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Offer Lieberman*

May, 1992

Abstract

In the present paper, the saddle point approximations to the density and tail probability of a ratio of quadratic forms in normal variables are derived. A numerical exposition via the Durbin-Watson test statistic reveals several desirable features. Beyond their accuracy and robustness to extreme scenarios, the approximations, which only involve a limited number of computable functions, provide the practitioner with an accessible and a very powerful tool.

Key Words : Cgf; Fourier transform; Quadratic forms; Saddle point approximation; Steepest descent.

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1. INTRODUCTION

A large number of statistics can be expressed as a ratio of quadratic forms in normal variables, namely

$$r = \frac{x'Fx}{x'Gx},$$

where the elements of the column vector x are independently and normally distributed, each with a zero mean and variance σ^2, F and G are (nxn) non-stochastic matrices and G is assumed to be positive semi-definite. Although the exact distribution of r is only known in some special cases, the computation of its cdf is made possible by the availability of numerical procedures such as Imhof (1961), Davies (1973), or Shively et al. (1990). Nevertheless, given the abundance of such statistics and their frequent use in applied statistics and econometrics, there is undoubtedly a need for tractable and accurate approximations to their density and distribution functions.

Unless G is an identity or an idempotent matrix, any attempt to provide such approximations by methods of matching moments or curve fitting techniques is doomed to a hopeless computational task, as then, the moments of r are expressed as infinite sums of invariant polynomials with multiple matrix arguments, see Smith (1989). Evans and King (1985) have shown that the beta approximation to the size of the Durbin-Watson test statistic is reasonably accurate. The approximate powers, however, could not have been obtained, due to the algebraic complexity of the moments under the alternative hypothesis.

In this paper, the saddle point method is used in deriving a compact and accurate formula to the distribution of a ratio of quadratic forms in normal variables. Unlike their direct Edgeworth counterparts, expansions based on the saddle point method contain relative rather than absolute errors, are non-negative everywhere (for the density function), and are almost uniformly superior to the former in the tails of the distribution, where most of the statistical interest lies. Ample evidence is available, see for instance, Easton and Ronchetti (1986), McCullagh (1987), Reid (1988), or Barndorff-Nielsen and Cox (1989).

Following Barndorff-Nielsen and Cox's (1979) development of the tilted Edgeworth expansion or, equivalently, the saddle point expansion, this method received a dramatic surge in popularity in the literature. Consequent applications to related problems have mostly utilized the probabilistic notion of exponential tilting, rather than the more traditional complex variable technique of steepest descent. Daniels (1987) showed that both approaches lead to identical density approximations, but not so for the tail probabilities.

The saddle point approximation to the density function is derived in section 2. The relative error is $O(n^{-1})$. This approximation is of particular importance, as unlike the cdf, the density function of r is not provided by any existing algorithm. It is demonstrated that by a straightforward transformation, an extension follows for the case of a ratio of a bilinear form to a quadratic form. The Lugannani and Rice (1980) formula is applied in obtaining an explicit expression for the cdf in section 3. This expression is shown to be remarkably simple. A brief numerical exposition via the power of the Durbin-Watson test statistic follows in section 4 and section 5 concludes the paper.

2. SADDLE POINT APPROXIMATION TO THE DENSITY FUNCTION

Although the exact mgf of r is known, its algebraic form as demonstrated by Smith (1989) is sufficient to deter one from any practical attempt to directly invert the Fourier transform

$$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{k_r(\omega)-\omega r}\, d\omega, \qquad (2.1)$$

where $f(r)$, $k_r(\omega)$, are the density and cgf of r respectively and c is a real positive constant in the convergence strip, that is assumed to contain in its interior the origin. Instead, we embark on an approach previously taken by Daniels (1956) and Holly and Phillips (1979) in their derivations of the saddle point approximations to the densities of the serial correlation coefficient and the k class estimator in a simultaneous system, respectively. The starting point in both is Geary's (1944) inversion formula for the density of a ratio.

