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SOME FURTHER EXACT RESULTS FOR STRUCTURAL EQUATION ESTIMATORS

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ABSTRACT

In the context of the single structural equation model, we derive a number of exact results that extend and/or simplify results hitherto available. First, we obtain expressions for both the conditional and unconditional densities of the limited information maximum likelihood estimator for the coefficients of the endogenous variables. The unconditional result is considerably simpler than the corresponding result obtained earlier by Phillips (1985), and we indicate how this result can be used to obtain distribution results for the coefficients of the exogenous variables in exactly the manner used in Phillips (1984a) for the ordinary least squares and two-stage least squares estimators. Next, we obtain expressions for the mean square error of the ordinary least squares/two-stage least squares estimators for the coefficients of the exogenous variables. Finally, a number of generalizations of these results are indicated, and we explain briefly how these results can contribute to further attempts to understand the general problems of inference in this model.

PROPOSED RUNNING HEAD: Structural Equation Estimators

1. INTRODUCTION

The study of exact distribution theory for the statistics used in structural econometric models was pioneered by Basmann (1961) and Bergstrom (1962) nearly thirty years ago. The subject is notoriously difficult and, notwithstanding a surge of progress in the 1980's, it is probably fair to say that our understanding of the properties of the various estimators and tests that are used remains far from satisfactory. Vastly improved computing technology may soon improve matters - in particular, by bringing higher-order asymptotic results within reach - but the challenge to provide a coherent structure for inference in this model remains. Moreover, the lessons to be learned from these efforts are not confined to the structural model, for this model has much in common with the multivariate time-series models that are currently of such interest to econometricians generally - see Phillips (1988) and (1989).

This paper extends the results available for the single structural equation model in two ways. First, we derive both a conditional and an unconditional version of the density of the limited information maximum likelihood (LIML) estimator for the coefficients of the endogenous variables. Primitive versions of these results were initially given in Hillier (1987). The unconditional result provides an alternative to the expression given in Phillips (1985) for the density of the LIML estimator for these coefficients, and permits an immediate extension of the results in Phillips (1984a) for the ordinary least squares (OLS) and two stage least squares (TSLS) estimators of the coefficients of the exogenous variables to the LIML case. Second, we provide exact expressions for the mean square error of the OLS and TSLS estimators of the coefficients of the exogenous variables. These results supplement the results in Phillips (1984a), Skeels (1989a) and (1989b), and Hillier (1985b).

The conditional result for LIML hinges on a decomposition for positive definite symmetric matrices - Theorem 1 in section 4 - that is apparently new. This is a key result for this model, for the following reason. The single structural equation model is an example of a curved exponential model - that is, a model in which the dimension of the minimal sufficient statistic exceeds that of the parameter space, see Efron (1975), Barndorff-Nielson (1980), Hillier (1987), and Hosoya, Tsukuda, and Terui (1989). In such models marginal distributions of the type traditionally studied for this model are not the 'relevant' bases for judging the merits of the procedures of interest, but should be replaced by particular conditional distributions. Theorem 1 in section 4 provides a means of obtaining both a variety of joint distributions, and the conditional distributions relevant in this context. In section 8 we briefly indicate how these results follow from Theorem 1, but a detailed study of the curved exponential character of this model is not attempted here.

2. MODEL AND CANONICAL FORMS OF ESTIMATORS

We consider a single (normalized) structural equation

(1) $$y = Y\beta^* + Z_1\gamma^* + u ,$$

with corresponding reduced form

(2) $$(y,Y) = (Z_1,Z_2)\begin{bmatrix} \pi_1 & \Pi_1 \\ \pi_2 & \Pi_2 \end{bmatrix} + (v,V) .$$

Here y is N × 1, Y is N × n, Z_1 is N × K_1 and Z_2 is N × K_2. We assume that $K_2 \geq n$, so that (1) is apparently identified, that $Z = (Z_1, Z_2)$ is fixed and of full column rank $K = K_1 + K_2$, and that the rows of (v,V) are independent normal vectors with mean zero and common covariance matrix

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{21}' \\ \omega_{21} & \Omega_{22} \end{pmatrix} \begin{matrix} 1 \\ n \end{matrix}$$

We shall also refer to an unnormalized version of (1):

(3) $$(y, Y)\beta_\Delta = Z_1\gamma^* + u .$$

The compatibility of (1) with (2) entails the relations

(4) $$\pi_1 - \Pi_1\beta^* = \gamma^* \qquad ((\pi_1, \Pi_1)\beta_\Delta = \gamma^*)$$

(5) $$\pi_2 - \Pi_2\beta^* = 0 \qquad ((\pi_2, \Pi_2)\beta_\Delta = 0)$$

and $u = v - V\beta^*$ $(u = (v,V)\beta_\Delta)$, where the relations in brackets refer to the unnormalized equation (3). We shall assume that (1) (and (3)) are compatible with (2), but defer for the moment any assumption on the identification of (1) and (3).

The OLS and TSLS estimators for γ^* in (1) are of the form

(6) $$g = (Z_1'Z_1)^{-1}Z_1'(y - Yb) ,$$

where b denotes the OLS or TSLS estimator for β^* in (1). The LIML estimator for γ^* is given by

(7) $$g_1 = (Z_1'Z_1)^{-1}Z_1'(y, Y)\tilde{\beta}_\Delta ,$$

where $\tilde{\beta}_\Delta$ is any (n+1) × 1 vector that maximizes

(8) $$\beta'_\Delta(y,Y)'P(y,Y)\beta_\Delta/\beta'_\Delta(y,Y)'(P_1-P)(y,Y)\beta_\Delta ,$$

where $P = I-Z(Z'Z)^{-1}Z'$ and $P_1 = I-Z_1(Z_1'Z_1)^{-1}Z_1'$. Now, $\tilde{\beta}_\Delta$ is determined by (8) only up to a scalar multiple, or, to put this another way, only the direction of $\tilde{\beta}_\Delta$ is determined, not its length. Thus, g_1 in (7) is well-defined only when $\tilde{\beta}_\Delta$ is normalized in some way, and $\tilde{\beta}_\Delta$ can, but need not, be normalized to correspond with (1). Partitioning $\tilde{\beta}_\Delta$ as $\tilde{\beta}_\Delta = (\tilde{\beta}_{\Delta1},\tilde{\beta}'_{\Delta2})'$, with $\tilde{\beta}_{\Delta2}$ nx1, we set $b = -\tilde{\beta}_{\Delta1}^{-1}\tilde{\beta}_{\Delta2}$ (note that b is uniquely defined even though $\tilde{\beta}_\Delta$ is not), and define the LIML estimator for γ^* in the normalized equation, (1), as in (6). Since this normalization for the LIML estimator is probably the most common, (6) covers all of the estimators for γ^* that are in common use, and is the version of the LIML estimator that we shall discuss. However, it would be of considerable interest to study the impact (if any) of the normalization rule adopted for $\tilde{\beta}_\Delta$ on the estimator g_1 in (7), but that is a subject for another paper. The effect of the normalization rule on the properties of the estimators for β^* and β_Δ is studied in Hillier (1990).

