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FRACTIONAL MATRIX CALCULUS

INTRODUCTION, APPLICATIONS AND FUTURE DEVELOPMENTS.

by

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ABSTRACT: Fractional Calculus gives a generalisation of the common techniques of integration and differentiation. Although the origin of Fractional Calculus lies more than 150 years behind us, it is still quite unknown.

Recently however, Phillips (1984,1985) used fractional calculus to derive the distributions of the Stein-Rule and SUR estimator and Knight (1986b) used it to find the moments of the 2SLS estimator.

This paper gives an introduction to fractional calculus by showing the ideas behind the definitions, giving examples and some applications. At the end an attempt is made to show the relevance of this calculus for the different fields of Econometrics and to put it in a more general mathematical perspective.

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1. INTRODUCTION

Fractional Calculus is the generalisation of integration and differentiation. In the familiar calculus of integration and differentiation the order of the operations is restricted to integer values larger than zero. Differentiating a function n times, $n \in \mathbb{N}$, gives the n-th derivative and integrating a function n times leads to the n-fold integral. The relation between integration and differentiation is given by the fundamental theorem of calculus from which it follows that the n-th derivative of the n-fold integral is equal to the original function. Hence differentiation can be seen as the inverse of integration.

The question arises if an operator can be found that extends the order of the operators to non-integer values and combines both integration and differentiation into one operator. An operator that extends the order from integers to fractions, or real numbers, is called a Fractional Differential Operator.

It is shown in this paper that a fractional operator can be found and that it makes sense. It is a natural extension of the classical operators, meaning that it has the same properties as the original ones, and even more important, it can be used in Econometrics.

After deriving two different one dimensional operators we develop two multivariate analogues. Examples show how the definitions are applied and that the results are in line with what we expect. To show the relevance of fractional calculus in Econometrics, a few applications are given and some ideas on possible use are presented. Since success and development of the calculus depend on knowledge in a range of allied subjects, an attempt to put fractional calculus in a more general mathematical perspective concludes this paper.

2. GENERAL CONDITIONS ON FRACTIONAL OPERATORS

Since fractional calculus gives an extension of the ordinary techniques of integration and differentiation, the properties of the extended operator ought to be similar to the original operators. Using these properties we impose five conditions on a fractional operator.

Let D_a^μ denote the fractional operator; μ stands for the order of the operation, $\mu>0$ meaning differentiation, $\mu<0$ meaning integration, and a stands for the lower bound of the integration.[1]

$f:\mathbb{R}\to\mathbb{R}$ is a real function which, apart from existence assuring regularity conditions, is arbitrary.

Any fractional operator D_a^μ should satisfy the following five conditions:

(a) $$D_a^\mu f(x) = \frac{d^n}{dx^n} f(x) , \qquad \text{if } \mu= n \in \mathbb{N}$$

$$= \int_a^x \ldots \int_a^{x_1} f(x_0)\, dx_0 \ldots dx_{-n-1} , \qquad \text{if } \mu=-n \in \mathbb{N}.$$

For integer values of μ, D_a^μ reduces to ordinary differentiation if μ is positive and ordinary integration if μ is negative.

(b) $$D_a^0 f(x) = f(x) ;$$

---

1 For differentiation this bound is irrelevant, but for α-fold integration it is an essential part of the operator since the difference $(D_a^\mu f(x)-D_b^\mu f(x))$, $\mu<0$ does not only depend on a and b but also varies with x.

D_a^0 is the identity (mapping).

(c) $D_a^{\mu}[\kappa f(x) + \lambda g(x)] = \kappa\ D_a^{\mu}[f(x)] + \lambda\ D_a^{\mu}[g(x)]$.

The operator D_a^{μ} is linear.

(d) For every analytic function f(x), $D_a^{\mu}[f(x)]$ is also analytic.

A function f(x) is (real) analytic if it can be written as a power series in x.

(e) $D_a^{\lambda}\ D_a^{\mu}[f(x)] = D_a^{\lambda+\mu}[f(x)]$; unless $\lambda<0$ and $\beta>0$.

This is referred to as the Exponential Law.

In cases where λ and μ are both positive (negative) integers, condition (e) says that differentiating (integrating) μ times followed by differentiating (integrating) λ times equals $(\lambda+\mu)$ times differentiating (integrating). If λ and μ have different signs it will in general matter if we integrate first and then differentiate ($\mu < 0$, $\lambda > 0$) or differentiate first and then integrate. A simple example illustrates this fact: Choose $f(x) = 4$ and the lowerbound $a = 0$. Then

$$D_0^1 D_0^{-1}[f(x)] = D_0^1[4x] = 4 = D_0^0[f(x)] ,$$

which is not equal to:

$$D_0^{-1} D_0^1[f(x)] = D_0^{-1}[0] = 0 \neq D_0^0[f(x)]$$

which furthermore shows that integration is not the inverse of differentiation and that $\lambda<0$ and $\beta>0$ simultaneously should be excluded from the exponential law.

