MONASH

WP# 1/92

MONASH

UNIVERSITY

AUSTRALIA

TESTING FOR ARMA(1,1) DISTURBANCES IN THE LINEAR REGRESSION MODEL

Shahidur Rahman and Maxwell L. King

Working Paper No. 1/92

January 1992

DEPARTMENT OF ECONOMETRICS

ISSN 1032-3813

ISBN 0 86746 983 8

TESTING FOR ARMA(1,1) DISTURBANCES IN THE LINEAR REGRESSION MODEL

Shahidur Rahman and Maxwell L. King

Working Paper No. 1/92

January 1992

DEPARTMENT OF ECONOMETRICS, FACULTY OF ECONOMICS COMMERCE & MANAGEMENT

MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA.

TESTING FOR ARMA(1,1) DISTURBANCES

IN THE LINEAR REGRESSION MODEL*

Shahidur Rahman and Maxwell L. King

Department of Econometrics
Monash University
Clayton, Victoria, 3168
Australia

ABSTRACT

Serious alternatives to the AR(1) disturbance model in econometric applications of linear regression include MA(1) disturbances and the sum of independent white noise and AR(1) disturbance components. All three are special cases of ARMA(1,1) processes. This paper reports an empirical power comparison of tests for AR(1), MA(1) and ARMA(1,1) disturbances assuming ARMA(1,1) disturbances. Tests compared include the Durbin-Watson test, the locally best invariant test and various point optimal invariant (POI) tests. The results suggest that the power of the POI test is largely invariant to the choice of AR(1) parameter value at which power is maximized. This conclusion is strengthened by a theoretical result.

KEY WORDS: autoregressive-moving average processes; Durbin-Watson test; locally best invariant test; point optimal invariant test; power.

* This work is supported by a grant from the Australian Research Council.

1. INTRODUCTION

Since the seminal work of Cochrane and Orcutt (1949) and Durbin and Watson (1950, 1951), the first-order autoregressive (AR(1)) error process has played an important role in econometric applications of the linear regression model. Lately, increased attention has been paid to the first-order moving average (MA(1)) error process as an alternative (see for example Nicholls, Pagan and Terrell (1975), King (1983a) and King and McAleer (1987)). There is also the disturbance process based on the sum of independent white noise and AR(1) components proposed by Revankar (1980). When one examines the usual reasons for including an error term in the regression model; these being to account for the effects of omitted regressors, errors in the measurement of the dependent variable, arbitrary human behaviour and functional approximation; Revankar's proposal is appealing because some of the effects are typically autocorrelated while others are independent (see King (1986)). All three error processes have one thing in common - they are special cases of ARMA(1,1) processes.

This suggests we need to have confidence in the power properties of tests for independence of regression errors against ARMA(1,1) alternatives rather than just AR(1) or possibly MA(1) alternatives. The most popular test, the DW test, is approximately locally best invariant (LBI) against AR(1) alternatives (Durbin and Watson (1971)) and MA(1) alternatives (King (1983b)) and is approximately uniformly LBI against ARMA(1,1) alternatives (King and Evans (1988)). While this property ensures close to optimum power in the neighbourhood of the null hypothesis (H_0) of independent errors, it guarantees nothing about the test's power some distance from H_0 when wrongly accepting H_0 might have damaging consequences. King (1983b, 1985a, 1987) has demonstrated that an attractive alternative is to use a point-optimal invariant (POI) test, i.e., a test that optimizes power at a pre-determined point under the alternative hypothesis.

In an empirical power comparison, King (1983b) showed that POI tests for MA(1) disturbances have a clear power advantage over the DW test against MA(1) alternatives some distance from H_0. It seems that the choice of point at which power is optimized plays an important role in determining the power of the POI test. Some suggestions about optimal choices of points are made by King (1985b). A different picture emerges from empirical power comparisons of tests against AR(1) disturbances (see King (1985a)). Here, for most typical econometric applications, POI tests have a very slight power advantage over the DW test and are largely invariant to the choice of point at which power is optimized. However, a few important cases have been found in which there is a substantial power advantage from using a POI test in place of the DW test. It would seem that the ordinary least squares (OLS) estimator is relatively inefficient in such cases making a powerful test even more desirable.

The aim of this paper is to see whether existing tests designed to have high power against either AR(1) disturbances or MA(1) disturbances, have good power against ARMA(1,1) disturbances. Are there clear advantages in using a POI test for ARMA(1,1) disturbances or can we recommend the use of one of the existing tests? An empirical power comparison is conducted in order to answer to this question.