Joint sufficient statistics for (π_1,Π_1), (π_2,Π_2), and Ω are $(\tilde{\pi}_1,\tilde{\Pi}_1) = (Z_1'Z_1)^{-1}Z_1'(y,Y)$, $(\hat{\pi}_2,\hat{\Pi}_2) = (Z_2'P_1Z_2)^{-1}Z_2'P_1(y,Y)$, and $S^* = (y,Y)'P(y,Y)$. Let

$$T = \begin{bmatrix} 1/\omega & , & 0 \\ -\alpha/\omega & , & \Omega_{22}^{-1/2} \end{bmatrix} ,$$

where $\omega^2 = \omega_{11}-\omega'_{21}\Omega_{22}^{-1}\omega_{21}$ and $\alpha = \Omega_{22}^{-1}\omega_{21}$, so that $T'\Omega T = I_{n+1}$, and define

(9) $$(x_1,X_1) = (Z_1'Z_1)^{1/2}(\tilde{\pi}_1,\tilde{\Pi}_1)T ,$$

(10) $$(x,X) = (Z_2'P_1Z_2)^{1/2}(\hat{\pi}_2,\hat{\Pi}_2)T ,$$

(11) $$S = T'S^*T .$$

It is straightforward to check that (x_1,X_1), (x,X), and S are mutually independent, that $S \sim W_{n+1}(m, I_{n+1})$, where m = N-K, and that the rows of both (x_1,X_1) and (x,X) are independent normal vectors with covariance matrix I_{n+1}, with

(12) $$E(x_1,X_1) = (Z_1'Z_1)^{-1/2}Z_1'Z(\pi,\Pi)T = (m_1,M_1), \text{ say,}$$

and

(13) $$E(x,X) = (Z_2'P_1Z_2)^{1/2}(\pi_2,\Pi_2)T = (\mu,M), \text{ say,}$$

where $\pi = (\pi_1', \pi_2')'$ and $\Pi = (\Pi_1', \Pi_2')'$.

The canonical forms of the OLS, LIML, and TSLS estimators for β^* are

(14) $$r_0 = (X'X+S_{22})^{-1}(X'x+s_{21})$$

(15) $$r_1 = -h_1^{-1}h_2$$

(16) $$r_2 = (X'X)^{-1}X'x$$

respectively, where S has been partitioned to conform with the partition of Ω above, and $h = (h_1, h_2')'$ is any characteristic vector satisfying $[S - f_1(x,X)'(x,X)]h = 0$, with f_1 the largest root of $|S - f(x,X)'(x,X)| = 0$. It is easy to check that, in each case, r and b are related by

(17) $$r = \Omega_{22}^{1/2}(b - \alpha)/\omega.$$

Defining $s = (Z_1'Z_1)^{1/2}g/\omega$, the canonical statistics to be studied below for the normalized case are of the form

(18) $$s_i = x_1 - X_1 r_i \ ; \ i = 0,1,2.$$

The statistics r_i (i=0,1,2), and hence the s_i in (18), are well-defined as long as $N-K \geq n+1$ and $K_2 \geq n+1$, whether or not equation (1) (or (3)) is identified. In what follows special cases of some results are given under various simplifying assumptions, some of which imply lack of identification. The implications of these assumptions will be discussed below, but for the moment we note that if $\text{rank}(\Pi_2) = n$ and $\text{rank}(\pi_2,\Pi_2) = n$, both β^* and β_Δ in (5) are uniquely defined except that β_Δ is determined only up to scale. Equation (4) is then a *definition* of γ^* as a function of (π_1,Π_1) and β^* (or β_Δ), and hence as a function of (π_1,Π_1) and (π_2,Π_2). Defining

(19) $$\beta = \Omega_{22}^{1/2}(\beta^* - \alpha)/\omega \qquad (\eta = T^{-1}\beta_\Delta)$$

and $\gamma = (Z_1'Z_1)^{1/2}\gamma^*/\omega$, these too are uniquely defined and, in (12) and (13), we have

(20) $$m_1 = E(x_1) = \gamma + M_1\beta \qquad (\gamma = (m_1,M_1)\eta)$$

and $\mu = E(x) = M\beta$ $((\mu,M)\eta = 0)$. Note that $\beta = 0$ if and only if Y is independent of u in (1).

3. SOME CONDITIONAL RESULTS

Since (x_1,X_1) is independent of (x,X) and S, it is independent of all functions of (x,X) and S, including the three statistics r_0, r_1, and r_2, and the largest root, f_1, associated with the LIML procedure.

Using r to denote any one of r_0, r_1, r_2, and setting $s = x_1 - X_1 r$, it follows at once that, given r,

$$(21) \qquad s|r \sim N(\ m_1 - M_1 r\ ,\ (1+r'r)I).$$

The densities of r_0, r_1, and r_2 are known - see sections 4 and 5 below - so that the joint density pdf(s,r) is easily obtained, and the marginal density of s can, in principle, be obtained from the relation

$$(22) \qquad \mathrm{pdf}(s) = \int_{R^n} \mathrm{pdf}(s|r)\ \mathrm{pdf}(r)\ (dr).$$

This is the procedure followed in Phillips (1984a), who gives results for the OLS and TSLS estimators. Skeels (1989a) and (1989b) also obtains results for the OLS/TSLS estimators using a somewhat different approach. In section 4 we derive an expression for the density of the LIML estimator r_1 that makes it possible to evaluate the integral in (22) for the LIML case by exactly the method that Phillips (1984a) uses to evaluate (22) for the OLS/TSLS estimators. In the remainder of this section we derive some simple consequences of (21) and (22).

First, it is known that the densities of the OLS and TSLS estimators r_0 and r_2 are identical except in respect of a single 'degrees of freedom' parameter, ν, which is $N-K_1$ for OLS, K_2 for TSLS (see equation (53) below). Hence, (22) implies that pdf(s) also differs between these two cases only in respect of ν, and we shall henceforth make no distinction between these two estimators.

