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MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA.

A New Test for Structural Change in Dynamic Models

Brett Inder and Hao Kang

Department of Econometrics
Monash University
Clayton
Victoria 3168
AUSTRALIA

Abstract

This paper considers the linear regression model with the lagged dependent variable as a regressor. It argues that the Dynamic CUSUM test may not be ideal in testing for a structural change at an unknown point in time. A new test is proposed, with critical values based on small disturbance asymptotics. A Monte Carlo study shows some power improvement. The new test also provides far more reliable estimates of the location of the structural break.

Key Words: CUSUM Test, Small Disturbance Asymptotics, Locating Structural Breaks, Lagged Dependent Variable.

1. INTRODUCTION

When an econometric model is used for forecasting or policy simulation, parameter stability plays a very important role. The problem of testing for the stability of an economic relationship over time has been discussed by many authors. Among the various test statistics, Brown, Durbin and Evans' (1975) (henceforth BDE) CUSUM test and CUSUM of squares tests, which are based on recursive residuals, have become the standard diagnostic tests in linear regression models when the possible timing and the type of instability are unknown.

However, relationships among economic variables are often dynamic, and lagged dependent variables are included in regression models. In this case, the CUSUM test is not applicable because of the presence of a stochastic regressor. In this case, Dufour (1982) suggested replacing the coefficients of the lagged dependent variables by their consistent estimates from the full sample, and hoping that the resulting resursive residuals and any tests based on them will have approximately the same properties as those based on the true coefficients of lagged dependent variables.

Ploberger, Kramer and Alt(1989) showed that under some regularity conditions, Dufour's approach is asymptotically valid if the coefficients of lagged dependent variables are estimated under the null hypothesis of no structural change. Kramer, Ploberger and Alt (1988) (henceforth KPA) also showed that asymptotically one can disregard the dynamic character of the regression and proceed with the CUSUM test as in the static model. Any choice between Dufour's approach and Kramer et al.'s approach is a matter of power and of the accuracy of the actual size of the test. KPA's Monte Carlo (MC) experiment shows that the dynamic CUSUM test can perform much better than Dufour's approach.

Although it outperforms Dufour's approach, the dynamic CUSUM test suffers from some serious drawbacks. The first is that it is only valid asymptotically; its power performance for small sample size is not satisfactory. The second is that it has low power when the shift is orthogonal to the mean regressors. Finally, as with many other tests, it possesses low power when the coefficients change late in the sample period.

Small disturbance asymptotics has been used profitably in a number of recent studies involving dynamic models. For example, Nankervis and Savin (1987) report the small disturbance distribution (SDD) of F and t statistics for testing linear restrictions on coefficients in the dynamic model; Inder (1986) and King and Wu (1991) find the SDD of the Durbin-Watson statistic in these models. In each case the SDDs are the same as the exact distributions of the statistics in a regression with the lagged dependent variable replaced by its mean.

In this paper we are interested in detecting structural change taking the form of a discrete jump in one or more of the parameters. We argue that the CUSUM test may not be ideal for this, and propose an alternative test, called the Q-step multiple t test. Critical values for this test are based on the statistic's SDD. The theory is given in section 2, with a MC comparison of this test with the dynamic CUSUM test being discussed in section 3. This study suggests that the new test is quite powerful when the structural change takes place in the latter part of the sample.

One of the benefits of CUSUM-type tests is that they can not only detect instability, but also provide information about where any structural breaks may take place. In section 4 we evaluate the Dynamic CUSUM and Q-step multiple-t procedures as estimators of the change point. Evidence suggests that the Dynamic CUSUM is quite inferior for this purpose.

2. THEORY

Consider the dynamic linear model

$$y_t = y_{t-1}\alpha + x_t'\beta_t + u_t \qquad (t = 1, \ldots, T), \qquad (1)$$

where y_t is the t^{th} observation on the dependent variable, x_t is a $k \times 1$ vector of observations on the exogenous variables at time t, α and β_t are unknown parameters, and u_t is a stochastic disturbance with $u_t \sim IN(0, \sigma^2)$. We are interested in testing for a discrete jump in at least one of the elements of β_t at an unknown point in time. Specifically, the null hypothesis can be formulated as

$$H_0: \beta_1 = \beta_2 = \ldots = \beta_T = \beta ,$$

with the alternative being H_1: at least one equality does not hold.

