MONASH 17/90

MONASH UNIVERSITY

AUSTRALIA

THE LOCALLY UNBIASED TWO-SIDED DURBIN-WATSON TEST

Simone D. Grose and Maxwell L. King

Working Paper No. 17/90

November 1990

DEPARTMENT OF ECONOMETRICS

ISSN 1032-3813

ISBN 0 86746 940 4

THE UNBIASED TWO-SIDED DURBIN-WATSON TEST

Simone D. Grose and Maxwell L. King

Working Paper No. 17/90

November 1990

DEPARTMENT OF ECONOMETRICS, FACULTY OF ECONOMICS, COMMERCE & MANAGEMENT

MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA.

THE LOCALLY UNBIASED TWO-SIDED DURBIN-WATSON TEST

by

Simone D. Grose and Maxwell L. King

Department of Econometrics
Monash University
Clayton, Victoria 3168
Australia

Abstract

An algorithm for constructing locally unbiased two-sided critical regions for the Durbin-Watson test is presented. It can also be applied to other two-sided tests. Empirical calculations suggest that, at least for the Durbin-Watson test, the current practice of using equal-tailed critical values yields approximately locally unbiased critical regions.

Address for Correspondence

Maxwell L. King
Department of Econometrics
Monash University
Clayton, Victoria 3168
Australia.

1. INTRODUCTION

For many decades, the Durbin-Watson (DW) test has been a standard diagnostic test in econometric regression analysis. Although originally designed as a bounds test, advances in computer hardware and numerical algorithms now allow exact p-values to be calculated as a matter of routine. For example, the SHAZAM computer package (White (1978)) will calculate the DW p-value upon request.

The DW test may either be applied in a one-sided or two-sided manner. It is standard practice for the two-sided test to be applied as two one-sided tests of equal size. For an α-level test, this involves either applying consecutive $\alpha/2$-level tests for positive and negative autocorrelation, respectively, or rejecting the null hypothesis if the calculated p-value is less than $\alpha/2$ or greater than $1 - \alpha/2$. While this is a computationally convenient way to proceed, statistical theory suggests that the criteria of local unbiasedness should be used to construct the critical region. This paper presents a practical algorithm for constructing locally unbiased two-sided critical regions for the DW test. The approach has wider appeal in that it can easily be adopted to any two-sided testing problem with a test statistic whose size and power can readily be calculated. We also report the results of a power comparison designed to discover whether the locally unbiased DW test has advantages over the standard two-sided test.

2. THE CONSTRUCTION OF LOCALLY UNBIASED CRITICAL REGIONS

The underlying model is the linear regression model

$$y = X\beta + u \tag{1}$$

where y is an $n \times 1$ vector, X is an $n \times k$ nonstochastic matrix with full column rank, β is a $k \times 1$ vector of unknown parameters and u is an $n \times 1$ vector of disturbances suspected of being generated by the stationary first-order autoregressive (AR(1)) process

$$u_t = \rho u_{t-1} + \varepsilon_t \tag{2}$$

in which $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)' \sim N(0, \sigma^2 I_n)$. The DW test is designed to test $H_0 : \rho = 0$ and is based on the statistic

$$d = e'A_1e/e'e = \sum_{t=2}^{n} (e_t - e_{t-1})^2 / \sum_{t=1}^{n} e_t^2$$

in which $e = (e_1, \ldots, e_n)'$ is the ordinary least squares (OLS) residual vector from (1) and A_1 is the tridiagonal $n \times n$ matrix whose off-diagonal elements are -1 and whose main-diagonal elements are 2 except for the first and last elements which are 1.