The plan of the paper is as follows. In Section 2 we give details of the model under test while POI tests for ARMA(1,1) disturbances are introduced in Section 3. The empirical power comparison is outlined in

Section 4 and its results are reported in Section 5. The results suggest that the power properties of the POI test are largely invariant to the choice of AR(1) parameter value at which power is maximized. This conclusion is strengthened by a theorem which shows that for certain design matrices, the POI test is approximately Uniformly Most Powerful Invariant (UMPI) along a ray in the ARMA(1,1) parameter space. The final section contains some concluding remarks.

2. THE MODEL

Consider the linear regression model

$$y = X\beta + u \tag{1}$$

where y is $n \times 1$, X is an $n \times k$ nonstochastic matrix of rank $k < n$, β is a $k \times 1$ vector of parameters and u is the $n \times 1$ disturbance vector. It is assumed that the components of u are generated by the stationary ARMA(1,1) process,

$$u_t = \rho u_{t-1} + \varepsilon_t + \gamma\varepsilon_{t-1}, \quad t = 1, \ldots, n, \tag{2}$$

where ρ and γ are parameters such that $|\rho|<1$, $|\gamma|<1$ and $\varepsilon_t \sim IN(0,\sigma^2)$, $t = 0, \ldots, n$. Thus $u \sim N(0, \sigma^2\Omega(\rho,\gamma))$ where

$$\Omega(\rho,\gamma) = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 & \cdots & \omega_{n-1} \\ \omega_1 & \omega_0 & \omega_1 & & \omega_{n-2} \\ \omega_2 & \omega_1 & \omega_0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \omega_{n-1} & \omega_{n-2} & \cdots & \cdots & \omega_0 \end{bmatrix}$$

in which

$$\omega_0 = (1 + \gamma^2 + 2\gamma\rho)/(1 - \rho^2) \tag{3}$$

$$\omega_1 = (1 + \rho\gamma)(\rho + \gamma)/(1 - \rho^2) \tag{4}$$

and

$$\omega_j = \rho\omega_{j-1}, \qquad j = 2, 3, \ldots, n-1 .$$

Note that when $\rho = 0$, (2) reduces to a MA(1) process while when $\gamma = 0$, it reduces to a stationary AR(1) process. Also observe that if $\rho > -\gamma$ then $\omega_1 > 0$ which implies positive first-order autocorrelation while $\rho < -\gamma$ implies negative first-order autocorrelation. When $\rho = -\gamma$, $u \sim N(0, \sigma^2 I_n)$.

Our interest is in testing

$$H_0 : \rho = -\gamma$$

against

$$H_a^+ : \rho > -\gamma \qquad (\text{or } H_a^-: \rho < -\gamma) .$$

We focus on one-sided alternatives because typically these are of greatest interest. Omitted effects that are correlated over time lead to positive first-order autocorrelation in the errors (H_a^+) while differencing the dependent variable often leads to negative autocorrelation (H_a^-).

3. POI TESTS FOR ARMA(1,1) DISTURBANCES

From King (1987), the POI test of H_0, which optimizes power at $(\rho,\gamma) = (\rho_0,\gamma_0)$, is to reject H_0 for small values of

$$s(\rho_0,\gamma_0) = \hat{u}'\Omega(\rho_0,\gamma_0)^{-1}\hat{u}/\hat{e}'\hat{e} \tag{5}$$

where $\hat{u}$ is the generalized least squares (GLS) residual vector from (1) assuming covariance matrix $\Omega(\rho_0,\gamma_0)$ and $\hat{e}$ is the OLS residual vector. In order to calculate (5), note that it can also be written as

$$s(\rho_0,\gamma_0) = \hat{e}^{*\prime}\hat{e}^*/\hat{e}'\hat{e}$$

where $\hat{e}^*$ are the OLS residuals from the transformed regression

$$G^{-1}y = G^{-1}X\beta + G^{-1}u$$

in which G is an $n \times n$ matrix such that

$$\Omega(\rho_0,\gamma_0) = GG'.$$

Following Ansley (1979), the G^{-1} transformation can be applied conveniently as follows. First define

$$W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\rho_0 & 1 & 0 & & 0 \\ 0 & -\rho_0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & 1 & 0 \\ 0 & 0 & \cdots & -\rho_0 & 1 \end{bmatrix}$$

and $V = W\Omega(\rho_0,\gamma_0)W'$. It is straightforward to show that

$$V = \begin{bmatrix} a & \gamma_0 & 0 & \cdots & 0 \\ \gamma_0 & 1+\gamma_0^2 & \gamma_0 & & 0 \\ 0 & \gamma_0 & 1+\gamma_0^2 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & 1+\gamma_0^2 & \gamma_0 \\ 0 & 0 & \cdots & \gamma_0 & 1+\gamma_0^2 \end{bmatrix}$$