The recursive residuals, which are a basis for the CUSUM test, are given by

$$w_r = [y_r - (y_{r-1}, x_r')b_{r-1}]/f_r \qquad (r = k + 2, \ldots, T), \quad (2)$$

where b_{r-1} is the ordinary least squares (OLS) estimator of (α, β') using the first $r-1$ observations, and

$$f_r = [1 + (y_{r-1}, x_r')[(Y_{-1}^{r-1}, X^{r-1})'(Y_{-1}^{r-1}, X^{r-1})]^{-1}(y_{r-1}, x_r')']^{1/2},$$

with

$$Y_{-1}^{r-1} = (y_0, y_1, \ldots, y_{r-2}) \text{ and } X^{r-1} = (x_1, x_2, \ldots, x_{r-1})$$

The Dynamic cusum test is based on the statistic

$$S = \max_{k+2 \le r \le T} \left| \frac{W^{(r)}}{\sqrt{T-k-1}} \right| \Big/ \left[1 + \frac{2(r-k-1)}{T-k-1} \right] , \qquad (3)$$

where $W^{(r)} = \hat{\sigma}^{-1} \sum_{t=k+2}^{r} w_t$ and $\hat{\sigma}^2 = (T-k-2)^{-1} \sum_{t=k+2}^{T} (w_t - \bar{w})^2$, with $\bar{w}$ being $(T-k-1)^{-1} \sum_{t=k+2}^{T} w_t$. H_0 is rejected for large values for S, with, for example, the 5% critical value being 0.948 (see BDE and KPA for details).

It is clear from (3) that the key to discriminating between null and alternative hypotheses for the CUSUM test is the term $W^{(r)}$, which is the sum of all recursive residuals up to time period r. If the structural break occurs at the point m, the w_t's will have zero mean up to the point m, and non-zero means subsequently. The inclusion of W_t for $t = k + 2, \ldots, m$ in the test statistic will thus not be helpful in detecting the change. They are likely to lead to a loss of power. This may well explain the poor power of the CUSUM test when m is close to the end of the sample period, as many w_t's with zero mean are included in the test statistic.

The above discussion leads us to consider an alternative to the CUSUM test which relies on more than just one-step ahead forecast errors, which are the basis of the CUSUM test. We propose a test which involves averaging, at each point in time r, forecasting errors from one-step to q-steps ahead. We can then apply a sequential testing procedure at each point in time, rejecting the null hypothesis if any of the separate tests lead to rejection.

It is helpful to apply a regression interpretation to the forecasting errors. The q forecasting errors can be calculated as the coefficients of a set of dummy variables added to the regressor matrix. Specifically, if forecasting q periods ahead from a point r, one can regress Y^{r+q} on $\left[Z^{r+q} \vdots D_{r,q}\right]$, where $Z^{r+q} = \left[Y_{-1}^{r+q} \vdots X^{r+q}\right]$, and $D_{r,q}$ is an $(r+q) \times q$ matrix of dummy variables given by

$$D_{r,q} = \begin{bmatrix} 0_{r\times q} \\ \hline I_q \end{bmatrix},$$

with $0_{r\times q}$ being an $r\times q$ null matrix, and I_q represents a $q\times q$ identity matrix. The forecasting errors are the estimated coefficients of $D_{r,q}$, represented in the vector $\hat{\gamma}_{r,q}$. A test of the hypothesis that the average of these q errors is zero is easily performed, as this amounts to a test of linear restrictions on the coefficients in the above regression. Using results on partitioned matrices, the test statistic is given by

$$g_{r,q} = \frac{\ell' \hat{\gamma}_{r,q}}{\hat{\sigma}\left[\ell' \left(D'_{r,q} \bar{P}_{Z^{r+q}} D_{r,q}\right)^{-1} \ell\right]^{1/2}}, \tag{4}$$

where ℓ' is a vector all of whose elements is q^{-1}, $\bar{P}_Z = I - Z(Z'Z)^{-1}Z'$, and $\hat{\sigma}^2$ is a consistent estimate of the error variance in the above regression. The statistic $g_{r,q}$ could be calculated for all values of r from $k+2$ to $T-q$, and each statistic compared with appropriate critical values.