Against $H_a : \rho \neq 0$, H_0 is rejected when $d < d_1^*$ or $d > d_2^*$ where d_1^* and d_2^* are such that

$$\Pr\left[d < d_1^* \text{ or } d > d_2^* \mid H_0\right] = \alpha \tag{3}$$

in which α is the desired level of significance. Provided $d_1^* < d_2^*$, (3) can also be written as

$$\Pr\left[d < d_1^* \mid H_0\right] + \Pr\left[d > d_2^* \mid H_0\right] = \alpha \, . \tag{4}$$

Let

$$\Omega(\rho) = \begin{bmatrix} 1 & \rho & \rho^2 & . & . & . & \rho^{n-1} \\ \rho & 1 & \rho & & & & . \\ \rho^2 & \rho & 1 & . & & & . \\ . & & & & . & & . \\ . & & & & & 1 & \rho \\ \rho^{n-1} & . & . & . & . & \rho & 1 \end{bmatrix}$$

denote the covariance matrix of u generated by (2) up to a scalar constant. Under (1) and (2), we can write (see for example King (1987, p.30))

$$\Pr\left[d < d^*\right] = \Pr\left[\zeta'(\Omega(\rho)^{1/2})'M(A_1 - d^*I_n)M\Omega(\rho)^{1/2}\zeta < 0\right]$$

$$= \Pr\left[\sum_{i=1}^{n-k} \gamma_i \zeta_i^2 < 0\right] \qquad (5)$$

where $M = I_n - X(X'X)^{-1}X'$, γ_i are the eigenvalues of $M(A_1 - d^*I_n)M\Omega(\rho)$ excluding k zero roots and $\zeta_i \sim IN(0, 1)$. Algorithms for calculating (5) are given or described by Koerts and Abrahamse (1969), Davies (1980), Farebrother (1980, 1990), Palm and Sneek (1984) and Shively, Ansley and Kohn (1989).

The standard DW test against H_a involves finding d_1^* and d_2^* such that

$$\Pr\left[d < d_1^* \mid H_0\right] = \alpha/2$$

and

$$\Pr\left[d > d_2^* \mid H_0\right] = 1 - \Pr\left[d < d_2^* \mid H_0\right]$$
$$= \alpha/2.$$

Let $\Pi(\rho)$ denote the probability of being in the critical region, i.e.

$$\Pi(\rho) = \Pr\left[d < d_1^*\right] + \Pr\left[d > d_2^*\right] .$$

The critical region for the locally unbiased DW test involves finding d_1^* and d_2^* such that (4) holds and

$$\left.\frac{\partial\Pi(\rho)}{\partial\rho}\right|_{\rho=0} = 0 . \qquad (6)$$

Equation (4) is known as the size condition while (6) is the local unbiasedness condition. Unfortunately we do not have an analytical expression for $\Pi(\rho)$ but we can calculate its value for any value of ρ

through (5). This suggests that we should use numerical methods to calculate (approximately) the left-hand side of (6).

For sufficiently small $h > 0$, the left-hand side of (6) can be approximated by the numerical derivative

$$D(h)' = \left\{\Pi(h) - \Pi(-h)\right\} / 2h \ .$$

The problem of finding the locally unbiased critical region involves solving (4) and

$$D(h) = 0 \tag{7}$$

jointly for d_1^* and d_2^*. In the calculations reported below, we used a modified version of Koerts and Abrahamse's (1969) FQUAD subroutine to calculate (5) and the IMSL subroutine ZSCNT to solve (4) and (7) simultaneously. The latter subroutine is designed to solve a system of q homogeneous equations in q unknowns using the secant method. The value of h was set to 0.001 because smaller values were found not to affect the resultant solution.

3. EMPIRICAL POWER COMPARISON

Given that locally unbiased DW critical values can be constructed, an obvious question is whether the extra computation yields worthwhile improvements in small-sample power. Others (see for example Abrahamse and Koerts (1969) and Koerts and Abrahamse (1969)) have found that there can be considerable asymmetry in the power function of the DW test over $H_a^+ : \rho > 0$ and $H_a^- : \rho < 0$. A possible cause could be local biasedness in which case the locally unbiased version of the test might be a better test. In order to investigate this conjecture, we compared power curves of the two DW critical regions, namely the locally unbiased (LU) and the equal-tailed (ET) two-sided critical regions.