Next, consider an arbitrary fixed linear combination of r and s:

$$(23) \qquad w = a_1'r + a_2's = a_2'x_1 + (a_1 - X_1'a_2)'r$$

with $a_2 \neq 0$, and a_2 normalized, if necessary, so that $a_2'a_2 = 1$. It follows from (21) that, given r,

(24) $w|r \sim N(\, a_2' m_1 - (M_1' a_2 - a_1)' r,\ (1+r'r))$.

The unconditional density of w can be obtained from (24) by the analogue of (22), and it follows from this that the density of an arbitrary linear combination of r and s has precisely the same form as the density of s itself upon making the substitutions

$$m_1 \to a_2' m_1 \quad ; \quad M_1 \to (M_1' a_2 - a_1)' \quad ; \quad K_1 \to 1$$

for terms that originate from pdf(s|r). The results in Phillips (1984a) and Skeels (1989a, 1989b) therefore cover not just the density of s for the OLS/TSLS estimators, but also the densities of arbitrary linear combinations of r and s. The results below for the LIML estimator extend in the same way, but we leave the details to the reader.

It is clear from (21) and (24) that the LIML estimator s_1 has no moments, because r_1 has none, and that for the OLS and TSLS estimators the moments of s exist up to the same order as those of the r from which it constructed (i.e., ν-n). Note also that, for OLS and TSLS,

(25) $E(w) = a_2' m_1 - (M_1' a_2 - a_1)' E(r)$.

In the case of an identified equation it follows from (20) that w is an unbiased estimator for $w_0 = a_1'\beta + a_2'\gamma$ only if either $E(r) = \beta$, which is true only if $\beta = 0$, or $a_1 = M_1' a_2$, a very special case. The moments of w are discussed in detail in Skeels (1989a) and will not be considered further here.

For an identified, normalized, equation we have $m_1 = \gamma + M_1\beta$, and (21) can be written as

$$(26) \qquad s|r \sim N(\gamma + M_1(\beta - r),\ (1+r'r)I).$$

Hence, in this case, given r,

$$(27) \qquad [(s-\gamma)'(s-\gamma)/(1+r'r)]|r \sim \chi'^2(K_1, (r-\beta)'M_1'M_1(r-\beta)/(1+r'r)).$$

Defining, for the identified case, $q = (s-\gamma)'(s-\gamma)$, the conditional density of q given r can be found easily from (27), and the unconditional density from a formula analogous to (22). The unconditional distribution function of q gives the probability content of spherical regions centered at γ. Some properties of q for the OLS/TSLS estimators are given in section 6.

The mean of q (i.e., MSE(s) in the identified case) does not exist for the LIML estimator, but for the OLS/TSLS estimators is given by (if $\nu - n > 2$)

$$(28) \qquad MSE(s) = E_r[K_1(1+r'r) + (r-\beta)'M_1'M_1(r-\beta)].$$

Several properties of the OLS/TSLS estimators can be deduced fairly easily from (28). First, it is clear that $MSE(s) \geq K_1$, its value in the simple regression ($\beta^* = 0$) case. However, a greater lower bound can be obtained by completing the square in r in (28) and noting that the expection of the resulting quadratic term in r cannot be negative. This gives the lower bound

$$(29) \qquad MSE(s) \geq K_1 + \beta'M_1'[I_{K_1} - M_1A_1^{-1}M_1']M_1\beta,$$

where $A_1 = K_1I_n + M_1'M_1$. Second, it follows from (16) and the properties of the matrix (x,X) that, given X, $r_2|X \sim N((X'X)^{-1}X'M\beta,\ (X'X)^{-1})$. Evaluating the expectation in (28) conditionally, given X, we see that

$$(30) \qquad MSE(s) = E_X\left\{ K_1 + tr[A_1(X'X)^{-1}] + \beta'\left[K_1 M'X(X'X)^{-2}X'M + (X - M)'X(X'X)^{-1}M_1'M_1(X'X)^{-1}X'(X - M)\right]\beta \right\}.$$

Since the matrix $\left[..\right]$ in the third term here is positive semi-definite for all X, and the density of X does not depend on β, it follows that, for the TSLS estimator, MSE(s) is necessarily larger when $\beta \neq 0$ (i.e., when Y is correlated with u in (1)) than it is when $\beta = 0$. The same result is easily obtained for the OLS estimator.

In some of the cases to be considered below the structural equation is not identified, and in such cases q should be defined as either $q = (s-m_1)'(s-m_1)$, with a suitable definition of m_1, or, if $m_1 = 0$, simply as $q = s's$.

4. DENSITY OF THE LIML ESTIMATOR FOR ENDOGENOUS COEFFICIENTS

Phillips (1985) has obtained an expression for the density of r_1 in an operator form. In this section we derive an expression for $pdf(r_1)$ that is analogous to results obtained earlier for the OLS/TSLS estimators (see section 5 below). The approach used is quite different from that used in Phillips (1984b, 1985), and is based on a decomposition for positive definite symmetric matrices that is of independent interest (Theorem 1 below). Symmetrically normalized analogues of the OLS/TSLS estimators for β_Δ in the unnormalized equation (3) may be defined and have densities identical to that of the LIML estimator except for certain numerical coefficients - see Hillier (1990). The results below for LIML can be extended to cover these estimators as well.

Since r_1 is a function of the independent matrices S and (x,X) we shall first find the conditional density of r_1 given the matrix (x,X). This can then be converted into the unconditional density by averaging with respect to the density of (x,X). Let

$$Q = \begin{bmatrix} t_2 & , & 0 \\ (X'X)^{1/2}r_2, & & (X'X)^{1/2} \end{bmatrix},$$

with r_2 as in (16) and $t_2^2 = x'[I-X(X'X)^{-1}X']x$, so that $Q'Q = (x,X)'(x,X) = W$, say, and define $R = (QS^{-1}Q')^{-1}$. Since S is independent of (x,X), it follows from Muirhead (1982, Theorem 3.2.11) that, conditional on (x,X), $R|(x,X) \sim W_{n+1}(m, V)$, with $V = (QQ')^{-1}$. That is,

$$\text{(31)} \qquad \text{pdf}(R|(x,X)) = C_1 \text{etr}\{-\tfrac{1}{2}V^{-1}R\}|R|^{(m-n-2)/2}|V|^{-m/2},$$

where $C_1 = [2^{m(n+1)/2}\Gamma_{n+1}(m/2)]^{-1}$.