Small disturbance asymptotics provides a simple means of obtaining reliable critical values for the $g_{r,q}$ tests. Given the above regression interpretation of the test statistic, the results of Nankervis and Savin (1987) can be applied to show that the $g_{r,q}$

statistic has an SDD which follows the t distribution with $r-k-1$ degrees of freedom. This result is seen by observing that the regressions involve $r+q$ observations, with $k+1+q$ regressors. We thus have a multiple hypothesis testing situation as discussed by Savin (1984). The acceptance region of an individual θ-level two-tailed test is given by

$$|g_{r,q}| \leq t_{\theta/2}(r-k-1) \qquad (r = k + 2, \ldots, T - q), \qquad (5)$$

where $t_{\theta/2}(r-k-1)$ is the upper $\theta/2$ significance point of a t distribution with $r-k-1$ degrees of freedom. For the multiple testing problem H_0 is accepted if $g_{r,q}$ falls in the acceptance region (5) for all values of r.

Because of the dependence that exists between each of the test statistics, we are only able to obtain an inequality on the small disturbance asymptotic significance level of this multiple testing procedure. Again following Savin (1984), we use Sidak's (1967) inequality to choose a significance level θ for each individual t test according to

$$\theta = 1 - (1 - \delta)^{T-k-q}, \qquad (6)$$

where δ is the nominal significance level of the multiple testing procedure. Critical values with different implications of θ and r are given by Games (1977).

As with the CUSUM test, there is a reasonable computational burden in executing the above testing procedure. OLS regressions need to be performed for each value of r, and a test statistic calculated according to (4). To complicate matters further, each value of r

(see (5)). This whole procedure could, of course, be automated without difficulty using any reasonable econometric software. There are also alternative ways of representing the forecast errors and updating regression estimates with new observations which would vastly improve computational speed. These are not discussed here as our focus is on the intuition behind and performance of the testing procedure.

The question of how many forecast errors to include in the test statistics (a value of q) remains. If q is too large, the testing must stop too soon before the end of the sample period, but a value of q which is too small will leave the test vulnerable to outliers or other temporary perturbations. Our preliminary MC analysis has led us to choose $q = 3$, and results on the performance of the test are presented in the next section with this value. This practical issue does, however, need further investigation.

3. COMPARISON OF THE TESTS

Same intuition has been presented in section 2 for why one may expect the multiple t test to be more powerful than the dynamic CUSUM test in detecting structural change. It is the purpose of this section to test this claim by performing a MC comparison of the tests. We consider their performance under the null hypothesis (closeness of actual size to nominal size) and the alternative (power).

For ease of comparison, the design of the MC study presented here is consistent with KPA's. The model generating the data is given by equation (1), with the first observation generated as

$$y_0 = x_0'\beta/(1-\alpha) + u_0/(1-\alpha^2)^{1/2} .$$

This start-up condition is commonly used to allow for a random y_0, but to ensure it has a realistic mean and variance.

There are two sample sizes used with each X matrix: small (T=31) and medium sized (T=58). We allow the structural shift to affect only the β parameters. Initially $\beta' = (10, 2)$ and $\alpha = -0.5$ or 0.5, and the structural shift in $\delta' = (\alpha, \beta')$ is given by

$$\Delta\delta' = \frac{b}{T^{1/2}} (0, \cos(\theta + \phi), \sin(\theta + \phi)),$$

where ϕ is 0°, 30°, 60° and 90°, and b = 0 under H_0 and 8 under H_1. θ is the angle between the mean regressor and constant term, ϕ is the angle between shift vector and mean regressor.

The structural shift occurs at time $m = zT + 1$, where z takes the values 0.3, 0.5 and 0.7.

The following X matrices were used:

X1: $x_t = (-1)^t$, with $x_0 = 1$.

X2: $x_t = 1 + \varepsilon_t$, with $x_0 = 1$, and $\varepsilon_t \sim IN(0, 1)$.

X3: $x_t = 1 + 4t/T + \varepsilon_t$, with $x_0 = 1$, and $\varepsilon_t \sim IN(0, 1)$.

Two values of σ, the error standard deviation, were used in the MC study: $\sigma = 0.5$ and $\sigma = 1$. This effectively gives us 24 sets of results: three X matrices, each with two sample sizes, two values of σ, and two values of α. Results are based on 3000 replications for each set of parameter values.

The results of this MC study are given in Tables 1 to 4. Table 1 presents the probabilities of a type I error, which it is hoped will be as close as possible to the nominal significance level of 5%.