in which $a = (1 + \gamma_0^2 + 2\gamma_0\rho_0)/(1 - \rho_0^2)$. Therefore V has the Cholesky decomposition

$$V = HH'$$

where H is an $n \times n$ matrix whose only non-zero elements are those in the main diagonal and the lower off-diagonal. If these elements are denoted by $h_{i,i}$ and $h_{i,i-1}$, respectively, i=1, ..., n, then they can be computed by the recursive scheme:

$$h_{1,1} = \sqrt{a} ,$$

$$h_{i,i-1} = \gamma_0 / h_{i-1,i-1} ,$$

$$h_{i,i} = \left\{\left(1 + \gamma_0^2\right) - h_{i,i-1}^2\right\}^{1/2} , \qquad i = 2, \ldots, n .$$

Suppose z denotes an $n \times 1$ vector (either y or a column of X) to be transformed to $z^* = G^{-1}z$. Because $G^{-1} = H^{-1}W$, this can be achieved by performing two transformations, the first being $\tilde{z} = Wz$ and the second being $z^* = H^{-1}\tilde{z}$. The latter can be performed recursively by:

$$z_1^* = \tilde{z}_1 / h_{1,1}$$

$$z_i^* = \left(\tilde{z}_i - h_{i,i-1} \; z_{i-1}^*\right) \Big/ h_{i,i} , \qquad i = 2, \ldots, n .$$

If $m = n - k$, $X^* = G^{-1}X$ and $M^* = I_n - X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$, then from King (1987), critical values for $s(\rho_0,\gamma_0)$ can be found by solving

$$\Pr\left[s(\rho_0,\gamma_0) < c \mid u \sim N(0,\sigma^2 I_n)\right]$$

$$= \Pr\left[\sum_{i=1}^{m}(\lambda_i - c)\xi_i^2 < 0 \mid \xi \sim N(0,I_m)\right] = \alpha$$

for c, where α is the desired significance level and $\lambda_1, \ldots, \lambda_m$ are the non-zero eigenvalues of $\Delta = (G^{-1})'M^*G^{-1}$. This can be solved iteratively using Imhof's (1961) algorithm, coded versions of which have been published by Koerts and Abrahamse (1969) or Davies (1980). The power of the test when $u \sim N(0, \sigma^2\Sigma)$ can be calculated in a similar manner by noting that

$$\Pr\left[s(\rho_0,\gamma_0) < c \mid u \sim N(0,\sigma^2\Sigma)\right]$$

$$= \Pr\left[\sum_{i=1}^{m}\upsilon_i\xi_i^2 < 0 \mid \xi \sim N(0,I_m)\right]$$

where $\upsilon_1, \ldots, \upsilon_m$ are the eigenvalues of $(\Delta - cM)\Sigma$ excluding k zero roots and $M = I_n - X(X'X)^{-1}X'$.

4. AN EMPIRICAL POWER COMPARISON

An empirical power comparison was conducted in order to see whether existing tests, designed to have high power against either AR(1) disturbances or MA(1) disturbances, have good power against ARMA(1,1) disturbances. A related question is whether there are clear advantages in using a POI test for ARMA(1,1) disturbances.

Powers were calculated under both H_a^+ and H_a^-. Tests included were the DW test; King's (1981) modified DW (MDW) test, which is a true LBI test against AR(1) and MA(1) disturbances and a true uniformly LBI test against ARMA(1,1) disturbances (King and Evans (1988)); King's (1985a) recommended POI test for AR(1) disturbances (s(0.5,0) against H_a^+ and s(-0.5,0) against H_a^-); King's (1983b) recommended POI test for MA(1) disturbances (s(0,0.5) against H_a^+ and s(0,-0.5) against H_a^-) and the POI test for ARMA(1,1) disturbances based on s(0.5,0.5) against H_a^+ and s(-0.5,-0.5) against H_a^-. The following design matrices were used in the comparison:

X1: ($n \times 3$; n = 20, 60). The first n observations of Durbin and Watson's (1951, p.159) consumption of spirits example.

X2: ($n \times 4$; n = 20, 60). The eigenvectors corresponding to the four smallest eigenvalues of the DW first differencing matrix A_1. Note that the eigenvector corresponding to the zero root of A_1 is the constant dummy regressor.

X3: ($n \times 4$; n = 20, 60). The eigenvectors corresponding to the zero and three largest eigenvalues of A_1.

X4: ($n \times 5$; n = 20, 60). A constant, three quarterly seasonal dummy variables and the quarterly Australian Consumers' Price Index commencing 1959(1).

