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LOCALLY OPTIMAL TESTING WHEN A NUISANCE PARAMETER IS PRESENT ONLY UNDER THE ALTERNATIVE

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Abstract: We consider hypothesis testing problems in which a nuisance parameter is present only under the alternative hypothesis. Standard asymptotic tests, such as likelihood ratio, Lagrange multiplier and Wald tests, are difficult to apply because of problems incurred in obtaining their asymptotic distributions. To overcome this difficulty, we reparameterize the testing problem to one for which an exact small sample test can be constructed using existing hypothesis testing procedures. The reparameterization technique is applied to two examples from the econometiics literature, and an empirical power comparison shows that our test has better power properties than tests previously proposed in the literature. Further, p-values for our test can be computed in O(n) operations so the test can be implemented efficiently.

1. Introduction

We consider hypothesis testing problems in which a nuisance parameter is present only under the alternative hypothesis. Problems of this nature often occur in the economics literature, particularly in the area of testing for stochastic coefficients in time series regression models. Recent examples include Andrews (1990), Bera and Higgins (1990), Shively (1988), and Watson and Engle (1985). Davies (1977) provides several examples in the statistics literature. As an illustration, suppose that the distribution of a random variable X depends on the parameters γ and δ, and that we want to test H_0: $\gamma = 0$ against the alternative H_A: $\gamma > 0$. If the distribution of X does not depend on δ when $\gamma = 0$, then δ is a nuisance parameter present only under the alternative. Standard asymptotic tests, such as likelihood ratio, Lagrange multiplier and Wald tests, are difficult to apply because of problems incurred in obtaining their asymptotic distributions. Therefore, tests need to be developed that do not rely on these testing principles.

Davies (1977) constructs a test for problems when a nuisance parameter is present only under the alternative and shows how to obtain a bound on the significance level. Watson and Engle (1985) use a slightly modified version of Davies' test statistic to test for a stochastic coefficient in Rosenberg's (1973) return to normalcy model while Shively (1988) derives the small sample equivalent of Watson and Engle's (SSEWE) asymptotic test.

We propose an alternative method of constructing tests when a nuisance parameter is present only under the alternative. We begin by noting that a multi-dimensional testing problem typically can be reparameterized into polar coordinates so that the resulting testing problem is one-dimensional and has nuisance parameters only under the alternative. For example, suppose we want to test H_0: $\theta = (\theta_1, \theta_2)' = 0$ against H_A: $\theta \neq 0$ based on the $n \times 1$ random vector X whose density function is $f(x|\theta)$. To reparameterize into polar coordinates, set $\theta =$

$\{r\cos(\phi),\ r\sin(\phi)\}'$ where $r = (\theta'\theta)^{1/2} \geq 0$ and $\phi = \tan^{-1}(\theta_2/\theta_1)$ with $0 \leq \phi < 2\pi$. H_0: $\theta = 0$ is now equivalent to H_0: $r = 0$ while $\theta \neq 0$ if and only if $r > 0$. Thus, our testing problem becomes one of testing H_0: $r = 0$ against H_A: $r > 0$ where ϕ is a nuisance parameter present only under the alternative.

This suggests that one way of constructing tests when a nuisance parameter is present only under the alternative is to reparameterize to a higher dimensional testing problem that can be handled using existing testing techniques. For example, one of the problems considered below can be reparameterized from $(\gamma, \delta)'$ to $\theta = (\theta_1, \theta_2)'$ in such a way that (i) θ_1 and θ_2 are both identified under the null hypothesis; and (ii) the testing problem reduces to testing H_0: $\theta = 0$ against H_A: $\theta_1 > 0$, $\theta_2 \geq 0$. Given this reparameterization, the locally most mean powerful (LMMP) test proposed by King and Wu (1990) can be applied.

This technique must be applied on a case-by-case basis because it may not always be possible to reparameterize in such a way as to allow the application of a known testing procedure. Our procedure is best illustrated by considering specific examples, but the general reparameterization technique is quite straightforward and can be applied to a variety of problems. Two important advantages of our technique for the problems considered below are that we can (i) obtain the exact small sample properties of our test statistic rather than only their asymptotic properties; and (ii) compute exact small sample p-values in O(n) operations using the technique of Shively, Ansley and Kohn (1990).

