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# THE LOCALLY UNBIASED TWO-SIDED DURBIN-WATSON TEST

by

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## Abstract

An algorithm for constructing locally unbiased two-sided critical regions for the Durbin-Watson test is presented. It can also be applied to other two-sided tests. Empirical calculations suggest that, at least for the Durbin-Watson test, the current practice of using equal-tailed critical values yields approximately locally unbiased critical regions.

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## 1. INTRODUCTION

For many decades, the Durbin-Watson (DW) test has been a standard diagnostic test in econometric regression analysis. Although originally designed as a bounds test, advances in computer hardware and numerical algorithms now allow exact p-values to be calculated as a matter of routine. For example, the SHAZAM computer package (White (1978)) will calculate the DW p-value upon request.

The DW test may either be applied in a one-sided or two-sided manner. It is standard practice for the two-sided test to be applied as two one-sided tests of equal size. For an  $\alpha$ -level test, this involves either applying consecutive  $\alpha/2$ -level tests for positive and negative autocorrelation, respectively, or rejecting the null hypothesis if the calculated p-value is less than  $\alpha/2$  or greater than  $1 - \alpha/2$ . While this is a computationally convenient way to proceed, statistical theory suggests that the criteria of local unbiasedness should be used to construct the critical region. This paper presents a practical algorithm for constructing locally unbiased two-sided critical regions for the DW test. The approach has wider appeal in that it can easily be adopted to any two-sided testing problem with a test statistic whose size and power can readily be calculated. We also report the results of a power comparison designed to discover whether the locally unbiased DW test has advantages over the standard two-sided test.

## 2. THE CONSTRUCTION OF LOCALLY UNBIASED CRITICAL REGIONS

The underlying model is the linear regression model

$$y = X\beta + u \quad (1)$$

where  $y$  is an  $n \times 1$  vector,  $X$  is an  $n \times k$  nonstochastic matrix with full

column rank,  $\beta$  is a  $k \times 1$  vector of unknown parameters and  $u$  is an  $n \times 1$  vector of disturbances suspected of being generated by the stationary first-order autoregressive (AR(1)) process

$$u_t = \rho u_{t-1} + \varepsilon_t \quad (2)$$

in which  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)' \sim N(0, \sigma^2 I_n)$ . The DW test is designed to test  $H_0 : \rho = 0$  and is based on the statistic

$$d = e' A_1 e / e' e = \sum_{t=2}^n (e_t - e_{t-1})^2 / \sum_{t=1}^n e_t^2$$

in which  $e = (e_1, \dots, e_n)'$  is the ordinary least squares (OLS) residual vector from (1) and  $A_1$  is the tridiagonal  $n \times n$  matrix whose off-diagonal elements are  $-1$  and whose main-diagonal elements are  $2$  except for the first and last elements which are  $1$ .

Against  $H_a : \rho \neq 0$ ,  $H_0$  is rejected when  $d < d_1^*$  or  $d > d_2^*$  where  $d_1^*$  and  $d_2^*$  are such that

$$\Pr[d < d_1^* \text{ or } d > d_2^* \mid H_0] = \alpha \quad (3)$$

in which  $\alpha$  is the desired level of significance. Provided  $d_1^* < d_2^*$ , (3) can also be written as

$$\Pr[d < d_1^* \mid H_0] + \Pr[d > d_2^* \mid H_0] = \alpha. \quad (4)$$

Let

$$\Omega(\rho) = \begin{bmatrix} 1 & \rho & \rho^2 & . & . & . & \rho^{n-1} \\ \rho & 1 & \rho & & & & . \\ \rho^2 & \rho & 1 & & & & . \\ . & & & . & & & . \\ . & & & & . & 1 & \rho \\ \rho^{n-1} & . & . & . & . & \rho & 1 \end{bmatrix}$$

denote the covariance matrix of  $u$  generated by (2) up to a scalar constant. Under (1) and (2), we can write (see for example King (1987, p.30))