If $\tilde{h} = (\tilde{h}_1, \tilde{h}_2')'$ (with $\tilde{h}_2$ nx1) is a characteristic vector satisfying $[S - f_1W]\tilde{h} = 0$, with f_1 the largest root of $|S - fW| = 0$, then $r_1 = -\tilde{h}_1^{-1}\tilde{h}_2$. On the other hand, if $h = (h_1, h_2')'$ is a characteristic vector satisfying $[R - f_1I]h = 0$, then $Q^{-1}h \propto \tilde{h}$ and, defining $r = -h_1^{-1}h_2$, we have $r = t_2^{-1}(X'X)^{1/2}(r_1 - r_2)$, or

$$\text{(32)} \qquad r_1 = r_2 + t_2(X'X)^{-1/2}r.$$

That is, for fixed (x,X), r_1 is a simple function of r. Thus, we shall first find the conditional joint density of r, f_1, and another matrix B (defined in Theorem 1 below), given (x,X), and then transform to $\text{pdf}(r_1,f_1,B|(x,X))$ using (32). The following Theorem and its corollaries (proved in the Appendix) give the details of the transformation $R \to (f_1, r, B)$, the corresponding transformation of the measure

$$(dR) = \bigwedge_{i \le j} dR_{ij},$$

and various joint and marginal densities when $R \sim W_{n+1}(k, V)$. Here, and throughout the paper, we use the notational shorthand for various volume elements explained in Muirhead (1982, Chapter 2).

Theorem 1:

(a) *Let* R $((n+1)x(n+1))$ *be a positive definite symmetric matrix, let* $F = \text{diag}\{f_1, f_2, \ldots, f_{n+1}\}$, *with* $f_1 > f_2 > \ldots > f_{n+1} > 0$, *be a diagonal matrix containing the ordered characteristic roots of* R, *and let* $H \in O(n+1)$ *be an orthogonal matrix with columns the orthonormal characteristic vectors of* R *corresponding to the roots* $f_1, f_2, \ldots, f_{n+1}$ *respectively, with the elements in the first row of* H *positive. Partition* F *and* H *as*

$$F = \begin{pmatrix} f_1 & , 0 \\ 0 & , F_2 \end{pmatrix}, \qquad H = \left(\begin{matrix} h_1 \\ h_2 \end{matrix} \middle| \, H_2 \right),$$

with F_2 nxn *and* h_2 $nx1$, *and define* $r = -h_1^{-1} h_2$ *and*

$$B = f_1^{-1} (I+rr')^{-1/2} (r, I) H_2 F_2 H_2' (r, I)' (I+rr')^{-1/2},$$

so that $r \in R^n$ *and* $0 < B < I$. *Then, the transformation* $R \to (f_1, r, B)$ *is one-to-one, and the volume element transforms as*

$$(33) \qquad (dR) = f_1^{n(n+3)/2} (1+r'r)^{-(n+1)/2} |I-B| (df_1) \left(\prod_{i=1}^{n} dr_i \right) (dB)$$

(b) *If* $R \sim W_{n+1}(m, V)$, *then the joint density of* (f_1, r, B) *is given by*

$$(34) \; \text{pdf}(f_1, r, B) = C_1 f_1^{m(n+1)/2-1} |B|^{(m-n-2)/2} |I-B| (1+r'r)^{-(n+1)/2} |V|^{-m/2}$$

$$\mathrm{etr}\left\{-\frac{1}{2}f_1V^{-1}\left[(1+r'r)^{-1}\begin{pmatrix}1\\-r\end{pmatrix}\begin{pmatrix}1\\-r\end{pmatrix}' + (r,I)'(I+rr')^{-1/2}B(I+rr')^{-1/2}(r,I)\right]\right\}$$

Note that, if $V = I$, r is independent of f_1 and B because the last term in (34) does not depend on r.

<u>Corollary 1.1:</u> *Assuming* $R \sim W_{n+1}(m,V)$, *the marginal density of* (r,f_1) *is given by*

(35) $$\mathrm{pdf}(r,f_1) = C_1^* \, f_1^{m(n+1)/2-1}(1+r'r)^{-(n+1)/2}\mathrm{etr}\{-\tfrac{1}{2}f_1V^{-1}\}|V|^{-m/2}$$

$$_1F_1((n+3)/2,\ (m+n+2)/2;\ \tfrac{1}{2}f_1(I+rr')^{-1/2}(r,I)V^{-1}(r,I)'(I+rr')^{-1/2}),$$

where $C_1^* = [C_1\Gamma_n((m-1)/2)\Gamma_n((n+3)/2)/\Gamma_n((m+n+2)/2)]$.

<u>Corollary 1.2:</u> *Assuming* $R \sim W_{n+1}(m,V)$, *the marginal density of* f_1 *is:*

(36) $$\mathrm{pdf}(f_1) = C_1^{**} \, f_1^{m(n+1)/2-1}\mathrm{etr}\{-\tfrac{1}{2}f_1V^{-1}\}|V|^{-m/2}$$

$$_2F_2(\ n/2,\ (n+3)/2;\ (m+n+2)/2,\ (n+1)/2;\ \tfrac{1}{2}f_1V^{-1}),$$

where $C_1^{**} = 2\pi^{(n+1)/2}C_1^*/\Gamma((n+1)/2)$.

By applying Corollary 1.2 to $\mathrm{pdf}(f_1|W)$, this result can be used to obtain the unconditional density of f_1. Rhodes (1981) has obtained a result of this type for f_1, but his expression for $\mathrm{pdf}(f_1)$ involves unknown coefficients; we leave it to the reader to confirm that the expression obtained by using Corollary 1.2 does not (set $V^{-1} = W$ in (36), multiply by $\mathrm{pdf}(W)$ in equation (40) below, and then integrate over $W>0$).

Applying Corollary 1.1 to the matrix $R = (QS^{-1}Q')^{-1}$, with $V = (QQ')^{-1}$, we obtain the conditional joint density $\mathrm{pdf}(r,f_1|(x,X))$.