X5: ($n \times 3$; n = 20, 60). A constant dummy, quarterly Australian household disposable income and private consumption expenditure (both seasonally adjusted) commencing 1967(2).

X1, X4 and X5 were included as examples of design matrices from annual and quarterly time series applications. The DW test against AR(1) disturbances is approximately UMPI for X2 and X3, the former design matrix being comprised of smoothly evolving sine curves while the nonconstant regressors of the latter are sine curves with periods of two, three and four observations, respectively.

For each design matrix and against H_a^+, powers were calculated at (ρ,γ) = (-0.5, 0.5), (0, 0), (0.5, -0.5), (-0.4, 0.8), (-0.2, 0.6), (0, 0.4), (0.2, 0.2), (0.4, 0), (0.6, -0.2), (0.8, -0.4), (0, 0.8), (0.2, 0.6), (0.4, 0.4), (0.6, 0.2), (0.8, 0), (0.4, 0.8) using exact critical values at the five per cent significance level. Against H_a^-, powers were calculated at the analogous set of points with the signs of the ρ and γ values reversed. All powers and critical values were computed using a modified version of Koerts and Abrahamse's (1969) FQUAD subroutine with maximum integration and truncation errors of 10^{-6}.

5. RESULTS

Selected calculated powers for X2 and X5 against H_a^+ are presented in Tables 1 and 2, respectively, while the corresponding powers against H_a^- are given in Tables 3 and 4. In Tables 1 and 2, powers for (ρ,γ) = (0.4,0.8), (0.6,0.6), (0.7,0.4) when n = 60 are omitted because they are always 1.000. Corresponding powers in Tables 3 and 4 were also omitted for the same reason.

Against H_a^+, a striking feature is that the trio of AR(1) tests, namely the DW, MDW and s(0.5,0) tests, always have very similar powers, particularly for n = 60. Power differences are rarely greater than 0.01. The largest power differences occur between the MDW and s(0.5,0) tests for n = 20 and large values of $\rho+\gamma$. With some minor exceptions for X2, X3 and X4, the s(0.5,0) test is typically slightly more powerful than the DW test and seems to have the best overall power of these three tests.

The remaining two tests, which are based on s(0,0.5) and s(0.5,0.5) and therefore test for an MA(1) component, also have very similar powers. Power differences greater than 0.01 are extremely uncommon, particularly when n = 60, and neither test dominates the other in terms of power performance. The s(0,0.5) test is slightly more powerful for (ρ,γ) values near (0,0.5) while the s(0.5,0.5) test has a slight power advantage for parameter values closer to (0.5,0.5).

The powers of the trio of AR(1) tests typically increase as ρ increases while $\rho+\gamma$ is held constant, although X2, particularly when n = 20, provides a notable exception. On the other hand, the powers of the remaining two tests typically decrease as ρ increases with $\rho+\gamma$ held constant although the reverse is the case for X3.

Comparing the two groups of tests, we find the AR(1) tests have a power advantage when ρ is positive and γ is negative or near zero. For positive γ values and ρ negative or near zero, the two tests which recognise an MA(1) component dominate the other three tests. Taking the s(0.5,0) and s(0,0.5) tests as representative of the AR(1) and MA(1) component tests, respectively, maximum power differences when n = 60 in favour of the s(0,0.5) test range up to 0.24. On the other hand, maximum differences that favour the s(0.5,0) test when n = 60 range from 0.17 for X2 to 0.1 for X3. The s(0,0.5) and s(0.5,0.5) tests do appear to have better overall power than the AR(1) tests.

Analogous patterns occurred for the powers against H_a^- with the following exceptions. The power of the DW test was dominated by that of the MDW and s(-0.5,0) tests with a noticeable power difference when n = 20. Also, the powers of all tests show a greater tendency to increase as ρ becomes more negative while $\rho+\gamma$ is held constant, particularly when n = 20. In this regard, the behaviour of the MA(1) component tests, namely s(0,-0.5) and s(-0.5,-0.5), differs from that of their H_a^+ analogues when n = 20.

Overall, the results strongly suggest that the power properties of the $s(\rho_0,\gamma_0)$ test are largely invariant to the choice of ρ_0 but not to the choice of γ_0 value. This conclusion is strengthened by the following theorem, a proof of which is given in the appendix.