Let $a_1,\dots,a_n$ denote the eigenvectors corresponding to the eigenvalues of A_1 arranged in ascending order, i.e.,

$$a_1 = n^{-1/2}(1,1,\dots,1)'$$

$$a_i = (2/n)^{1/2}(\cos[\pi(i-1)/2n],\ \cos[3\pi(i-1)/2n],\dots,$$
$$\cos[(2n-1)\pi(i-1)/2n])',$$
$$i = 2,\dots,\ n\ .$$

Powers of the LU and ET DW tests using exact one per cent critical values calculated as outlined above were computed at $\rho = -0.9\ (0.1)\ 0.9$ for the following $n \times k$ X matrices with n = 15,25,50 and k = 2,3 as well as k = 4,5,6 for n = 25,50.

X1 : The k regressors are $a_1,\dots,a_k$.

X2 : The k regressors are $a_1,\ a_{n-k+2},\dots,a_n$.

X3 : The k regressors are $a_1,\ (a_2+a_n)/2^{1/2},\dots,(a_k+a_{n-k+2})/2^{1/2}$.

Each of the X matrices includes a_1 which acts as an intercept term. X1 and X2 are the X matrices which result in the upper and lower bounding distributions, respectively, of the DW statistic and were included in the expectation that they would exhibit extreme behaviour. X3 is known as Watson's (1955) X matrix. OLS is (approximately) least efficient relative to generalised least squares and the DW test has been found to have extremely poor power properties (see for example Tillman (1975) and King (1985)) for this X matrix. One can also show that the distribution of d for X3 under H_0 is symmetric about 2. Hence upper tail critical values can be obtained by subtracting the corresponding lower tail critical values from 4 and *vice versa*.

Selected calculated critical values for the two versions of the two-sided DW test are presented in Table 1 while selected calculated powers for n = 15, k = 3 and n = 25, k = 6 are given in Tables 2 and 3.

There is considerable evidence of asymmetry in the power curves, particularly for X1 and X2 which have greatest power against H_a^- and H_a^+, respectively. In contrast, there is virtually no evidence of bias in the ET test. For X3, the ET and LU critical regions coincide exactly. For the remaining X matrices, forcing local unbiasedness resulted in only very small changes to the ET critical values and hence the power curves for the ET and LU critical regions are almost identical. Any differences are greatest for small n and large k, *ceteris paribus*. Furthermore, where there are differences, the LU power curves are typically more asymmetric than their ET counterparts. Our results also demonstrate (see for example Table 3) the extreme influence the design matrix can have on the power of the two-sided DW test.

4. CONCLUDING REMARKS

This paper discusses a practical algorithm for constructing locally unbiased two-sided critical regions for the DW test. The approach has wider appeal in that it can also be adopted to any two-sided testing problem with a test statistic whose size and power can readily be calculated. For the case of the DW test, however, our empirical investigations suggest that the extra computation involved in finding LU critical values is not rewarded by improvements in small-sample power. It seems that the current practice of using ET critical values gives approximately LU critical regions.

REFERENCES

Abrahamse, A.P.J. and J. Koerts, 1969, A comparison between the power of the Durbin-Watson test and the power of the BLUS test, Journal of the American Statistical Association 64, 938-948.

Davies, R.B., 1980, Algorithm AS155. The distribution of a linear combination of χ^2 random variables, Applied Statistics 29, 323-333.

Farebrother, R.W., 1980, Algorithm AS153. Pan's procedure for the tail probabilities of the Durbin-Watson statistic, Applied Statist ics 29, 224-227.

Farebrother, R.W., 1990, Algorithm AS256. The distribution of a quadratic form in normal variables, Applied Statistics 39, 294-309.

King, M.L., 1985, A point optimal test for autoregressive disturbances, Journal of Econometrics 27, 21-37.

King, M.L., 1987, Testing for autocorrelation in linear regression models: A survey, in M.L. King and D.E.A. Giles, (eds.), Specification analysis in the linear model, (Routledge and Kegan Paul, London) 19-73.