3. FRACTIONAL CALCULUS

This section contains the derivation of two fractional operators that satisfy the five conditions. Two examples show how they are applied and that the results coincide with our intuition.

The Grünwald definition starts from first principles with the definition of differentiation and integration. Define the increment $\delta_N = \frac{x-a}{N}$. The derivative of f(x) is then by definition equal to:

$$\frac{d}{dx} f(x) = \lim_{\delta_N \to 0} \frac{-[f(x)-f(x-\delta_N)]}{\delta_N} = \lim_{N\to\infty} \delta_N^{-1} [f(x)-f(x-\delta_N)]$$

and with the same definition of the increment one can define the integral as the limit of the Riemann-sum:

$$D_a^{-1} f(x) = \int_a^x f(y)\, dy = \lim_{N\to\infty} \sum_{j=0}^{N-1} \delta_N^1 f(x-j\,\delta_N) .$$

Repetition of these definitions leads to expressions involving binomial coefficients. Replacing factorials by gamma functions gives

The Grünwald definition

$$D_a^{\mu} f(x) = \lim_{N\to\infty} \left(\frac{x-a}{N}\right)^{-\mu} \sum_{j=0}^{N-1} \frac{\Gamma(j-\mu)}{\Gamma(-\mu)\Gamma(j+1)} f\left(x-j\left(\frac{x-a}{N}\right)\right) . \qquad (1)$$

We have used the Gauss or Euler limit definition of the gamma function, which allows the argument to be negative. See Oldham and Spanier (1974) for further details.

It is clear that the Grünwald definition is not very attractive to use for more complicated functions. The relevance of the Grünwald definition is, in the first place, the natural way in which integration and differentiation are extended from first principles. A second advantage is the generality of μ, which can take any value.

The Riemann-Liouville definition on the other hand, divides μ into $\mu=n-\alpha$, $n\in\mathbb{N}$ and $\alpha\in\mathbb{R}^+$, integrates to the order α and thereupon differentiates n times in the usual way.

Consider Cauchy's or Dirichlet's formula for repeated integration

$$D_a^{-n}[f(x)] = \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y)\, dy \, .$$

If we replace (n-1)! by $\Gamma(n)$ it is no longer necessary to limit n to integer values. Substitution of $-n$ by $-\alpha$ and bearing in mind the exponential law, the following definition is valid:

The Riemann-Liouville Definition

$$D_a^{\mu}[f(x)] = D_a^{n}[D_a^{-\alpha}[f(x)]]; \quad n \in \mathbb{N}; \quad \alpha \in \mathbb{R}; \quad \mu = n - \alpha$$

$$D_a^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, dy$$

$$D_a^{n} = \frac{d^n}{dx^n}$$

$$n = \begin{cases} 0 & \mu < 0 \\ [+\mu] + 1 & \mu \geq 0 \, ; \end{cases} \quad [\ ] \text{ the entier of its argument}$$

$$\alpha = n - \mu \, .$$

This definition coincides with a number of different 'traditional' fractional operators depending on the value of a. The definition coincides with Riemann's definition for a=0, with both Liouville's definition and with one of the Weyl definitions for $a=\infty$, and with the second Weyl definition for $a=-\infty$ and integrating from x to a. See Ross (1975) for further details and history.

Examples illustrate how the Riemann-Liouville definition is applied:

Example 1 $\quad D_0^{\mu}[x^p] = \dfrac{\Gamma(p+1)}{\Gamma(p+1-\mu)} x^{p-\mu} \quad ; \qquad p>-1 \, .$

Proof: $\quad D_0^{-\alpha}[x^p] = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-y)^{\alpha-1}\, y^p dy \, .$

Substitution of x = y/x gives

$$= \frac{1}{\Gamma(\alpha)} x^{p+\alpha} B(\alpha,p+1); \quad \alpha>0,\ p>-1 \text{ and } B(,) \text{ the Beta function.}$$

Expressing the Beta function in terms of the Gamma function, and n-fold differentiation, leads to

$$= \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} x^{p-\mu} .$$

A formula that reduces for integer values of $\mu \equiv n \in \mathbb{N}$ to

$$p(p-1)(p-2) \ldots (p-n)\, x^{p-n}$$

as one would expect.

Example 2 $\quad D^{\mu}_{-\infty}\, e^{cx} = c^{\mu} e^{cx} \quad ; \qquad c > 0$

Proof: $\quad D^{-\alpha}_{-\infty}[e^{cx}] = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-y)^{\alpha-1}\, e^{cy} dy .$

Substitution of $z = c(x-y)$ gives

$$= \frac{c^{-\alpha}}{\Gamma(\alpha)} e^{cx} \int_{0}^{\infty} z^{\alpha-1} e^{-z} dz .$$

The last integral is just another definition of the gamma function and hence equal to $\Gamma(\alpha)$. Therefore we obtain after differentiating to the order n:

$$D^{\mu}_{-\infty}[e^{cx}] = c^{-\alpha} c^{n} e^{cx} = c^{\mu} e^{cx} .$$

A sensible result.

