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DUALITY AND ELASTICITIES OF SUBSTITUTION I: THEORETICAL CONSIDERATIONS*

by J. VAN ZYL**

ABSTRACT

An alternative approach to the estimation of functional specifications from physical input data, involves estimation of factor share equations from cost data. This is done by using the duality that exists between the production function and the cost function along the expansion path.

The basic dual input definition of elasticity of substitution renders a number of alternative concepts possible. The Allen, Morishima and Shadow measures are particularly useful.

Although specific production functions frequently embody assumptions that may have important disadvantages with respect to the substitutability of inputs, the translog specification represents relaxations of these maintained hypotheses.

The translog functional specification is thus the production function of choice for empirical estimation of elasticities of substitution between input pairs when little information about the production process other than cost data, is available.

INTRODUCTION

Elasticity is the ratio between the proportional change in one variable and proportional change in another. It is no more than a useful descriptive summary of the characteristics of a relationship between two variables, and might be described as the derivative of one natural log with respect to another.

The elasticity of substitution is a pure number that indicates the extent to which one input substitutes for another (Henderson and Quant, 1971). If there are two inputs, X_1 and X_2, the elasticity of substitution between X_1 and X_2 is usually defined as:

E_s = % change in (X_2/X_1)/% change in $MRS_{X_1X_2}$ .... (1)

The elasticity of substitution provides an indication of the shape of an isoquant. A large elasticity of substitution indicates that the entrepreneur has a high degree of flexibility in dealing with input price variation. If a high elasticity of substitution exists between a pair of factors, the manager can quickly adjust the input mix in response to changing relative prices. However, if the elasticity of substitution is small, the input mix can hardly be altered even in the face of large relative shifts in prices. The extent to which a farmer adjusts the input mix to changing relative prices thus indicates the magnitude of the elasticity of substitution between input pairs (Hicks, 1932; Varian, 1978). Technological change which increases the elasticity of substitution between input pairs will give farmers additional flexibility in dealing with input price variation.

DUALITY CONCEPTS

General

As an alternative to the estimation of functional specifications from physical input data, a contemporary approach frequently involves estimation of factor share equations from cost data (Fuss, McFadden & Mundlak, 1978). This is an advantage for agricultural economics research in that cost data are usually more readily available than physical input data, and frequently also more reliable.

Production functions have corresponding dual cost functions or perhaps correspondences. The term dual as used in this context means that all of the information needed to obtain the corresponding cost function is contained in the production function, and, conversely, the cost function contains all of the information needed to derive the underlying production function. Cost functions are usually expressed in monetary rather than in physical terms.

Single input cost functions are not normally thought of as arising from an optimalisation procedure. However, it is widely accepted that any point on a single input production function represents a technical maximum output (Y) for the specific level of input use (X) associated with the point. Each point on the inverse cost function is optimal in the sense that it represents the lowest cost method of producing the specific amount of output at the chosen point. However, if the underlying production function is not always monotonically increasing, and as a result the dual is a correspondence, a point on the dual cost correspondence is not necessarily a least cost point for the chosen level of output. For example, if the production function is the familiar neoclassical three-stage production function, the resultant cost is a correspondence, but not a function, for two values of X exist for at least some values for Y.

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*This article, for some parts, makes use of two papers, one by Fuss, McFadden and Mundlak (1978) and another by McFadden (1978), from a book "*Production Economics: A Dual Approach to Theory and Application*" edited by Fuss and McFadden (1978). It also relies on a publication by Devertin and Pagoulatos (1985)

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In a multifactor setting, the duality of the production function and the corresponding cost function becomes more complicated. Suppose that a production function for an output Y is given by Y = f(X), where X = a vector of inputs treated as variable. The corresponding dual cost function exists under the following specific set of conditions (McFadden, 1978:8):

- Marginal products of the inputs are non-negative. This implies free availability of inputs.
- Marginal rates of substitution (MRS) between input pairs are non-increasing. In the dual factor case this implies that $d(dX_2/dX_1)/dX_1$ is non-positive.