Our examples are testing for a stochastic coefficient in Rosenberg's (1973) return to normalcy model and testing for two stochastic coefficients generated by random walk processes with correlated innovation variances. The return to normalcy model has been applied frequently in the literature. For example, Bos and Newbold (1984) used it to test whether the systematic risk of an asset or portfolio is varying through time while Watson and Engle (1985) used it to check for stability in an efficient market for gold and silver prices. A discussion of multiple random walk coefficient models and their applications in economics can be found in LaMotte and McWhorter (1978).

The paper is organized as follows. Sections 2 and 3 apply the reparameterization technique to the return to normalcy and the multiple random walk coefficient models, respectively. Section 4 reports an empirical example and Section 5 contains some concluding remarks.

2. Testing the return to normalcy model

Rosenberg's return to normalcy model is given by

$$y_t = x_t\alpha_t + z_t'\beta + \varepsilon_t \tag{1}$$

$$\alpha_t - \mu = \rho(\alpha_{t-1} - \mu) + a_t \tag{2}$$

$$0 \leq \rho < 1 \tag{3}$$

$t = 1, ..., n$, where x_t is an independent scalar regressor, z_t is a $k \times 1$ vector of independent regressors, β is a $k \times 1$ vector of unknown constant coefficients, and ε_t and a_t are independent disturbances with $\varepsilon_t \sim IN(0, \sigma^2)$, $a_t \sim IN(0, \lambda\sigma^2)$ and $\lambda \geq 0$. The parameter ρ is restricted to $0 \leq \rho < 1$ because in typical economic applications one would expect α_t to be positively correlated. Testing for a stochastic coefficient in this model is equivalent to testing $H_0: \lambda = 0$ (constant coefficient) against the alternative $H_A: \lambda > 0, 0 \leq \rho < 1$. In this problem ρ is a nuisance parameter that is present only under the alternative.

We can rewrite (1) and (2) as

$$y_t = x_t\mu + z_t'\beta + v_t \tag{4}$$

where $v_t = x_t(\alpha_t - \mu) + \varepsilon_t$. In matrix notation, (4) can be written as

$$y = (X \;\; Z)\begin{pmatrix}\mu \\ \beta\end{pmatrix} + v \tag{5}$$

where $y = (y_1, ..., y_n)'$, $X = (x_1, ..., x_n)'$, $Z = (z_1, ..., z_n)'$ and $v = (v_1, ..., v_n)'$. It is assumed that $(X \;\; Z)$ has full column rank. Note that $v \sim N[0, \sigma^2\{I + \lambda\Omega(\rho)\}]$ with the (s,t)th element of $\Omega(\rho)$ given by

$$\Omega_{st}(\rho) = x_s x_t \, \rho^{|s-t|} / (1 - \rho^2). \tag{6}$$

The problem of testing $H_0: \lambda = 0$ against $H_A: \lambda > 0, 0 \le \rho < 1$ is invariant to transformations of the form

$$y^* = cy + (X \;\; Z)\begin{pmatrix}a \\ b\end{pmatrix} \tag{7}$$

where a and c are scalars, and b is a $k \times 1$ vector.

We reparameterize the problem from the parameter space $(\lambda,\rho)'$ to a parameter space $\theta = (\theta_1,\theta_2)'$ using a transformation involving polar coordinates. Because $\lambda \ge 0$ and we wish to test $H_0: \lambda = 0$, a natural transformation is

$$\lambda = (\theta'\theta)^{1/2}. \tag{8}$$

Note that testing $H_0: \lambda = 0$ is equivalent to testing $H_0: (\theta'\theta)^{1/2} = 0$, i.e. testing whether the length of the θ vector is zero.

To complete the reparameterization from $(\lambda,\rho)'$ to the vector θ, we need to determine the angle θ makes with the θ_1-axis. Let ϕ represent this angle. Because λ defines the length of θ, ρ should determine ϕ. There are several mappings of ρ onto ϕ that one might consider. One possibility is to map ρ onto $0 \le \phi < \pi/2$, another is to map ρ onto $0 \le \phi < \pi$, and a third is to map ρ onto $0 \le \phi < 2\pi$. We rule out the third mapping immediately because it would require the limit as ρ tends to one to give the same covariance matrix of v as $\rho = 0$. Of the remaining two transformations, we contend that the better reparameterization is to map ρ onto $0 \le \phi < \pi/2$. The original alternative parameter space $\lambda > 0, 0 \le \rho < 1$ is a subset of the positive quadrant, so a natural transformation from $(\lambda,\rho)'$ to $(\theta_1,\theta_2)'$ is one that results in the new alternative parameter space lying in the positive quadrant (i.e. $0 \le \phi < \pi/2$) rather than in half the two-dimensional plane (i.e. $0 \le \phi < \pi$). We consider both reparameterizations below, but will focus primarily on the mapping of ρ onto $0 \le \phi < \pi/2$.

