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by

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ABSTRACT

The Neyman-Pearson Lemma usually implies that the best critical region (BCR) for testing a given hypothesis varies with the alternative. For this reason, most commonly used tests are justified either by replacing the objective of maximum power by a weaker objective such as maximum slope of the power function at the null, or by resorting to some other testing principle, such as a likelihood ratio test. However, BCRs that vary with the alternative may sometimes in fact vary very little, and in this paper I suggest a measure of the variability of such regions. Some examples confirm the conjecture that there are cases where the BCRs vary very little.

Section 4 of the paper applies this idea to the family of one-parameter curved exponential models, and relates it to Efron's [4] notion that inference in highly curved models is likely to be more difficult that it is in uncurved or linear exponential models.

Finally, we discuss testing tactics in problems where it can be shown that the BCRs for the problem vary very little.

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1. INTRODUCTION

For most hypothesis testing problems of practical interest application of the Neyman-Pearson Lemma (NPL) produces the negative conclusion that there is no uniformly best critical region (BCR), because the best region for testing the null hypothesis against a particular point alternative varies with the alternative. This conclusion usually persists even when the problem is reduced to sufficient statistics, is reduced by invariance, or is reduced by imposing similarity.

For such problems it is common to compromise by replacing the (evidently too demanding) objective of maximum power by a weaker objective such as maximum slope of the power function at the null (in the case of one-parameter one-sided alternatives), or (in multi-parameter and/or two sided problems) by some other measure of the local behaviour of the power surface - see, e.g., Cox and Hinkley [3], Chapter 5. Alternatively, one may simply resort to a general testing principle that has reasonable properties in a wide variety of applications, e.g., a likelihood ratio or score test.

In this paper I shall argue that, in certain circumstances, these responses to a negative verdict from the NPL can be inappropriate, in the sense that it may be possible to find a critical region that is very close to being optimal, and may differ from the critical region implied by any of these compromises.

The essence of the argument is that, while it may be technically true that the BCR for a given problem varies with the alternative, it is plausible that, for some problems at least, the BCR does not in fact vary very much. In problems of this type it is reasonable to expect that the problem might admit a solution that, while not uniformly optimal, is close to being so. Conversely, of course, in cases where the BCRs do vary considerably, no single critical region (including those chosen by the compromises mentioned above) will perform well over the whole range of alternatives. Thus, our first objective will be to define a quantitative measure of the variability of the best critical regions for a particular problem. This is the subject of section 2, and in section 3 we apply the ideas introduced in section 2 to a number of examples, including cases where the BCRs vary considerably with the alternative, and also cases where they vary very little.

Two of the examples dealt with in section 3 are instances of models that, after reduction by sufficiency, are members of the class of curved exponential models (Efron [4], Barndorff-Nielson [1]), and in section 4 we examine the variability of BCRs for hypothesis testing problems in general models of this type. The one-parameter (linear) exponential model essentially characterizes the class of models for which standard likelihood-based inference procedures yield neat solutions. This led Efron [4] to the notion of the 'statistical curvature' of a model, and to the conjecture that the adequacy of standard (linear) methods could be related to a measure of that curvature. In section 8 of [4], Efron argues that the difference between the locally best critical region and the BCR for a nearby point alternative can be related to his measure of statistical curvature. That is, Efron uses his curvature measure as a local measure of the variability of the BCRs for testing problems in this context.

Our second purpose in this paper is to examine the *global* behaviour of BCRs for testing problems involving one-parameter curved exponential models, and in section 4 we shall show that the measure of the variability of BCRs introduced in section 2 depends on the global properties of the statistical manifold generated by such a model. This result lends support to Efron's notion that inference is more difficult in models with large curvature, but, because the analysis involves global concepts, rather than local, Efron's (local) curvature measure is not implicated here.

To simplify matters we shall abstract from the problem of nuisance parameters, although we tacitly assume that the problem has already been reduced by sufficiency, invariance, or similarity, so this may not be as serious a restriction as it may at first appear.

2. THE VARIABILITY OF BEST CRITICAL REGIONS

Suppose that the testing problem of interest can be reduced to one involving a random vector x defined on a sample space $\mathcal{X}$, a parameter vector θ defined on a parameter space Θ, and a null hypothesis H_0 that specifies a particular point $\theta_0 \in \Theta$. If the density of x is $pdf_0(x)$ under H_0, and is $pdf_1(x;\theta)$ under the point alternative H_θ, the NPL says that the BCR of size α for testing H_0 against H_θ consists of those values of $x \in \mathcal{X}$ for which

$$pdf_1(x;\theta) > c^*_\alpha(\theta)pdf_0(x), \tag{1}$$

where $c^*_\alpha(\theta)$ is a constant that, in general, depends on θ, and must be chosen so that the size of the test is α.

Frequently the inequality (1) is equivalent to an inequality of the form

$$f(x;\psi(\theta)) > c_\alpha(\psi(\theta)), \tag{2}$$

where $\psi(\theta) \in \Psi$ is some function of θ defined on a space Ψ that may differ from Θ. Clearly, (1) is a version of (2), but, as the examples below will show, the distinction between (1) and (2) can be useful.

For each fixed θ, let

$$\omega_\alpha(\theta) = \{x;\ pdf_1(x;\theta) > c^*_\alpha(\theta) pdf_0(x)\} \cap \mathcal{X}$$

$$= \{x;\ f(x;\psi(\theta)) > c_\alpha(\psi(\theta))\} \cap \mathcal{X} \quad (3)$$

denote the BCR of size α for testing H_0 against H_θ, where $c^*_\alpha(\theta)$ and $c_\alpha(\psi(\theta))$ satisfy $\Pr\{x \in \omega_\alpha(\theta)\} = \alpha$. The question we want to address is whether $\omega_\alpha(\theta)$ varies much with θ, and a natural way to measure this is to use the *union* of the regions (3) as θ varies over the parameter space under the alternative. That is, to consider

$$ER(\alpha) = \{\bigcup_{\theta\in\Theta_1} \omega_\alpha(\theta)\} \cap \mathcal{X}, \quad (4)$$

where $\Theta_1 = \Theta - \{\theta_0\}$. The region (4) clearly contains every critical region of size α for the given problem, and we shall therefore refer to it as the *enveloping region* for the problem.

The next question is how to measure the *size* of the enveloping region $ER(\alpha)$, and here it seems natural to use the probability content of $ER(\alpha)$ under H_0, since this is the usual measure of the size of each of the constituent regions $\omega_\alpha(\theta)$. That is, we shall use the number

$$\delta(\alpha) = \Pr\{x \in ER(\alpha) | H_0\} \quad (5)$$

to measure the size of $ER(\alpha)$. Clearly, $\alpha < \delta(\alpha) < 1$, with a number near α indicating that the regions $\omega_\alpha(\theta)$ vary very little as θ varies, and a number near 1 indicating that $\omega_\alpha(\theta)$ can be almost anywhere in $\mathcal{X}$. For a one-parameter two-sided problem in which a BCR of size α exists for each one-sided alternative, $\delta(\alpha) = 2\alpha$, and it is useful to think of this as a benchmark against which to compare the value of $\delta(\alpha)$ for other problems.