For notational convenience and parsimony, the distinction between random variables and given values will not always be entertained. The interpretation should be clear from the context though.

Given the assumptions above, the joint mgf of $x'Fx$, $x'Gx$ is

$$M(\omega_1,\omega_2) = Ee^{\omega_1 x'Gx + \omega_2 x'Fx}$$

$$= |I - 2(\omega_1 G + \omega_2 F)|^{-1/2},$$

see for example, Johnson and Kotz (1970). From Geary's (1944) extension to Cramér's theorem, the inversion formula for the density of r may be written as

$$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\partial M(u-r\omega_2,\omega_2)}{\partial u}\bigg|_{u=0} d\omega_2, \tag{2.2}$$

where the change of variables, $\omega_1 = u - r\omega_2$, has occurred, the assumption that G is positive semi-definite was used and the contour of integration runs along the imaginary axis in the ω_2 plane. Let $D = F - rG$, $R = I - 2(uG + \omega_2 D)$. Using the result,

$$\frac{\partial |R|^{-1/2}}{\partial u} = |R|^{-1/2}\, tr(R^{-1}G),$$

equation (2.2) becomes

$$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} tr[(I - 2\omega D)^{-1} G]\ |I - 2\omega D|^{-1/2}\, d\omega$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} tr[(I - 2\omega D)^{-1} G]\ e^{-1/2\sum_{i=1}^{n} \ln(1-2\omega d_i)}\, d\omega, \tag{2.3}$$

where the d_i's, $i = 1,\dots,n$, are the ordered eigenvalues of D, $d_1 \geq d_2 \geq \dots \geq d_n$, and the subscript have been omitted. Now, with $B(\omega) = tr[(I - 2\omega D)^{-1} G]$, $n\psi(\omega) = -1/2\sum_{i=1}^{n}\ln(1 - 2\omega d_i)$, equation (2.3) is of the general form of integrals

$$h(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B(\omega)\, e^{n\psi(\omega)} d\omega, \tag{2.4}$$

for which, the saddle point method can be readily applied. The key idea behind the technique is to find an admissible deformation of the path of integration in the complex plane, such that the contribution to the value of h(r) outside the immediate neighborhood of the saddle point of the integrand is negligible. It follows directly from the Cauchy-Reimann equations that if the chosen contour is a straight line parallel to the imaginary axis, then it intersects the real axis orthogonally at the saddle point. The point c becomes the real saddle point, $\hat{\omega}$, satisfying $\psi'(\hat{\omega}) = 0$. The approximation is, then

$$h(r) = \frac{B(\hat{\omega})e^{n\psi(\hat{\omega})}}{(2\pi n\psi''(\hat{\omega}))^{1/2}}\left[1 + O(n^{-1})\right].$$

For a detailed and readable account, the reader is refered to De Bruijn (1958). Applying the method to the problem in hand, the saddle point defining equation is seen to be

$$\sum_{i=1}^{n}(d_i/(1-2\hat{\omega}d_i)) = 0, \tag{2.5}$$

with

$$f(r) = \frac{tr[(I - 2\hat{\omega}D)^{-1}G]\ \exp\{-1/2\sum_{i=1}^{n}\ln(1 - 2\hat{\omega}d_i)\}}{\left\{4\pi\sum_{i=1}^{n}[d_i^2/(1-2\hat{\omega}d_i)^2]\right\}^{1/2}}\left[1 + O(n^{-1})\right], \tag{2.6}$$

which agrees with the general form of the saddle point approximation for the density of a ratio, given in Daniels (1954, sec. 9). The modulus of the integrand decreases most rapidly on either side of the saddle point along the contour of steepest descent. Since this curve has a sharp peak at $\hat{\omega}$, the saddle point method embodies in it exactly the same principle as the method of Laplace. De Bruijn (1958) illustrates this point with an intuitive explanation.

