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# A LAGRANGE MULTIPLIER TEST FOR GARCH MODELS

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*Key Words and Phrases : Lagrange multiplier test; linear regression model; GARCH models; singular information matrix.*

## ABSTRACT

This paper extends the Lagrange multiplier (LM) test to testing white noise disturbances against GARCH disturbances in the linear regression model. The resulting LM test for the GARCH alternative is identical to the LM test for an ARCH alternative.

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## 1. INTRODUCTION

In recent years, there has been considerable interest in the autoregressive conditional heteroscedasticity (ARCH) disturbance model introduced by Engle (1982). Since their introduction, the ARCH model and its various generalizations, especially the generalized ARCH (GARCH) model introduced by Bollerslev (1986), have been particularly popular and useful in modelling the disturbance behaviour of the regression models of monetary and financial variables. An extensive survey of the theory and applications of these models is given by Bollerslev, Chou, Jayaraman and Kroner (1990).

To date there has been comparatively little emphasis in the literature on testing for the presence of ARCH disturbances, and in particular, testing for GARCH disturbances. Engle (1982) recommended the use of the Lagrange multiplier (LM) test for ARCH disturbances. Bollerslev (1986) observed that a difficulty with constructing the LM test for GARCH disturbances is that the block of the information matrix whose inverse is required, is singular. This problem is similar to that discussed by Godfrey (1978) who showed a similar difficulty with constructing the LM test of white noise against a full ARMA disturbance process in the context of the linear regression model.

The aim of this paper is to show that a LM test for testing white noise disturbances against GARCH disturbances in the linear regression model can be derived. The resulting LM test is shown to be equivalent to the LM test against ARCH disturbances in the linear regression model.

## 2. THEORY

The linear regression model with GARCH disturbances can be written as

$$y_t = X_t'\gamma + \varepsilon_t, \qquad t = r+1,\dots,n, \tag{1}$$

where $$\varepsilon_t \big| \psi_{t-1} \sim \mathrm{IN}(0, \sigma_t^2), \tag{2}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \tag{3}$$

$$r = \max(p,q),$$

in which $\psi_t$ is the information set available at time t, $X_t$ is a kx1 vector of observations on lagged endogenous and exogenous variables, and $\gamma$, $\alpha$ and $\beta$ are unknown parameter vectors. The GARCH process, (3), can be rewritten as

$$\sigma_t^2 = z_t'v, \tag{4}$$

where $z_t' = (w_t', h_t')$,

$$w_t' = (1, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2),$$

$$h_t' = (\sigma_{t-1}^2, \dots, \sigma_{t-p}^2), \text{ and}$$

$$v' = (\alpha_0, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p).$$

Our interest is in testing whether the error term is white noise against the alternative of a GARCH effect, (4). This testing problem can be parameterized as testing

$$H_o : \alpha_1 = \dots = \alpha_q = \beta_1 = \dots = \beta_p = 0$$

against the alternative

$$H_a : \text{there exists at least one } \alpha_i \text{ and } \beta_j > 0, \text{ for } i = 1,\dots,q, \; j = 1,\dots,p,$$

in the context of (1), (2) and (4).

The log likelihood for (1), (2) and (4) apart from a constant term is

$$L = -1/2 \sum_{t=r+1}^{n} \log \sigma_t^2 - 1/2 \sum_{t=r+1}^{n} \varepsilon_t^2/\sigma_t^2,$$

where $\varepsilon_t = y_t - X_t'\gamma$. Let $s = 1 + q + p$, i.e., the number of parameters in (4), let $\lambda$ be the sx1 score vector with respect to v and let $I$ be the sxs information matrix associated with v. We only need to consider the parameters in v because we can show that the elements in the off-diagonal block of the information matrix associated with $\gamma$ and v are zero.

The LM test against a GARCH(p,q) process, namely testing $H_0$ against $H_a$ in the context of (1), (2) and (4), would normally be

$$LM = \hat{\lambda}' \hat{I}^{-1} \hat{\lambda} \tag{5}$$

where $\hat{\lambda}$ and $\hat{I}$ are the score vector and information matrix, respectively, evaluated under the restricted estimates which in this case are the OLS estimates of (1). This form of the LM test requires $\hat{I}$ to be nonsingular. We can show that for our testing problem,

$$\hat{\lambda} = \frac{1}{2\hat{\sigma}^2} \sum_{t=r+1}^{n} \hat{z}_t \hat{f}_t,$$

where $\hat{f}_t = \left( \frac{e_t^2}{\hat{\sigma}^2} - 1 \right)$,

$$\hat{z}_t' = \left( 1, e_{t-1}^2, \ldots, e_{t-q}^2, \hat{\sigma}^2, \ldots, \hat{\sigma}^2 \right),$$

$e_t$ are the OLS residuals from (1), $\hat{\sigma}^2 = \sum_{t=r+1}^{n} e_t^2 /(n - r)$ is the maximum likelihood estimate of $\sigma^2$, and

$$\hat{I} = \frac{1}{2\hat{\sigma}^4} \sum_{t=r+1}^{n} \hat{z}_t \hat{z}_t'.$$

Furthermore, we can show that $\hat{I}$ is singular. In particular, all the columns (rows) associated with the $\beta$ parameters in $\hat{I}$ are linear combinations of the first column (row) of $\hat{I}$. This implies that $\hat{I}$ has rank (s - p) only. Therefore, as Bollerslev (1986) observed, we cannot derive a LM test for testing white noise disturbances against GARCH disturbances in a linear regression model, using the standard formula.