$$\begin{aligned} \Pr[d < d^*] &= \Pr[\zeta'(\Omega(\rho)^{1/2})'M(A_1 - d^*I_n)M\Omega(\rho)^{1/2}\zeta < 0] \\ &= \Pr\left[\sum_{i=1}^{n-k} \gamma_i \zeta_i^2 < 0\right] \end{aligned} \quad (5)$$

where  $M = I_n - X(X'X)^{-1}X'$ ,  $\gamma_i$  are the eigenvalues of  $M(A_1 - d^*I_n)M\Omega(\rho)$  excluding  $k$  zero roots and  $\zeta_i \sim IN(0, 1)$ . Algorithms for calculating (5) are given or described by Koerts and Abrahamse (1969), Davies (1980), Farebrother (1980, 1990), Palm and Sneek (1984) and Shively, Ansley and Kohn (1989).

The standard DW test against  $H_a$  involves finding  $d_1^*$  and  $d_2^*$  such that

$$\Pr[d < d_1^* \mid H_0] = \alpha/2$$

and

$$\begin{aligned} \Pr[d > d_2^* \mid H_0] &= 1 - \Pr[d < d_2^* \mid H_0] \\ &= \alpha/2. \end{aligned}$$

Let  $\Pi(\rho)$  denote the probability of being in the critical region, i.e.

$$\Pi(\rho) = \Pr[d < d_1^*] + \Pr[d > d_2^*].$$

The critical region for the locally unbiased DW test involves finding  $d_1^*$  and  $d_2^*$  such that (4) holds and

$$\left. \frac{\partial \Pi(\rho)}{\partial \rho} \right|_{\rho=0} = 0. \quad (6)$$

Equation (4) is known as the size condition while (6) is the local unbiasedness condition. Unfortunately we do not have an analytical expression for  $\Pi(\rho)$  but we can calculate its value for any value of  $\rho$

through (5). This suggests that we should use numerical methods to calculate (approximately) the left-hand side of (6).

For sufficiently small  $h > 0$ , the left-hand side of (6) can be approximated by the numerical derivative

$$D(h)' = \left\{ \Pi(h) - \Pi(-h) \right\} / 2h .$$

The problem of finding the locally unbiased critical region involves solving (4) and

$$D(h) = 0 \tag{7}$$

jointly for  $d_1^*$  and  $d_2^*$ . In the calculations reported below, we used a modified version of Koerts and Abrahamse's (1969) FQUAD subroutine to calculate (5) and the IMSL subroutine ZSCNT to solve (4) and (7) simultaneously. The latter subroutine is designed to solve a system of  $q$  homogeneous equations in  $q$  unknowns using the secant method. The value of  $h$  was set to 0.001 because smaller values were found not to affect the resultant solution.

### 3. EMPIRICAL POWER COMPARISON

Given that locally unbiased DW critical values can be constructed, an obvious question is whether the extra computation yields worthwhile improvements in small-sample power. Others (see for example Abrahamse and Koerts (1969) and Koerts and Abrahamse (1969)) have found that there can be considerable asymmetry in the power function of the DW test over  $H_a^+ : \rho > 0$  and  $H_a^- : \rho < 0$ . A possible cause could be local biasedness in which case the locally unbiased version of the test might be a better test. In order to investigate this conjecture, we compared power curves of the two DW critical regions, namely the locally unbiased (LU) and the



equal-tailed (ET) two-sided critical regions.

Let  $a_1, \dots, a_n$  denote the eigenvectors corresponding to the eigenvalues of  $A_1$  arranged in ascending order, i.e.,

$$a_1 = n^{-1/2}(1, 1, \dots, 1)',$$

$$a_i = (2/n)^{1/2}(\cos[\pi(i-1)/2n], \cos[3\pi(i-1)/2n], \dots, \cos[(2n-1)\pi(i-1)/2n])',$$

$$i = 2, \dots, n.$$

Powers of the LU and ET DW tests using exact one per cent critical values calculated as outlined above were computed at  $\rho = -0.9$  (0.1) 0.9 for the following  $n \times k$  X matrices with  $n = 15, 25, 50$  and  $k = 2, 3$  as well as  $k = 4, 5, 6$  for  $n = 25, 50$ .