The approximation has a *relative* error of $O(n^{-1})$. In contrast, the Edgeworth expansion to the density function, when available, contains generally only an *absolute* error of $O(n^{-1/2})$.

So far in the discussion, we have made an implicit assumption that $B(\omega)$ and $\psi(\omega)$ are analytic on the path. In view of (2.6) then, the necessary condition for $B(\omega)$ and $\psi(\omega)$ to be analytic in the neighborhood of $\hat{\omega}$, is that $\hat{\omega} \neq 1/2d_i$, $\forall i$, $i = 1,\ldots,n$. The sufficient conditions are that $\hat{\omega}$ lies in the open interval containing the origin (c_1, c_2), such that

$$\begin{cases} c_1 = 1/2d_n, \; c_2 = 1/2d_1 & \text{if } d_1 >\ldots 0 >\ldots> d_n \\ c_1 = -\infty, \; c_2 = 1/2d_1 & \text{if } d_1 >\ldots> d_n > 0 \, . \end{cases} \tag{2.7}$$

See also Rice (1980), for the case of a single quadratic form.

By the definition of $\hat{\omega}$, in (2.5), it will also be convenient to refer to the sufficient conditions, (2.7), as the range of the saddle point. Specifically, as r runs from $-\infty$ to ∞ and 0 to ∞, $\hat{\omega}$ runs from $1/2d_n$ to $1/2d_1$ and $-\infty$ to $1/2d_1$, respectively. A justification of (2.6) as a valid asymptotic expansion is established by the use of Watson's Lemma, see for example, Daniels (1954, sec. 2). Finally, by the convexity of $n\psi(\omega)$, $n\psi''(\hat{\omega}) = 2\sum_{i=1}^{n}[d_i^2/(1-2\hat{\omega}d_i)^2] > 0$, the uniqueness of the saddle point is also ensured.

2.1 Ratio of a Bilinear Form to a Quadratic Form

The expansion appearing in (2.6) is not restricted to a ratio of quadratic forms. Holly and Phillips (1979) derived the saddle point approximation to the density of $\hat{\beta}_k$, the k class estimator in a simultaneous equations model. In their notation, $\hat{\beta}_k = \dfrac{y_2' A_k y_1}{y_2' A_k y_2}$. From lemma 1 in Lye (1991), a ratio of a bilinear form to a quadratic form can be easily transformed into a ratio of quadratic forms. Writing then,

$$\hat{\beta}_k = \frac{z' B_1 z}{z' B_2 z} \; ,$$

where $z = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $B_1 = 1/2 \begin{pmatrix} 0 & A_k \\ A_k & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A_k \end{pmatrix}$,

the approximate density of $\hat{\beta}_k$ follows from (2.6), with the d_i's, $i = 1,\ldots,n$, now being the eigenvalues of $B_1 - \hat{\beta}_k B_2$. In their derivation, Holly and Phillips (1979) have made an explicit use of the joint mgf of a bilinear form and a quadratic form which, evidently, rendered the saddle point approximation non-uniform, the expression being dependent on the range of r. See p. 1535 of their paper.

There are, of course, numerous statistics which fall into the same category. A prominent example arises in the context of the first order stochastic difference equation,

$$y_t = \gamma y_{t-1} + X_t'\beta + u_t,$$

with $u_t = \rho u_{t-1} + e_t$, $e_t \sim N(0, \sigma^2)$, $t = 1,\ldots,T$,

where y_{t-1}, u_{t-1} are lag values of y_t, u_t, respectively, $|\rho| < 1$, β is a (kx1) coefficient vector and X_t is (kx1). The OLS estimator of γ is

$$\hat{\gamma} = \frac{y'_{t-1} M\, y_t}{y'_{t-1} M\, y_{t-1}},$$

where $M = I - X(X'X)^{-1}X'$. As is well known, $\hat{\gamma}$ is biased. Knowledge of the approximate density enables us to calculate the moments and bias of $\hat{\gamma}$. To retain the asymptotic properties of the saddle point approximation, the kth moment of r about the origin can be obtained by numerical integration of (2.6), namely $E(r^k) = \int_{-\infty}^{\infty} r^k f(r)\, dr$. This important feature is not entailed by any of the procedures mentioned in the introduction.