Now, to be able to calculate $\delta(\alpha)$ for any particular problem we need a more explicit definition of the region $ER(\alpha)$. Consider first the simple case where $c^*_\alpha(\theta)$ in (1) (and therefore $c_\alpha(\psi(\theta))$ in (2)) does not depend on θ. We have:

Lemma 1: Let $\omega_\alpha(\theta) = \{x;\ pdf_1(x;\theta) > c_\alpha pdf_0(x)\} \cap \mathcal{X}$, with c_α free of θ, and let

$$LR(c_\alpha) = \{x;\ \sup_{\theta\in\Theta_1} pdf_1(x;\theta) > c_\alpha pdf_0(x)\} \cap \mathcal{X}. \tag{6}$$

Then, $LR(c_\alpha) = ER(\alpha)$. That is, the enveloping region is simply the region determined by the maximum likelihood ratio. Equivalently, using (2),

$$ER(\alpha) = \{x;\ \sup_{\psi\in\Psi} f(x;\psi) > c_\alpha\} \cap \mathcal{X}. \tag{7}$$

Proof: Let $\bar{\omega}_\alpha(\theta)$ denote the complement of $\omega_\alpha(\theta)$ in $\mathcal{X}$, with corresponding meanings for $\overline{ER}(\alpha)$ and $\overline{LR}(c_\alpha)$, so that

$$\overline{ER}(\alpha) = \{\bigcap_{\theta\in\Theta_1} \bar{\omega}_\alpha(\theta)\} \cap \mathcal{X}.$$

If $x \in \overline{ER}(\alpha)$ then $pdf_1(x;\theta) < c_\alpha pdf_0(x)$ for all $\theta\in\Theta$, which implies that $\sup_{\theta\in\Theta_1} pdf_1(x;\theta) < c_\alpha pdf_0(x)$. That is, $x \in \overline{ER}(\alpha) \Rightarrow x \in \overline{LR}(c_\alpha)$, or $x \in LR(c_\alpha) \Rightarrow x \in ER(\alpha)$. Conversely, $\sup_{\theta\in\Theta_1} pdf_1(x;\theta) < c_\alpha pdf_0(x)$ implies $pdf_1(x;\theta) < c_\alpha pdf_0(x)$ for all $\theta\in\Theta_1$, which implies that $x \in \bar{\omega}_\alpha(\theta)$ for all $\theta\in\Theta_1$, i.e., that $x \in \overline{ER}(\alpha)$. That is, $x \in \overline{LR}(c_\alpha) \Rightarrow x \in \overline{ER}(\alpha)$, or $x \in ER(\alpha) \Rightarrow x \in LR(c_\alpha)$. □

A corollary of Lemma 1 is that, under our present assumptions, if a uniformly BCR of size α exists it is given by the critical region for the likelihood ratio test.

As a consequence of Lemma 1, if $c^*_\alpha(\theta)$ and $c_\alpha(\theta)$ in (1) and (2) do not vary with θ, then

$$\delta(\alpha) = \Pr\{\ x \in LR(c_\alpha)\}. \tag{8}$$

Thus, in cases where the critical values do not vary with the alternative, the number $\delta(\alpha)$ is reasonably easy to calculate. Examples 1 and 2 in section 3 are of this type.

In most cases, of course, the density of the statistic $f(x;\psi)$ in (2) will depend on $\psi = \psi(\theta)$, so that the critical value in (2) will be a function of θ. In such cases the problem of defining $ER(\alpha)$, and hence of calculating $\delta(\alpha)$, is more difficult. However, it is often possible to approximate the distribution of a suitably standardized version of $f(x;\psi)$ by a standard distribution (e.g., standard Normal, Chi-square, or perhaps Beta), and in such cases an approximation to $\delta(\alpha)$ that is adequate for the purposes we have in mind here can easily be found.

For instance, let $\mu_0(\psi) = E_0[f(x;\psi)]$ and $\sigma_0^2(\psi) = var_0[f(x;\psi)]$ be the mean and variance of $f(x;\psi(\theta))$ under H_0, and suppose that the statistic

$$g(x;\psi) = (f(x;\psi) - \mu_0(\psi))/\sigma_0(\psi) \qquad (9)$$

is approximately standard normal under H_0. Then, if z_α denotes the $(1-\alpha)$th quantile of the standard normal distribution, $c_\alpha(\theta)$ in (2) is approximately equal to

$$\mu_0(\psi) + z_\alpha \sigma_0(\psi). \qquad (10)$$

A slight modification of the argument in the proof of Lemma 1 then shows that $\delta(\alpha)$ is approximately $Pr\{ x \in ALR(z_\alpha)\}$, where

$$ALR(\alpha) = \{x;\ \sup_{\psi\in\Psi} g(x;\ \psi) > z_\alpha\} \cap \mathcal{X}. \qquad (11)$$

That is, an approximation to $\delta(\alpha)$ can often be obtained by maximizing a suitably standardized version of the likelihood, and then finding the probability that x is in the region thus defined.

3. EXAMPLES

Example 1: Testing Regression Coefficients - Unrestricted Alternatives

Suppose that $y \sim N(X\beta + Z\gamma, \sigma^2 I)$, where y is $n \times 1$, X is $n \times k$, Z is $n \times p$, β is $k \times 1$, and γ is $p \times 1$, and we wish to test H_0: $\gamma = 0$ against an unrestricted alternative H_1: $\gamma \neq 0$. The class of similar tests may be characterized by the statistics

$$b = y'M_xZ(Z'M_xZ)^{-1}Z'M_xy/y'M_xy \qquad (0 < b < 1),$$

with $M_x = I_n - X(X'X)^{-1}X'$, and

$$g = (Z'M_xZ)^{-1/2}Z'M_xy/(y'M_xZ(Z'M_xZ)^{-1}Z'M_xy)^{1/2},$$

so that $g'g = 1$. The vector g lies on the surface of a p-dimensional unit sphere and is a measure of the direction of the OLS estimator, $\hat{\gamma}$, for γ. The joint density of (b, g) (with respect to Lebesgue measure for b and the invariant measure on the unit sphere for g) is given by

$$pdf_1(b,g) = c(m,p)\ b^{p/2-1}(1-b)^{(m-p)/2-1}e^{-\lambda/2} \sum_{j=0}^{\infty} a_m(j)[(\lambda b)^{1/2}g'\mu]^j \qquad (12)$$

where $\lambda = \gamma'Z'M_xZ\gamma/\sigma^2$, $\mu = (Z'M_xZ)^{1/2}\gamma/(\gamma'Z'M_xZ\gamma)^{1/2}$, $m = n-k$, $c(m,p) = [\Gamma(m/2)/2\pi^{p/2}\Gamma((m-p)/2)]$, and $a_m(j) = \Gamma(((j+m)/2)/j!$. Note that μ also lies on the surface of the unit p-sphere and is a measure of the direction of γ under the alternative hypothesis. Under H_0 (12) reduces to

$$pdf_0(b,g) = c(m,p)\ b^{p/2-1}(1-b)^{(m-p)/2-1}, \qquad (13)$$

so that, under H_0, b and g are independent, $b \sim B(p/2,(m-p)/2)$, and g is uniformly distributed on the surface of the unit p-sphere.

Since the infinite series in (12) is a monotone increasing function of $b^{1/2}g'\mu$ for all $\lambda > 0$, the ratio $pdf_1(b,g)/pdf_0(b,g)$ is large just if

$b^{1/2}g'\mu$ is large. Hence, for fixed γ the best similar region (BSR) of size α for testing H_0 against this γ has the form

$$b^{1/2}g'\mu > c_\alpha(\mu), \tag{14}$$

which corresponds to (2) with $x \equiv (b, g)$, $\theta \equiv \gamma$, $\psi(\theta) \equiv \mu$, and $f(x;\psi) \equiv b^{1/2}g'\mu$. Since the region (14) (a subset of $(0,1) \times S_p$, where S_p denotes the surface of the unit p-sphere) depends on the direction of γ, as measured by μ, the BSR of size α varies with μ. However, the critical values $c_\alpha(\mu)$ do not, in fact, vary with μ, so that the regions $\omega_\alpha(\mu)$ are of the form

$$\omega_\alpha(\mu) = \left\{ (b,g);\ b \in (0,1),\ g \in S_p,\ b^{1/2}g'\mu > c_\alpha \right\} \tag{15}$$

where c_α is determined so that $\Pr\{(b,g) \in \omega_\alpha(\mu)\} = \alpha$.

Since μ is unrestricted under the alternative hypothesis, and $\sup_{\mu \in S_p}(g'\mu) = 1$ (when $\mu = g$), it follows from Lemma 1 that the enveloping region has the form

$$ER(\alpha) = \left\{(b,g);\ b \in (0,1),\ g \in S_p;\ b > c_\alpha^2\right\}, \tag{16}$$

and the size of this region is easily evaluated under H_0 because b and g are independent under H_0 and $b \sim B(p/2, (m-p)/2)$. Values of $\delta(\alpha)$ are given in Table 1 for selected values of m and p and $\alpha = .05$.