If we map ρ onto $0 \le \phi < \pi/2$, a test statistic can be constructed that has a desirable power property in the $(\theta_1,\theta_2)'$ parameter space. If we map ρ onto $0 \le \phi < \pi$, we can construct an intuitively appealing test statistic that is identical to the test statistic constructed when ρ is mapped onto $0 \le \phi < \pi/2$. In particular, the two tests are the locally most powerful invariant (LMPI) test of $H_0: \lambda = 0$ against $H_1: \lambda > 0, \rho = 0.5$.

Consider first the mapping of ρ onto $0 \le \phi < \pi/2$. Then θ is written as

$$(\theta_1,\theta_2)' = (\lambda\cos(\rho\pi/2), \lambda\sin(\rho\pi/2))',$$

with $\phi = \rho\pi/2$ determining the angle θ makes with the θ_1-axis. The testing problem becomes one of testing $H_0: \theta = 0$ against $H_A: \theta_1 > 0, \theta_2 \ge 0$. Note that $\rho = 0$ corresponds to the positive θ_1-axis, and as ρ approaches one the vector θ approaches the θ_2-axis. The null hypothesis implies that θ has zero length, while the alternative implies that θ lies in the first quadrant, excluding the θ_2-axis. Using this transformation and (6), the distribution of v is

$$v \sim N[0, \sigma^2\{ I + (\theta'\theta)^{1/2}\, \Omega(2\tan^{-1}(\theta_2/\theta_1)/\pi)\}]. \qquad (9)$$

because $\lambda = (\theta'\theta)^{1/2}$, and $\rho\pi/2 = \phi = \tan^{-1}(\theta_2/\theta_1)$ implies $\rho = 2\tan^{-1}(\theta_2/\theta_1)/\pi$.

The one-sided nature of the reparameterized problem indicates that a multivariate one-sided testing procedure is needed. We could use one-sided versions of the likelihood ratio, Wald or Lagrange multiplier tests. However, these are asymptotic tests and it is not clear how they behave in small samples. In addition, their asymptotic distributions under the null hypothesis are probability mixtures of chi-squared distributions and the degenerate distribution at zero. King and Wu (1990) recently proposed an alternative procedure that provides a test statistic whose exact small sample distribution can be obtained. They constructed a LMMP invariant (LMMPI) test (where invariance is with respect to transformations of the form (7)) which is a multivariate one-sided test of the disturbance covariance matrix in the linear regression model. Within the class of invariant tests of the same size, the LMMPI test maximizes the mean slope of the power curve over all permissible directions from the null hypothesis. A summary of King and Wu's results is given in the appendix. King and Wu show that the LMMPI test is equivalent to the LMPI test in the direction $\theta_1 = \theta_2 > 0$. Setting $\theta_1 = \theta_2$ in (9) gives

$$v \sim N[0, \sigma^2 \{ I + \sqrt{2}\, \theta_1 \Omega(0.5)\}].$$

The LMPI test in the direction $\theta_1 = \theta_2 > 0$, and therefore the LMMPI test, is to reject H_0 for large values of

$$e'\Omega(0.5)e \,/\, e'e \qquad (10)$$

where e is the vector of ordinary least squares (OLS) residuals from (4). This is also the LMPI test of $H_0\!: \lambda = 0$ against $H_1\!: \lambda > 0, \rho = 0.5$.

Exact small sample p-values for this test can be computed using the results in Shively, Ansley and Kohn (1990). They show how to compute p-values in O(n) operations for a large class of invariant test statistics, including (10).