These assumptions imply that isoquant maps consisting of concentric rings are ruled out, and that positive slopes on isoquants are not allowed. Each isoquant is weakly convex to the origin.

The cost function that corresponds to the production function is $C(Y;P) = \min (P^1X:f(X)Y)$. If the assumptions or conditions mentioned above are met, then this minimum cost function that corresponds to the production function displays the following characteristics: (a) it exists; (b) it is non-increasing for each price in the input price vector; (c) it is continuous; (d) it is homogeneous to degree one in all variable input prices; and (e) it is concave in each input price for a given level of output (Y*). Detailed proofs of statements (a)-(e) are provided by McFadden (1978:10-13).

The isoquant maps needed for the existence of a corresponding dual cost function are not necessarily more plausible in an applied setting than other isoquant maps, but are rather a matter of mathematical convenience.

Consider a particular class of production functions known as homothetic production functions, which include both homothetic production functions and monotonic transformations of homogeneous production functions. A key characteristic of the homothetic production functions is that a line of constant slope drawn from the origin of the corresponding isoquant map will connect points of constant slope. Hence, homothetic production functions have linear expansion paths. Moreover, any isocline drawn from the origin will have a constant slope. An isocline of constant slope represents all points in which the ratio of the inputs remains fixed or constant, and can be referred to as a factor beam (Beattie and Taylor, 1985:42).

Now consider the factor beam for the homothetic production function representing the expansion path, or at least lowest cost combination of inputs. The production surface arising above the expansion path represents the production function for the use of the optimal input bundle as defined by the lowest expansion path conditions. Therefore, every point on the production surface directly above the expansion path is optimal in that it represents the minimum cost of producing a given level of output.

The cost function that is dual to the production function represented by the expansion path conditions along the factor beam can be obtained by making use of expansion path conditions. Detailed elicitation procedures can be found in Debertin and Pagoulatos (1985:8-11).

Any point on the dual cost function representing a particular quantity of output is optimal in the sense that it represents the minimum cost, or least cost combination of inputs to produce that output. However, at most only one point on the dual cost function represents universal optimality where the marginal cost of producing the incremental unit of output, using the least cost combination of factors, is exactly equal to the marginal revenue obtained from the sale of this incremental unit of output.

If total product is increasing at a decreasing rate along the expansion path, then costs are increasing at an increasing rate. If total product along the expansion path is increasing at an increasing rate, then costs are increasing at a decreasing rate. If total product along the expansion path is increasing at a constant rate, then costs are also increasing at a constant rate. If the product sells for a fixed price, that price is a constant marginal revenue (MR). Marginal revenue (MR) can be equated to the least cost marginal cost (MC*) only if MC* is increasing. With fixed input prices and elasticities of production, this can happen only if the cost elasticity is greater than one, which means that the function coefficient for the underlying production function is strictly less than one.

Duality theorems

The two most important theorems relating to duality are Hotelling's lemma and Shephard's lemma (Beattie and Taylor, 1985:Ch.6). Both are specific applications of a mathematical theorem known as the envelope theorem.

Shephard's lemma states that for the cost function arising from expansion path conditions with respect to change in the price of the i-th factor, evaluated at any particular point (output level) on the least cost total cost function, the change in cost is equal to the least cost quantity of the i-th factor that is used (Shephard, 1970).

Hotelling's lemma makes use of the envelope theorem with respect to profit, rather than cost functions, and can be applied to the product supply or to the factor demand side. When applied to product supply, Hotelling's lemma states that the change in the indirect profit function arising from the expansion path with respect to the k-th product price is equal to the optimal quantity of the k-th output that is produced. When applied to the factor demand side, the lemma states that the change in the indirect profit function with respect to a change in the k-th factor price is equal to the negative of the optimal quantity of the k-th input as indicated by expansion path conditions.

Hotelling's and Shephard's lemmas are of considerable importance for empirical research. If the undertaking is operating according to the assumptions embodied in the expansion path conditions on both the factor and product sides, then product supply and factor demand equations can be obtained without any need for estimating the production function from physical input data.