Fortunately the results of Silvey (1959) and Aitchison and Silvey (1960) allow one to derive a LM test for a testing problem which has a singular information matrix. Their results suggest that if the rank of $\hat{I}$ is (s - p), a test may be based upon the modified LM test statistic

$$LM_m = \hat{\lambda}'[\hat{I} + F_1F_1']^{-1}\hat{\lambda} \tag{6}$$

where $[\hat{I} + F_1F_1']^{-1}$ is a generalized (g)-inverse of $\hat{I}$. $F_1$ is an s x p submatrix of F which is the matrix of partial derivatives of the restrictions imposed under the null hypothesis. The g-inverse of $\hat{I}$ follows directly from Rao (1973, p. 34 problem 5). However the g-inverse of a matrix is generally not unique. In the case of $[\hat{I} + F_1F_1']^{-1}$, Rao and Mitra (1971, Lemma 2.2.4 (ii) p. 21) showed that a quadratic form as in (6) is invariant to the choice of g-inverse if $\hat{\lambda}$ is contained in the column space of $\hat{I}$. For this to be true there must exist a vector r such that $\hat{\lambda} = \hat{I}r$ (see Breusch (1978, p 24). We show in Appendix 1 that $\hat{\lambda}$ is indeed contained in the column space of $\hat{I}$. Therefore, using the Aitchison-Silvey results, one can derive a modified LM test for GARCH disturbances based on (6).

Poskitt and Tremayne (1980) showed that if the restrictions implicit in the null hypothesis are simple exclusion restrictions, the form of F is such that the matrix $F_1F_1'$ is null apart from p ones placed appropriately along the diagonal, that is,

$$F_1F_1' = \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix} \tag{7}$$

where the submatrices on the diagonal are of dimension $(s - p)x(s - p)$ and $p \times p$. Let $\hat{I}$ be partitioned as

$$\begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{bmatrix} \tag{8}$$

where $\hat{I}_{11}$ is the $(s - p)x(s - p)$ block of the information matrix associated with the $\alpha$ parameters. Observe that $\hat{I}_{11}$ is nonsingular.

We can partition the score vector, $\hat{\lambda}$, into

$$\hat{\lambda} = \begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{bmatrix} \tag{9}$$

where $\hat{\lambda}_1$ is the $(s - p)x1$ score vector with respect to $\alpha$ and $\hat{\lambda}_2$ is the px1 score vector with respect to $\beta$. The j th element of the score vector, $\hat{\lambda}_2$, is

$$\frac{\partial L}{\partial \beta_j} = \frac{1}{2} \sum_{t=r+1}^{n} \left( \frac{e_t^2}{\hat{\sigma}^2} - 1 \right), \text{ for } j = 1,\ldots,p,$$

which is zero under the null because

$$\hat{\sigma}^2 = \sum_{t=r+1}^{n} e_t^2 /(n - r).$$

This implies that the score vector, (9), can be written as

$$\begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \hat{\lambda}_1 \\ 0 \end{bmatrix}. \tag{10}$$

Note that the first element of $\hat{\lambda}_1$ which is the score associated with $\alpha_0$ is

$$\frac{\partial L}{\partial \alpha_0} = \frac{1}{2\hat{\sigma}^2} \sum_{t=r+1}^{n} \left( \frac{e_t^2}{\hat{\sigma}^2} - 1 \right)$$

which is also zero.

Applying the Silvey-Aitchison modified LM test, (6), for testing a GARCH(p,q) process, we can show that

$$\hat{\lambda}'[\hat{I} + F_1F_1']^{-1}\hat{\lambda} = \hat{\lambda}_1'[\hat{I}_{11}]^{-1}\hat{\lambda}_1$$

given (7), (8) and (10). This implies that the LM test of the null of white noise against a GARCH(p,q) process, (4), for the disturbance terms in (1) is equivalent to a LM test of white noise against an ARCH(q) process. This LM test has the form

$$LM_{ARCH/GARCH} = 1/2f^{0\prime}\hat{W}(\hat{W}'\hat{W})^{-1}\hat{W}'f^0 \tag{11}$$

where $\hat{W}' = \left(\hat{w}_{q+1}, \ldots, \hat{w}_n\right)$,

$\hat{w}_t' = \left(1, e^2_{t-1}, \ldots, e^2_{t-q}\right)$,

$f^{0\prime} = \left(e^2_{q+1}/\hat{\sigma}^2 - 1, \ldots, e^2_n/\hat{\sigma}^2 - 1\right)$.