The second reparameterization we consider is the mapping of ρ onto $0 \leq \phi < \pi$. This transformation implies

$$(\theta_1,\theta_2)' = (\lambda\cos(\rho\pi), \lambda\sin(\rho\pi))'.$$

Thus, the testing problem becomes one of testing $H_0: \theta = 0$ against $H_A: \{(\theta_1,\theta_2)': \theta_1 > 0, \theta_2 \geq 0\} \cup \{(\theta_1,\theta_2)': \theta_1 \leq 0, \theta_2 > 0\}$. As before, the null hypothesis implies that the length of θ is 0, while the alternative implies that θ lies somewhere in the first or second quadrant, excluding the negative θ_1-axis and $\theta = 0$. Using this transformation and (6), the distribution of v is

$$v \sim N[0, \sigma^2\{ I + (\theta'\theta)^{1/2}\, \Omega(\tan^{-1}(\theta_2/\theta_1)/\pi)\}]. \qquad (11)$$

This reparameterization is no longer a multivariate one-sided testing problem so we cannot construct a LMMPI test. Recall that for the previous reparameterization, the LMPI test in the direction $\theta_1 = \theta_2 > 0$, which splits the alternative parameter space in half, has desirable power properties. Thus a reasonable test of $H_0: \theta = 0$ against $H_A: \{(\theta_1,\theta_2)': \theta_1 > 0, \theta_2 \geq 0\} \cup \{(\theta_1,\theta_2)': \theta_1 \leq 0, \theta_2 > 0\}$ is the LMPI test in the direction of the θ_2-axis because the θ_2-axis splits the alternative space.

The covariance matrix in (11) does not exist when $\theta_1 = 0$. However,

$$\lim_{\theta_1=\theta_2\to 0} \tan^{-1}(\theta_2/\theta_1) = \pi/2.$$

Thus, we construct the LMPI test of $H_0: v \sim N(0, \sigma^2 I)$ against $H_A: v \sim N[0, \sigma^2\{ I + \sqrt{2}\,\theta_2\Omega(0.5)\}]$. This test is given by (10). Therefore, the tests constructed for our two different reparameterizations are identical.

We conducted an empirical power comparison of our suggested test based on (10) with Shively's (1988) SSEWE test. While it is possible to compute exact small sample powers of our test, the powers of the SSEWE test must be simulated. Therefore, to make the comparisons fair, 1000 repetitions were used to simulate the power of each test against specific values of $(\lambda,\rho)'$ in the context of (1) and (2). For model M1, $k = 1$, $n = 20$, x_t is the U.S. monthly consumer price index for the period October 1977 to May 1979 and $z_t = 1$, $t = 1, \ldots, 20$. For model M2, $k = 2$, $n = 30$, x_t is log of annual income for 1869-1898 and z_t are observations of the remaining regressors in Durbin and Watson's (1951) consumption of spirits example. Both tests are based on ratios of quadratic forms in OLS residuals and, therefore, are invariant to the values of the regression coefficients, which were set to zero, and the error term variance, which was set to one. A step interval length of 0.1 was used to construct the SSEWE test; see Shively (1988).

Table 1, in which our test is denoted as the LMMPI test, provides the estimated powers of the two tests. It is clear that, except for ρ=0.1, our LMMPI test has the same or higher power than the SSEWE test. The differences in power between the two tests are small, especially when ρ=0.1.

Our test has other advantages besides almost always having more power. First it is an exact test and its critical values can be easily computed. The critical values for the SSEWE test can only be approximated, although simulations indicate that the approximation is very good (see Shively, 1988). Secondly, these approximate critical values of the SSEWE test are extremely burdensome to compute compared to the critical values of our test which can be calculated in O(n) operations.

3. Multiple Random Walk Coefficients

Our second example involves testing for two stochastic coefficients generated by random walk processes with correlated innovations. The model for this problem is

$$y_t = x_t\alpha_t + w_t\gamma_t + z_t'\beta + \varepsilon_t \tag{12}$$

$$\alpha_t = \alpha_{t-1} + a_{1t} \tag{13}$$

$$\gamma_t = \gamma_{t-1} + a_{2t} \tag{14}$$

$t = 1, \ldots, n$, where x_t and w_t are independent scalar regressors, z_t is a $k \times 1$ vector of independent regressors, and β is a $k \times 1$ vector of constant coefficients. The disturbances ε_t and $a_t = (a_{1t}, a_{2t})'$ are independent with $\varepsilon_t \sim IN(0, \sigma^2)$ and $a_t \sim IN(0, \sigma^2\Lambda)$ where