The evidence in this table suggests both tests perform reasonably well under H_0: sizes are generally a little low, but in most cases are acceptably close to .05. Both tests perform the worst with the X1 matrix, with some sizes of around 2%.

Turning to power, there are a few points which can be drawn from the MC simulation.

The performance of both tests very much depends on ϕ the angle between the shift vector and the projection of the mean regressor on the β plane. When ϕ increases, the power of both tests decreases dramatically. Actually, KPA proved that when $\phi = 90^\circ$, the dynamic CUSUM test has trivial local power. For the QSMT test, the results show that it posesses non-trivial power in this case although its performance is still poor.

When α is negative, then for small sample size (T = 31) the dynamic CUSUM test has better power than QSMT test for all three X matrices if z = 0.3. On the other hand, the QSMT test outperforms the dynamic CUSUM test for all X matrices if z = 0.7. When z = 0.5, the QSMT test is preferred with X1 and X3, and the two tests have similar performance for X2.

If the sample size is increased to 59, the dynamic CUSUM test is still preferred with $T^* = 0.3$, but QSMT test is much more powerful with $T^* = 0.7$. When $T^* = 0.5$, however, the situation is a little bit different. For the X1 matrix, the dynamic CUSUM test has better performance for large σ. For the small σ value, there is little difference between the tests. For X2, the dynamic CUSUM test dominates the QSMT test for both σ values, and with X3 both tests have good power when σ is small, while QSMT is slightly better for the large σ value.

A more reasonable assumption is that α takes positive values. In this case, QSMT test shows some advantage over the dynamic CUSUM test. For convenience, we assume that power differences of less than 0.1 are not considered important or substantial.

When the sample size is small, for the X1 matrix there are only four points at which the dynamic CUSUM test is more powerful than QSMT. However, the improvement is small with all under 0.2. On the other hand, there are 14 points at which the QSMT test outperforms the dynamic CUSUM test, and for half of them, the power improvement exceeds 0.5. A similar situation occurs with X2. For the X3 matrix, QSMT test is superior to dynamic CUSUM test in virtually every case. It is not uncommon to have situations where the dynamic CUSUM test has virtually no power, while the QSMT test has power close to 1.

Overall, it seems that QSMT test deserves to be recommended as an alternative to the dynamic CUSUM test, especially for small sample sizes.

4. ESTIMATION OF THE BREAK POINT

One of the alleged advantages of the CUSUM test over other tests for structural change is that it can provide information about the location of such a change. While there is no formal treatment of this, it is seen that the point in the sample at which the test becomes significant could be used as an estimator of the break point. In this section we evaluate this estimator for the dynamic model, and also compare it to the analogous estimator using the QSMT test.

Specifically, the change point in the sample, T^*, is estimated by the smallest value of r at which the Dynamic CUSUM statistic exceeds its critical value (see equation (3)). Similarly, for the QSMT estimator, the smallest value of r which leads to a rejection of the null hypothesis using equation (5) is used as the estimator.

Table 5 gives the results of this comparison, using a similar MC design to section 3. In each case, the σ value is 1 and the structural break takes place around half-way through the sample (T^* = 17 for the smaller and 31 for the larger sample). An estimator is calculated only if the test leads to rejection of the null hypothesis. We calculate the mean absolute errors in the estimates of T^*, as well as their root mean square errors (RMSE), and also the percent of estimates within <u>one</u> of the correct T^* value (PN1).

It is obvious from Table 5 that the Dynamic CUSUM estimator performs very poorly. For the X1 and X2 matrices, the dynamic CUSUM test detects the structural change an average of 5 to 8 periods after the true breaking point for the small sample size, and 10 to 15 periods for the medium sample size. On the other hand, the QSMT test estimates the break point quite accurately. When the sample size is small, the estimated break point is around 18 - 20 while T^* = 17; when sample size is medium, the estimated break point is around 33 - 37 while T^* = 31. For X3 the QSMT test provides an almost unbiased estimate for the break point while the performance of the dynamic CUSUM estimator varies with the sign of α.

In all the three experiments, the largest PN1 value of the dynamic CUSUM test is around 24%, with more than half being below 4%. In contrast, half of the PN1 values for the QSMT test are over 40%, with values as large as 90%.