4. FRACTIONAL MATRIX CALCULUS

In Section 3 we were only concerned with real functions of one variable and a fair amount of literature exists on this topic. See Ross (1974). In this section we shall extend the definitions to real functions f with multivariate arguments. Some of the mathematical background necessary to extend to Multivariate Fractional Calculus is given in the appendix.

Notation: $\mathbb{M}_{km}$ is the space of all real (kxm) matrices of maximal rank. $\mathbb{S}_m=\{S\in\mathbb{M}_{mm}|S'=S\}$ is the set of all non-singular symmetric matrices. $X'=X^t$ is the transpose of X. $\mathbb{V}_{mk} = \{H_1\in\mathbb{M}_{km}|H_1'H_1=I_m\}$ the set of all semi-orthogonal kxm matrices, k>m (the Stieffel manifold). $\mathbb{O}_m = \mathbb{V}_{mm}$ the orthogonal group. $\det(\ )^q = (\det(\ ))^q$ the qth power of the determinant, $tr(X) = trace(X) = \Sigma X_{ii}$ and $etr(X)=\exp(tr(X))$. X>0 denotes that X is positive definite. $X>Y \Leftrightarrow (X-Y)>0$. All $\lambda_i(X)>0$ denotes that all the eigenvalues of the possibly non-symmetric X are larger than 0. $p=\frac{m+1}{2}=\dim(\mathbb{S}_m)$ and finally define $\rho_{ij}=1$ if i=j and $\rho_{ij}=\frac{1}{2}$ if i≠j.

There are at least three different ways leading to a definition of $D^\mu = \det(\frac{\partial}{\partial X})^\mu$ for arbitrary values of μ. (See appendix for the meaning of D^μ and an explanation of differential operators in general). All three methods use the division of $\mu=n-\alpha$, leave $D^n=\det(\frac{\partial}{\partial X})^n$ unchanged and find a fractional integral operator $D^{-\alpha}$, similar to the one dimensional Riemann-Liouville definition.

Method 1 generalises the different terms that appear in the Riemann-Liouville definition to $\mathbb{R}^{mm}$. Replace $\Gamma(\alpha)$ by the multivariate Gamma function $\Gamma_m(\alpha)$ and hence also $\alpha>0$ by $\alpha>\frac{m-1}{2}=p-1$ (assuring existence of Γ_m), $\alpha-1$ by $\alpha-p$ and the distance $(x-y)$ by the generalised distance $\det(X-Y)$.[2] Finally change the integration area to 0<Y<X for the Riemann definition and to (X-Y)>0 for the Liouville definition. It remains unclear however why det(X-Y) is chosen and not some other generalised distance. The following two methods do make this explicit and give a connection with the differential operator $\det(\frac{\partial}{\partial X})$.

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2 Det(X-Y) can be reduced to the Lorentz distance. See Gårding (1951)

Method 2 rewrites the polynomial $P(\partial)$ in a formal way. Meaning that if the equation $P(\xi)^{-\alpha} = Q(\alpha;\xi)$ holds, we replace ξ by $(\frac{\partial}{\partial X})$ and it follows:

$$P(\frac{\partial}{\partial X})^{-\alpha}[f(X)] = Q(\alpha;\frac{\partial}{\partial X})[f(X)] \ . \tag{4.1}$$

In our case where $P(\frac{\partial}{\partial X}) = \det(\frac{\partial}{\partial X})$, it follows from (A.8) that:

$$\det(\xi)^{-\alpha} = \frac{1}{\Gamma_m(\alpha)} \int_{S>0} \det(S)^{\alpha-p} \mathrm{etr}(-S'\xi)dS \equiv Q(\alpha;\xi); \quad a>p-1 \tag{4.2}$$

Replacing ξ by $(\frac{\partial}{\partial X})$ leads to:

$$\begin{aligned} D^{-\alpha}[f(X)] &= \frac{1}{\Gamma_m(\alpha)} \int_{S>0} \det(S)^{\alpha-p} \mathrm{etr}(-S'\frac{\partial}{\partial X})\ dS\ f(X) \\ &= \frac{1}{\Gamma_m(\alpha)} \int_{S>0} \det(S)^{\alpha-p} \mathrm{etr}(-S'\frac{\partial}{\partial X})\ f(X)\ dS \ . \end{aligned}$$

Using Taylor's theorem (A.12), $\mathrm{etr}(-S'\frac{\partial}{\partial X})\ f(X) = f(X-S)$ we obtain

The Liouville definition:

If $f: \mathbb{M}_{mm} \to \mathbb{R}$ is a symmetric function then:

$$D^{-\alpha}[f(X)] := \frac{1}{\Gamma_m(\alpha)} \int_{S>0} f(X-S) \det(S)^{\alpha-p}\ dS \tag{4.3}$$

$$D^{n} := \det(\frac{\partial}{\partial X})^{n} \ .$$

This is exactly the approach Phillips (1985) uses. Phillips does not give conditions under which D^{μ} exists and a remark is in order. Formula (4.2) is valid for positive definite ξ. The determinant is a symmetric function and therefore the Laplace transform $Q(\alpha,\xi)$ of $\det(S)^{\alpha-p}$ is also symmetric. By remark (iii) in the appendix, we may extend the definition to all square matrices $X \in \mathbb{M}_{mm}$. If however the symmetry condition on $S(=S')$ is not changed then f has to remain a symmetric function. We shall not develop this point.