$$\Lambda = \begin{bmatrix} \lambda_1^2 & \rho\lambda_1\lambda_2 \\ \rho\lambda_1\lambda_2 & \lambda_2^2 \end{bmatrix} \tag{15}$$

and $|\rho| < 1$. Testing for stochastic variation in the two coefficients is equivalent to testing $H_0: \lambda_1^2 = \lambda_2^2 = 0$ against the alternative $H_A: \lambda^2 = (\lambda_1^2, \lambda_2^2)' \neq 0, |\rho| < 1$. Again, ρ is a nuisance parameter that is present only under the alternative. The model (12), (13) and (14) can be rewritten as

$$y_t = x_t\alpha_0 + w_t\gamma_0 + z_t'\beta + v_t \tag{16}$$

where

$$v_t = \varepsilon_t + \textstyle\sum_{i=1}^{t} x_t a_{1i} + \sum_{i=1}^{t} w_t a_{2i}.$$

In matrix notation, (16) can be written

$$y = (X \;\; W \;\; Z)\begin{pmatrix}\alpha_0 \\ \gamma_0 \\ \beta\end{pmatrix} + v \tag{17}$$

where $W = (w_1, \ldots, w_n)'$ and y, X, Z and v are as defined in (5). It is assumed that $(X \;\; W \;\; Z)$ has full column rank. Note that

$$v \sim N[0, \sigma^2\{I + \lambda_1^2 E_1 + \lambda_2^2 E_2 + \rho\lambda_1\lambda_2 E_3\}]$$

with the (i,j)th element of E_1, E_2 and E_3 given by

$$E_1^{(i,j)} = \min(i,j)\, x_i x_j,$$
$$E_2^{(i,j)} = \min(i,j)\, w_i w_j,$$
$$E_3^{(i,j)} = \min(i,j)\, [x_i w_j + x_j w_i],$$

respectively.

As with the return to normalcy model, we can often use economic arguments to impose the condition $0 \le \rho < 1$ on (15), rather than the weaker condition $|\rho| < 1$, and therefore increase the power of the test when $0 \le \rho < 1$. We will show how to reparameterize the model when $0 \le \rho < 1$. A similar reparameterization can be used when $|\rho| < 1$.

We will reparameterize the problem from the parameter space $(\lambda_1^2, \lambda_2^2, \rho)'$ to a parameter space $\theta = (\theta_1, \theta_2, \theta_3)'$ using a transformation involving polar coordinates. A polar coordinate representation usually specifies the length of the vector, the angle ϕ_1 the vector makes with the $\theta_1\theta_2$-plane, and the angle ϕ_2 the projection of the vector into the $\theta_1\theta_2$-plane makes with the θ_1-axis. We will consider a slightly different, but equivalent, polar coordinate representation.

Because $\lambda_1^2, \lambda_2^2 \ge 0$ and we wish to test H_0: $\lambda_1^2 = 0, \lambda_2^2 = 0$, a reasonable transformation is

$$\theta'\theta = \lambda_1^4 + \lambda_2^4.$$

Therefore, the length of θ is determined by λ_1^2 and λ_2^2. Testing H_0: $\lambda_1^2 = 0, \lambda_2^2 = 0$ is equivalent to testing H_0: $\theta_1^2 + \theta_2^2 + \theta_3^2 = 0$, i.e. testing whether the length of θ is zero. Thus the reparameterized testing problem provides a natural method of testing H_0: $\lambda_1^2 = \lambda_2^2 = 0$ by checking whether $\theta'\theta = \lambda_1^4 + \lambda_2^4 > 0$. Now consider a mapping of $(\lambda_1^2, \lambda_2^2, \rho)'$ into the first quadrant of the $(\theta_1, \theta_2, \theta_3)'$ parameter space. A transformation that accomplishes this is

$$(\theta_1, \theta_2, \theta_3)' = (\lambda_1^2\cos(\rho\pi/2), \lambda_2^2\cos(\rho\pi/2), (\lambda_1^4+\lambda_2^4)^{1/2}\sin(\rho\pi/2))'.$$

The testing problem now becomes one of testing $H_0: \theta = 0$ against $H_A: \theta_1 > 0, \theta_2 > 0, \theta_3 \geq 0$, and King and Wu's (1990) one-sided multiparameter testing technique can be used.