Koerts, J. and A.P.J. Abrahamse, 1969, On the theory and application of the general linear model, (Rotterdam University Press, Rotterdam).

Palm, F.C. and J.M. Sneek, 1984, Significance tests and spurious correlation in regression models with autocorrelated errors, Statistische Hefte 25, 87-105.

Shively, T.S., C.F. Ansley and R. Kohn, 1989, Fast evaluation of the distribution of the Durbin-Watson and other invariant test statistics in regression, paper presented at the Australasian Meeting of the Economic Society, Armidale.

Tillman, J.A., 1975, The power of the Durbin-Watson test, Econometrica 43, 959-974.

Watson, G.S., 1955, Serial correlation in regression analysis I, Biometrika 42, 327-341.

White, K.J. 1978, A general computer program for econometric methods - SHAZAM, Econometrica 46, 239-240.

Table 1: Calculated critical values, d_1^* and d_2^* for the ET and LU two-sided Durbin-Watson tests at the one per cent level.

k	Test	X1 d_1^*	X1 d_2^*	X2 d_1^*	X2 d_2^*	X3 d_1^*	X3 d_2^*
				n = 15			
2	ET	0.9732	3.2759	0.7241	3.0268	0.9258	3.0742
	LU	0.9694	3.2726	0.7274	3.0306	0.9258	3.0742
3	ET	1.1537	3.3793	0.6207	2.8463	1.0546	2.9454
	LU	1.1530	3.3788	0.6264	2.8537	1.0546	2.9454
				n = 25			
2	ET	1.1267	3.0248	0.9752	2.8733	1.0886	2.9114
	LU	1.1251	3.0233	0.9767	2.8749	1.0886	2.9114
4	ET	1.3230	3.1677	0.8323	2.6770	1.2021	2.7979
	LU	1.3180	3.1636	0.8364	2.6820	1.2021	2.7979
6	ET	1.5607	3.3101	0.6899	2.4393	1.3502	2.6498
	LU	1.5527	3.3042	0.6958	2.4474	1.3502	2.6498
				n = 50			
2	ET	1.3385	2.7390	1.2610	2.6615	1.3137	2.6863
	LU	1.3380	2.7386	1.2614	2.6620	1.3137	2.6863
4	ET	1.4253	2.8165	1.1835	2.5747	1.3485	2.6515
	LU	1.4239	2.8151	1.1849	2.5761	1.3485	2.6515
6	ET	1.5215	2.8959	1.1041	2.4785	1.3899	2.6101
	LU	1.5191	2.8938	1.1062	2.4809	1.3899	2.6101

Table 2: Calculated powers of the ET and LU two-sided Durbin-Watson tests for X1, X2 and X3 with n = 15 and k = 3 at the one per cent level.

ρ	X1		X2		X3
	ET	LU	ET	LU	ET = LU
-0.9	0.6809	0.6814	0.2275	0.2216	0.0418
-0.8	0.5156	0.5162	0.1992	0.1941	0.0585
-0.7	0.3624	0.3629	0.1665	0.1619	0.0638
-0.6	0.2317	0.2322	0.1283	0.1245	0.0593
-0.5	0.1353	0.1356	0.0908	0.0877	0.0486
-0.4	0.0735	0.0737	0.0590	0.0568	0.0359
-0.3	0.0384	0.0385	0.0357	0.0343	0.0247
-0.2	0.0203	0.0203	0.0207	0.0199	0.0164
-0.1	0.0121	0.0121	0.0127	0.0123	0.0116
0.0	0.0100	0.0100	0.0100	0.0100	0.0100
0.1	0.0127	0.0126	0.0121	0.0124	0.0116
0.2	0.0206	0.0205	0.0201	0.0209	0.0163
0.3	0.0352	0.0351	0.0375	0.0391	0.0243
0.4	0.0580	0.0577	0.0709	0.0735	0.0352
0.5	0.0889	0.0886	0.1287	0.1327	0.0476
0.6	0.1258	0.1254	0.2174	0.2228	0.0586
0.7	0.1639	0.1634	0.3351	0.3414	0.0649
0.8	0.1974	0.1969	0.4676	0.4739	0.0644
0.9	0.2211	0.2206	0.5935	0.5991	0.0578

Table 3: Calculated powers of the ET and LU two-sided Durbin-Watson tests for X1, X2 and X3 with n = 25 and k = 6 at the one per cent level.