X1 : The  $k$  regressors are  $a_1, \dots, a_k$ .

X2 : The  $k$  regressors are  $a_1, a_{n-k+2}, \dots, a_n$ .

X3 : The  $k$  regressors are  $a_1, (a_2 + a_n)/2^{1/2}, \dots, (a_k + a_{n-k+2})/2^{1/2}$ .

Each of the X matrices includes  $a_1$  which acts as an intercept term. X1 and X2 are the X matrices which result in the upper and lower bounding distributions, respectively, of the DW statistic and were included in the expectation that they would exhibit extreme behaviour. X3 is known as Watson's (1955) X matrix. OLS is (approximately) least efficient relative to generalised least squares and the DW test has been found to have extremely poor power properties (see for example Tillman (1975) and King (1985)) for this X matrix. One can also show that the distribution of  $d$  for X3 under  $H_0$  is symmetric about 2. Hence upper tail critical values can be obtained by subtracting the corresponding lower tail critical values from 4 and *vice versa*.

Selected calculated critical values for the two versions of the two-sided DW test are presented in Table 1 while selected calculated powers for  $n = 15$ ,  $k = 3$  and  $n = 25$ ,  $k = 6$  are given in Tables 2 and 3.

There is considerable evidence of asymmetry in the power curves, particularly for  $X_1$  and  $X_2$  which have greatest power against  $H_a^-$  and  $H_a^+$ , respectively. In contrast, there is virtually no evidence of bias in the ET test. For  $X_3$ , the ET and LU critical regions coincide exactly. For the remaining  $X$  matrices, forcing local unbiasedness resulted in only very small changes to the ET critical values and hence the power curves for the ET and LU critical regions are almost identical. Any differences are greatest for small  $n$  and large  $k$ , *ceteris paribus*. Furthermore, where there are differences, the LU power curves are typically more asymmetric than their ET counterparts. Our results also demonstrate (see for example Table 3) the extreme influence the design matrix can have on the power of the two-sided DW test.

#### 4. CONCLUDING REMARKS

This paper discusses a practical algorithm for constructing locally unbiased two-sided critical regions for the DW test. The approach has wider appeal in that it can also be adopted to any two-sided testing problem with a test statistic whose size and power can readily be calculated. For the case of the DW test, however, our empirical investigations suggest that the extra computation involved in finding LU critical values is not rewarded by improvements in small-sample power. It seems that the current practice of using ET critical values gives approximately LU critical regions.

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Table 1: Calculated critical values,  $d_1^*$  and  $d_2^*$  for the ET and LU two-sided Durbin-Watson tests at the one per cent level.

k	Test	X1		X2		X3	
		d* <sub>1</sub>	d* <sub>2</sub>	d* <sub>1</sub>	d* <sub>2</sub>	d* <sub>1</sub>	d* <sub>2</sub>
n = 15							
2	ET	0.9732	3.2759	0.7241	3.0268	0.9258	3.0742
	LU	0.9694	3.2726	0.7274	3.0306	0.9258	3.0742
3	ET	1.1537	3.3793	0.6207	2.8463	1.0546	2.9454
	LU	1.1530	3.3788	0.6264	2.8537	1.0546	2.9454
n = 25							
2	ET	1.1267	3.0248	0.9752	2.8733	1.0886	2.9114
	LU	1.1251	3.0233	0.9767	2.8749	1.0886	2.9114
4	ET	1.3230	3.1677	0.8323	2.6770	1.2021	2.7979
	LU	1.3180	3.1636	0.8364	2.6820	1.2021	2.7979
6	ET	1.5607	3.3101	0.6899	2.4393	1.3502	2.6498
	LU	1.5527	3.3042	0.6958	2.4474	1.3502	2.6498
n = 50							
2	ET	1.3385	2.7390	1.2610	2.6615	1.3137	2.6863
	LU	1.3380	2.7386	1.2614	2.6620	1.3137	2.6863
4	ET	1.4253	2.8165	1.1835	2.5747	1.3485	2.6515
	LU	1.4239	2.8151	1.1849	2.5761	1.3485	2.6515
6	ET	1.5215	2.8959	1.1041	2.4785	1.3899	2.6101
	LU	1.5191	2.8938	1.1062	2.4809	1.3899	2.6101