Note that if $\phi_1 = \rho\pi/2$, then ϕ_1 is the angle between the vector $(\theta_1, \theta_2, \theta_3)' = (\lambda_1^2\cos(\rho\pi/2), \lambda_2^2\cos(\rho\pi/2), (\lambda_1^4+\lambda_2^4)^{1/2}\sin(\rho\pi/2))'$ and the $\theta_1\theta_2$-plane. Considering the projection of $(\theta_1, \theta_2, \theta_3)'$ onto the $\theta_1\theta_2$-plane, we see that if ϕ_2 represents the angle the projection makes with the θ_1-axis then λ_1 and λ_2 determine ϕ_2. In particular, $\phi_2 = \tan^{-1}(\lambda_2/\lambda_1)$.

In the context of (17),

$$v \sim N[0, \sigma^2\{I + \mu_1(\theta)E_1 + \mu_2(\theta)E_2 + \mu_3(\theta)E_3\}]$$

when $\theta_1^2 + \theta_2^2 > 0$, where

$$\mu_i(\theta) = \theta_i h(\theta), \qquad i = 1, 2,$$

$$\mu_3(\theta) = 2(\theta_1\theta_2)^{1/2} h(\theta) \cos^{-1}\{h(\theta)^{-1}\} / \pi$$

and

$$h(\theta) = \left\{\theta'\theta / (\theta_1^2 + \theta_2^2)\right\}^{1/2}.$$

Observe that $\mu_i(\theta)$, $i = 1, 2, 3$, are discontinuous functions at $\theta_1 = \theta_2 = 0$. However, upon noting that

$$\lim_{\theta\to 0} \mu_i(\theta) = 0, \qquad i = 1, 2, 3,$$

a heuristically reasonable modification is to use the limiting value of $\mu_i(\theta)$ (i.e. 0) as the function value at $\theta = 0$. The hypothesis testing problem has now been reduced to one that can be handled using King and Wu's (1990) one-sided multiparameter LMMPI test. Since the LMMPI test is equivalent to the LMPI test in the direction $\theta_1 = \theta_2 = \theta_3 > 0$, we set $\theta_1 = \theta_2 = \theta_3$ in the expression $\mu_i(\theta)$, $i = 1, 2, 3$, to give

$$\mu_1(\theta) = \mu_2(\theta) = \theta_1\sqrt{(3/2)} \qquad\qquad \mu_3(\theta) = \theta_1\sqrt{(3/2)}d$$

where $d = 2[\cos^{-1}\{\sqrt{(2/3)}\}]/\pi$. The LMPI test in the direction $\theta_1 = \theta_2 = \theta_3 > 0$, and thus the LMMPI test statistic of $H_0: \theta = 0$ against $H_A: \theta_1 > 0, \theta_2 > 0, \theta_3 \geq 0$, is to reject H_0 for large values of

$$e'(E_1 + E_2 + dE_3)e \,/\, e'e$$

where e is the OLS residual vector from (16). Exact small sample p-values for this test statistic can be computed using the results in Shively, Ansley and Kohn (1990).

4. An Empirical Example

In this section, we consider an empirical example first discussed by Watson and Engle (1985) and also used by King (1987). Watson and Engle used Rosenberg's return to normalcy model to check for stability in an efficient market for gold and silver prices. The underlying model is

$$R_t = \alpha + \beta r_t + \varepsilon_t$$

where R_t is the one-period holding yield on gold or silver, and r_t is the risk free rate of return as measured by the return on 90-day Treasury bills with one week remaining until maturity, and ε_t, $t = 1, ..., n$, are $IN(0, \sigma^2)$ random variables. Let

$$\beta_t - \mu = \rho(\beta_{t-1} - \mu) + a_t$$

where a_t, $t = 1, \ldots, n$ are $IN(0, \lambda\sigma^2)$ random variables independent of ε_t, $t = 1, \ldots, n$. A test for stability involves testing $H_0: \lambda = 0$ against $H_A: \lambda > 0, 0 \leq \rho < 1$. We conducted this test using the test statistic derived in Section 2, i.e. $e'\Omega(0.5)e / e'e$, and Watson and Engle's data set of 208 weekly observations on gold and silver prices over the period 1975-1979.