Method 3 uses the Laplace transform and its properties. The properties that will be used are given in the appendix. Under suitable conditions, ensuring existence, we have

$$P_\omega\left(\frac{\partial}{\partial X}\right)^\alpha[f(X)] = P_\omega\left(\frac{\partial}{\partial X}\right)^\alpha \mathcal{L}^{-1}\{\mathcal{L}\{f(X)\}\} = \mathcal{L}^{-1}\{P(S)^\alpha \mathcal{L}\{f(X)\}\} \tag{4.4}$$

and

$$P_\omega\left(\frac{\partial}{\partial X}\right)^\alpha \mathcal{L}^{-1}\{P(S)^{-\alpha}\ \mathcal{L}\{f(x)\}\} = f(X)\ . \tag{4.5}$$

Hence $g(\alpha;X) \equiv \mathcal{L}^{-1}\{P(S)^{-\alpha}\mathcal{L}\{f(x)\}\}$ is a solution to the partial differential equation:

$$P_\omega\left(\frac{\partial}{\partial X}\right)^\alpha[g(\alpha;X)] = f(X)\ . \tag{4.6}$$

In this way we have constructed an inverse for the differential operator $P_\omega(\frac{\partial}{\partial X})^\alpha$. For obvious reasons we call this operator $D^{-\alpha}$. If we set $\Omega = \mathcal{S}$ and $P(S) = \det(S)$ we have as conjugate operator $P_\omega(\frac{\partial}{\partial X}) = \det(\rho_{ij}\frac{\partial}{\partial x_{ij}})$, (see appendix (A.4)). $D^{-\alpha}$ takes the form:

$$D^{-\alpha}[f(X)] = \frac{2^{m(p-1)}}{(2\pi i)^{mp}} \int_{\substack{Im(S)\in\mathcal{S} \\ Re(S)=U_0^m>0}} \int_{Y>0} \det(S)^{-\alpha} f(Y) etr(-S'Y)dY \quad etr(S'X)dS$$

$$= \int_{Y>0} f(Y)\{\frac{2^{m(p-1)}}{(2\pi i)^p} \int_{U=U_0>0} \det(S)^{-\alpha} etr(S'(X-Y))dS\ \}\ dY\ . \tag{4.7}$$

With $Z=(X-Y)$ we recognise the inner integral as Cauchy inverse of $\det(S)^{-\alpha}$ and hence by (A.7) equal to $\det(Z)^{\alpha-p}/\Gamma_m(\alpha)$; $Z>0$ and 0 elsewhere. Substitution in (4.7) gives

<u>The Riemann definition</u>

$$D^{-\alpha}[f(X)] = \frac{1}{\Gamma_m(\alpha)} \int_{(X-Y)>0;Y>0} f(Y) \det(X-Y)^{\alpha-p} dY \tag{4.8}$$

$$= \frac{1}{\Gamma_m(\alpha)} \int_0^X f(X-S) \det(S)^{\alpha-p} dS \qquad \text{(ignoring sign)}$$

$$D^n = \det(\rho_{ij} \frac{\partial}{\partial X_{ij}})\ .$$

<u>Examples</u> show how the definitions can be applied. The first illustrates the Liouville definition. Recall that in this case $D^n = \det(\frac{\partial}{\partial X})^n$ and

$X \in \mathbb{M}_{mm}$.

Example 1

$$D^{\mu}[\mathrm{etr}(B'X)] = \det(B)^{\mu}\,\mathrm{etr}(B'X); \qquad \text{All } \lambda_i(B)>0 \tag{4.9}$$

Proof: $$D^{-\alpha}[\mathrm{etr}(AXA')] = \frac{1}{\Gamma_m(\alpha)} \int_{S>0} \mathrm{etr}(A(X-S)A')\,\det(S)^{\alpha-p}\,dS; \quad \alpha>p-1$$

$$= \mathrm{etr}(AXA')\frac{1}{\Gamma_m(\alpha)} \int_{S>0} \mathrm{etr}(-A'AS)\,\det(S)^{\alpha-p}dS$$

$$= \mathrm{etr}(AXA')\,\frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha)}\,\det(S)^{-\alpha}\,. \tag{4.10}$$

in view of (A.7). Differentiating (4.10) gives $D^n\,\mathrm{etr}(A'AX)\,\det(A'A)^{-\alpha} = \det(A'A)^n\,\mathrm{etr}(A'AX)\,\det(A'A)^{-\alpha}$. Since all these functions are symmetric we can extend (4.10) to the desired result (4.9). (See Herz (1955)). The condition that all eigenvalues of B should be larger than zero, all $\lambda_i(B)>0$, is necessary and sufficient for convergence of the integral.