Theorem: Let D be the n $\times$ n matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdots & & \cdots & 0 \\ 1 & 0 & 0 & & & & 0 \\ 0 & 1 & 0 & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & 0 & 0 \\ 0 & 0 & \cdots & & \cdots & 1 & 0 \end{bmatrix}.$$

If the column space of X is spanned by k of the eigenvectors of D + D′, then the POI test, based on rejecting H_0 against H_a^+ (H_a^-) for small values of $s(\rho_0,\gamma_0)$ where $\rho_0 + \gamma_0 > 0$ ($\rho_0 + \gamma_0 < 0$), is approximately UMPI along the ray $\gamma = \gamma_0$ in the (ρ,γ) parameter space under H_a^+ (H_a^-).

A similar result holds for the DW test in the context of testing against AR(1) disturbances. In fact the above theorem can be regarded as a generalization of the DW result. For the DW problem, empirical power comparisons suggest (see for example King (1985a)) that the DW test is approximately UMPI in most economic applications although exceptions do exist. We conjecture that a similar picture holds for testing against ARMA(1,1) disturbances. Despite the fact that the above theorem only mentions very specific X matrices, we expect the $s(\rho_0,\gamma_0)$ test to be approximately UMPI along the ray $\gamma = \gamma_0$ in the majority of typical economic applications. Our empirical results support this conjecture. However, we acknowledge the existence of X matrices for which the $s(\rho_0,\gamma_0)$ test does not have this property.

6. CONCLUDING REMARKS

In the introduction we argued that tests for autocorrelation in linear regression disturbances need to have good power against ARMA(1,1) disturbances. Our empirical power comparison, together with the above theoretical result, shows that POI tests have good power properties. Because the power of a POI test is largely invariant to the choice of ρ_0 value at which power is maximized, the choice of test therefore reduces to a choice of γ_0 value in $s(\rho_0,\gamma_0)$. Setting γ_0 to zero is equivalent to choosing a test designed to test for AR(1) disturbances. Our investigations suggest a nonzero value for γ_0 - say 0.5 when testing for positive autocorrelation or -0.5 when testing for negative autocorrelation. Alternatively, γ_0 might be selected in a less arbitrary manner. In the case of testing for MA(1) disturbances, King (1985b) recommends choosing γ_0 so that the optimized power is 0.65. For our problem, this suggests a γ_0 value such that the power at $(\rho,\gamma) = (\gamma_0,\gamma_0)$ is 0.65.

APPENDIX

Proof of Theorem

In order to prove the theorem, we need to show that for X matrices whose columns can be written as linear combinations of k of the eigenvectors of D + D′, critical regions defined by $s(\rho_0,\gamma_0)$ are (approximately) invariant to the choice of ρ_0 value. This would imply that for any ρ_0 value, we get (approximately) the same critical region so that the test optimizes power at all points along the ray $\gamma = \gamma_0$ in the (ρ,γ) parameter space.

Observe that $W = I_n - \rho_0 D$ so that

$$\begin{aligned}\Omega(\rho_0,\gamma_0) &= (I_n - \rho_0 D)^{-1} V (I_n - \rho_0 D')^{-1} \\ &\approx (I_n - \rho_0 D)^{-1}(I_n + \gamma_0 D)(I_n + \gamma_0 D')(I_n - \rho_0 D')^{-1} \\ &= (I_n + \gamma_0 D)(I_n - \rho_0 D)^{-1}(I_n - \rho_0 D')^{-1}(I_n + \gamma_0 D').\end{aligned}$$

The last equality follows from

$$(I_n - \rho_0 D)^{-1} = I_n + \rho_0 D + \rho_0^2 D^2 + \ldots + \rho_0^{k-1} D^{k-1}.$$

Also observe that when $\rho \neq 0$, $\Omega(\rho,\gamma)$ can be written as

$$\Omega(\rho,\gamma) = (\omega_0 - \omega_1/\rho) I_n + (1 - \rho^2)\omega_1 \Omega(\rho,0) \,/\, \rho ,$$

where ω_0 and ω_1 are given by (3) and (4). Furthermore,

$$\Omega(\rho,0) = \left[(1 + \rho^2) I_n - \rho(D + D') - \rho^2 C\right]^{-1}$$

where $C = \mathrm{diag}(1,0,0,\ldots,0,1)$. $\Omega(\rho,\gamma)$ can therefore be approximated by

$$\Omega^*(\rho,\gamma) = (\omega_0 - \omega_1/\rho) I_n + (1 - \rho^2)\omega_1 \left[(1 + \rho^2) I_n - \rho(D + D')\right]^{-1} \Big/ \rho$$

when $\rho \neq 0$ and define

$$\Omega^*(0,\gamma) = \Omega(0,\gamma) = (1 + \gamma^2)I_n + \gamma(D + D') .$$

Note that because of the form of $\Omega^*(\rho,\gamma)$, eigenvectors of $D + D'$ are also eigenvectors of $\Omega^*(\rho,\gamma)$.

