-Watson statistics and system log-likelihood values.

The estimation of each model under the traditional and combined approaches does not yield many startling differences: there appear to be little if any differences in the parameter estimates. This suggests that, in using the traditional approach to estimating systems of demand share equations, parameter estimates may be only marginally different. Thus we require another way of comparing the two approaches. The method that we use is to test the validity of the two different distributional assumptions made regarding the shares.

Under the 'traditional' approach, the maintained hypothesis is that the shares are distributed as multivariate normal (and thus each share has a marginal distribution that is univariate normal). The 'combined' approach, however, assumes that the logratio shares follow a multivariate normal distribution and thus the shares themselves have an additive logistic normal distribution.

Although a multivariate normal distribution implies that the marginal distributions are all univariate normal, the reverse is not necessarily true (for further discussion, see *inter alia* Seber (1977, 1984), Pierce

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and Dykstra (1969), Hogg and Craig (1965) and Anderson (1958)). This means that for us to test the validity of the normal/additive logistic normal distributional assumptions, we are unable to use tests for univariate normality, such as the Jarque-Bera test (see Jarque and Bera (1987) for further details) as they are not strictly appropriate. However, the literature provides us with "an embarrassingly large battery (of tests for multivariate normality, with) varying claims to appropriateness" (Aitchison (1986a, p.143)). We have thus relied on Aitchison's (1986a) choice of the Anderson-Darling form of empirical distribution function test. For further details regarding this test, see *inter alia* Anderson and Darling (1954), Stephens (1974) and Aitchison (1986a).

Thus, we are able to test both the assumption of normality of the shares (directly) and the assumption of additive logistic normality of the shares (by testing for normality of the log-ratio shares). This is achieved by testing for normality of the estimation residuals under each of the two approaches using the radius test (Aitchison (1986a, p.146)). To calculate the test statistic, we first compute the radii as:

$$\lambda_t = \hat{u}_t \, \hat{\psi}^{-1} \, \hat{u}_t{}' \qquad t = 1, \ldots, T \qquad (16)$$

where $\hat{u}_t$ is a $1 \times n$ (= 3) vector of estimation residuals for the three estimated equations at time t, $\hat{\psi}$ is the estimated variance-covariance matrix of the residuals (ψ is labelled Ω in the traditional approach and Σ in the combined approach).

The next step is to evaluate the $\chi^2_{(3)}$ distribution at λ_t ($t = 1, \ldots, T$). This provides upper tail areas corresponding to the T (= *93*) different arguments. Arranging these upper tail areas in

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ascending order, we refer to these statistics as θ_t (t = 1, ... , T). To test for significant departures from multivariate normality under the two approaches, we compare the θ_t's to the order statistics of a uniform distribution on (0,1). This is achieved by calculating the Anderson-Darling test statistic and comparing it with tabulated critical values (see Aitchison (1986a, p. 146)). The test statistic is defined as:

$$Q_A = -(1/T)\sum_{t=1}^{T}(2t - 1)\{\log \theta_t + \log(1 - \theta_{T+1-t})\} - T. \qquad (17)$$

The five percent asymptotic critical value is 2.492 and we reject multivariate normality for large values of Q_A, since this implies that the departure from multivariate normality is statistically significant.

Although this is a large sample test, "empirical study suggests that the asymptotic (critical) value is reached very rapidly, and it appears safe to use the asymptotic value for a sample size as large as *40*" (Anderson and Darling (1954, p.766)). Thus, with our sample of *93* observations, we should be able to use the asymptotic critical values for the Anderson-Darling test for multivariate normality. The results of this test are given in Table 5. In all cases, the Anderson-Darling test is unable to reject the null hypothesis of normality under the 'traditional' approach or of additive logistic normality under the 'combined' approach. This implies that the test is unable to discriminate between the two distributions, as they are almost the same.