TABLE 1 ABOUT HERE

Clearly, this is a case where the location of the BSR can vary considerably with the alternative. The test usually used is, of course, the F-test, with critical region of the form $b > c$, where c is chosen so that the size of *this* region is α. On the evidence in Table 1, this region will envelop all BCRs of size α^*, say, only if α^* is very small indeed. In this sense the F-test is, within the class of similar tests, itself a compromise; and a very unsatisfactory one. On the other hand, the F-test can, of course, be shown to be optimal within the smaller class of invariant tests, and this stronger restriction on the class of tests considered might be judged an acceptable compromise.

Example 2: Testing Regression Coefficients: One-Sided Alternatives

With the same assumptions as in example 1, suppose that we wish to test H_0: $\gamma=0$ against the one-sided alternative H_1^+: $\gamma > 0$, where $\gamma > 0$ means $\gamma_i \geq 0$ for $i=1,\ldots,p$, with strict inequality for at least one i. The BSR for fixed γ is again given by (14), but since the direction of γ, μ, is restricted under the alternative, (16) no longer gives the enveloping region for the problem. Instead, the supremum of $g'\mu$ must be obtained subject to the restriction $\gamma > 0$, and this is precisely the problem discussed in Hillier [8].

The results in [8] show that the region $LR(c_\alpha)$ depends on $\rho = z_1'M_xz_2/(z_1'M_xz_1z_2'M_xz_2)^{1/2}$ in the case $p = 2$, and on $\rho_{ij} = z_i'M_xz_j/(z_i'M_xz_iz_j'M_xz_j)^{1/2}$, $i\neq j=1,2,3$, in the case $p = 3$. Equations (7) and (8) in [8] are easily adapted to permit the calculation of $\delta(\alpha) = \Pr\{(b,g) \in LR(c_\alpha)\}$, and some results of such calculations are given in Table 2(a) for the case $p = 2$ (for various values of m and ρ), and in Table 2(b) for the case $p = 3$ (for various values of m and the ρ_{ij}'s).

TABLES 2(a) AND 2(b) ABOUT HERE

It is clear from Tables 2(a) and 2(b) that, as one would expect intuitively, information on the signs of the coefficients being tested can significantly reduce the possible variation in the BSRs for the problem. When $p = 2$ and ρ is large and positive the size of the enveloping region $ER(\alpha)$ is quite close to that of any individual BSR, and the situation is not much worse than that for a one-parameter two-sided problem for any positive ρ. When p = 3 information on the signs of the coefficients is less restrictive on the BSRs, as one might expect on dimension grounds, but again the situation is not much worse than for a one-parameter two-sided problem for any configuration of positive ρ_{ij}'s.

Example 3: Standard Symmetric Multivariate Normal Distribution

Suppose that the $p \times 1$ vectors y_i $(i=1,\ldots,n)$ are independent $N(0,\Sigma(\rho))$ variates, with $\Sigma(\rho) = (1-\rho)I_p + \rho\ell\ell'$, where ℓ is a $p \times 1$ vector of ones, and $-1/(p-1) < \rho < 1$ (so that $\Sigma(\rho)$ is positive definite). It is easy to check that the statistics $V_1 = p^{-1}\Sigma_{i=1}^{n}(\ell' y_i)^2$ and $V_2 = \Sigma_{i=1}^{n} y_i' y_i - V_1$ are minimal sufficient for ρ, are independent of one-another, and that $V_1/(1+(p-1)\rho) \sim \chi^2(n)$, $V_2/(1-\rho) \sim \chi^2(n(p-1))$. Hence, the joint density of V_1 and V_2 is given by:

$$\mathrm{pdf}(V_1,V_2) = [2^{np/2}\Gamma(n/2)\Gamma(n(p-1)/2)]^{-1}(1-\rho)^{-n(p-1)/2}[1+(p-1)\rho]^{-n/2}$$

$$\exp\{-V_1/2(1+(p-1)\rho)-V_2/2(1-\rho)\}\ V_1^{n/2-1}\ V_2^{n(p-1)/2-1} \qquad (17)$$

For the problem of testing H_0: $\rho = 0$ against a specific alternative H_ρ, with $\rho > 0$, the NPL says that the BCR of size α consists of the values of $V_1 > 0$, $V_2 > 0$ for which

$$(p-1)(1-\rho)V_1 - (1 + (p-1)\rho)V_2 > c_\alpha(\rho), \qquad (18)$$

where $c_\alpha(\rho)$ is a constant to be chosen so that the size of the region is α. Since the regions (18) depend on ρ, there is no BCR of size α for testing H_0 against H_ρ.

The critical regions (18) are shown graphically in Figure 1 for the case n = 20, p = 5, α = .05. The region (18) with $\rho = 0$ is the locally best similar region (Gokhale and Sen Gupta [7]), and the critical value

$c_\alpha(0)$ is obtainable from Table 2 in [7]. The regions (18) are bounded below by the lines

$$V_1 = a_\alpha(\rho) + b(\rho)V_2, \tag{19}$$

with intercept $a_\alpha(\rho) = c_\alpha(\rho)/(p-1)(1-\rho)$ and slope $b(\rho) = (1+(p-1)\rho)/(p-1)(1-\rho)$. The structure of the enveloping region is clear from Fig. 1, and it is immediately apparent that the size of ER(.05) in this case is very little more than the size of any single region $\omega_\alpha(\rho)$. In section 4 below we give an approximate formula for the size of ER(α) in this problem, and for other problems with a similar structure, and Table 3 gives some (approximate) values of $\delta(\alpha)$ for various values of p and α. The results in Table 3 confirm the impression in Fig. 1 that $\delta(\alpha)$ is very near α for this problem.

FIGURE 1, FOLLOWED BY TABLE 3, ABOUT HERE

Another point that is evident from Fig. 1 is that the locally BCR is itself a reasonable approximation to ER(α). This explains the satisfactory power characteristics of the locally best test noted by Sen Gupta [11].

Example 4: Pure Autoregressive Process

Assume that the n × 1 vector $y \sim N(0, \Sigma(\rho))$, with

$$\Sigma(\rho) = \{\ (1-\rho^2)^{-1}\rho^{|i-j|},\ i,j = 1,2,\ldots,n\ \},\quad |\rho| < 1,$$

so that $\{y_t\}$ is a stable autoregressive process with parameter ρ. The density of y is

$$\mathrm{pdf}(y;\rho) = (2\pi)^{-n/2}(1-\rho^2)^{1/2}\exp\{-\tfrac{1}{2}[(1+\rho^2)s_1 - 2\rho d + s_2]\} \tag{20}$$

where $s_1 = \Sigma_{t=2}^{n-1}y_t^2$, $s_2 = (y_1^2+y_n^2)$, and $d = \Sigma_{t=2}^{n}y_t y_{t-1}$. Hence, the statistics s_1 and d are minimal sufficient for ρ.

We consider the problem of testing H_0: $\rho = 0$ against H_1^+: $\rho > 0$. Under H_0 the density of y reduces to $(2\pi)^{-n/2}\exp\{-(s_1+s_2)/2\}$, so that the BCR for testing H_0 against the point alternative H_ρ (with $\rho > 0$) consists of those values of s_1 and d for which

$$\{2d - \rho s_1\} > c_\alpha(\rho), \tag{21}$$

where $c_\alpha(\rho)$ must be chosen so that the size of the test is α. Since the regions (21) (in (d,s_1)-space) vary with ρ, there is no uniformly best test for H_0. The regions (21) are evidently the regions above the straight lines $d = c_\alpha(\rho)/2 + (\rho/2)s_1$, with intercept $c_\alpha(\rho)/2$ and slope $\rho/2$.

Now, it is straightforward to check that, under H_0, $E_0(d) = 0$, $var_0(d) = (n-1)$, $E_0(s_1) = (n-2)$, $var_0(s_1) = 2(n-2)$, and $cov_0(s_1,d) = 0$. Hence, the variate $(2d-\rho s_1)$ has mean $-(n-2)\rho$ and variance $2[2(n-1)+(n-2)\rho^2]$ under H_0, and it is easy to see that the higher-order cumulants of $(2d-\rho s_1)$ are $O(n)$ as $n \to \infty$. Hence, the standardized variate $[2d-\rho s_1+(n-2)\rho]/\{2[2(n-1)+(n-2)\rho^2]\}^{1/2}$ is approximately standard normal for large n, and $c_\alpha(\rho)$ in (21) is therefore approximately equal to $z_\alpha\{2[2(n-1)+(n-2)\rho^2]\}^{1/2} - (n-2)\rho$. Using this approximation, the lines (21) are shown in Figure 2 for the case $n = 40$, $\alpha = .05$, and several values of ρ. The enveloping region, $ER(\alpha)$, is also indicated in Fig. 2, and it is again clear that the size of this region, $\delta(\alpha)$, is not much greater than that of any individual BCR. Some (approximate) values of $\delta(\alpha)$ are given in Table 4 for various values of n and α, and again these reveal that the BCRs for this problem vary very little with the alternative.