5. SUMMARY

The CUSUM test is probably the most well-known diagnostic test for structural change in regression models. Recent work by KPA and others have provided a justification for its use in models which include a lagged dependent variable. This paper has proposed an alternative to the Dynamic CUSUM test which involves q-step ahead forecast errors rather than recursive residuals, which are essentially one-step ahead forecast errors. There is reason to expect that the QSMT test will yield a power improvement for changes that occur late in the sample period, and the MC evidence supports this.

We have also shown in this paper how small disturbance asymptotics can be used to obtain critical values for the QSMT test. This approach could also be used profitably with other tests which are applicable to dynamic models, such as the "multiple F test" approach of Andrews (1991).

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Table 1. Probability of Rejecting a True Null Hypothesis[a]

(Nominal Size = 0.05)

	σ	α	Test	X matrix X_1	X_2	X_3
T = 31	0.5	-0.5	DCUSUM	0.036	0.019	0.040
			QSMT	0.045	0.029	0.039
		0.5	DCUSUM	0.037	0.043	0.036
			QSMT	0.039	0.047	0.043
	1.0	-0.5	DCUSUM	0.036	0.016	0.036
			QSMT	0.045	0.023	0.039
		0.5	DCUSUM	0.043	0.047	0.038
			QSMT	0.045	0.047	0.043
T = 59	0.5	-0.5	DCUSUM	0.030	0.022	0.040
			QSMT	0.037	0.031	0.044
		0.5	DCUSUM	0.035	0.039	0.040
			QSMT	0.039	0.044	0.048
	1.0	-0.5	DCUSUM	0.029	0.017	0.040
			QSMT	0.040	0.031	0.046
		0.5	DCUSUM	0.041	0.045	0.041
			QSMT	0.047	0.049	0.051

a: DCUSUM is the Dynamic CUSUM test, and QSMT is the q-step Multiple t test. q = 3 for these results.

Table 2. Power of the tests (X_1 matrix)

			α = -0.5				α = 0.5			
		φ	0°	30°	60°	90°	0°	30°	60°	90°
			Dynamic CUSUM							
		z=.3	.745	.527	.196	.017	.319	.308	.164	.043
		z=.5	.498	.260	.072	.016	.284	.170	.071	.020
		z=.7	.165	.073	.028	.034	.127	.055	.033	.037
	σ=1		QSMT							
		z=.3	.241	.237	.131	.027	.158	.172	.124	.044
		z=.5	.576	.497	.245	.046	.531	.519	.274	.060
		z=.7	.712	.617	.223	.064	.652	.564	.211	.075
T=31			Dynamic CUSUM							
		z=.3	.996	.932	.386	.006	.589	.525	.289	.008
		z=.5	.908	.556	.090	.009	.611	.353	.082	.008
		z=.7	.333	.090	.015	.012	.246	.069	.016	.013
	σ=.5		QSMT							
		z=.3	.586	.582	.415	.034	.424	.424	.381	.041
		z=.5	.995	.984	.807	.128	.988	.982	.832	.129
		z=.7	1.000	.999	.886	.132	1.000	.998	.861	.116
			Dynamic CUSUM							
		z=.3	.872	.732	.252	.029	.807	.685	.239	.034
		z=.5	.758	.584	.155	.032	.620	.523	.160	.033
		z=.7	.274	.197	.068	.031	.148	.141	.052	.045
	σ=1		QSMT							
		z=.3	.277	.252	.094	.041	.299	.301	.118	.043
		z=.5	.441	.342	.151	.060	.418	.375	.188	.073
		z=.7	.591	.498	.237	.049	.515	.448	.221	.055
T=59			Dynamic CUSUM							
		z=.3	1.000	1.000	.676	.016	1.000	1.000	.660	.014
		z=.5	.999	.987	.523	.008	.993	.981	.544	.012
		z=.7	.793	.572	.165	.021	.521	.371	.128	.021
	σ=.5		QSMT							
		z=.3	.909	.889	.437	.040	.904	.893	.462	.043
		z=.5	.976	.960	.627	.100	.972	.963	.670	.113
		z=.7	.999	.999	.913	.106	.998	.997	.877	.111

Table 3. Power of the tests (X_2 matrix)