The second example illustrates the Riemann definition. Recall that $X \in \mathbb{S}_m$ and $D_s^n = \det(\rho_{ij}\frac{\partial}{\partial x_{ij}})^n$.

Example 2

$$D_s^{\mu}\,[\det(X)^q] = \frac{\Gamma_m(p+q)}{\Gamma_m(p+q-\mu)}\,\det(X)^{q-\mu}; \qquad \mu>p+q-1$$

Proof: $$D_s^{-\alpha}[\det(X)^q] = \frac{1}{\Gamma_m(\alpha)} \int_0^X \det(S)^q\,\det(X-S)^{\alpha-p}\,dS\,.$$

Through substitution of $Z = X^{-1/2}SX^{-1/2}$ and $dS = \det(X^{1/2})^{m+1}dZ$ the expression reduces to

$$= \det(X)^{q+\alpha}\,\frac{1}{\Gamma_m(\alpha)}\,B_m(p+q,\ \alpha)\,.$$

After expressing the multivariate beta function B_m as a quotient of gamma functions we obtain

$$D_s^{-\alpha}[\det(X)^q] = \det(X)^{q+\alpha}\,\frac{\Gamma_m(p+q)}{\Gamma_m(p+q+\alpha)}\,. \tag{4.11}$$

If we rewrite $\det(X)^{q+\alpha}$ as the Cauchy inverse of $\det(S)^{-q-\alpha-p}$ (see (A.8)) and differentiate to the order n we obtain:

$$D_S^{\mu}[\det(X)^q] = \frac{\Gamma_m(p+q)}{\Gamma_m(p+q+\alpha)}\Gamma_m(q+p+\alpha)\frac{2^{m(p-1)}}{(2\pi i)^{mp}} \int_{Re(S)=U_o>0} D_S^n \operatorname{etr}(S'X)\det(S)^{-q-\alpha-p} dS$$

$$= \Gamma_m(p+q) \frac{2^{m(p-1)}}{(2\pi i)^{mp}} \int_{Re(S)=U_o>0} \operatorname{etr}(S'X) \det(S)^{n-q-\alpha-p}\, dS$$

$$= \frac{\Gamma_m(p+q)}{\Gamma_m(p+q-n+\alpha)} \det(X)^{q-n+\alpha}$$

5. APPLICATIONS

In this section we present two applications and some ideas to show that Fractional Calculus is more than an exquisite mathematical nicety.

Application 1: Wilks' generalised variance.

A result that generalises Wolfe (1975) and which is useful in Econometrics is given by:

Theorem 1

Let X be a random matrix with distribution function F(X), Moment Generating Function $M_X(T)$, and let the support S_u of X be defined as $S_u=\{X\in M_{mm} | \text{all } \lambda_i(X)>0\}$ then

i) $D^{\mu}[M_X(T)]|_{T=0} < \infty \Leftrightarrow E[\det(X)^{\mu}] < \infty$

If both expressions converge then:

ii) $E[\det(X)^{\mu}] = D^{\mu}[M_X(T)]|_{T=0}$

Proof: We start with ii) and i) follows naturally.

ii) If both expressions converge we have for $\alpha > p - 1$:

$$D^{-\alpha}[M_X(T)] = \frac{1}{\Gamma_m(\alpha)} \int_{S>0}\left\{\int_{X\in S_u} \operatorname{etr}[(T-S)'X]d\, f(X)\right\} \det(S)^{\alpha-p}\, dS \quad (5.1)$$

Because the integral converges, we may interchange the order of integration by Fubini's theorem and we obtain:

$$= \int_{X\in S_u} \frac{1}{\Gamma_m(\alpha)} \int_{S>0} \operatorname{etr}[(T-S)'X] \det(S)^{\alpha-p}\, dS\, d\,F(X). \quad (5.2)$$

But the inner integral is just the fractional integral of etr(T'X). Note that T is the variable and $X \in S_u$ is the constant satisfying the condition in Example 1, Section 4. Using that example it follows:

$$= \int_{X\in S_u} \operatorname{etr}(T'X) \det(X)^{-\alpha}\, d\,F(X)\ .$$

After n-fold differentiation, we obtain

$$D^{\mu}[M_X(T)] = \int_{X\in S_u} \operatorname{etr}(T'X) \det(X)^{\mu}\, dF(X)\ . \quad (5.3)$$

Letting $T \to 0$, it follows from Lebesgues monotonic convergence theorem (since $\lim_{T\to 0} \det(X)^{\mu} \operatorname{etr}(T'X) = \det(X)^{\mu}$) that

$$D^{\mu}M_X(T)\big|_{T=o} = \int_{X\in S_u} \det(X)^{\mu}\, dF(X) = E[\det(X)^{\mu}] \quad (5.4)$$

ii) It follows from Fubini's theorem that $E[\det(X)^{\mu}]$ converges if $D^{\mu}[M_X(0)]$ converges and vice versa. Hence if one converges the other cannot diverge.

Remarks:

- The support S_u is chosen to assure existence of the integrand in (5.2)
- X does not need to be symmetric since all functions are symmetric (remark (iii) in the appendix).