Assume that our testing problem is one in which

$$u \sim N(0,\sigma^2\Omega^*(\rho,\gamma)) .$$

When the column space of X is spanned by k of the eigenvectors of $D + D'$, the OLS and GLS (assuming covariance matrix $\Omega^*(\rho_0,\gamma_0)$) estimates of β in (1) coincide (Watson (1967)). If $\hat{u}^*$ denotes the GLS residual vector from (1) assuming covariance matrix $\Omega^*(\rho_0,\gamma_0)$, the POI test statistic can be written as

$$\begin{aligned}
&\hat{u}^{*\prime}\Omega^*(\rho_0,\gamma_0)^{-1}\hat{u}^* \,/\, \hat{e}'\hat{e} \\
&= \hat{e}'\Omega^*(\rho_0,\gamma_0)^{-1}\hat{e} \,/\, \hat{e}'\hat{e} \\
&\approx \hat{e}\left[(I_n + \gamma_0 D)(I_n - \rho_0 D)^{-1}(I_n - \rho_0 D')^{-1}(I_n + \gamma_0 D')\right]^{-1}\hat{e} \,/\, \hat{e}'\hat{e} \\
&= \tilde{e}'(I_n - \rho_0 D')(I_n - \rho_0 D)\tilde{e} \,/\, \tilde{e}'(I_n + \gamma_0 D')(I_n + \gamma_0 D)\tilde{e}
\end{aligned}$$

where $\tilde{e} = (I_n + \gamma_0 D)^{-1}\hat{e}$. Therefore the critical region

$$\hat{u}^{*\prime}\Omega^*(\rho_0\gamma_0)^{-1}\hat{u}^* \,/\, \hat{e}'\hat{e} < c_1$$

is approximately equivalent to

$$\tilde{e}'(I_n - \rho_0 D')(I_n - \rho_0 D)\tilde{e} < c_1\tilde{e}'(I_n + \gamma_0 D')(I + \gamma_0 D)\tilde{e}$$

which is approximately equivalent to

$$\tilde{e}'D\tilde{e} \,/\, \tilde{e}'\tilde{e} > c_2$$

assuming $\gamma_0 + \rho_0 > 0$ or

$$\tilde{e}'D\tilde{e} \,/\, \tilde{e}'\tilde{e} < c_2$$

if $\gamma_0 + \rho_0 < 0$, where c_2 is an appropriately chosen critical value. In both cases the approximate critical region is invariant to ρ_0 as required.

References

Ansley, C.F. (1979). An algorithm for the exact likelihood of a mixed autoregressive-moving average process. *Biometrika* 66, 59-65.

Cochrane, D. and G.H. Orcutt (1949). Applications of least squares regression to relationships containing autocorrelated error terms. *Journal of the American Statistical Association* 44, 32-61.

Davies, R.B. (1980). Algorithm AS155. The distribution of a linear combination of χ^2 random variables. *Applied Statistics* 29, 323-333.

Durbin, J. and G.S. Watson (1950). Testing for serial correlation in least squares regression I. *Biometrika* 37, 409-428.

Durbin, J. and G.S. Watson (1951). Testing for serial correlation in least squares regression II. *Biometrika* 38, 159-178.

Durbin, J. and G.S. Watson (1971). Testing for serial correlation in least squares regression III. *Biometrika* 58, 1-19.

Imhof, P.J. (1961). Computing the distribution of quadratic forms in normal variables. *Biometrika* 48, 419-426.

King, M.L. (1983a). Testing for autoregressive against moving average errors in the linear regression model. *Journal of Econometrics* 21, 35-51.

King, M.L. (1983b). Testing for moving average regression disturbances. *Australian Journal of Statistics*, 25, 23-34.

King, M.L. (1985a). A point optimal test for autoregressive disturbances. *Journal of Econometrics* 27, 21-37.

King, M.L. (1985b). A point optimal test for moving average regression disturbances. *Econometric Theory* 1, 211-222.

King, M.L. (1986). Efficient estimation and testing of regressions with a serially correlated error component. *Journal of Quantitative Economics* 2, 231-247.

King, M.L. (1987). Towards a theory of point-optimal testing. *Econometric Reviews* 6, 169-218.

King, M.L. and M.A. Evans (1988). Locally optimal properties of the Durbin-Watson test. *Econometric Theory* 4, 509-516.

King, M.L. and M. McAleer (1987). Further results on testing AR(1) against MA(1) disturbances in the linear regression model. *Review of Economic Studies* 54, 649-663.