Transforming $r \to r_1$ using (32) (Jacobian $t_2^{-n}|X'X|^{1/2}$) and simplifying, we obtain

(37) $$\mathrm{pdf}(r_1,f_1|(x,X)) = C_1^* \,\mathrm{etr}\{-\tfrac{1}{2}f_1W\}f_1^{m(n+1)/2-1}|W|^{(m+1)/2}$$

$$(1+r_1'r_1)^{-(n+1)/2}\,[h'Wh]^{-(n+1)/2}\,{}_1F_1((n+3)/2,\ (m+n+2)/2;\ \tfrac{1}{2}f_1(G'W^{-1}G)^{-1}),$$

where we have put

(38) $$h = \begin{pmatrix} 1 \\ -r_1 \end{pmatrix}(1+r_1'r_1)^{-1/2}, \qquad G = \begin{pmatrix} r_1' \\ I_n \end{pmatrix}(I+r_1r_1')^{-1/2},$$

and $W = (x,X)'(x,X)$. Note that $\mathrm{pdf}(r_1,f_1|(x,X))$ depends upon (x,X) only through W, and that

(39) $$(G'W^{-1}G)^{-1} = (I+r_1r_1')^{1/2}[(X'X)^{-1} + t_2^{-2}(r_1-r_2)(r_1-r_2)']^{-1}(I+r_1r_1')^{1/2}$$

$$= (I+r_1r_1')^{1/2}X'[I-(x-Xr_1)[(x-Xr_1)'(x-Xr_1)]^{-1}(x-Xr_1)']X(I+r_1r_1')^{1/2}.$$

The conditional result in equation (37) can be used to obtain a number of interesting results that are useful for inference in this model. For example, on multiplying (37) by pdf(W), and then transforming $W \to (r_2,\ t_2^2,\ X'X)$, we obtain $\mathrm{pdf}(r_1,\ r_2,\ f_1,\ t_2^2,\ X'X)$, and from this we can obtain, for instance, the joint density of the TSLS and LIML estimators, $\mathrm{pdf}(r_1,r_2)$. Some further results that follow from (37) are mentioned in section 8 below

The matrix $W = (x,X)'(x,X)$ has the non-central Wishart distribution with K_2 degrees of freedom and non-centrality matrix $(\beta,I)'M'M(\beta,I)$. That is,

$$\text{(40)} \qquad \mathrm{pdf}(W) = C_2 \mathrm{etr}(-\tfrac{1}{2}W)|W|^{(K_2-n-2)/2} {}_0F_1(K_2/2,\ (\beta,I)'M'M(\beta,I)W/4),$$

where $C_2 = [\mathrm{etr}\{-M'M(I+\beta\beta')/2\}/[2^{(n+1)K_2/2}\Gamma_{n+1}(K_2/2)]]$. Now,

$$\text{(41)} \qquad \mathrm{pdf}(r_1,f_1) = \int_{W>0} \mathrm{pdf}(r_1,f_1|W)\ \mathrm{pdf}(W)\ (dW)\ ,$$

with $\mathrm{pdf}(r_1,f_1|W)$ given by (37), and $\mathrm{pdf}(W)$ by (40). To evaluate the integral we first transform to $\bar{W} = (h,G)'W(h,G)$ (the Jacobian is unity because $(h,G) \in O(n+1)$). Partitioning $\bar{W}$ as

$$\bar{W} = \begin{bmatrix} \bar{w}_{11} & \bar{w}_{21}' \\ \bar{w}_{21} & \bar{W}_{22} \end{bmatrix} \begin{matrix} 1 \\ n \end{matrix}\ ,$$

we then put $U = \bar{W}_{22} - \bar{w}_{11}^{-1}\bar{w}_{21}\bar{w}_{21}'$, $z = \bar{w}_{21}$, and $w = \bar{w}_{11}$, the Jacobian again being unity. Note that, in (37), $h'Wh = w$ and, because $[(h,G)'W^{-1}(h,G)] = [(h,G)'W(h,G)]^{-1}$, $(G'W^{-1}G)^{-1} = U$. The argument of the ${}_0F_1$ function in (40) becomes

$$\bar{M}(\beta,I)'(h,G)\begin{bmatrix} w & z' \\ z & U + w^{-1}zz' \end{bmatrix}(h,G)'(\beta,I)\bar{M}'/4,$$

where $\bar{M}$ is any nxn matrix such that $\bar{M}'\bar{M} = M'M$. We can write this function as an inverse Laplace transform:

$$a_n\Gamma_n(K_2/2) \int_{\mathrm{Re}(Z)>0} \mathrm{etr}(Z)|Z|^{-K_2/2}\ \mathrm{etr}\{A_{22}U/2\}$$

$$\exp\{(wa_{11} + 2z'a_{21} + w^{-1}z'A_{22}z)/2\}(dZ)\ ,$$

where $a_n = [2^{n(n-1)/2}/(2\pi i)^{n(n+1)/2}]$ and we have put

$$A = \begin{bmatrix} a_{11} & a_{21}' \\ a_{21} & A_{22} \end{bmatrix} = (h,G)'(\beta,I)'\bar{M}'Z^{-1}\bar{M}(\beta,I)(h,G)/2$$

(cf. Herz (1955), Constantine (1963), James (1964)). Since $|W| = |\bar{W}| = w|U|$, and $tr(W) = tr(\bar{W}) = w + w^{-1}z'z + tr(U)$, it is reasonably straightforward to integrate over $U > 0$ (using Constantine (1963, Theorem 1)), and over $z \in R^n$ (by completing the square). This leaves

(42) $$\text{pdf}(r_1,f_1,w) = [(2\pi)^{n/2}2^{n(m+K_2)/2}\Gamma_n((m+K_2)/2)C_1^*C_2]$$

$$f_1^{m(n+1)/2-1}(1+f_1)^{-n(m+K_2+1)/2}(1+r_1'r_1)^{-(n+1)/2}\ w^{(m+K_2-n)/2-1}$$

$$a_n\Gamma_n(K_2/2)\int_{Re(Z)>0} etr(Z)|Z|^{-K_2/2}\ |I_n-A_{22}/(1+f_1)|^{-(m+K_2+1)/2}$$

$${}_2F_1((m+K_2)/2,\ (n+3)/2;\ (m+n+2)/2;\ f_1[(1+f_1)I_n-A_{22}]^{-1})$$

$$\exp\left\{-\ \tfrac{1}{2}w(1+f_1-a_{11}-a_{21}'[(1+f_1)I_n-A_{22}]^{-1}a_{21})\right\}(dZ).$$

The term in the exponent of the last term in the integrand in (42) may be written as

$$-\tfrac{1}{2}w(1+f_1)|I_n - \bar{M}(I+\beta\beta')\bar{M}'Z^{-1}/2(1+f_1)|/|I_n - A_{22}/(1+f_1)|$$

Hence, integrating over $w > 0$, we obtain an expression which can be written in the form

(43) $$\text{pdf}(r_1,f_1) = C_3f_1^{m(n+1)/2-1}(1+f_1)^{-(m+K_2)(n+1)/2}(1+r_1'r_1)^{-(n+1)/2}$$

$$[a_n\Gamma_n(K_2/2)/\Gamma_n((n+1)/2)] \int_{Re(Z)>0} \int_{R>0} etr(Z)|Z|^{-K_2/2}$$

$$etr\{-[I - G'(\beta,I)'\bar{M}'Z^{-1}\bar{M}(\beta,I)G/2(1+f_1)]R\}$$

$$|I - \bar{M}(I+\beta\beta')\bar{M}'Z^{-1}/2(1+f_1)|^{-(m+K_2-n)/2}$$

$${}_2F_2((m+K_2)/2,\ (n+3)/2;\ (m+n+2)/2,\ (n+1)/2;\ f_1R/(1+f_1))(dR)(dZ)\ ,$$

where $C_3 = [2^{(m+K_2)(n+1)/2}\Gamma_{n+1}((m+K_2)/2)C_1^*C_2]$.