3. APPROXIMATE TAIL PROBABILITIES

We proceed by fitting an expression to the tail probability. If normalization is required, then the approximate density needs to be numerically integrated. However, as will be demonstrated in the next section, for the case considered, this will generally be unnecessary. Using Geary's (1944) formula for the density, the tail probability is, formally,

$$\mathrm{pr}(r \geq c^*) = \int_{c^*}^{\infty} f(r)dr$$

$$= \frac{1}{2\pi i} \int_{c^*}^{\infty} \int_{c-i\infty}^{c+i\infty} \mathrm{tr}\{(I - 2\omega(F - rG))^{-1} G]\,|I - 2\omega(F - rG)|^{-1/2}\, d\omega\, dr$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_{c^*}^{\infty} \mathrm{tr}[(I - 2\omega(F - rG))^{-1} G]\,|I - 2\omega(F - rG)|^{-1/2}\, dr\, d\omega$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |I - 2\omega(F - c^*G)|^{-1/2} \frac{d\omega}{\omega}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-1/2 \ln |I-2\omega D^*|} \frac{d\omega}{\omega}, \tag{3.1}$$

where $D^* = F - c^*G$. Now, $-1/2\ln|I - 2\omega D^*|$ is the cgf of the quadratic form $Q^* = x'D^*x = \sum_{i=1}^{n} d_i^* \chi^2_{i(1)}$, where the d_i^*'s are the eigenvalues of D^* and the $\chi^2_{i(1)}$'s, $i = 1,\ldots,n$, are independent central Chi-squared variates, each with one degree of freedom. By the additivity property of cgfs,

$$k_{Q^*}(\omega) = \sum_{i=1}^{n} k_{d_i^* \chi^2_{i(1)}}(\omega),$$

where $K_z(\omega)$ is the cgf of Z. Thus, (3.1) is, conveniently,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\left\{\sum_{i=1}^{n} k_{d_i^* \chi^2_{i(1)}}(\omega)\right\} \frac{d\omega}{\omega}. \tag{3.2}$$

There are several different versions of saddle point approximations to integrals of the form of (3.2), of which, distinctively, the most appealing in terms of its simplicity is due to Lugannani and Rice (1980). Applied to (3.2), the approximation is

$$\mathrm{pr}(r \geq c^*) = 1 - \Phi(\hat{\xi}) + \phi(\hat{\xi})\left[\frac{1}{\hat{z}} - \frac{1}{\hat{\xi}}\right], \tag{3.3}$$

where Φ, ϕ, are the standard normal cdf and pdf respectively, $\hat{z} = \hat{\omega}\left\{2\sum_{i=1}^{n}(d_i^{*2}/(1-2\hat{\omega}d_i^*)^2)\right\}^{1/2}$, $\hat{\xi} = \left\{\sum_{i=1}^{n}\ln(1 - 2\hat{\omega}d_i^*)\right\}^{1/2}\mathrm{sgn}(\hat{\omega})$, and $\hat{\omega}$ as defined by (2.5), is in the open interval (c_1^*, c_2^*), with c_j^*, $j = 1,2$, given by (2.7), except that the d_i^*'s replace the d_i's everywhere, $i = 1,\ldots,n$. $\mathrm{Sgn}(\hat{\omega})$ is $-$, 0, + when Q^* is $\leq$, $=$, $\geq$, $E(Q^*)$, respectively. See the appendix for rationale. When $Q^* = E(Q^*)$, equation (3.3) can be shown to collapse to