Koerts, J. and A.P.J. Abrahamse (1969). *On the Theory and Application of the General Linear Model*. Rotterdam University Press: Rotterdam.

Nicholls, D.F., A.R. Pagan and R.D. Terrell (1975). The estimation and use of models with moving average disturbance terms: A survey. *International Economic Review* 16, 113-134.

Revankar, N.S. (1980). Analysis of regressions containing serially correlated and serially uncorrelated error components. *International Economic Review* 21, 185-199.

Watson, G.S. (1967). Linear least squares regression. *Annals of Mathematical Statistics* 38, 1679-1699.

Table 1: Powers against H_a^+ for X2 at the five percent significiance level

		Test				
ρ	γ	DW	MDW	s(0.5,0)	s(0,0.5)	s(0.5,0.5)
				n = 20		
-0.4	0.8	0.275	0.282	0.280	0.404	0.398
-0.2	0.6	0.322	0.328	0.327	0.411	0.406
0.0	0.4	0.341	0.347	0.346	0.378	0.374
0.2	0.2	0.338	0.341	0.342	0.329	0.327
0.4	0.0	0.311	0.311	0.313	0.272	0.271
0.6	-0.2	0.257	0.255	0.258	0.208	0.208
0.8	-0.4	0.179	0.177	0.179	0.140	0.141
0.0	0.8	0.648	0.651	0.654	0.839	0.839
0.2	0.6	0.720	0.719	0.724	0.836	0.837
0.4	0.4	0.732	0.727	0.734	0.775	0.776
0.6	0.2	0.689	0.680	0.690	0.667	0.669
0.8	0.0	0.580	0.569	0.579	0.509	0.512
0.4	0.8	0.889	0.882	0.890	0.970	0.972
0.6	0.6	0.908	0.899	0.908	0.958	0.961
0.7	0.4	0.865	0.854	0.864	0.891	0.895
				n = 60		
-0.4	0.8	0.682	0.686	0.686	0.923	0.922
-0.2	0.6	0.791	0.795	0.794	0.925	0.924
0.0	0.4	0.840	0.842	0.842	0.890	0.888
0.2	0.2	0.860	0.861	0.861	0.839	0.838
0.4	0.0	0.863	0.864	0.865	0.783	0.782
0.6	-0.2	0.853	0.852	0.854	0.722	0.721
0.8	-0.4	0.810	0.808	0.810	0.636	0.635
0.0	0.8	0.994	0.994	0.994	1.000	1.000
0.2	0.6	0.999	0.999	0.999	1.000	1.000
0.4	0.4	1.000	1.000	1.000	1.000	1.000
0.6	0.2	1.000	1.000	1.000	0.999	0.999
0.8	0.0	0.999	0.999	0.999	0.997	0.997

Table 2: Powers against H_a^+ for X5 at the five percent significiance level

ρ	γ	DW	MDW	s(0.5,0)	s(0,0.5)	s(0.5,0.5)
				Test		
				n = 20		
-0.4	0.8	0.265	0.271	0.267	0.386	0.374
-0.2	0.6	0.320	0.327	0.323	0.410	0.400
0.0	0.4	0.358	0.364	0.363	0.399	0.393
0.2	0.2	0.384	0.388	0.391	0.377	0.373
0.4	0.0	0.398	0.399	0.406	0.350	0.349
0.6	-0.2	0.397	0.394	0.403	0.318	0.318
0.8	-0.4	0.366	0.357	0.368	0.270	0.272
0.0	0.8	0.649	0.650	0.657	0.842	0.840
0.2	0.6	0.752	0.748	0.761	0.867	0.869
0.4	0.4	0.808	0.800	0.817	0.852	0.855
0.6	0.2	0.831	0.819	0.839	0.818	0.823
0.8	0.0	0.824	0.808	0.829	0.765	0.771
0.4	0.8	0.919	0.910	0.926	0.981	0.984
0.6	0.6	0.956	0.947	0.960	0.985	0.987
0.7	0.4	0.950	0.939	0.954	0.965	0.968
				n = 60		
-0.4	0.8	0.684	0.687	0.685	0.924	0.919
-0.2	0.6	0.796	0.798	0.797	0.928	0.924
0.0	0.4	0.848	0.849	0.849	0.896	0.892
0.2	0.2	0.872	0.873	0.873	0.852	0.849
0.4	0.0	0.882	0.883	0.884	0.807	0.805
0.6	-0.2	0.886	0.886	0.889	0.769	0.768
0.8	-0.4	0.883	0.882	0.887	0.738	0.741
0.0	0.8	0.994	0.994	0.994	1.000	1.000
0.2	0.6	0.999	0.999	0.999	1.000	1.000
0.4	0.4	1.000	1.000	1.000	1.000	1.000
0.6	0.2	1.000	1.000	1.000	1.000	1.000
0.8	0.0	1.000	1.000	1.000	0.999	0.999