For gold prices, the value of our test statistic is 32.40 with a corresponding p-value of 0.001. Watson and Engle also obtained a p-value of 0.001 so the two tests strongly reject the null hypothesis of a stable relationship between gold prices and the risk free rate of return. Similarly, for silver prices, the value of our test statistic is 53.45 with a p-value of 0.990 while Watson and Engle obtained a p-value of 0.75. Thus, both tests indicate that there is a stable relationship between silver prices and the risk free rate of return.

5. Conclusion

We have provided a new approach for handling testing problems when a nuisance parameter is present only under the alternative. It involves reparameterizing to a higher dimensional problem that can be handled using existing hypothesis testing techniques. It is often possible to provide a natural reparameterization of the problem in such a way that LMMPI tests, which optimize average power in the neighborhood of the null hypothesis, can be constructed. For the return to normalcy model, our test has better power properties than the computationally inefficient SSEWE test based on Davies' (1977) procedure. In addition, our test is an exact small sample test for which we can compute p-values in O(n) operations.

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Appendix

This appendix provides a summary of King and Wu's (1990) unpublished results.

Suppose X is an $n \times 1$ random vector with density function $f(x|\theta)$ where θ is a $p \times 1$ vector of unknown parameters and we wish to test H_0: $\theta = 0$ against H_A: $\theta \neq 0$ or H_A^+: $\theta > 0$. For testing H_0 against H_A, SenGupta and Vermeire (1986) introduced the class of LMMP unbiased tests. These tests maximize the mean curvature of the power hypersurface in the neighborhood of $\theta = 0$ subject to local unbiased conditions.

King and Wu showed that the LMMP test of H_0 against H_A^+ rejects H_0 for large values of

$$\sum_{i=1}^{p} \partial \ln f(x|\theta)/\partial\theta_i\Big|_{\theta=0};$$

i.e., the sum of the scores evaluated at H_0. They also showed that this test is locally most powerful in the direction given by

$$\theta_1 = \theta_2 = \ldots = \theta_p > 0. \qquad \text{(A.1)}$$

Consider the linear regression model

$$y = Z\beta + u, \qquad u \sim N(0, \sigma^2\Omega(\theta)),$$

where y is an $n \times 1$ vector, Z is an $n \times q$ nonstochastic matrix, β is a $q \times 1$ vector of unknown parameters, Ω is a known $n \times n$ positive definite matrix function, and θ is an unknown $p \times 1$ vector. Without loss of generality, we assume $\Omega(0) = I_n$. Testing $H_0: \theta = 0$ against $H_A^+: \theta > 0$ is invariant to mean and scale transformations. Letting

$$A_i = -\partial\Omega(\theta)/\partial\theta_i\big|_{\theta=0},$$

King and Wu showed that the LMMPI test of H_A^+ rejects H_0 for small values of

$$\sum_{i=1}^{p} e'A_ie/e'e = e'Ae/e'e$$

where $A = \sum_{i=1}^{p} A_i$ and e is the OLS residual vector. This test is also LMPI in the direction given by (A.1).

TABLE 1

Simulated powers for the return to normalcy model

ρ	$\lambda = 1.0$		$\lambda = 0.5$		$\lambda = 0.3$		$\lambda = 0.1$	
	LMMPI	SSEWE	LMMPI	SSEWE	LMMPI	SSEWE	LMMPI	SSEWE
				M1				
0.9	**0.90**	**0.90**	**0.90**	0.88	**0.89**	0.86	**0.74**	0.69
0.7	**0.77**	0.76	**0.75**	0.71	**0.71**	0.67	**0.57**	0.52
0.5	**0.54**	**0.54**	**0.49**	0.46	**0.48**	0.45	**0.35**	0.33
0.3	**0.29**	0.27	**0.26**	0.24	**0.24**	0.22	**0.18**	0.17
0.1	**0.10**	**0.10**	0.08	**0.09**	0.09	**0.11**	**0.08**	**0.08**
				M2				
0.9	**0.89**	0.88	**0.82**	0.80	**0.73**	0.71	**0.46**	0.44
0.7	**0.76**	0.74	**0.68**	0.63	**0.57**	0.53	**0.32**	0.27
0.5	**0.53**	0.50	**0.44**	0.43	**0.36**	0.33	**0.17**	0.14
0.3	**0.28**	0.26	**0.22**	0.21	**0.18**	0.16	**0.12**	0.11
0.1	**0.10**	0.09	**0.09**	**0.09**	0.07	**0.10**	0.06	**0.07**

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