The theorem states that we can find the moments of the density function F(X) to any order using Fractional Calculus. In the univariate case Theorem 1 ii) reduces to $E(x^{\mu}) = D^{\mu}[M_x(0)]$ which is just a generalisation of the classic result $E(x^n) = \frac{d^n}{dx^n} [M_x(0)]$, i.e. the n^{th} moment can be found by differentiating the MGF to the appropriate order and setting t equal to zero.

Application 2: Estimation using Fractional Moments.

Marzoug and Ahmad (1985) use fractional moments to estimate the parameters of a Burr (type XII) distribution. They equate two fractional moments of the theoretical distribution with corresponding fractional sample moments and solve for the two parameters α and β, analogous to a method of moments estimator.

The Burr distribution is univariate, but with the help of theorem 1 we could apply the same method to multivariate distributions. With fractional moments we can estimate a larger number of parameters without the need to use very high moments.

Application 3: Anti Projection.

Consider a random (mxn) matrix X, m≤n with (joint) density function f(X). If we partition X as X=(Y,Z), with Y (mxk) and Z (m x(n-k)), m≤n-k, we can find the (marginal) distribution of Y by "integrating Z out". Richards (1984) shows that for a specific class of density functions we can reverse the integration by means of fractional calculus.

If f is a hyperspherical distribution, i.e. $f_{mn}(X) = h_{mn}(XX')$ and rapidly decreasing on $\mathbb{S}$, i.e. f(X) and all its partial derivatives decrease faster than any polynomial in the entries of X then

$$f_{mk}(Y) \propto \int_{S>YY'} \det(S-YY')^{(n-k-m-1)/2} g_{mn}(S)dS = g_{mk}(YY') \tag{5.5}$$

and the major result that g is uniquely determined by g_{mk} through the relationship:

$$g_{mn}(XX') \propto \int_{T>XX'} (D^{\ell}_{SW} g_{mk}(T)) \det(T - XX')^{\ell - \frac{(n-k+m+1)}{2}} dT \tag{5.6}$$

where $\propto$ means proportional to, $D^{\ell}_{SW} = (-1)^{\ell} D^{\ell}_{S}$ and ℓ is any non negative integer such that $D^{\ell}_{SW}[g_{mk}(T)]$ is absolutely integrable on {T>0}.

Remark: Richards uses a Weyl definition of the fractional operator and therefore $D^{\ell}_{SW} = (-1)^{\ell} D^{\ell}_{S}$ in our notation.

Example: If $Y \sim N_{mk}(0, \Sigma, I_k)$ then $X \sim N_{mn}(0, \Sigma, I_n)$

Or: if all columns of Y are i.i.d. $N_m(0, \Sigma)$ then so are the columns of X.

Proof: $f_{mk}(Y) = g_{mk}(YY') \propto \operatorname{etr}(-\frac{1}{2}\Sigma^{-1}YY')$

If we substitute T for YY' then T>0 a.s. if $\Sigma^{-1}>0$

$$D^{\ell}_{SW}\, g_{mk}(T) \propto P_{SW}(-\tfrac{1}{2}\Sigma^{-1})\ \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}T)$$

$$D_{SW}^{-\ell+(n-k)/2}[D^{\ell}_{SW} g_{mk}(T)] \propto \int_{T>XX'} \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}T)\det(T-XX')^{\ell-\frac{n-1}{2}-p}\, dT \quad (5.7)$$

$$\propto \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}XX')$$

And it follows that $X \sim N(0, \Sigma, I_n)$.

There are many other applications. Phillips (1984,1985) and Knight (1986a, 1986b) use fractional calculus in a number of articles on exact distributions of estimators in small samples. It needs no explanation to realise how important this is in econometrics. Another application that one might think of is in large sample theory. Some theorems require that the variance or the third moment of a random variable exists where it suffices that $EX^{1\frac{1}{2}}$ or $EX^{2\frac{2}{3}}$ exist. With fractional calculus we may prove $EX^{2\frac{2}{3}} < \infty$ (without deriving the distribution) or find $\lim_{\mu\to 2} E(X^{\mu})$.

If we shift towards the more economic fields of econometrics, we might think of dynamic models where differential equations play an important role. Another application arises if we apply Fractional Calculus to diffusion of, for instance, money in the economy. Oldham and Spanier (1974) solve a number of diffusion problems that arise in chemistry and physics, although the application of physics to economic problems is not always very fortunate.

In operations research a large number of problems boil down to solving integral equations. For instance loss, profit and risk functions (in a non deterministic situation) are all integral transforms. And more specifically in semi-Markov programming we minimize $q(.) = \int h(\ )Q(\ )\ dX$ under the condition $\int q(\ )\ dX = 1$. Fractional calculus can help in solving these equations.

7. CONCLUSION

In the last section we saw that Fractional Calculus can be useful in Economics and Econometrics. It can on the other hand be related to many mathematical areas. Differential operators and equations, Greens function and fundamental solutions, pseudo differential operators, integral transformations, innerproducts, special functions are all directly linked to Fractional Calculus. At a fundamental level we should have a closer look at integration and differentiation through the theory of exterior differential forms. These forms can be integrated and differentiated and there exists a generalisation of the fundamental law of calculus through "Stokes (general) Theorem". Differential forms are in turn closely related to manifolds. Calculus on manifolds can and will be an important tool in multivariate statistics, etc.