Table 3: Powers against H_a^- for X2 at the five percent significiance level

		Test				
ρ	γ	DW	MDW	s(0.5,0)	s(0,0.5)	s(0.5,0.5)
				n = 20		
0.4	-0.8	0.114	0.115	0.114	0.122	0.120
0.2	-0.6	0.168	0.170	0.169	0.183	0.180
0.0	-0.4	0.234	0.239	0.238	0.251	0.247
-0.2	-0.2	0.309	0.318	0.319	0.316	0.314
-0.4	0.0	0.389	0.404	0.407	0.377	0.377
-0.6	0.2	0.478	0.498	0.504	0.439	0.441
-0.8	0.4	0.592	0.618	0.627	0.533	0.538
0.0	-0.8	0.363	0.374	0.374	0.428	0.422
-0.2	-0.6	0.533	0.556	0.560	0.613	0.612
-0.4	-0.4	0.690	0.722	0.732	0.754	0.759
-0.6	-0.2	0.811	0.845	0.856	0.846	0.853
-0.8	0.0	0.898	0.925	0.934	0.911	0.918
-0.4	-0.8	0.776	0.811	0.822	0.870	0.875
-0.6	-0.6	0.897	0.928	0.938	0.953	0.959
-0.7	-0.4	0.925	0.951	0.959	0.963	0.968
				n = 60		
0.4	-0.8	0.503	0.509	0.508	0.716	0.712
0.2	-0.6	0.678	0.687	0.686	0.825	0.822
0.0	-0.4	0.786	0.795	0.796	0.846	0.844
-0.2	-0.2	0.850	0.859	0.860	0.841	0.839
-0.4	0.0	0.892	0.899	0.900	0.833	0.832
-0.6	0.2	0.924	0.930	0.931	0.839	0.838
-0.8	0.4	0.958	0.962	0.962	0.883	0.882
0.0	-0.8	0.977	0.980	0.981	0.999	0.999
-0.2	-0.6	0.996	0.998	0.998	1.000	1.000
-0.4	-0.4	0.999	1.000	1.000	1.000	1.000
-0.6	-0.2	1.000	1.000	1.000	1.000	1.000
-0.8	0.0	1.000	1.000	1.000	1.000	1.000

Table 4: Powers against H_a^- for X5 at the five percent significiance level

ρ	γ	DW	MDW	s(0.5,0)	s(0,0.5)	s(0.5,0.5)
				n = 20		
0.4	-0.8	0.172	0.174	0.170	0.206	0.196
0.2	-0.6	0.237	0.241	0.237	0.279	0.267
0.0	-0.4	0.299	0.308	0.305	0.331	0.321
-0.2	-0.2	0.361	0.373	0.375	0.368	0.362
-0.4	0.0	0.423	0.440	0.447	0.398	0.396
-0.6	0.2	0.492	0.513	0.526	0.435	0.437
-0.8	0.4	0.589	0.615	0.633	0.511	0.516
0.0	-0.8	0.496	0.515	0.515	0.638	0.628
-0.2	-0.6	0.642	0.671	0.681	0.764	0.766
-0.4	-0.4	0.755	0.789	0.808	0.833	0.840
-0.6	-0.2	0.838	0.871	0.892	0.875	0.884
-0.8	0.0	0.901	0.929	0.945	0.915	0.923
-0.4	-0.8	0.843	0.877	0.896	0.946	0.955
-0.6	-0.6	0.922	0.949	0.965	0.978	0.984
-0.7	-0.4	0.936	0.962	0.975	0.977	0.983
				n = 60		
0.4	-0.8	0.562	0.568	0.567	0.800	0.797
0.2	-0.6	0.720	0.728	0.728	0.868	0.866
0.0	-0.4	0.812	0.820	0.820	0.869	0.867
-0.2	-0.2	0.865	0.872	0.873	0.853	0.851
-0.4	0.0	0.899	0.906	0.906	0.839	0.837
-0.6	0.2	0.928	0.933	0.934	0.840	0.839
-0.8	0.4	0.959	0.963	0.963	0.881	0.881
0.0	-0.8	0.984	0.987	0.987	1.000	1.000
-0.2	-0.6	0.998	0.998	0.998	1.000	1.000
-0.4	-0.4	1.000	1.000	1.000	1.000	1.000
-0.6	-0.2	1.000	1.000	1.000	1.000	1.000
-0.8	0.0	1.000	1.000	1.000	1.000	1.000