ρ	X1		X2		X3
	ET	LU	ET	LU	ET = LU
-0.9	0.9075	0.9102	0.2442	0.2368	0.0111
-0.8	0.7907	0.7960	0.2332	0.2264	0.0228
-0.7	0.6226	0.6302	0.2048	0.1985	0.0338
-0.6	0.4288	0.4373	0.1665	0.1609	0.0398
-0.5	0.2550	0.2623	0.1238	0.1191	0.0393
-0.4	0.1325	0.1375	0.0830	0.0795	0.0332
-0.3	0.0621	0.0650	0.0498	0.0474	0.0248
-0.2	0.0278	0.0292	0.0270	0.0256	0.0170
-0.1	0.0135	0.0140	0.0143	0.0137	0.0118
0.0	0.0100	0.0100	0.0100	0.0100	0.0100
0.1	0.0143	0.0137	0.0134	0.0140	0.0118
0.2	0.0269	0.0256	0.0276	0.0291	0.0170
0.3	0.0496	0.0472	0.0615	0.0644	0.0247
0.4	0.0825	0.0791	0.1306	0.1356	0.0331
0.5	0.1231	0.1185	0.2501	0.2574	0.0393
0.6	0.1658	0.1602	0.4187	0.4273	0.0405
0.7	0.2045	0.1983	0.6059	0.6137	0.0356
0.8	0.2345	0.2279	0.7670	0.7727	0.0264
0.9	0.2537	0.2468	0.8757	0.8791	0.0168

MONASH UNIVERSITY

DEPARTMENT OF ECONOMETRICS

WORKING PAPERS

1990

1/90 P. Burridge. "The Functional Central Limit Theorem: An Introductory Exposition with Application to Testing for Unit Roots in Economic Time Series".

2/90 Maxwell L. King and Ping X. Wu. "Locally Optimal One-Sided Tests for Multiparameter Hypotheses."

3/90 Grant H. Hillier. "On Multiple Diagnostic Procedures for the Linear Model."

4/90 Jean-Marie Dufour and Maxwell L. King. "Optimal Invariant Tests for the Autocorrelation Coefficient in Linear Regressions with Stationary or Nonstationary AR(1) Errors."

5/90 Keith R. McLaren. "A Reappraisal of the Neoclassical Approach to Modelling Business Investment."

6/90 Francis Vella. "Non-Wage Benefits in a Simultaneous Model of Wages and Hours: Labor Supply Functions of Young Females."

7/90 Francis Vella. "A Simple Estimator for Simultaneous Models with Censored Endogenous Regressors."

8/90 Nicola J. Crichton and Timothy R.L. Fry. "An Analysis of the Effect of an Offender's Employment Status on the Type of Sentence Chosen by the Magistrate."

9/90 Paramsothy Silvapulle and Maxwell L. King. "Testing Moving Average Against Autoregressive Disturbances in the Linear Regression Model."

10/90 Asraul Hoque and Brett A. Inder. "Structural Unemployment in Australia."

11/90 Maxwell L. King, Chandra Shah and Kees Jan van Garderen. "Tutoring in Economic Statistics: The Monash experience."

12/90 R.D. Snyder. Why Kalman Filter?

13/90 Grant H. Hillier and Christopher L. Skeels. "Some further exact results for structural equation estimators."

14/90 Robert W. Faff, John H.H. Lee and Tim R.L. Fry. "Time stationarity of systematic risk: some Australian evidence."