Table 2: Calculated powers of the ET and LU two-sided Durbin-Watson tests for X1, X2 and X3 with  $n = 15$  and  $k = 3$  at the one per cent level.

$\rho$	X1		X2		X3
	ET	LU	ET	LU	ET = LU
-0.9	0.6809	0.6814	0.2275	0.2216	0.0418
-0.8	0.5156	0.5162	0.1992	0.1941	0.0585
-0.7	0.3624	0.3629	0.1665	0.1619	0.0638
-0.6	0.2317	0.2322	0.1283	0.1245	0.0593
-0.5	0.1353	0.1356	0.0908	0.0877	0.0486
-0.4	0.0735	0.0737	0.0590	0.0568	0.0359
-0.3	0.0384	0.0385	0.0357	0.0343	0.0247
-0.2	0.0203	0.0203	0.0207	0.0199	0.0164
-0.1	0.0121	0.0121	0.0127	0.0123	0.0116
0.0	0.0100	0.0100	0.0100	0.0100	0.0100
0.1	0.0127	0.0126	0.0121	0.0124	0.0116
0.2	0.0206	0.0205	0.0201	0.0209	0.0163
0.3	0.0352	0.0351	0.0375	0.0391	0.0243
0.4	0.0580	0.0577	0.0709	0.0735	0.0352
0.5	0.0889	0.0886	0.1287	0.1327	0.0476
0.6	0.1258	0.1254	0.2174	0.2228	0.0586
0.7	0.1639	0.1634	0.3351	0.3414	0.0649
0.8	0.1974	0.1969	0.4676	0.4739	0.0644
0.9	0.2211	0.2206	0.5935	0.5991	0.0578

Table 3: Calculated powers of the ET and LU two-sided Durbin-Watson tests for X1, X2 and X3 with  $n = 25$  and  $k = 6$  at the one per cent level.

$\rho$	X1		X2		X3
	ET	LU	ET	LU	ET = LU
-0.9	0.9075	0.9102	0.2442	0.2368	0.0111
-0.8	0.7907	0.7960	0.2332	0.2264	0.0228
-0.7	0.6226	0.6302	0.2048	0.1985	0.0338
-0.6	0.4288	0.4373	0.1665	0.1609	0.0398
-0.5	0.2550	0.2623	0.1238	0.1191	0.0393
-0.4	0.1325	0.1375	0.0830	0.0795	0.0332
-0.3	0.0621	0.0650	0.0498	0.0474	0.0248
-0.2	0.0278	0.0292	0.0270	0.0256	0.0170
-0.1	0.0135	0.0140	0.0143	0.0137	0.0118
0.0	0.0100	0.0100	0.0100	0.0100	0.0100
0.1	0.0143	0.0137	0.0134	0.0140	0.0118
0.2	0.0269	0.0256	0.0276	0.0291	0.0170
0.3	0.0496	0.0472	0.0615	0.0644	0.0247
0.4	0.0825	0.0791	0.1306	0.1356	0.0331
0.5	0.1231	0.1185	0.2501	0.2574	0.0393
0.6	0.1658	0.1602	0.4187	0.4273	0.0405
0.7	0.2045	0.1983	0.6059	0.6137	0.0356
0.8	0.2345	0.2279	0.7670	0.7727	0.0264
0.9	0.2537	0.2468	0.8757	0.8791	0.0168

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