In the totally unidentified case with $\bar{M} = 0$ we obtain at once from (43)

$$\mathrm{pdf}(r_1,f_1) = \mathrm{pdf}(r_1)\ \mathrm{pdf}(f_1)$$

with

(44) $$\mathrm{pdf}(r_1) = \Gamma((n+1)/2)[\pi(1+r_1'r_1)]^{-(n+1)/2}$$

(cf. Phillips (1984b, (1985)), and

(45) $$\mathrm{pdf}(f_1) = C_4f_1^{m(n+1)/2-1}(1+f_1)^{-(m+K_2)(n+1)/2}$$

$${}_2F_1((m+K_2)/2,\ (n+3)/2;\ (m+n+2)/2;\ f_1I_n/(1+f_1))$$

with

$$C_4 = \frac{\pi^{(n+1)/2}\Gamma_n((m-1)/2)\Gamma_n((n+3)/2)\Gamma_{n+1}((m+K_2)/2)}{\Gamma((n+1)/2)\Gamma_n((m+n+2)/2)\Gamma_{n+1}(m/2)\Gamma_{n+1}(K_2/2)}\ .$$

Thus, in this leading case r_1 and f_1 are independent. We shall see shortly that this is not true in general.

In the general case, first notice that (43) is invariant under $G \to GH$, $H \in O(n)$, because on making this substitution we can then transform $R \to HRH'$ and leave the integral unchanged. Hence, replacing G by GH and averaging over O(n), the term $\mathrm{etr}\{G'(\beta,I)'\bar{M}'Z^{-1}\bar{M}(\beta,I)GR/2(1+f_1)\}$ in (43) is replaced by the generalized hypergeometric series ${}_0F_0^{(n)}(\bar{M}(\beta,I)GG'(\beta,I)'\bar{M}'Z^{-1}/2(1+f_1), R)$. We may then use results from Davis (1979) to evaluate the inverse Laplace transform and obtain

$$(46)\quad \mathrm{pdf}(r_1,f_1) = C_4\Gamma((n+1)/2)[\pi(1+r_1'r_1)]^{-(n+1)/2}\mathrm{etr}\{-M'M(I+\beta\beta')/2\}$$

$$f_1^{m(n+1)/2-1}(1+f_1)^{-(m+K_2)(n+1)/2}$$

$$\sum_{\alpha,\kappa;\phi} \frac{((m+K_2-n)/2)_\kappa}{j!\ k!\ (K_2/2)_\phi}\, \theta_\phi^{\alpha,\kappa} C_\phi^{\alpha,\kappa}(\bar{M}(\beta,I)GG'(\beta,I)'\bar{M}'/2,\ \bar{M}(I+\beta\beta')\bar{M}'/2)$$

$$(1+f_1)^{-(j+k)}\mathcal{G}_\alpha(f_1),$$

where

$$(47)\quad \mathcal{G}_\alpha(f_1) = [\Gamma_n((n+1)/2)]^{-1}\int_{R>0} \mathrm{etr}(-R)[C_\alpha(R)/C_\alpha(I)]$$

$${}_2F_2((m+K_2)/2,\ (n+3)/2;\ (m+n+2)/2,\ (n+1)/2;\ f_1R/(1+f_1))(dR)$$

$$= \sum_{\ell=0}^{\infty}\sum_{\lambda} \frac{((m+K_2)/2)_\lambda\ ((n+3)/2)_\lambda}{\ell!\,((m+n+2)/2)_\lambda((n+1)/2)_\lambda}\ [f_1/(1+f_1)]^\ell\ g(\alpha,\lambda),$$

with

$$(48) \qquad g(\alpha,\lambda) = \sum_{\rho\in\alpha.\lambda} \left\{ \sum_{\rho^*\equiv\rho} (\theta_{\rho^*}^{\alpha,\lambda})^2 \right\} ((n+1)/2)_\rho C_\rho(I)/C_\alpha(I).$$

The notation used here is explained in detail in Davis (1979) and Chikuse and Davis (1986).

The marginal density of r_1 itself may be obtained at once from equation (46) by simply integrating over $f_1 > 0$. This gives the density in the form

$$(49) \qquad \text{pdf}(r_1) = C_5\Gamma((n+1)/2)[\pi(1+r_1'r_1)]^{-(n+1)/2} \text{ etr}\{-M'M(I+\beta\beta')/2\}$$

$$\sum_{\alpha,\kappa;\phi} \frac{a(\alpha,\kappa;\phi)}{j!k!} C_\phi^{\alpha,\kappa}(\bar{M}(\beta,I)GG'(\beta,I)'\bar{M}'/2,\ \bar{M}(I+\beta\beta')\bar{M}'/2),$$

with $C_5 = [\Gamma(m(n+1)/2)\Gamma(K_2(n+1)/2)C_4/\Gamma((m+K_2)(n+1)/2)]$, and

$$(50) \qquad a(\alpha,\kappa;\phi) = \frac{((m+K_2-n)/2)_\kappa (K_2(n+1)/2)_{j+k}}{(K_2/2)_\phi ((m+K_2)(n+1)/2)_{j+k}} \theta_\phi^{\alpha,\kappa}$$

$$\sum_{\ell=0}^{\infty} \sum_{\lambda} \frac{((m+K_2)/2)_\lambda ((n+3)/2)_\lambda (m(n+1)/2)_\ell g(\alpha,\lambda)}{\ell!((m+n+2)/2)_\lambda ((n+1)/2)_\lambda (j+k+(m+K_2)(n+1)/2)_\ell}.$$

Equation (49) is exactly analogous to equation (53) below for the OLS/TSLS estimators.

When n = 1 (49) simplifies to

$$(51) \qquad \text{pdf}(r_1) = C_5[\pi(1+r_1^2)]^{-1}\exp\{-d^2/2\}$$

$$\sum_{j,k=0}^{\infty} \frac{a(j,k)}{j!k!} (d^2/2)^{j+k}[(1+r_1\beta)^2/(1+r_1^2)(1+\beta^2)]^j,$$

where, from (50),

(52) $$a(j,k) = [((m+K_2-1)/2)_k (K_2)_{j+k} (1)_j / (K_2/2)_{j+k} (m+K_2)_{j+k}]$$

$$_4F_3((m+K_2)/2,\ 2,\ m,\ j+1;\ (m+3)/2,\ 1,\ j+k+m+K_2;\ 1)\ ,$$

and we have put $d^2 = M'M(1+\beta^2)$, a scalar in this case.