			α = -0.5				α = 0.5			
		φ	0°	30°	60°	90°	0°	30°	60°	90°
						Dynamic CUSUM				
		z=.3	.379	.378	.162	.011	.203	.244	.188	.051
		z=.5	.314	.215	.074	.009	.167	.152	.091	.014
		z=.7	.123	.060	.015	.007	.117	.073	.035	.015
	σ=1					QSMT				
		z=.3	.054	.112	.081	.024	.056	.088	.105	.067
		z=.5	.173	.202	.150	.046	.177	.175	.178	.076
		z=.7	.254	.258	.156	.056	.275	.263	.189	.093
T=31						Dynamic CUSUM				
		z=.3	.744	.872	.638	.012	.239	.281	.526	.040
		z=.5	.654	.621	.333	.006	.165	.128	.171	.003
		z=.7	.312	.126	.041	.002	.127	.062	.028	.001
	σ=.5					QSMT				
		z=.3	.168	.341	.261	.045	.198	.265	.288	.095
		z=.5	.625	.771	.563	.090	.632	.649	.614	.153
		z=.7	.820	.911	.692	.149	.810	.818	.734	.200
						Dynamic CUSUM				
		z=.3	.528	.458	.156	.029	.285	.268	.155	.052
		z=.5	.398	.317	.092	.023	.237	.206	.090	.028
		z=.7	.126	.098	.033	.015	.103	.079	.051	.027
	σ=1					QSMT				
		z=.3	.086	.085	.066	.046	.062	.082	.086	.066
		z=.5	.152	.167	.109	.047	.124	.146	.117	.072
		z=.7	.198	.166	.101	.046	.173	.170	.121	.070
T=59						Dynamic CUSUM				
		z=.3	.945	.969	.676	.024	.335	.426	.532	.045
		z=.5	.914	.893	.444	.017	.285	.308	.255	.016
		z=.7	.503	.362	.116	.004	.163	.131	.066	.004
	σ=.5					QSMT				
		z=.3	.259	.403	.248	.056	.241	.288	.245	.099
		z=.5	.578	.679	.414	.073	.498	.508	.409	.138
		z=.7	.730	.802	.502	.091	.589	.634	.517	.143

Table 4. Power of the tests (X_3 matrix)

		ϕ	α = -0.5 0°	30°	60°	90°	α = 0.5 0°	30°	60°	90°
						Dynamic CUSUM				
		z=.3	.777	.761	.518	.028	.040	.061	.107	.048
		z=.5	.549	.547	.418	.030	.064	.079	.121	.030
		z=.7	.363	.357	.252	.032	.030	.055	.082	.047
	σ=1					QSMT				
		z=.3	.422	.300	.166	.033	.201	.147	.086	.048
		z=.5	.967	.921	.541	.050	.846	.721	.369	.051
		z=.7	1.000	1.000	.944	.059	.999	.999	.818	.060
T=31						Dynamic CUSUM				
		z=.3	.989	.989	.947	.026	.008	.033	.165	.057
		z=.5	.788	.866	.852	.026	.027	.039	.155	.033
		z=.7	.599	.600	.562	.045	.004	.011	.093	.037
	σ=.5					QSMT				
		z=.3	.986	.903	.447	.042	.771	.588	.247	.070
		z=.5	1.000	1.000	.995	.077	1.000	1.000	.945	.083
		z=.7	1.000	1.000	1.000	.113	1.000	1.000	1.000	.113
						Dynamic CUSUM				
		z=.3	.905	.814	.411	.031	.202	.194	.137	.038
		z=.5	.812	.773	.413	.030	.185	.208	.130	.040
		z=.7	.609	.550	.315	.037	.160	.159	.129	.039
	σ=1					QSMT				
		z=.3	.605	.461	.105	.044	.540	.416	.106	.064
		z=.5	.950	.880	.335	.052	.897	.756	.278	.066
		z=.7	1.000	1.000	.784	.055	.995	.987	.671	.044
T=59						Dynamic CUSUM				
		z=.3	1.000	1.000	.946	.039	.377	.349	.331	.041
		z=.5	1.000	.998	.943	.040	.271	.343	.351	.051
		z=.7	.970	.945	.826	.047	.247	.269	.305	.055
	σ=.5					QSMT				
		z=.3	.999	.993	.591	.058	.998	.989	.519	.068
		z=.5	1.000	1.000	.987	.081	1.000	1.000	.969	.084
		z=.7	1.000	1.000	1.000	.116	1.000	1.000	1.000	.101