FIGURE 2, FOLLOWED BY TABLE 4, ABOUT HERE

Example 5: Testing for Non-Sphericity in Regression

Suppose that $y \sim N(X\beta, \sigma^2\Sigma(\theta))$, where X is n×p, and θ (r×1) is such that Σ(θ) is positive definite symmetric. We consider the problem of testing H_0: $\theta = \theta_0$ against some class of alternatives, and assume (without loss of generality) that $\Sigma(\theta_0) = I_n$. The classes of similar tests and tests invariant under the transformations $y \to \gamma_0 y + X\gamma$ ($\gamma_0 > 0$, $\gamma \in R^p$) are both characterized by the statistic (cf. Hillier [9])

$$v = C'y/\{y'CC'y\}^{1/2}, \qquad (22)$$

where C (n×(n-k)) is such that $C'X = 0$ and $C'C = I_m$, with $m = n-k$. Since $v'v = 1$, v is defined on the surface of the unit m-sphere, S_m, and the density of v with respect to the invariant measure on S_m is

$$\mathrm{pdf}_1(v;\theta) = A^{-1}(m)\left|C'\Sigma(\theta)C\right|^{-1/2}[v'(C'\Sigma(\theta)C)^{-1}v]^{-m/2}, \qquad (23)$$

where $A(m) = 2\pi^{m/2}/\Gamma(m/2)$ is the surface content of S_m. Under H_0 (23) reduces to $\mathrm{pdf}_0(v) = A^{-1}(m)$, the uniform distribution on S_m, so that, by the NPL, the BSR of size α for testing H_0 against the point alternative H_θ has critical region

$$\omega_\alpha(\theta) = \{v;\ v \in S_m,\ v'(C'\Sigma(\theta)C)^{-1}v < c_\alpha(\theta)\}, \qquad (24)$$

where $c_\alpha(\theta)$ must be chosen so that the size of the test is α.

For most testing problems of this type (and there are many; e.g., tests for: autocorrelation of first or higher order, moving average errors, heteroscedasticity (of any form), combinations of these, etc.), the regions (24) vary with the alternative, so that there is no uniformly best critical region. Typically the critical values $c_\alpha(\theta)$ in (24) themselves vary, so in order to calculate the size of ER(α) it is again necessary to approximate this critical value.

Thus, let $s(\theta) = v'(C'\Sigma(\theta)C)^{-1}v$. It is easy to show that

$$E_0(s(\theta)) = m^{-1}tr[C'\Sigma(\theta)C]^{-1} = \mu_0(\theta), \text{ say,}$$

and

$$var_0(s(\theta)) = 2[m^{-1}tr(C'\Sigma(\theta)C)^{-2} - m^{-2}(tr(C'\Sigma(\theta)C)^{-1})^2]/(m+2) = \sigma_0^2(\theta),$$

say. The variate

$$s^*(\theta) = (s(\theta)) - \mu_0(\theta)) / \sigma_0(\theta) \tag{25}$$

can be treated as approximately standard normal for large n (cf. Evans and King [6]), so that $c_\alpha(\theta)$ in (24) is approximately $-z_\alpha\sigma_0(\theta)+\mu_0(\theta)$. Note, though, that the density of $s(\theta)$ depends on X, so the accuracy of this approximation may vary with X.

Using this approximation, and applying Lemma 1, the size of the enveloping region $ER(\alpha)$ is approximately $\delta(\alpha) = Pr\{v \in ALR(z_\alpha)\}$, where

$$ALR(z_\alpha) = \{v; v \in S_m, \sup_\theta\{-s^*(\theta)\} > z_\alpha\}, \tag{26}$$

where the supremum is taken over the region of interest under the alternative hypothesis.

As an illustration we have evaluated the (approximate) value of $\delta(\alpha)$ by this method (by simulation) for the case in which $\theta \equiv \rho$, and $\Sigma(\rho)$ is as in example 4 (i.e., for tests for first-order autocorrelation in the linear model). For the case $p = 3$, $n = 40$, $\alpha = .05$, and tests of the hypothesis $\rho = 0$ against the one-sided alternative $\rho > 0$ we find that, as expected, the value of $\delta(\alpha)$ varies with X, but only between about .066 (for strongly seasonal data) and .092 (for highly correlated explanatory variables). Thus, once again the BSRs may be seen to vary little with the alternative.

4. ONE-PARAMETER CURVED EXPONENTIAL MODELS

Examples 3 and 4 in section 3 are examples of one-parameter curved exponential models (see Efron [4], Efron and Hinkley [5], Barndorff-Nielson [1], [2]). In this section we generalize those examples, and consider arbitrary one-parameter exponential models for which the minimal sufficient statistic has dimension $k > 1$. The ideas introduced in section 2 will enable us to shed further light on Efron's [4] conjecture that inference in models with small curvature is, in some sense, more straightforward than it is in models with large curvature.

Suppose that, after reduction by sufficiency, invariance, or similarity, the class of tests of interest is characterized by the $k\times 1$ vector x defined on a sample space $\mathcal{X} \subset R^k$, and assume that the density of x (with respect to some dominating measure on $\mathcal{X}$) has the form

$$\mathrm{pdf}(x;\theta) \;=\; g(x)\,\exp\{\eta_\theta' x - n\kappa(\theta)\}\ , \tag{27}$$

where $\eta_\theta = \eta(\theta)$ is a vector-valued function of the scalar parameter θ, and $\theta \in \Theta$, where Θ is some subset of the real line. We assume that x is minimal sufficient in (27), so that the one-dimensional manifold $\mathcal{M} = \{\eta(\theta);\ \theta \in \Theta\}$ cannot be embedded in a Euclidean space of dimension less than k. The coefficient n attached to the cumulant transform, $\kappa(\theta)$, in (27) is included so that, in the approximations necessary later, x behaves like a sum of independent, identically distributed, random variables. It can be replaced by any coefficient that is $O(n)$ as $n \to \infty$.

We consider the problem of testing H_0: $\theta = \theta_0$ against the alternative that θ is in some subset of $\Theta_1 = \Theta - \{\theta_0\}$ (e.g., $\theta > \theta_0$, or $\theta < \theta_0$), and write $\eta_0 = \eta(\theta_0)$ and $\kappa_0 = \kappa(\theta_0)$. The NPL applied to (27) yields that the BCR of size α for testing H_0 against the point alternative H_θ has critical region

$$\{ x \; ; \; (\eta_\theta - \eta_0)'x > c_\alpha(\theta) \} \cap \mathcal{X} , \tag{28}$$

where $c_\alpha(\theta)$ must be chosen so that

$$\int_{(\eta_\theta - \eta_0)'x > c_\alpha(\theta)} g(x) \exp\{\eta_0'x - n\kappa_0\}(dx) = \alpha \tag{29}$$

For fixed θ the region (28) is the region 'above' the hyperplane $(\eta_\theta - \eta_0)'x = c_\alpha(\theta)$. The normal to the hyperplane, $(\eta_\theta - \eta_0)$, determines its orientation in R^k, and the constant $c_\alpha(\theta)$ determines its location. As θ varies, the orientation of the hyperplane also varies, and this aspect of the variation in BCRs is governed entirely by the behaviour of $(\eta_\theta - \eta_0)$ as θ varies. Changes in the orientation of the hyperplane must, however, be compensated for by changes in its position in such a way as to satisfy equation (29) (see Figs 1 and 2 above).