One final area we address is that of hypothesis testing. As outlined in section 2 of this paper, the logratio transformation still allows us to use the conventional hypothesis testing techniques. As an example, we choose to test the symmetry restrictions in the A.I.D.S. model under

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each of the two approaches. In the context of the 'traditional' approach, testing of the symmetry restrictions involves the estimation of the A.I.D.S. model in both its restricted and unrestricted forms and performing a likelihood ratio test. Similarly, testing the symmetry restrictions under the 'combined' approach to estimation involves the same form of likelihood ratio test: we compare the restricted and unrestricted log-likelihood values. Using the results shown previously in Tables 1 and 2 (symmetry constrained) and below in Tables 6 and 7 (symmetry unconstrained), we see that under both the 'traditional' and 'combined' approaches symmetry is rejected.

This rejection of symmetry is not, of itself, the most important part of these results. The most important point to note is that the calculated values for the likelihood ratio tests under the 'traditional' and 'combined' approaches are *not* the same (calculated values of 25.49 and 24.009, respectively, as against a critical value of 12.5916). In this case, although different, the test statistics lead to the same outcome for the test: a rejection of symmetry. However, it is plausible that there may be cases in which we marginally reject an hypothesis under one approach and yet fail to reject it under the other. This would distinguish the two approaches, since, although they both give virtually the same parameter estimates, hypothesis tests using the log-likelihood values (for example, Likelihood Ratio, Lagrange Multiplier and Wald tests) may result in very different conclusions being drawn. If this is the case, then this should provide sufficient incentive for applied researchers to adopt the 'combined' approach in their investigations.

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5. Conclusions.

In this paper we have discussed an alternative stochastic specification to use in estimating systems of share equations. The approach is based upon combining the economic specification of the deterministic component of the model with modelling the stochastic component using the compositional data analysis approach used by statisticians in other disciplines. The new 'combined' approach solves the theoretical problem with the 'traditional' approach used in economics, namely, the non-zero probability of obtaining a share outside the unit simplex.

Furthermore our use of the 'combined' approach based upon the stochastic assumption of additive logistic normality of the shares turns out to be less restrictive than previous attempts (Woodland (1979)) to bind shares using the Dirichlet distribution. This is because the Dirichlet turns out to be based upon a particularly restrictive independence assumption which is unlikely to be appropriate for the data encountered in economics and because the Dirichlet is not as flexible as the class of additive logistic normal distributions used.

The efficacy of using the 'combined' approach in practice is investigated in an empirical example using Australian data. We find that the new approach is easy to fit in an existing econometrics package but for the data and deterministic (AIDS and Indirect Addilog) specifications used there appears to be no large differences in the parameter estimates. This finding is in common with Woodland (1979) who finds no apparent difference between the 'traditional' use of multivariate normality and the use of the Dirichlet distribution.

An implication of our findings is that there is little to choose between the 'combined' and 'traditional' approaches. This is confirmed by our results of tests of additive logistic normality of shares in the 'combined' approach and of multivariate normality of shares in the 'traditional' approach. In both cases we are unable to reject the distributional assumption which underlies the approach. Thus for this data and these demand systems the two approaches are similar. However, this is unlikely to always be the case. It is not sufficient to use multivariate normality as an approximation to additive logistic normality, as it is theoretically inappropriate. We therefore recommend that future empirical studies adopt a 'combined' approach.

It is possible that a reason for the apparent similarity in the results is due to the high level of aggregation and high signal to noise ratio in the data used. Thus in investigations using micro-level or disaggregated data where we are likely to find observations which are sufficiently "close" to zero or one to lead to a difference between the approaches we would argue that the 'combined' approach is more suitable. Although the use of this type of data may well reveal any differences between the two approaches, the 'combined' approach, as outlined and used in this paper, is unable to cope with zero observations. Bacon-Shone (1991) and Aitchison (1986a) propose ad hoc modifications which can be made to the CODA approach to handle zeros. However, even if we adopt their suggestions to cope with zeros in our 'combined' approach, the task still remains to modify the 'traditional' approach to solve, in effect, the same problem (see Heien and Wessells (1990) for one such attempt). Such extensions are the subject of a future paper.

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Table 1: Maximum Likelihood Estimates for A.I.D.S. (Traditional Approach, Symmetry Constrained).