We could continue, but the general idea is clear: success and development of fractional calculus depend on our knowledge in a range of mathematical subjects. On the other hand development of fractional calculus will generate insight in these subjects. This symbiosis may prove very useful. Together with significant applications this might develop fractional calculus from an unknown mathematical theory into a useful tool of econometrics.

APPENDIX.

Definition: Differential Operator.

A mapping (or operator) M of a function space F to a function space G is called a differential operator if the image g of f, g = M(f) ($f \in F$, $g \in G$), in every point X of an open set Ω is determined by

- a finite number of (partial) derivatives of f,
- the value of $X \in \Omega$ through functions $a_\alpha(X)$.

We shall only concern ourselves with linear differential operators of the form:

$$P(X;\partial) = \sum_{[\alpha]\le r} a_\alpha(X)\partial^\alpha = \sum_{\alpha_1+\ldots+\alpha_n\le r} a_{\alpha_1,\ldots,\alpha_n}(X)\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\ldots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \quad (A.1)^{3}$$

P must be regarded as working on a function $f \in F$ that succeeds P : $P(X;\partial)f(X)$. The terms $\partial^\alpha[f(x)]$ represent the partial derivatives as meant in the definition. The simplest example of differential operator is, of course, $\frac{\partial}{\partial x}$.

Some criteria:

- We call P an ordinary differential operator if dimension $(\Omega) = 1$ and a partial differential operator if dimension $(\Omega) > 1$.
- The order of P is the largest value of $[\alpha]$ associated with a function a_α not identically zero. In (3,1) it is assumed to be r.
- The sum of all terms with $[\alpha] = r$ is called the principal part P_r

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3 $\alpha = (\alpha_1, \ldots, \alpha_n)$; a multi-index and,
$[\alpha] = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ its "length".
$a_\alpha(X)$: analytic functions on an open set $\Omega(X \in \Omega)$ of dimension n.
$\partial^\alpha = \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} = \left((-i)\frac{\partial}{\partial X_1}\right)^{\alpha_1} \left((-i)\frac{\partial}{\partial X_2}\right)^{\alpha_2} \ldots \left((-i)\frac{\partial}{\partial X_n}\right)^{\alpha_n}$; is often omitted.

of P : $P_r(X;\partial) = \sum_{[\alpha]=r} a_\alpha(X)\partial^\alpha$.

- P has constant coefficients if $P(X;\partial) = P(\partial)$, i.e. does not depend on X.
- If we substitute for every $\partial_i^{\alpha_i}$ a real variable ξ_i then $P(X,\xi)$ is a polynomial in ξ. $P_r(X,\xi)$ is called the characteristic polynomial.

  A differential operator with constant coefficient is called:

  Elliptic if $P(\xi) = 0$ has no real roots apart from $\xi = 0$

  Parabolic if $P(\xi) = 0$ has only the solution $\xi = 0$

  Hyperbolic with respect to $\xi \neq 0$ if

  $P_r(\xi) \neq 0 \wedge |\mathrm{Im}(t)|>t_0>0 \Rightarrow P(t\xi+\eta)\neq 0$; for certain $t_0\in\mathbb{R}$, $\eta \in \Omega$.

  The Hyperbolic definition implies that $P(t\xi + \eta) = 0$ only has real roots t (if P is homogeneous).

Examples of operators that are very common in econometrics, mathematics and physics are:

$\Delta : = \frac{\partial^2}{\partial x_1^2} + \ldots \frac{\partial^2}{\partial x_n^2}$; the Laplace operator is elliptic.

$\frac{\partial^m}{\partial x^m}$ is parabolic since $P(\xi)=0$ has the m-fold solution $\xi=0$

$w : = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$; the wave operator is hyperbolic with respect to $(1,0,\ldots,0)$.

$D : = \det(\frac{\partial^2}{\partial x_{ij}})$; if $\Omega = \mathbb{M}_{mm}$.

$D_s : = \det(\rho_{ij} \frac{\partial}{\partial x_{ij}})$; if $\Omega = \mathbb{S}_{mm}$ with $\rho_{ij} = \begin{cases} 1 & i=j \\ \frac{1}{2} & i\neq j \end{cases}$ is hyperbolic with respect to $-I_m$

We shall proof that D_s is hyperbolic. $\det(\rho_{ij}\xi_{ij}) = \det(-I_m) = (-1)^m$ and $\det(\rho_{ij}t\xi + \rho_{ij}\eta_{ij}) = \det(\rho_{ij}\eta - tI_m) = 0$ has only real roots since $\rho_{ij}\eta_{ij}$ is a symmetric matrix and t are its eigenvalues.