Intuitively, the geometric structure of the BCRs in (28) suggests that the structure of the enveloping region for the problem is intimately related to the *global* structure of the one-dimensional manifold in R^k generated by η_θ. This contrasts with Efron's [4] discussion of locally most powerful tests where, naturally, only the *local* behaviour of the manifold $\eta(\theta)$ near θ_0 is relevant. To make further progress with the general case, and to explicate this intuition, we essentially need to approximate the dependence of the critical value $c_\alpha(\theta)$ in (28) upon θ.

Let $E_0(x) = n\lambda_0$ and $\mathrm{cov}_0(x) = n\Sigma_0$ denote the mean and covariance matrix of x under H_0, and define

$$z = (n\Sigma_0)^{-1/2}(x - n\lambda_0). \tag{30}$$

It is straightforward to see that, as $n \to \infty$, $z \to N(0, I_k)$ under H_0, so that the standardized variate

$$(\eta_\theta-\eta_0)'(x-n\lambda_0)/\{n(\eta_\theta-\eta_0)'\Sigma_0(\eta_\theta-\eta_0)\}^{1/2} \qquad (31)$$

has a limiting standard normal distribution as $n \to \infty$. Hence, $c_\alpha(\theta)$ in (28) is approximately

$$c_\alpha(\theta) \doteq n\lambda_0'(\eta_\theta-\eta_0) + z_\alpha\{n(\eta_\theta-\eta_0)'\Sigma_0(\eta_\theta-\eta_0)\}^{1/2},$$

and we can rewrite (28) in terms of z as

$$\beta_\theta' z > z_\alpha , \qquad (32)$$

where

$$\beta_\theta = \Sigma_0^{1/2}(\eta_\theta-\eta_0)/\{(\eta_\theta-\eta_0)'\Sigma_0(\eta_\theta-\eta_0)\}^{1/2} \qquad (33)$$

is a point on the surface of the unit k-sphere, S_k.

The region (32) is the region in z-space 'above' the hyperplane $\beta_\theta' z = z_\alpha$, and β_θ in (33) is the unit normal to that hyperplane. As θ varies the orientation of the hyperplane changes, but each such hyperplane is tangential, at a point in the direction of the unit normal β_θ, to the k-dimensional sphere of radius z_α centred at the origin. The situation is illustrated in Figure 3 for the case k = 2: the hyperplanes $\beta_\theta' z = z_\alpha$ are straight lines tangential to a circle centred at the origin of radius z_α.

FIGURE 3 ABOUT HERE

In the case k = 2 the (approximate) size of the enveloping region $ER(\alpha)$ can be deduced quite easily from the geometry of Fig. 3 and the symmetry of the (approximate) distribution of z. For, suppose that β_θ varies (on the circumference of the unit circle) continuously with θ, and traces out a continuous arc, a, of length a as θ varies over the parameter space under the alternative (we shall see shortly that this is so only for tests against one-sided alternatives, but we defer those matters for the moment). Since the probability content of the region outside the circle $z'z = z_{\alpha}^2$ is given by $\Pr\{\chi^2(2) > z_{\alpha}^2\}$, the section of $ER(\alpha)$ that is in the direction of the arc a has probability content (under H_0) approximately equal to $(a/2\pi)\Pr\{\chi^2(2) > z_{\alpha}^2\}$. Also, the symmetry of the approximate $N(0, I_k)$ distribution for z implies that the combined probability content of the remaining two sections of $ER(\alpha)$ is simply α, so that, approximately,

$$\delta(\alpha) \doteq \alpha + (a/2\pi)\Pr\{\chi^2(2) > z_{\alpha}^2\}$$

$$= \alpha + (a/2\pi)\exp\{-z_{\alpha}^2/2\}. \qquad (34)$$

Thus, in the case k = 2, $\delta(\alpha)$ is an increasing function of the length of the arc a. This arc is generated by projecting the points $\Sigma_0^{1/2}(\eta_\theta - \eta_0)$ onto the circumference of the unit circle, so that its length, a, directly reflects the *global* behaviour of the curve $\eta(\theta)$ as θ varies. Equation (34) therefore expresses, for the case k = 2, the fact that the global behaviour of the curve $\eta(\theta)$ has a direct bearing on the variability of the BCRs for testing a point hypothesis in these models. Note, too, that a (and hence $\delta(\alpha)$) may depend on the hypothesis of interest through Σ_0 and η_0.

Now, let $\dot{\eta}_0 = \partial\eta(\theta)/\partial\theta|_{\theta=\theta_0}$, and let

$$\beta_0 = \Sigma_0^{1/2}\dot{\eta}_0/\{\dot{\eta}_0'\Sigma_0\dot{\eta}_0\}^{1/2}. \qquad (35)$$

It is easy to see that, for any k, $\beta_\theta \to \beta_0$ as $\theta \to \theta_0$ from above, and $\beta_\theta \to -\beta_0$ as $\theta \to \theta_0$ from below. Hence, for the case k = 2, and for alternatives $\theta > \theta_0$, a in (34) is the largest angle between β_θ and β_0 as θ varies over the region $\theta > \theta_0$. Correspondingly, for alternatives $\theta < \theta_0$, a is the largest angle between β_θ and $-\beta_0$ as θ varies over $\theta < \theta_0$. In general, therefore (unless this angle is π in both cases), (34) applies only for one-sided alternatives, and becomes (in an obvious notation)

$$\delta(\alpha) \doteq 2\alpha + [(a^+ + a^-)/2\pi]\exp\{-z_\alpha^2/2\} \qquad (36)$$

for two-sided alternatives. The angle a is usually easy to calculate in any particular case.

Example 3 continued:

Here $k = 2$, $\theta \equiv \rho$, $\theta_0 = 0$,

$$\eta_\rho = -\tfrac{1}{2}\begin{pmatrix} [1+(p-1)\rho]^{-1} \\ (1-\rho)^{-1} \end{pmatrix}; \qquad \eta_0 = -\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

$$\lambda_0 = \begin{pmatrix} 1 \\ p-1 \end{pmatrix}; \quad \Sigma_0 = 2\begin{pmatrix} 1 & 0 \\ 0 & p-1 \end{pmatrix}, \text{ and}$$

$$\beta_\rho = \begin{pmatrix} (1-\rho)(p-1)^{1/2} \\ -[1+(p-1)\rho] \end{pmatrix}\{p[1+(p-1)\rho^2]\}^{-1/2}.$$

Hence, $\beta_\rho \to \beta_0 = p^{-1/2}\begin{pmatrix} (p-1)^{1/2} \\ -1 \end{pmatrix}$ as $\rho \to 0$, and $\beta_\rho \to \beta_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ as $\rho \to 1$, so that $a^+ = \cos^{-1}[p^{-1/2}]$. Some approximate values of $\delta(\alpha)$ for the case of alternatives $\rho > 0$, calculated using (34), are given in Table 3 for various values of p and α. For alternatives $-1/(p-1) < \rho < 0$, $\beta_\rho \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $\rho \to -1/(p-1)$, so that $a^- = \sin^{-1}[p^{-1/2}]$.

Example 4 continued:

Here, $k = 2$, $\theta \equiv \rho$, $\theta_0 = 0$,

$$\eta_\rho = -\tfrac{1}{2}\begin{pmatrix} 1+\rho^2 \\ -2\rho \end{pmatrix}; \qquad \eta_0 = -\tfrac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

$$n\lambda_0 = \begin{pmatrix} 0 \\ (n-2) \end{pmatrix}; \qquad n\Sigma_0 = \begin{pmatrix} (n-1) & 0 \\ 0 & 2(n-2) \end{pmatrix}, \text{ and}$$

$$\beta_\rho = \begin{pmatrix} -\rho(n-1)^{1/2} \\ (8(n-2))^{1/2} \end{pmatrix} \{(n-1)\rho^2+8(n-2)\}^{-1/2}.$$

Hence, $\beta_\rho \to \beta_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $\rho \to 0$, and $\beta_\rho \to \beta_1$ as $\rho \to 1$, where

$$\beta_1 = \begin{pmatrix} (n-1)^{1/2} \\ (8(n-2))^{1/2} \end{pmatrix} [9n-17]^{-1/2}.$$

Therefore, $a^+ = \cos^{-1}[(8(n-2)/(9n-17))^{1/2}]$, and the length of the arc a depends only on the sample size, n. It is easy to see that, for alternatives $\rho < 0$, $a^- = a^+$. Table 4 contains some values of $\delta(\alpha)$ for the case of alternatives $\rho > 0$, and for various values of n and α.