$$\mathrm{pr}(\ r \geq c^*) = 1/2 - \frac{\rho_3}{6(2\pi)}^{1/2}, \tag{3.4}$$

where ρ_3 is the third standardized cumulant of Q^*, given by

$$\rho_3 = 8\sum_{i=1}^{n}d_i^{*3}/(2\sum_{i=1}^{n}d_i^{*2})^{3/2}.$$

3.1 Brief Discussion

Unlike other saddle point approximations to the tail area, which account in different ways to the pole in the origin in (3.2), Lugannani and Rice's formula is valid over the entire range of the statistic. See for comparisons; Holly and Phillips (1979, appendix B), Robinson (1982), or Daniels (1987). Practically, the formula is very simple to execute, merely requiring the calculation of the eigenvalues of D^* and a grid search solution for the saddle point, by means such as described by Easton and Ronchetti (1986). Most standard computer packages provide these facilities. The only drawback of the technique arises in the vicinity of $E(Q^*)$. As $K'(0) = \mu$, the mean of the distribution, it follows from (2.5) that when Q^*

is close to $E(Q^*)$, $\hat{\omega}$ is close to zero, hence, from (3.3), the terms $1/\hat{z}$, $1/\hat{\xi}$ may become exceedingly large, resulting in a deterioration of the approximation. The problem is handled by tightening the convergence tolerance of (2.5).

4. NUMERICAL EXAMPLE

The method applies for a wide class of test statistics for autocorrelation and heteroscedasticity. Shively et al. (1990) discuss the subclasses consisting of locally best invariant and most powerful invariant tests, all of which can be expressed as a ratio of quadratic forms in time series regression residuals. The Durbin-Watson test statistic which attracted a large number of studies over the years, is probably the most popular test for autocorrelation and so, it is chosen for our exposition.

Consider the classical linear model,

$$y = X\beta + u,$$

where y is $(n\times1)$, X is $(n\times k)$ non-stochastic and of rank k, β is $(k\times1)$ and u is an $(n\times1)$ disturbance vector. The Durbin-Watson test statistic for a first order autocorrelation in the OLS residuals is given by

$$d = \frac{u'MAMu}{u'Mu},$$

where M is defined above, and A is the usual first differencing matrix. Under the alternative hypothesis of a positive AR(1) process, $u \sim N(0, \Sigma)$, with

$$\Sigma = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & \rho & \dots & \rho^{n-1} \\ \rho & 1 & \rho \dots & \rho^{n-2} \\ & & & \\ \rho^{n-1} & \dots & \rho & 1 \end{pmatrix},$$

and ρ is the autoregressive coefficient. The power of this test is, usefully,

$$1 - pr(d \ge c^*) = 1 - pr\left[\frac{v'(\Sigma^{1/2})'MAM(\Sigma^{1/2})v}{v'(\Sigma^{1/2})'M(\Sigma^{1/2})v} \ge c^* \right],$$

where $v \sim N(0, I)$. In this particular case, $D^* = \Sigma M(A - c^*I)M$. The experiment is specifically designed to examine the technique's robustness to extreme situations, as well as is conventional, to conform with typical econometric scenarios. The data sets, containing intercepts, are;

EVL (EVS); The six largest (smallest) eigenvectors of the A matrix. Their use results in the bounding extremes of the distribution of d.

WTS; Watson's matrix, consisting of $(\tilde{a}_2 + \tilde{a}_{n-1})/\sqrt{2}$ and $(\tilde{a}_3 + \tilde{a}_{n-2})/\sqrt{2}$ as the regressors, the $\tilde{a}_i$'s, $i = 1,..,n$, being the eigenvectors corresponding to the roots of of A, sorted in increasing order. The distribution of d under the null with this data matrix is symmetric around the mean of two.

SPIRIT; Log real income per head and log relative price of spirits in the U.K., commencing 1870, as in Durbin and Watson (1950).

TREND; A linear time trend.

POP; Australian household population, households and household headship ratio, 1966 and 1971.