Conjugate Operator $P_\omega(\partial)$

Conjugate Operator $P_{\omega}(\partial)$

An important role will be played by the function $f:\mathbb{R}^{nm}\to\mathbb{R}$ defined as:

$$f(X) \;=\; \mathrm{etr}(S'X) \;=\; e^{S_{11}X_{11}+\ldots+S_{nm}X_{nm}} . \qquad (A.2)$$

If the elements in X are independent and we let $P(\partial)$ "work" on $f(X)$ we obtain

$$P(\partial)f(X) \;=\; P(\partial)\,\mathrm{etr}(S'X) \;=\; P(S)f(X) . \qquad (A.3)$$

If the elements in X are dependent, this will show in the exponent and in the derivatives and (A.3) will no longer hold. We can, however, find for every polynomial P(S) and Ω (that satisfy some basic conditions) a corresponding operator $P_{\omega}(\partial)$ such that:

$$P_{\omega}(\partial)\,\mathrm{etr}(S'X) \;=\; P(S)\,\mathrm{etr}(S'X) \qquad (A.4)$$

We refer to $P_{\omega}(\partial)$ as the CONJUGATE operator of P(S).

It is easily checked that $P_{w}(\partial) = D_{s}$ for $\Omega = \mathbb{S}_{m}$ and $P(S) = \det(S)$.

Remarks: i) $D\,\det(X)^{\mu} = 0$ and $D_{s}\det(X)^{\mu} = c\,\det(X)^{\mu-1}$ (if $\dim\Omega > 1$)

ii) $D[F(X)] \neq J(F(X))$, the Jacobian.

Laplace Transforms

Without concerning ourselves with conditions or proofs we shall give some definitions, properties and examples that are important in deriving multivariate fractional operators. Let $S = U + iV$; $U, V \in \mathbb{S}$.

The Laplace transform $\mathcal{L}(f(X)) = \hat{f}(S) \equiv g(S)$ of a function f(X) is unique and defined as:

$$g(S) := \int_{X>0} \mathrm{etr}(-S'X)\, f(dX \qquad (A.5)$$

The Cauchy Inversion f(X) of a function g(S) has g(S) as its Laplace transform:

$$\frac{2^{m(p-1)}}{(2\pi i)^{mp}} \int_{\substack{U=U>0\\ V\in\mathbb{S}_m}} \mathrm{etr}(S'X)\, g(S)\, dS = \begin{cases} f(X) & ;\ X>0 \\ 0 & ;\ \text{elsewhere} \end{cases} \qquad (A.6)$$

Examples.

$$\mathcal{L}(H(X)) = \Gamma_m(p)\ \det(S)^{-p}; \qquad H(X) = \begin{cases} 1; & X > 0: \\ 0; & \text{elsewhere} \end{cases} \qquad \text{The Heaviside function} \tag{A.7}$$

$$\mathcal{L}(\det(X)^{a-p}) = \det(S)^{-a}\ \Gamma_m(A) \qquad \mathrm{Re}(a) > p-1$$

and hence by Cauchy's inversion formula

$$\frac{2^{m(p-1)}}{2(\pi i)^{mp}} \int_{U=U_0>0} \operatorname{etr}(S'X)\ \det(S)^{-a} ds = \frac{\det(X)^{a-p}}{\Gamma_m(a)}; \qquad X>0\ . \tag{A.8}$$

The Convolution Formula: The convolution product

$$f_3(X) = \int_0^X f_1(X-Y) f_2(Y) dY$$

of f_1 and f_2 has Laplace transform;

$$\mathcal{L}\{f_3(X)\} = \mathcal{L}\{f_1(X)\}\mathcal{L}\{f_2(X)\}\ . \tag{A.9}$$

Integration and $P_\omega(\partial)$-differentiation may be interchanged.

For instance:

$$P_\omega(\frac{\partial}{\partial S})\int \operatorname{etr}(-S'X) f(X) dX = \int_{X>0} P(-X) \operatorname{etr}(-S'X) f(X) dX\ . \tag{A.10}$$

Definition: $f(X)$ is a Symmetric function if $f(H\ X\ H') = f(X)$; $X \in \mathbb{S}_m$ for all orthogonal matrices H. $f(X)$ can be written as a complex analytic function of the m elementary symmetric functions

$$\sigma_1 = \operatorname{tr}(X); \quad \sigma_2 = \sum_{i<j} \det\begin{pmatrix} x_{ii} & x_{ij} \\ x_{ij} & x_{jj} \end{pmatrix}; \quad \ldots\ \sigma_m = \det(X)\ .$$

Remark: iii) Because $f(W^{1/2}XW^{1/2}) = f(XW) = f(WX)$ for every $W>0$ a symmetric function can be extended from $X \in \mathbb{S}_m$ to $X \in \mathbb{M}_{mm}$ as long as $f(X)$ is defined.

Taylor's Theorem:

$$f(X-S) = \sum_{k=0}^{\infty} \frac{1}{k!}\ [\operatorname{tr}(-S'\frac{\partial}{\partial X})]^k\ f(X) = \operatorname{etr}(-S'\frac{\partial}{\partial X}) f(X) \tag{A.11}$$

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