In the case $k > 2$ the argument leading to equation (34) can be generalized, but in this case provides an approximation to $\delta(\alpha)$ that is necessarily an overestimate. Essentially this is because the points $\beta_\theta \in S_k$, being the projections of the points $\Sigma_0^{1/2}(\eta_\theta-\eta_0)$ on a one-dimensional manifold in R^k onto S_k, themselves lie on a line on the surface S_k. If we let B be the smallest circular 'cap' on S_k that contains all points β_θ for $\theta > \theta_0$, the points β_θ clearly cannot cover B, so that not every hyperplane tangential to the k-sphere $z'z = z_\alpha^2$ at a point in the direction of B defines a BCR. However, by assuming that this is the case (as it is when $k = 2$) we can clearly obtain an upper bound on $\delta(\alpha)$ that is analogous to (34), and this is probably the best generalization of (34) possible in general. Of course, in any particular case the (approximate) value of $\delta(\alpha)$ can be found by applying Lemma 1 in (32), but this does not seem to lead to a theoretical result analogous to (34).

Assume, then, that B is the smallest circular cap on S_k that contains all points β_θ for $\theta > \theta_0$, and that this cap subtends an angle a at the centre of the unit k-sphere. As in the case $k = 2$, the angle a reflects the global behaviour of the curve $\eta(\theta)$ as θ varies over $\theta > \theta_0$.

The probablility content of the region outside the k-sphere $z'z = z_\alpha^2$ in the direction of B is simply $[C(B)/A(k)]\Pr\{\chi^2(k) > z_\alpha^2\}$, where $A(k) = 2\pi^{k/2}/\Gamma(k/2)$ is the surface content of S_k, and

$$C(B) = A(k-1)\int_0^{a/2} \sin^{k-2}x\, dx$$

is the surface content of the cap B.

The remaining region of interest is the region bounded below by the family of hyperplanes tangential to the k-sphere $z'z = z_\alpha^2$ at points in the direction of the 'rim' of B, and above by the surface of the cone of points in the direction of B but outside the k-sphere $z'z = z_\alpha^2$. A straightforward but tedious calculation yields, for the (approximate) probability content of this region,

$$\sum_{j=1}^{k-1} f_j(a)\Pr\{\chi^2(j) > z_\alpha^2\},$$

where

$$f_j(a) = \left[\binom{k-2}{j-1}\frac{\Gamma(j/2)\Gamma((k-j)/2)}{2\Gamma(1/2)\Gamma((k-1)/2)}\right]\sin^{j-1}(a/2)\cos^{k-j-1}(a/2). \qquad (37)$$

Hence, $\delta(\alpha)$ is approximately given by:

$$\delta(\alpha) \doteq [C(B)/A(k)]\Pr\{\chi^2(k) > z_\alpha^2\} + \sum_{j=1}^{k-1} f_j(a)\Pr\{\chi^2(j) > z_\alpha^2\} \qquad (38)$$

It is not difficult to see that the approximation (38) to $\delta(\alpha)$ is, for fixed α and k, an increasing function of a. As in the case k = 2, this expresses the fact that the variability of the BCRs depends on the global behaviour of the manifold generated by $\eta(\theta)$. In addition, it is clear from (38) that, *ceteris paribus*, $\delta(\alpha)$ is larger the larger is the dimension, k, of the full exponential model within which the curved model (27) is embedded. Thus, (38) provides a rough indication of the impact of two distinct aspects of 'model curvature' on one-parameter hypothesis testing problems: (i) the dimension of the full exponential model needed to accomodate the curved model, and (ii) the curvature, within that full model, of the manifold $\eta(\theta)$, as measured by the angle a.

5. CHOICE OF CRITICAL REGION WHEN BCRs VARY LITTLE

For problems like examples 2-5 in section 3, for which the BCRs of a given size do not vary greatly, the question arises how best to exploit this information - bearing in mind that no uniformly best test exists, so that any procedure chosen must involve some compromise. Two possible approaches suggest themselves here: (a) choosing as the critical region the enveloping region, ER(α), itself, or (b) choosing some subset of ER(α) of size α that is, in some sense, a good approximation to every BCR $\omega_{\alpha}(\theta)$.

The first of these suggestions would entail the compromise of using a critical region whose size, $\delta(\alpha)$, is larger than the originally intended size, α, but which is known to contain every BCR of size α. The power of the test implied by this choice is therefore not less than that of the best test of size α for any alternative. Provided $\delta(\alpha)$ is fairly close to α, and to compromise on size is not a serious matter for the problem at hand, this would seem to be a sensible strategy for such problems. Moreover, this strategy is reasonably easy to implement, at least approximately: if the critical values in (1) and (2) do not depend on θ it is simply the likelihood ratio test carried out at (the possibly unconventional) size $\delta(\alpha)$. If the critical values in (1) and (2) do depend on θ, the procedure can be implemented approximately as discussed in section 2.

Note that, in this approach, we are thinking of α as fixed *a priori*, so that $\delta(\alpha)$, the size of the test actually used, is determined by the problem. The second approach - choosing a subset of ER(α) of size α - would be appropriate for problems in which an increase in size would be unacceptable. This approach can be implemented in a number of ways, and we conclude by briefly discussing three of them: (i) choosing the enveloping region ER(α^*) for BCRs of size $\alpha^* < \alpha$ in such a way as to ensure that $\delta(\alpha^*) = \alpha$; (ii) choosing the BCR of size α corresponding to a particular point alternative $\theta = \theta^*$, i.e., choosing a single $\omega_\alpha(\theta^*)$; (iii) choosing a subset of ER(α) of size α that in some sense is a good approximation to ER(α).

The first of these approaches is a generalization of the approach usually adopted for one-parameter two-sided testing problems in which a BCR exists for each one-sided problem. In that case $\alpha^* = \alpha/2$. It is also essentially the same as the Type I procedure suggested by Roy [10], except that Roy did not impose the condition that $\alpha = \delta(\alpha^*)$ should be close to α^* in suggesting the procedure. Since we are thinking here of fixing α and allowing α^* to be determined by the problem through the relation $\alpha = \delta(\alpha^*)$, and the region ER($\alpha^*$) can claim only to contain all BCRs of size α^*, this condition is clearly important.

Now, normally (though see Stein [12]), $\alpha^* < \alpha$ will imply that ER(α^*) is a subset of ER(α). Thus, this approach can be thought of as a procedure for choosing a subset of ER(α) of size α. Viewed this way, though, it suffers from one striking drawback: ER(α^*) may not contain some points $x \in \mathcal{X}$ that are in every BCR of size α. An example of this is provided by superimposing on Fig. 3 another circle of radius z_{α^*}, with $\alpha^* < \alpha$. It is easy to see that, for a small and $(\alpha-\alpha^*)$ sufficiently large, ER(α^*) excludes some points that are in every $\omega_\alpha(\theta)$. Thus, this strategy is likely to produce a poor approximation to ER(α), and therefore involve an unacceptable compromise in terms of power.