Table 5. Performance of Estimators of the change point

	α	ϕ	Dynamic CUSUM 0°	30°	60°	90°	QSMT 0°	30°	60°	90°
X_1 T=31 (T*=17)	.5	$E\|\hat{T}^*-T^*\|$	9.1	8.7	7.9	4.7	2.3	2.6	3.4	4.0
		RMSE	9.91	9.44	8.57	5.74	3.02	3.13	3.95	5.54
		PN1	0.4	0.0	2.6	19.4	21.9	8.2	1.5	42.9
	-.5	$E\|\hat{T}^*-T^*\|$	8.9	8.8	7.8	4.8	2.3	2.7	3.2	3.9
		RMSE	9.49	9.31	8.44	6.19	3.05	3.29	3.90	5.71
		PN1	0.0	0.0	1.2	23.8	18.5	7.9	2.0	40.0
X_1 T=59 (T*=31)	.5	$E\|\hat{T}^*-T^*\|$	14.8	16.6	17.4	9.8	4.6	6.1	10.3	10.6
		RMSE	16.20	17.91	18.88	12.06	7.51	9.31	13.23	13.00
		PN1	1.9	1.4	3.2	7.7	50.5	46.0	28.6	9.3
	-.5	$E\|\hat{T}^*-T^*\|$	15.5	17.1	17.9	10.1	5.4	6.9	9.1	11.3
		RMSE	16.95	18.42	19.27	11.88	8.59	10.67	12.28	13.76
		PN1	1.6	1.6	2.0	8.0	45.8	44.4	31.1	9.7
X_2 T=31 (T*=17)	.5	$E\|\hat{T}^*-T^*\|$	6.1	7.1	7.8	6.5	1.7	2.0	3.5	4.5
		RMSE	7.05	8.18	8.67	7.79	2.95	3.17	4.81	5.83
		PN1	9.2	8.8	6.7	12.5	75.8	65.4	43.3	24.4
	-.5	$E\|\hat{T}^*-T^*\|$	8.1	8.8	8.7	5.2	2.6	3.1	3.2	3.6
		RMSE	8.81	9.48	9.49	6.53	4.17	4.36	4.45	5.22
		PN1	2.8	3.9	4.5	16.6	50.3	48.2	49.7	33.3
X_2 T=59 (T*=31)	.5	$E\|\hat{T}^*-T^*\|$	13.0	14.7	15.7	9.3	5.3	6.7	7.9	10.2
		RMSE	15.23	16.38	17.37	11.39	7.85	9.51	10.46	12.70
		PN1	6.3	1.6	2.4	12.0	36.2	30.5	25.3	13.3
	-.5	$E\|\hat{T}^*-T^*\|$	15.7	17.3	16.4	10.1	7.4	8.8	9.6	8.3
		RMSE	17.28	18.70	17.72	12.58	10.22	11.44	12.12	11.29
		PN1	3.0	0.3	0.0	5.3	27.1	21.4	18.2	21.6
X_3 T=31 (T*=17)	.5	$E\|\hat{T}^*-T^*\|$	3.7	3.7	4.9	5.1	0.6	0.6	1.4	5.1
		RMSE	3.89	3.86	5.56	6.39	1.14	1.16	2.05	6.06
		PN1	3.8	4.6	2.5	8.8	90.1	85.0	47.4	11.3
	-.5	$E\|\hat{T}^*-T^*\|$	7.2	7.3	7.3	5.8	0.6	0.6	1.4	4.5
		RMSE	7.97	8.05	8.02	6.73	1.07	1.30	2.16	5.69
		PN1	0.2	0.4	1.2	6.3	91.6	87.4	50.5	20.9
X_3 T=59 (T*=31)	.5	$E\|\hat{T}^*-T^*\|$	6.1	7.2	9.9	11.1	1.3	1.0	2.0	10.1
		RMSE	6.89	8.33	11.57	13.47	2.96	2.54	4.65	12.58
		PN1	4.8	5.2	7.1	8.7	84.0	87.4	76.1	13.2
	-.5	$E\|\hat{T}^*-T^*\|$	9.7	10.8	13.1	12.0	1.1	1.0	2.5	11.4
		RMSE	10.83	12.10	14.58	14.30	2.51	2.00	5.06	13.40
		PN1	1.0	1.4	2.9	4.9	84.4	85.5	67.2	9.8