ELASTICITIES OF SUBSTITUTION

A number of alternative definitions for the elasticity of substitution are possible. Each definition can be based on constant output cost or marginal cost (Fuss, McFadden & Mundlak, 1978:241). Furthermore, each alternative can also be evaluated assuming that prices of the remaining inputs remain constant. The other quantities of inputs can also be held constant or allowed to vary, thereby generating short and long-run elasticity of substitution estimates.

In the dual factor case, the elasticity of substitution will lie between zero and plus infinity. However, if there are more than two inputs, some input pairs may be complements to each other, thus leading to a potential negative elasticity of substitution between some input pairs. The definition of an elasticity of substitution in an n-factor case is futher complicated because a series of specific assumptions must be made with respect to the prices and input levels for those factors of production not directly involved in the elasticity of substitution calculations, and the calculated elasticity of substitution between inputs i and j will vary depending on these assumptions.

The "usual" definition of elasticity of substitution is attributed to Hicks (1932) and can be generalized to the n-factor case such that:

$$O_{ij} = [d \ln(X_j/X_i)] / [(d \ln (P_i/P_j)] = (d \ln X_j - d \ln X_i) / (d\ln P_i/d \ln P_j) \quad \ldots \quad (2)$$

Equation 2 is sometimes referred to as the two output-two price elasticity of substitution (TTES) (Fuss, McFadden and Mundlak, 1978; Ball and Chambers, 1982). Allen (1983:504) uses Hicksian (1932) definition of the elasticity of substitution, but also develops an alternative measure of his own which is linked to the own and cross price constant output factor demand elasticity.

$$O_{ij}^A = S_j E_{ij} \quad \ldots \quad (3)$$

where: S_j = the share of total cost attributable to the j-th input (P_jX_j/C^*)
and $E_{ij} = (d \ln X_i)/d \ln P_j)$ evaluated at constant output.

The Allen elasticity of substitution (AES) is of the one output-one input (OOES) variety (Ball and Chambers, 1982).

The AES concept forms the basis for still other elasticity of substitution concepts. For example, the Morishima elasticity of substitution (MES) (Koizumi, 1976) is an example of TOES (Two output-one input elasticity of substitution) and is defined in terms of the AES as:

$$O_{ij}^M = S_j (O_{ij}^A - O_{jj}^A) = E_{ij} - E_{jj} \quad \ldots \quad (4)$$

The Shadow elasticity of substitution (SES) (McFadden, 1963) is an example of TTES, and is therefore closer to the original Hicksian definition than is the Morishima (MES) or Allen (AES) definitions. SES allows all inputs not involved in the calculation to vary, and can thus be thought of as a long-run elasticity of substitution. SES can be expressed in terms of the Allen measure (AES) as:

$$O_{ij}^S = [(S_iS_j)/(S_i + S_j)] [2 O_{ij}^A - O_{ii}^A - O_{jj}^A] \ldots (5)$$

ELASTICITIES OF SUBSTITUTION AND FUNCTIONAL SPECIFICATION

General

Specific production functions frequently embody assumptions related with the functional form. Fuss, McFadden and Mundlak (1978) refer to these assumptions as maintained hypotheses. These maintained hypotheses are frequently not explicitly recognized by researchers, but do impose constraints on the possible outcomes that can be generated by the analysis.

An example of a maintained hypothesis is the assumption with regard to Hicksian elasticity of substitution that exists between input pairs when a Cobb-Douglas type functional form is chosen to represent the production process. The TTES for any functional form of the Cobb-Douglas type is equal to 1,0 as a maintained hypothesis (Henderson and Quandt, 1971: Ch 3; Debertin and Pagoulatos, 1985:27). This holds even if the production function is not linearly homogeneous, and the partial production elasticities sum to a number other than one (1,0). It can moreover be easily shown that the relationship holds for any factor pair if the function contains more than two inputs.

A maintained hypothesis that the elasticity of substitution between labour and capital is one (1,0) may be tolerable in a 19th-century study dealing with a production process representing the output of a society and utilizing capital and labour as inputs. As is empirically shown (Debertin & Pagoulatos, 1985), it is clearly intolerable in a study conducted in the 1980's dealing with substitutability between inputs.