This statistic is of the same form as Breusch and Pagan's (1979) LM test for heteroscedasticity in the disturbances of (1). Hillier (1991) showed that this test, in principle at least, is exact. Under normality, which is assumed here, it can be shown that plim $f^{0\prime}f^0/(n - q) = 2$. Thus an asymptotically equivalent statistic to (11) is

$$(n - q)f^{0\prime}W(W'W)^{-1}W'f^0/f^{0\prime}f^0 = (n - q)R^2$$

where $R^2$ is the squared multiple correlation between $f^0$ and W which is also the $R^2$ from the regression of $e^2_t$ on an intercept and q consecutive lagged values of $e^2_t$. Both test statistics have an asymptotic chi-squared distribution with q degrees of freedom under $H_0$.

The equivalence of the LM test for both ARCH and GARCH alternatives in fact can be obtained in a simpler way. The LM test in the form (5) can be considered as testing whether the score vector, $\lambda$, is significantly different from zero at the restricted parameter estimates. Since in our testing problem the score vector associated with the $\beta$ parameter is zero under the null hypothesis, this testing problem reduces to one of testing only whether the score vector with respect to $\alpha$, $\hat{\lambda}_1$, is significantly different from zero. The equivalence of the LM test may also be shown by writing the GARCH (p,q) process, (4), as an ARMA process for $\varepsilon_t^2$ as discussed by Bollerslev (1988) and apply the results of Poskitt and Tremayne (1980) to derive a LM test.

The equivalence of the LM test for both ARCH and GARCH alternatives is similar to Godfrey's (1978) results where he found equivalence in the LM test for testing white noise against AR(p) and MA(q) disturbances. This raise the question of how one would proceed if we reject the null hypothesis of well-behaved disturbances. However, we need to note that the ARCH(q) process is nested within the GARCH(p,q) process. So if we reject the null of white noise disturbances we can proceed by testing ARCH(q) disturbances against GARCH(p,q) disturbances. In the case of AR(p) and MA(q) disturbances, it is more difficult to distinguish between the two since they are non-nested.

## 3. AN EMPIRICAL EXAMPLE

In this section we report the application of the modified LM test to the empirical example considered by Engle and Bollerslev (1986). They estimated the model

$$\log (y_t / y_{t-1}) = \varepsilon_t$$

where $y_t$ are weekly observations on the exchange rate between the U.S.

dollar and the Swiss franc from July 1973 through August 1985 for a total of 632 observations and $\varepsilon_t$ is a disturbance which may follow a GARCH(1,1) process, that is

$$\varepsilon_t/\psi_{t-1} \sim N(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2.$$

If we apply the modified LM test to test for GARCH(1,1) disturbances or equivalently ARCH(1) disturbances in this model, the calculated value of the test statistic is 625.49109. When compared against the ninety-five percentile of the $\chi^2(1)$ distribution, namely 3.841, this test suggests clearly we should reject the null hypothesis of well-behaved disturbances.

## 4. CONCLUSION

This paper shows how a LM test of white noise disturbances against GARCH disturbances in a linear regression model can be derived. In particular, we find that the LM test of white noise disturbances against GARCH(p,q) disturbances in a linear regression model is equivalent to that against ARCH(q) disturbances. This implies that under the null of white noise, the GARCH(p,q) effect and the ARCH(q) effect are locally equivalent alternatives. The result of this paper can also be further extended to derive a LM test for testing the null of an ARCH(q) process against a GARCH($r_1$,q+$r_2$) alternative, where $r_1 > 0$ and $r_2 > 0$, since the inverse of the information matrix is also singular in this testing problem.

## APPENDIX 1

Breusch (1978 p. 24) showed that when $\lambda$ is normally distributed, one can represent $\lambda$ as a singular linear transformation from a smaller dimensioned vector of random variables with a non singular covariance matrix, for example,

$$\lambda = C'x$$

where $x \sim N(0, V)$. If this is the case and $I$ can be written as $I = C'C$, then $\lambda$ is contained within the column space of $C'$ which implies that there exists some vector r for which $\hat{\lambda} = \hat{I}r$.

The score vector $\lambda$ in our testing problem is

$$\lambda = \frac{1}{2} \sum_{t=r+1}^{n} \frac{1}{\sigma_t^2} z_t f_t,$$

where $f_t = \left( \frac{\varepsilon_t^2}{\sigma_t^2} - 1 \right),$

$$z_t' = \left(1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-q}^2, \sigma_{t-1}^2, \ldots, \sigma_{t-p}^2\right),$$

$$I = \frac{1}{2} \sum_{t=r+1}^{n} \frac{1}{\sigma_t^4} z_t z_t' .$$

The score vector $\lambda$ in our testing problem is of the form

$$\lambda = C'x$$

where $c_t = z_t'/(2\sigma_t^2)$ and $x_t = f_t$. Similarly, we can write $I$ as $C'C$. We can show that $E(f) = 0$ and $E(ff') = V$, where V is a nonsingular matrix. Since in our testing problem we can write $\lambda = C'x$ and $I = C'C$, it follows that there exists a vector r such that $\hat{\lambda} = \hat{I}r$.

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