Sample sizes of 20, 40 and ρ in the range 0 - .99 are considered. Approximate powers are reported to four decimal places in tables 1 and 2 and are compared with exact values obtained by the Davies (1973) algorithm. Numerically, $\hat{\omega}$ is chosen to satisfy $\left|\sum_{i=1}^{n}\{d_i^*/(1-2\hat{\omega}d_i^*)\}\right| \leq 10^{-6}$. This convergence criterion is generally only necessary around the mean of Q^*, where $\hat{\omega}$ is small. Otherwise, the powers are largely insensitive to drastic changes in it.

As demonstrated, at least for the range of data sets considered, the approximation is generally accurate to three decimal places in most practical situations. As is natural for an asymptotic expansion, accuracy increases uniformly with the sample size, but most importantly, also towards the tails of the distribution, where the interest is. The formula still performs reasonably well in the unlikely circumstances of the eigenvectors data, with 6 regressors and n = 20. This should provide the practitioner with a considerable amount of confidence in the technique, even if these extremes happen to eventuate.

5. FINAL REMARKS

The class of estimators and statistics that can be expressed either as a ratio of quadratic forms, or a ratio of a bilinear form to a quadratic form, encompasses virtually most relevant statistics in current use. For the general case, neither an expression, nor a numerical algorithm exists for the density function of these statistics. Procedures are only available for the computation of the cdf.

In this article, we have provided saddle point approximations to both the density and tail probability of a ratio of quadratic forms in normal variables. The requirement of a full knowledge of the cgf, always being the major hindrance of the technique, was circumvented by Geary's (1944) inversion formula for the density of the ratio. The futility of a direct inversion of the Fourier transform is immediately obvious from the form of the mgf as exhibited by Smith (1989).

The merits of the saddle point method and its superiority over the Edgeworth expansion have been extensively studied in the modern literature. The current research enforces previous findings. In particular, the technique's exceptional performance in the tails of the distribution and robustness to unusual events are certainly encouraging.

Table 1

Approximate powers of the Durbin-Watson test statistic.

n=40

Data	SPIRIT		TREND		POP	
ρ	Davies	Sp	Davies	Sp	Davies	Sp
0.00	.0500	.0500	.0500	.0500	.0501	.0500
0.10	.1382	.1381	.1422	.1420	.1381	.1380
0.20	.2987	.2983	.3127	.3124	.2986	.2981
0.30	.5101	.5095	.5369	.5363	.5111	.5103
0.40	.7129	.7122	.7458	.7452	.7163	.7155
0.50	.8580	.8573	.8864	.8859	.8636	.8629
0.60	.9387	.9383	.9574	.9571	.9449	.9444
0.70	.9757	.9755	.9859	.9857	.9808	.9806
0.80	.9905	.9904	.9955	.9954	.9941	.9940
0.90	.9960	.9960	.9984	.9984	.9983	.9983
0.99	.9978	.9977	.9991	.9991	.9994	.9994

Data	EVL		EVS		WTS	
ρ	Davies	Sp	Davies	Sp	Davies	Sp
0.00	.0500	.0499	.0500	.0501	.0500	.0500
0.10	.1241	.1237	.1215	.1215	.1308	.1307
0.20	.2484	.2476	.2559	.2557	.2752	.2748
0.30	.4091	.4079	.4532	.4528	.4678	.4672
0.40	.5730	.5715	.6693	.6687	.6594	.6585
0.50	.7096	.7080	.8410	.8404	.8043	.8034
0.60	.8067	.8052	.9397	.9393	.8898	.8892
0.70	.8679	.8666	.9815	.9813	.9276	.9281
0.80	.9028	.9016	.9952	.9951	.9266	.9282
0.90	.9202	.9191	.9989	.9988	.8604	.8620
0.99	.9257	.9247	.9996	.9996	.6699	.6736

Sp : Saddle point approximation.

Davies : The Davies algorithm.