5. DENSITIES OF OLS/TSLS ESTIMATORS AND SPECIAL CASES

The density functions of the OLS/TSLS estimators have the form (see (Phillips (1980), Phillips (1983), and Hillier (1985a)),

(53) $$\mathrm{pdf}(r) = C_6(1+r'r)^{-(\nu+1)/2}\ \mathrm{etr}\{-M'M(I+\beta\beta')/2\}$$

$$\sum_{\alpha,[k];\phi} \frac{c_\nu(\alpha,[k];\phi)}{j!\ k!}\ C_\phi^{\alpha,[k]}(M'M(I+\beta r')(I+rr')^{-1}(I+r\beta')/2,\ M'M\beta\beta'/2)$$

where $\nu = N-K_1$ for OLS, $\nu = K_2$ for TSLS, the constant C_6 is given by

$$C_6 = [\Gamma((\nu+1)/2)/\pi^{n/2}\Gamma((\nu-n+1)/2)],$$

and the numerical coefficients $c_\nu(\alpha,[k];\phi)$ are given by

(54) $$c_\nu(\alpha,[k];\phi) = ((\nu+1)/2)_\alpha((\nu-n)/2)_k\ \theta_\phi^{\alpha,[k]}/(\nu/2)_\phi.$$

The remaining notation is explained in the references above.

Equation (53) for the OLS/TSLS estimators is evidently analogous to equation (49) for the LIML estimator, except that the numerical coefficients $a(\alpha,k;\phi)$ are much more complicated than the $c_\nu(\alpha,[k];\phi)$ in (53), and the powers of $(1+r'r)$ in the leading terms differ.

The special cases to be considered below are:

Case A: $(m_1, M_1) = 0, \quad (\mu, M) = 0$

Here $E(y,Y) = 0$, equations (1) and (3) are totally unidentified, (21) becomes simply $s|r \sim N(0, (1+r'r)I)$, (44) applies and (53) simplifies to:

$$\text{(55)} \qquad \text{pdf}(r) = C_6(1+r'r)^{-(\nu+1)/2}.$$

In this case we take $q = s's$.

Case B: $(\mu, M) = 0, \ (m_1, M_1) \neq 0$

Here Z_2 is not involved in the reduced form, and again the model is totally unidentified. This case yields the leading terms in the series expansions of pdf(s) and pdf(q) for the general case, and hence embodies some of the important properties of s and q. Here (21) and(55) apply, and $q = (s-m_1)'(s-m_1)$, with $m_1 = (Z_1'Z_1)^{1/2}(\pi_1-\Pi_1\alpha)/\omega$ (not $\gamma+M_1\beta$, because β and γ are undefined).

Case C: $\beta = 0$ $(\text{rank}(\mu,M) = \text{rank}(M) = n)$

Here the model is identified but Y is independent of u in (1). The density of the OLS/TSLS estimators in (53) simplifies to

$$\text{(56)} \qquad \text{pdf}(r) = C_6(1+r'r)^{-(\nu+1)/2}\text{etr}\{-M'M/2\}$$
$$\qquad {}_1F_1(\ (\nu+1)/2,\ \nu/2;\ M'M(I+rr')^{-1}/2).$$

The density of the LIML estimator in (49) does not simplify greatly. Here we may take $q = (s-\gamma)'(s-\gamma)$ and $\gamma \equiv m_1$.

We also give a number of results for the special case $n = 1$, because the general expressions are exceedingly complicated and difficult to evaluate numerically except when $n = 1$, or perhaps $n = 2$.

6. MEAN AND DENSITY OF q: OLS/TSLS ESTIMATORS

Case A: $(m_1,M_1) = 0, \ (\mu,M) = 0$

Since, in this case, $s|r \sim N(0, (1+r'r)I)$,

$$(57) \ \mathrm{pdf}(q|r) = [2^{K_1/2}\Gamma(K_1/2)]^{-1}(1+r'r)^{-K_1/2}\exp\{-q/2(1+r'r)\}q^{K_1/2-1},$$

and pdf(r) is given by (55). Multiplying (57) by (55), setting $r = vt$ $(v = r(r'r)^{-1/2}, \quad t^2 = r'r, \quad (dr) = 2^{-1}(t^2)^{n/2-1}(dt^2)(v'dv))$, and integrating over $v'v = 1$ and $t^2>0$, we obtain

$$(58) \qquad \mathrm{pdf}(q) = [2^{K_1/2}\Gamma(K_1/2)]^{-1}\exp\{-q/2\} \ q^{K_1/2-1}$$

$$C(\nu,n)\,{}_1F_1(n/2, \ (\nu+K_1+1)/2; \ q/2),$$

where

$$(59) \ C(\nu,n) = \Gamma((\nu+1)/2)\Gamma((\nu+K_1-n+1)/2)/\Gamma((\nu+K_1+1)/2)\Gamma((\nu-n+1)/2).$$

Also, it is straightforward to show that

$$(60) \qquad E(q) = (\nu-1)K_1/(\nu-n-1).$$

CASE B: $(\mu,M) = 0, \ (m_1,M_1) \neq 0$

Using (21) we find that

$$(61) \ \mathrm{pdf}(q|r) = [2^{K_1/2}\Gamma(K_1/2)]^{-1}(1+r'r)^{-K_1/2}\exp\{-q/2(1+r'r)\} \ q^{K_1/2-1}$$

$$\exp\{-r'M_1'M_1r/2(1+r'r)\}\,{}_0F_1(K_1/2; \ qr'M_1'M_1r/4(1+r'r)^2).$$

Multiplying (61) by (55), the integration over $r \in R^n$ can be accomplished as in Case A above and we find

(62) $\text{pdf}(q) \;=\; [2^{K_1/2}\Gamma(K_1/2)]^{-1}\exp\{-q/2\}\, q^{K_1/2-1}$

$$C(\nu,n) \sum_{j,k=0}^{\infty} \frac{(-1)^k}{j!k!}\, a^*_\nu(j,k)\, (q/2)^j\, C_{j+k}(M_1'M_1/2)$$

$${}_1F_1(j+k+n/2,\; 2j+k+(\nu+K_1+1)/2;\; q/2),$$

with

(63) $\quad a^*_\nu(j,k) = (1/2)_{j+k}((\nu+K_1-n+1)/2)_j/(K_1/2)_j((\nu+K_1+1)/2)_{2j+k}.$