	Food	Tobacco	Alcohol	Other
γ_{1F}	0.092 (0.008)	.	.	.
γ_{1T}	-0.006 (0.002)	0.014 (0.001)	. .	.
γ_{1A}	-0.002 (0.004)	-0.015 (0.001)	-0.006 (0.004)	.
γ_{1O}	-0.083 (0.007)	0.007 (0.002)	0.023 (0.004)	0.053 (0.009)
α_1	0.190 (0.001)	0.027 (0.0002)	0.065 (0.0005)	0.718 (0.001)
β_1	-0.066 (0.003)	-0.040 (0.001)	-0.052 (0.002)	0.159 (0.004)
δ_{12}	-0.002 (0.001)	0.001 (0.0002)	-0.003 (0.0004)	0.005 (0.001)
δ_{13}	-0.002 (0.001)	0.001 (0.0002)	-0.003 (0.0004)	0.004 (0.001)
δ_{14}	0.004 (0.001)	0.003 (0.0002)	0.005 (0.0004)	-0.013 (0.001)
R^2	0.942	0.958	0.973	0.981
D.W.	0.643	1.052	1.159	0.850

L (system) = 1455.79

(Note: Standard Errors in parentheses.)

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Table 2: Maximum Likelihood Estimates for A.I.D.S.

(Combined Approach, Symmetry Constrained).

	Food	Tobacco	Alcohol	Other
γ_{1F}	0.092 (0.008)	.	.	.
γ_{1T}	-0.006 (0.002)	0.014 (0.001)	.	.
γ_{1A}	-0.002 (0.004)	-0.016 (0.001)	-0.004 (0.005)	.
γ_{1O}	-0.083 (0.007)	0.008 (0.002)	0.023 (0.004)	0.053 (0.009)
α_1	0.190 (0.001)	0.027 (0.0002)	0.065 (0.0005)	0.718 (0.001)
β_1	-0.067 (0.004)	-0.040 (0.001)	-0.051 (0.002)	0.158 (0.004)
δ_{12}	-0.002 (0.001)	0.0005 (0.0002)	-0.003 (0.0004)	0.005 (0.001)
δ_{13}	-0.002 (0.001)	0.001 (0.0002)	-0.002 (0.0004)	0.003 (0.001)
δ_{14}	0.004 (0.001)	0.003 (0.0002)	0.005 (0.0004)	-0.013 (0.001)
R^2	0.954	0.962	0.979	-
D.W.	0.661	0.981	1.180	-

L (system) = 627.343

(Note: Standard Errors in parentheses.)

Table 3: Maximum Likelihood Estimates for the Indirect Addilog Model.

(Traditional Approach).

	Food	Tobacco	Alcohol	Other
γ_1	-0.294 (0.044)	0.662 (0.032)	0.087 (0.038)	-1.286 (0.029)
$\alpha_1\gamma_1$	0.256 (0.003)	0.041 (0.0003)	0.093 (0.001)	1 (0)
δ_{11}	-0.002 (0.001)	-0.001 (0.0001)	-0.0001 (0.0005)	0.003 (0.001)
δ_{12}	-0.003 (0.001)	-0.0005 (0.0001)	-0.003 (0.0004)	0.006 (0.001)
δ_{13}	-0.001 (0.001)	-0.0004 (0.0001)	-0.003 (0.0004)	0.004 (0.001)
δ_{14}	0.006 (0.001)	0.002 (0.0001)	0.006 (0.0005)	-0.014 (0.001)
R^2	0.710	0.949	0.910	0.912
D.W.	0.170	0.950	0.285	0.901

L (system) = 1316.50

(Note: Standard Errors in parentheses.)

Table 4: Maximum Likelihood Estimates for the Indirect Addilog Model. (Combined Approach).