Since the original Hicks (1932) and Allen (1938) publications, economists have devoted considerable effort in remodeling the original Cobb-Douglas concept.

Constant elasticity of substitution specification

The constant elasticity of substitution specification (Arrow *et.al.*, 1961) was an effort to remodel the orginal Cobb-Douglas concept without the maintained hypothesis regarding the elasticity of substitution.

Henderson and Quandt (1971: Ch. 3) prove that the Cobb-Douglas production function is a special case of the constant elasticity of substitution specification. The constant elasticity of substitution function was an improvement if the interest centered on the elasticity of substitution within a production process that used only two inputs. However, if the function were extended to the n-input case, it resulted in a maintained hypothesis of the same elasticity of substitution for every input pair (Renvankar, 1971).

Because agricultural economists are usually interested in dividing input categories into more than two inputs, the constant elasticity of substitution specification was not extensively used.

The transcendental production function

Halter, Carter and Hocking (1957) proposed a transcendental production function to depict the three-stage production process as represented by neoclassical theory. The transcendental production function is actually a variable elasticity of substitution production function (Debertin and Pagoulatos, 1985: 29).

This function is readily estimable with data from agricultural production processes (Halter and Bradford, 1959). Allen (AES), Morishima (MES) and Shadow (SES) elasticities of substitution can be calculated from the production function.

Despite recognition of the transcendental functional form (Fuss, McFadden and Mundlak, 1978:242), it is not widely used for empirical purposes. The function is not monotonically increasing for at least certain parameter values, which means that the inverse or dual cost curve associated with it is a correspondence and not a function. As a result, parameters of the production process represented by the transcendental cannot be readily derived from the corresponding cost data (Debertin and Pagoulatos, 1985:30).

The translog production function

Although Diewert (1971) recognized that advances in computing technology made it possible to estimate functional forms that were non-linear in the parameters, he introduced the concept of linear in the parameters functional forms. The reason was that little if any new information about the production process would be gained by using more complex and computationally burdensome functional forms. He also recognized the close linkages that exist between various functional forms by looking at them in terms of Taylor's series expansions.

The translog production function was introduced by Christensen, Jorgenson and Lau (1971) and is simply a second order Taylor's series expansion of $\ln Y$ in $\ln X_i$, whereas the Cobb-Douglas is a first order expansion (Debertin and Pagoulatos, 1985:32).

The linearity in parameters makes parameter estimation simple. It is normally monotonically increasing with respect to the use of each input under the usual parameter assumptions. However, results depend upon the units in which the X_i are measured. If $O<X_i<1, \ln X_i<O$, and under certain positive parameter combinations, the function may not be increasing with respect to the i-th input (Debertin and Pagoulatos, 1985:33). There is no maintained hypothesis about the elasticity of substitution between input pairs, and the various elasticity of substitution measures can be derived either directly from the production function, or as is frequently more desirable, from a dual cost function of the translog form.

The production function is therefore suitable for empirical estimations of elasticities of substitution between input pairs if the information available is mainly production of cost data.

One can, instead of using the translog production function to derive elasticities of substitution between input pairs, rely on duality and begin with a dual cost function of the translog form. The translog cost function expresses cost as a function of all input prices and the quantity of output that is produced. For a given level of output Y*, the corresponding point on the cost function is assumed to be the minimum cost of producing Y* arising from the expansion path conditions.

The least cost translog function is:

$$\ln C^* = \Theta_o + \Sigma\, \Theta_i \ln P_i + \Sigma\, \Sigma\Theta_{ij} \ln P_i \ln P_j + \Theta_y \ln Y + \Sigma\, \Sigma\Theta_{iz} \ln P_i \ln Z_k^{\circ} + \Sigma\Theta_{yz} \ln Y \ln Z_k^{\circ} + \Sigma\Sigma\Theta_{jk} \ln Z_j^{\circ} \ln Z_k^{\circ} + \Sigma\Theta_z \ln Z_k^{\circ} + \Sigma\Theta_{yi} \ln Y \ln P_i \ \ldots \ (6)$$

Where $P = (P_1, \ldots\ldots, P_n)$ the vector of input prices
$z^o = (z_1, \ldots\ldots, z_n)$ the vector representing levels of the fixed inputs
Y = output
Θ = the parameter vector to be estimated.