Table 2

Approximate powers of the Durbin-Watson test statistic.

n=20

Data	SPIRIT		TREND		POP	
ρ	Davies	Sp	Davies	Sp	Davies	Sp
0.00	.0500	.0498	.0500	.0499	.0500	.0495
0.10	.0980	.0976	.1000	.0996	.0961	.0948
0.20	.1743	.1734	.1810	.1799	.1684	.1658
0.30	.2796	.2778	.2940	.2921	.2677	.2633
0.40	.4050	.4024	.4287	.4258	.3871	.3806
0.50	.5349	.5314	.5661	.5625	.5133	.5050
0.60	.6528	.6487	.6874	.6834	.6321	.6227
0.70	.7482	.7439	.7816	.7775	.7334	.7239
0.80	.8177	.8134	.8468	.8430	.8134	.8045
0.90	.8625	.8583	.8862	.8828	.8725	.8648
0.99	.8820	.8778	.9011	.8977	.9094	.9026

Data	EVL		EVS		WTS	
ρ	Davies	Sp	Davies	Sp	Davies	Sp
0.00	.0500	.0492	.0500	.0502	.0500	.0498
0.10	.0814	.0799	.0814	.0814	.0875	.0870
0.20	.1221	.1197	.1310	.1307	.1421	.1408
0.30	.1700	.1665	.2050	.2039	.2123	.2100
0.40	.2218	.2169	.3063	.3041	.2909	.2872
0.50	.2749	.2685	.4304	.4268	.3659	.3608
0.60	.3302	.3225	.5635	.5588	.4236	.4179
0.70	.3965	.3890	.6875	.6821	.4516	.4473
0.80	.4967	.4936	.7880	.7826	.4400	.4398
0.90	.6642	.6524	.8596	.8547	.3864	.3926
0.99	.9113	.8939	.9008	.8961	.3159	.3290

Sp : Saddle point approximation.

Davies : The Davies algorithm.

APPENDIX: THE LUGANNANI-RICE FORMULA

No attempt will be made here to rederive the Lugannanni-Rice formula, or justify its asymptotic properties. The reader is, however, encouraged to refer to Daniels (1987, sec. 4), or Barndorff-Nielsen and Cox (1989, p. 115), for alternative derivations of the same expression.

Let X_i, $i = 1,\dots,n$, be independent random variables, each with a cgf $K_{X_i}(\omega)$ and let $S = \sum_{i=1}^{n} X_i$. By the additivity property of cgfs, $K_S(\omega) = \sum_{i=1}^{n} k_{X_i}(\omega)$. The Fourier inversion formula for the tail probability may be written as

$$\Pr(S \geq s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\left\{\sum_{i=1}^{n}(k_{X_i}(\omega) - \omega x_i)\right\}\frac{d\omega}{\omega} \qquad (c > 0), \qquad (A.1)$$

where s, x_i, are given values of S and X_i, respectively. The Lugannani-Rice approximation to (A.1) is

$$1 - \Phi(\hat{\xi}) + \phi(\hat{\xi})\left[\frac{1}{\hat{z}} - \frac{1}{\hat{\xi}}\right],$$

where, in addition to the definitions in section 3, $\hat{\xi} = \left\{2\sum_{i=1}^{n}(\hat{\omega}x_i - K_{X_i}(\omega))\right\}^{1/2}\operatorname{sgn}(\hat{\omega})$ and $\hat{z} = \hat{\omega}\left\{\sum_{i=1}^{n} K''_{x_i}(\omega)\right\}^{1/2}$. For the problem in hand, $X_i \equiv d_i^{*}\chi^2_{i(1)}$, a weighted Chi-squared variate with one degree of freedom, $K_{d_i^{*}\chi^2_{i(1)}}(\omega) = -1/2\ln(1 - 2\omega d_i^{*})$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} d_i^{*}\chi^2_{i(1)} = 0$. The full expressions for $\hat{z}$ and $\hat{\xi}$ in (3.3) easily follow.

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