The second approach in this category - choosing the BCR for a particular point θ^* - is often suggested (cf. Cox and Hinkley [3], p. 102, Efron [4], §8). Here there is no compromise on size, but the power of the test is inevitably below that of the BCR $\omega_\alpha(\theta)$ at some points in the parameter space. Clearly, $\omega_\alpha(\theta^*)$ is necessarily a subset of $ER(\alpha)$, and the outstanding problem here is to choose θ^* so that $\omega_\alpha(\theta^*)$ is, in some sense, a good approximation to $ER(\alpha)$. For instance, a reasonable criterion would be to choose θ^* to minimize the maximum probability content of the part of $ER(\alpha)$ that is not in $\omega_\alpha(\theta^*)$ over all alternatives. That is, to choose θ^* so as to minimize

$$\lambda(\theta^*) = \sup_{\theta\in\Theta_1}\left[\Pr\{x \in ER(\alpha)|H_\theta\} - \Pr\{x \in \omega_\alpha(\theta^*)|H_\theta\}\right] \qquad (39)$$

This brings us to the third approach mentioned above, which is essentially just a generalization of the first two approaches in this category. Whatever rule is used to determine θ^*, the second approach discussed above confines attention to those subsets of $ER(\alpha)$ that are themselves BCRs for some θ. This restriction evidently simplifies the problem, but it is not obvious that it is innocuous in terms of power. Thus, we suggest generalizing this approach to allow arbitrary subsets ω of $ER(\alpha)$ of size α, so that in place of the criterion (39) we would have the problem: minimize

$$\lambda(\omega) = \sup_{\theta\in\Theta_1}\left[\Pr\{x \in ER(\alpha)|H_\theta\} - \Pr\{x \in \omega|H_\theta\}\right] \qquad (40)$$

over choices of ω satisfying $\omega \subset ER(\alpha)$ and $\Pr\{x \in \omega|H_0\} = \alpha$. This region, if it can be found, would have the desirable property of minimizing the maximum power lost by choosing a subset of $ER(\alpha)$. Clearly, this minimum loss may be large unless the BCRs for the problem vary little, so the compromise implicit here is again only likely to be acceptable in cases where $\delta(\alpha)$ is close to α. The technical difficulties involved in implementing either (39) or (40) are obviously formidable, and are beyond the scope of the present paper.

Of course, for some problems it may turn out that $\delta(\alpha)$ is very much larger than α (as in example 1 when $p = 4$ or 5), indicating that the BCRs do vary considerably. In such a case no compromise is likely to be satisfactory, and the only remedies likely to simplify the problem are (as in example 1): (i) to restrict the parameter space under the alternative, or (ii) to invoke some additional, and acceptable, restrictions on the class of tests considered. The measure of the variability of BCRs we have proposed has the merit of distinguishing cases of this sort from those in which satisfactory procedures can be devised without further restrictions.

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Table 1

Testing Regression Coefficients: Unrestricted Alternatives

Size of Enveloping Region (α = .05)

	p = 1	2	3	4	5
m = 10	.100	.281	.500	.701	.848
20	.100	.268	.466	.649	.791
30	.100	.266	.457	.635	.776
61	.100	.261	.447	.620	.759

Table 2(a)

Testing Regression Coefficients: One-sided Alternatives

Size of Enveloping Region: p = 2, α = .05

m =	10	20	30	61
ρ = .99	.067	.061	.059	.056
.90	.081	.074	.072	.069
.60	.103	.094	.092	.089
.30	.118	.110	.106	.104
.00	.131	.122	.119	.117
-.30	.145	.135	.132	.130
-.60	.160	.149	.146	.144
-.90	.181	.170	.166	.164
-.99	.195	.183	.180	.178

Table 2(b)

Testing Regression Coefficients: One-sided Alternatives

Size of Enveloping Region: $p = 3$, $\alpha = .05$

ρ_{12}	ρ_{13}	ρ_{23}	m = 20	41	61
.9	.9	.9	.0815	.0808	.0806
.9	.6	.2	.1104	.1091	.1087
.7	.5	.3	.1305	.1286	.1280
.7	.3	.1	.1435	.1412	.1405
.5	.3	.1	.1547	.1521	.1513
.9	−.4	−.4	.1645	.1618	.1610
.3	.2	.1	.1759	.1726	.1717
.7	−.4	−.4	.1878	.1843	.1832
0.0	0.0	0.0	.1964	.1925	.1914
.5	−.5	−.5	.2174	.2130	.2116
.9	.9	−.9	.2346	.2297	.2283
.5	−.7	−.7	.2527	.2471	.2454
−.5	−.3	−.1	.2545	.2488	.2471
−.6	−.4	−.1	.2756	.2691	.2672
−.4	−.4	−.4	.2864	.2795	.2775
−.7	−.6	−.1	.3348	.3261	.3236

Table 3

Symmetric Multivariate Normal Model: Approximate values of $\delta(\alpha)$

p	=	2	3	4	5
$\delta(.01)$	=	.0183	.0202	.0211	.0218
$\delta(.05)$	=	.0823	.0894	.0931	.0955
$\delta(.10)$	=	.1550	.1669	.1733	.1775

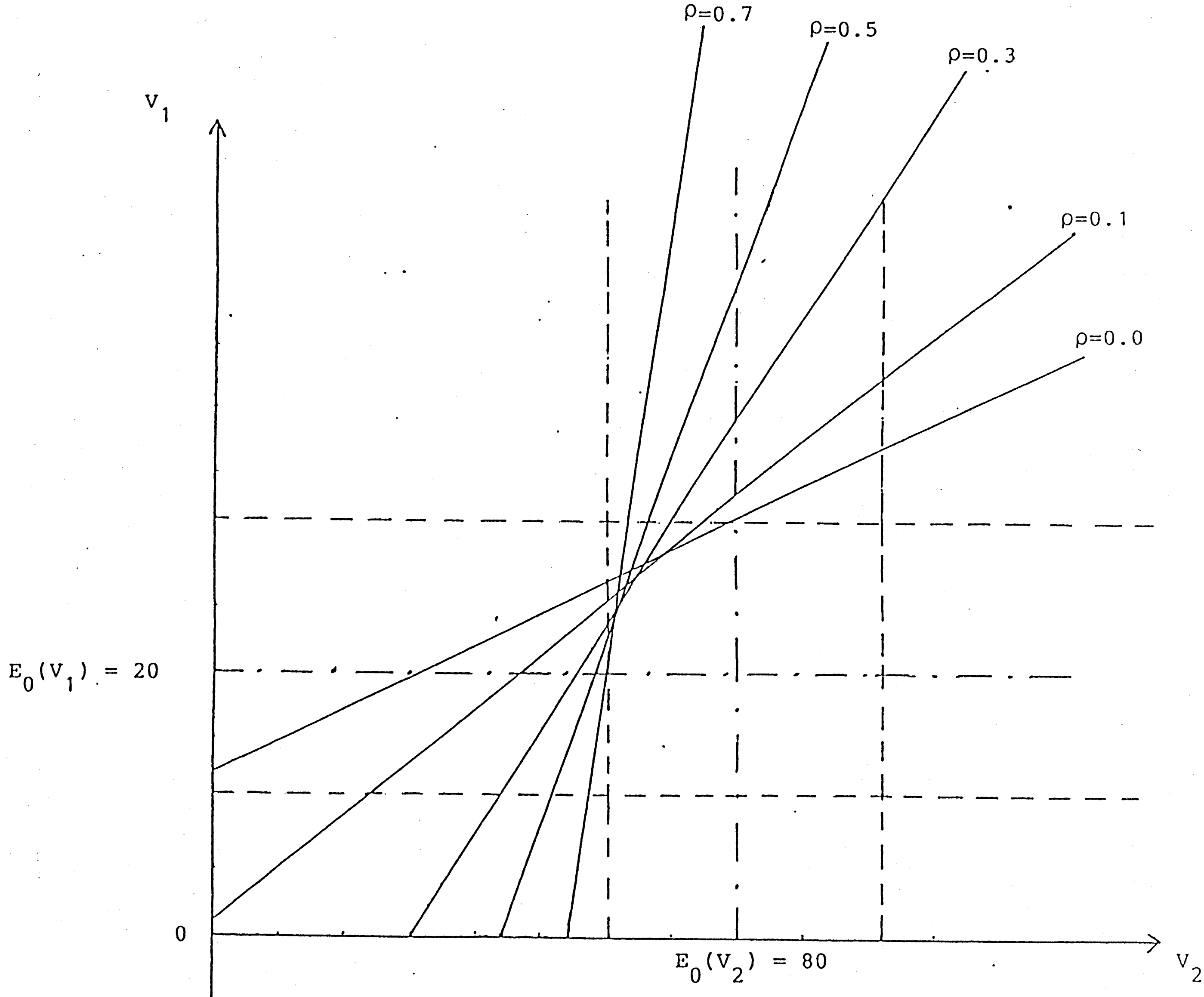
Table 4

Pure autoregressive process: Approximate values of $\delta(\alpha)$

n	=	20	40	60	100
$\delta(.01)$	=	.0137	.0136	.0136	.0136
$\delta(.05)$	=	.0643	.0641	.0640	.0640
$\delta(.10)$	=	.1244	.1241	.1240	.1240

Figure 1

Critical and Enveloping Regions; Symmetric Multivariate Normal Model: $n = 20$, $p = 5$, $\alpha = .05$



The critical region for each ρ is the region above the line indicated. Dashed lines are the 5% and 95% critical values for V_1 and V_2.