Equation 6 is normally estimated from a series of cost share equations (Debertin and Pagoulatos, 1985: 35), which is given by:

$$S_1 = \Theta_1 + \Sigma\Theta_{1j} \ln P_j + \Sigma\Theta_{1z} \ln Z_k + \Theta_{y1} \ln Y$$

$$S_i = \Theta_i + \Sigma\Theta_{ij} \ln P_j + \Sigma\Theta_{iz} \ln Z_k + \Theta_{yi} \ln Y$$

$$S_n = \Theta_n + \Sigma\Theta_{nj} \ln P_j + \Sigma\Theta_{nz} \ln Z_k + \Theta_{yn} \ln Y \ \ldots\ldots \ (7)$$

The cost share equations (Equation 7) are empirically estimated and include price and output variables and levels of fixed inputs that would normally be readily available from farm records or from census data. If data on the level of fixed inputs are not available, their combined impact is estimated as part of the intercept term.

ESTIMATION PROBLEMS AND RESTRICTIONS

Economic theory imposes a number of restrictions on the estimation process (Debertin and Pagoulatos, 1985). Firstly, total cost = ΣS_i. Thus, given total cost and any n-1 cost shares, the remaining cost share is known with certainty. Therefore, one equation is redundant, and the choice of the equation to be omitted is arbitrary, but the empirical results may not be invariant with respect to the choice of the omitted equation unless an iterative estimation procedure is used (Berndt and Wood, 1975).

As indicated earlier, any total cost function should be homogeneous of degree 1,0 in input prices. This restriction can be imposed by restricting $\Sigma\Theta_i = 1$ en $\Sigma\Theta_{ij} = o$. Since Young's theorem states that the order of differentiation makes no difference and the Θij are in reality partial derivatives, a symmetry restriction must also be imposed so that $\Theta ej = \Theta ij$ is applied for all i and j inputs. Finally, the cost share for the i-th input is also related to the cost share for the j-th input.

From the parameter estimates of the cost share equations, the corresponding AES between input pairs and the related measures can be derived. The usual approach is to insert the mean of the cost shares (S_i) for each input category in the data for the sample period in order to obtain the Allen estimates (AES). Once the AES are obtained, the corresponding MES and SES can then be calculated from Equations 4 and 5. Again, the mean of the factor shares for the sample data is introduced into the formulas along with the estimated AES.

The SES estimate obtained from this model is not quite the long-run measure envisaged by McFadden (1978) since inputs in the Z vector are treated as fixed. The true long-run measure suggested by McFadden (1978) could be obtained if all input categories were treated as part of the X vector (Debertin and Pagoulatos, 1985:37).

CONCLUSION

The elasticity of substitution is an indication of the extent to which one input substitutes for another, thus providing an indication of the shape of an isoquant. Technological change which increases the elasticity of substitution between input pairs would give farmers additional flexibility in dealing with input price variation.

Instead of the estimation of functional specifications on physical input data, a contemporary approach frequently involves the estimation of factor share equations from the cost data. This is done by making use of the duality that exists between the production function and the cost function along the expansion path.

From the basic dual input definition of elasticity of substitution, a number of alternative concepts are possible. Each definition can be evaluated in terms of constant output, cost or marginal cost, or each alternative can also be evaluated by assuming remaining inputs as constant. The Allen, Morishima and Shadow measures are particularly useful.

Specific production functions frequently embody assumptions related to the functional form. Although not developed for that purpose, the Cobb-Douglas production function was one of the first forms consistent with the required assumptions for the development of the dual cost function. However, it also has important disadvantages with respect to the substitutability of inputs. The translog specification represents relaxations of these maintained hypotheses.

The translog specification is thus the production function suitable for empirical estimations of elasticities of substitution between input pairs where little information about the production process other than cost data is available.

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