	Food	Tobacco	Alcohol	Other
γ_1	-0.345 (0.044)	0.650 (0.029)	0.124 (0.044)	-1.321 (0.030)
$\alpha_1 \gamma_1$	0.254 (0.003)	0.041 (0.0003)	0.094 (0.001)	1 (0)
δ_{11}	-0.002 (0.001)	-0.001 (0.0001)	-0.0005 (0.0005)	0.004 (0.001)
δ_{12}	-0.003 (0.001)	-0.0005 (0.0001)	-0.003 (0.0005)	0.006 (0.001)
δ_{13}	-0.001 (0.001)	-0.0004 (0.0001)	-0.002 (0.0005)	0.004 (0.001)
δ_{14}	0.006 (0.001)	0.002 (0.0001)	0.006 (0.0005)	-0.014 (0.001)
R^2	0.769	0.962	0.918	-
D.W.	0.184	1.020	0.251	-

L (system) = 484.017

(Note: Standard Errors in parentheses.)

Table 5: Results of the Anderson-Darling Test for Normality Under the Traditional and Combined Approaches.

Model	Anderson-Darling Test Statistic	
	Traditional Approach	Combined Approach
A.I.D.S.	0.782	1.055
Indirect Addilog	0.309	0.631

Critical Value (5%): 2.492

Table 6: Maximum Likelihood Estimates for A.I.D.S. (Traditional Approach, Symmetry Unconstrained).

	Food	Tobacco	Alcohol	Other
γ_{1F}	0.093 (0.007)	-0.005 (0.002)	-0.002 (0.004)	-0.087 (0.007)
γ_{1T}	-0.020 (0.003)	0.012 (0.001)	-0.019 (0.002)	0.027 (0.003)
γ_{1A}	0.028 (0.010)	-0.005 (0.002)	0.004 (0.005)	-0.027 (0.009)
γ_{1O}	-0.102 (0.009)	-0.002 (0.002)	0.017 (0.004)	0.087 (0.008)
α_1	0.190 (0.0009)	0.028 (0.0002)	0.066 (0.0005)	0.717 (0.001)
β_1	-0.054 (0.004)	-0.038 (0.001)	-0.048 (0.002)	0.139 (0.004)
δ_{12}	-0.003 (0.001)	0.0005 (0.0002)	-0.003 (0.0004)	0.006 (0.001)
δ_{13}	-0.002 (0.001)	0.001 (0.0002)	-0.003 (0.0004)	0.004 (0.001)
δ_{14}	0.003 (0.001)	0.003 (0.0002)	0.005 (0.0004)	-0.011 (0.001)
R^2	0.952	0.966	0.975	0.987
D.W.	0.810	1.182	1.208	1.240

L (system) = 1481.28

Symmetry Test: $2 (L^{(6)} - L^{(1)}) = 25.49$

5% critical value: $\chi^2_{(6)} = 12.592$

(Note: Standard Errors in parentheses.)

Table 7: Maximum Likelihood Estimates for A.I.D.S.

(Combined Approach, Symmetry Unconstrained).

	Food	Tobacco	Alcohol	Other
γ_{1F}	0.092 (0.007)	-0.004 (0.002)	-0.002 (0.004)	-0.087 (0.007)
γ_{1T}	-0.019 (0.003)	0.012 (0.001)	-0.020 (0.002)	0.028 (0.003)
γ_{1A}	0.027 (0.010)	-0.006 (0.002)	0.005 (0.005)	-0.026 (0.010)
γ_{1O}	-0.100 (0.009)	-0.002 (0.002)	0.016 (0.005)	0.086 (0.009)
α_1	0.190 (0.001)	0.028 (0.0002)	0.065 (0.0005)	0.717 (0.001)
β_1	-0.004 (0.004)	-0.037 (0.001)	-0.047 (0.002)	0.139 (0.004)
δ_{12}	-0.003 (0.001)	0.0004 (0.0002)	-0.003 (0.0004)	0.005 (0.001)
δ_{13}	-0.002 (0.001)	0.001 (0.0002)	-0.002 (0.0004)	0.004 (0.001)
δ_{14}	0.003 (0.001)	0.003 (0.0002)	0.005 (0.0004)	-0.011 (0.001)
R^2	0.963	0.972	0.982	-
D.W.	0.840	1.148	1.249	-

L (system) = 651.352

Symmetry Test: $2 (L^{(7)} - L^{(2)}) = 24.009$

5% critical value: $\chi^2_{(6)} = 12.592$

(Note: Standard Errors in parentheses.)