Figure 2

Critical and Enveloping Regions for tests of an Autoregressive Parameter: $n = 40$, $\alpha = .05$

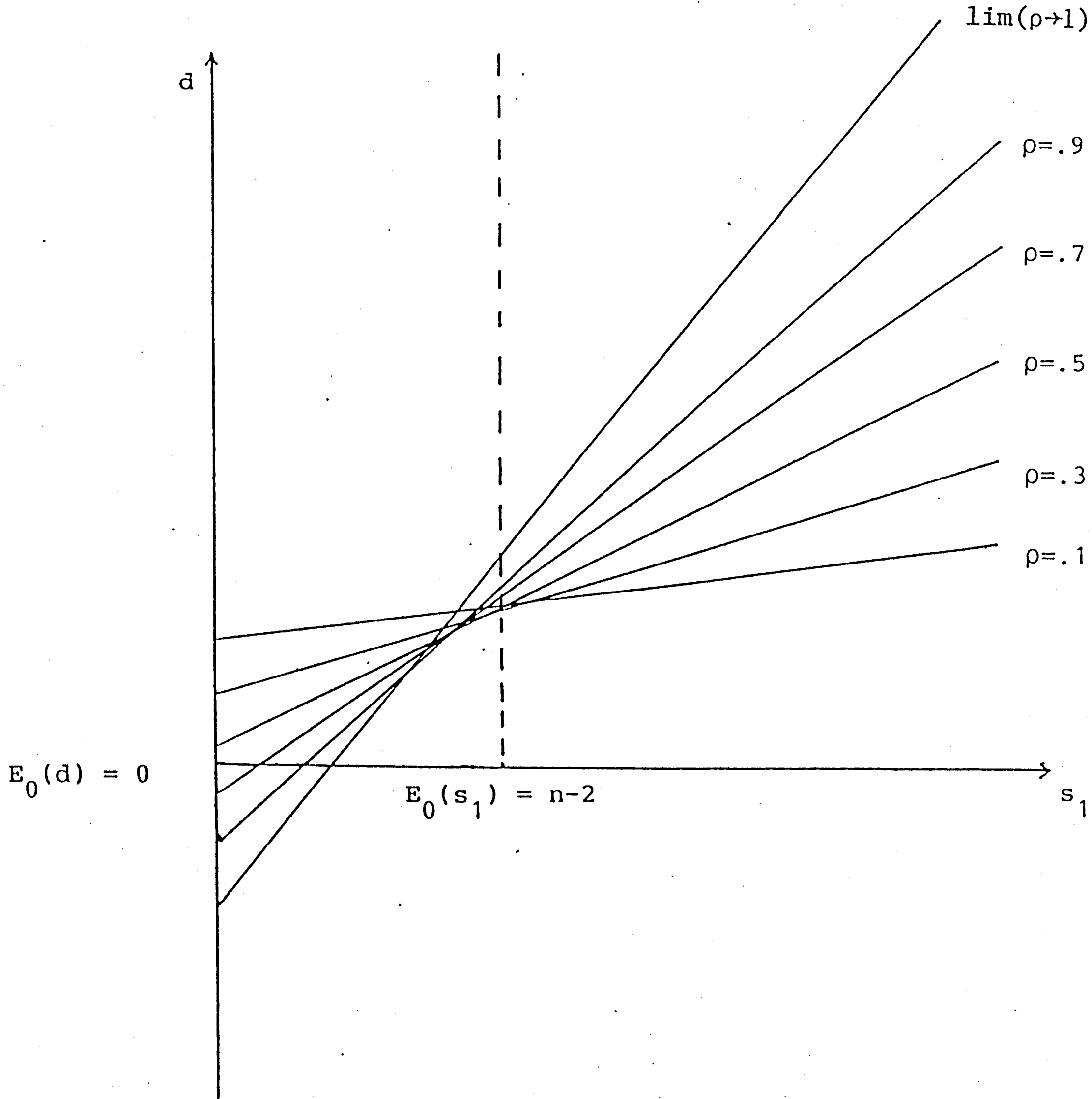
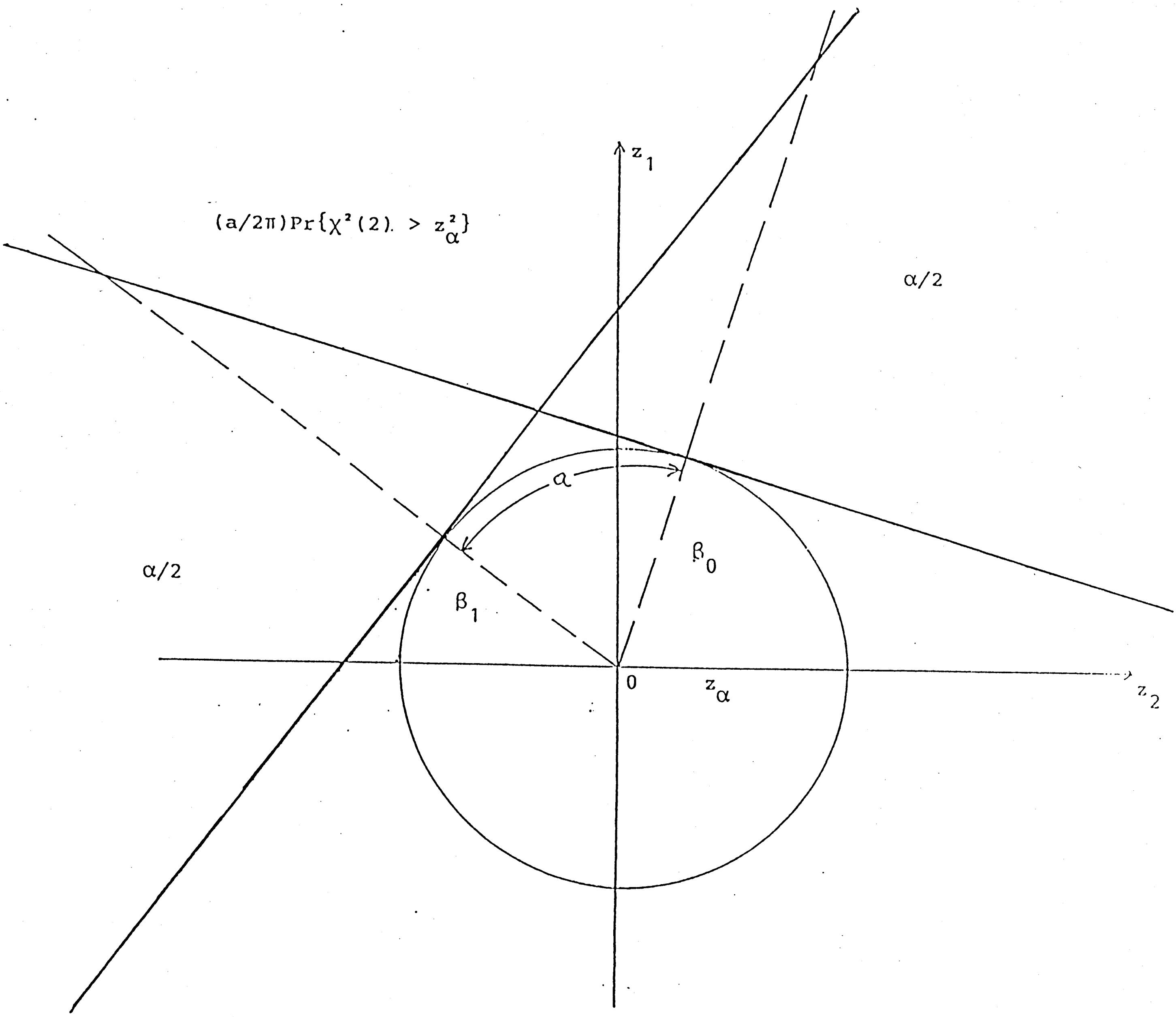


Figure 3

Approximate critical and enveloping regions in terms of standardised variables for Curved Exponential Models: $k = 2$.



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