The mean of q in this case is most easily obtained from

(64) $\quad E(q) \;=\; E_r[\, K_1(1+r'r) + r'M_1'M_1r].$

Multiplying (64) by (55) and integrating over r as before gives

(65) $\quad E(q) \;=\; [(\nu-1)K_1 + \text{tr}(M_1'M_1)]/(\nu-n-1).$

<u>CASE C</u>: $\beta = 0$

Consider first the mean of q (= MSE(s)). Rewrite (64) in the form

(66) $\quad E(q) \;=\; K_1 + E_r[(1+r'r)\text{tr}[A_1rr'(I+rr')^{-1}]].$

Multiplying (66) by pdf(r) in (56), the resulting integral over r is evidently invariant under the simultaneous transformations $A_1 \to H'A_1H$, $M'M \to H'M'MH$, $H \in O(n)$, because, after making these transformations, we can then transform $r \to Hr$, leaving the integral unchanged. Hence, averaging the integral over O(n), we find that the expectation of the second term in (66) is given by

(67) $\quad C_6\text{etr}\{-M'M/2\} \displaystyle\int_{r\in R^n} (1+r'r)^{-(\nu-1)/2}$

$$\sum_{\alpha,[1];\phi} \frac{((\nu+1)/2)_\alpha C_\phi^{\alpha,[1]}(M'M/2,\, A_1)C_\phi^{\alpha,[1]}((I+rr')^{-1},\, rr'(I+rr')^{-1})}{j!(\nu/2)_\alpha\, C_\phi(I_n)}(dr).$$

To evaluate the coefficients in (67) we use:

LEMMA 1: For $\mathrm{Re}(t) > -1$,

$$(68)\qquad \int_{x\in R^n} (1+x'x)^{-(t+n+1)/2} C_{\phi}^{\alpha,[k]}((I+xx')^{-1},\ xx'(I+xx')^{-1})\,(dx)$$

$$= \frac{\pi^{n/2}\Gamma((t+1)/2)((t+n)/2)_{\alpha}(1/2)_k}{\Gamma((t+n+1)/2)((t+n+1)/2)_{\phi}}\ \theta_{\phi}^{\alpha,[k]} C_{\phi}(I_n).$$

PROOF: The integral on the left in (68) may be written as the Laplace transform (with $p = (n+1)/2$):

$$\int_{R^n}\int_{S>0} \frac{\mathrm{etr}\{-S(I+xx')\}|S|^{(t+n+1)/2-p} C_{\phi}^{\alpha,[k]}(S,\ xx'S)}{\Gamma_n((t+n+1)/2)((t+n+1)/2)_{\phi}}\,(dS)(dx).$$

Transform to $z = S^{1/2}x$ and evaluate the Laplace transform using Davis (1979), equations (2.12) and (2.2), to obtain

$$\frac{\Gamma_n((t+n)/2)((t+n)/2)_{\alpha}(1/2)_k}{\Gamma_n((t+n+1)/2)((t+n+1)/2)_{\phi}(n/2)_k}\ \theta_{\phi}^{\alpha,[k]} C_{\phi}(I_n) \int_{R^n} \exp\{-z'z\}(z'z)^k (dz),$$

and this yields the right side of (68). □

Using (68) in (67) with $t = \nu-n-2$, $k = 1$, gives (for $\nu > n+1$)

$$(69)\qquad E(q) = K_1 + \frac{(\nu-1)}{(\nu-n-1)}\,\mathrm{etr}\{-M'M/2\}P(M'M,\ A_1),$$

where

$$(70)\qquad P(M'M,\ A_1) = \sum_{\alpha,[1];\phi} \frac{((\nu-2)/2)_{\alpha}((\nu+1)/2)_{\alpha}}{j!(\nu/2)_{\alpha}((\nu-1)/2)_{\phi}}\ \theta_{\phi}^{\alpha,[1]} C_{\phi}^{\alpha,[1]}(M'M/2, A_1/2)$$

is a polynomial in the matrices $M'M$ and A_1, invariant under $M'M \to H'M'MH$, $A_1 \to H'A_1H$, $H \in O(n)$. When $n = 1$ (69) simplifies to

$$(71) \qquad E(q) = K_1 + \left(\frac{\lambda_1+K_1}{\nu-2}\right) \exp\{-\lambda/2\}\ {}_1F_1((\nu-2)/2,\ \nu/2;\ \lambda/2),$$

where $\lambda_1 = M_1'M_1$ and $\lambda = M'M$. The second term in (71) is an increasing function of λ_1, a decreasing function of λ, and a decreasing function of ν (= K_2 for the TSLS estimator).

The density of $q = (s-\gamma)'(s-\gamma)$ can be obtained by much the same argument and we find:

$$(72) \qquad \mathrm{pdf}(q) = [q^{K_1/2-1}/2^{K_1/2}\Gamma(K_1/2)]\mathrm{etr}\{-M'M/2\}C(\nu,n)$$

$$\sum_{i,j,k,\ell=0}^{\infty} \frac{(-1)^{i+k}(q/2)^{i+\ell}}{i!j!k!\ell!} \sum_{\alpha,\phi\in\alpha.[k+\ell]} e_{\phi}^{i,\alpha,k,\ell} C_{\phi}^{\alpha,[k+\ell]}(M'M/2,\ M_1'M_1/2),$$

with coefficients $e_{\phi}^{i,\alpha,k,\ell}$ equal to

$$(73) \qquad \frac{((\nu+1)/2)_{\alpha}(i+\ell+(\nu+K_1)/2)_{\alpha}((\nu+K_1-n+1)/2)_{i+\ell}(1/2)_{k+\ell}}{(\nu/2)_{\alpha}(K_1/2)_{\ell}(i+\ell+(\nu+K_1+1)/2)_{\phi}((\nu+K_1+1)/2)_{i+\ell}}\ \theta_{\phi}^{\alpha,[k+\ell]}.$$

When $n = 1$ this reduces to

$$(74) \qquad \mathrm{pdf}(q) = \exp\{-q/2\}q^{K_1/2-1}/[2^{K_1/2}\Gamma(K_1/2)]C(\nu,1)$$

$$\sum_{k,\ell=0}^{\infty} \frac{(1/2)_{\ell}((\nu+K_1)/2)_{k+\ell}}{k!\ell!(K_1/2)_{\ell}((\nu+K_1+1)/2)_{k+2\ell}}(\lambda_1 q/4)^{\ell} p_k(\lambda,\lambda_1)$$

$${}_1F_1(\ell+1/2,\ k+2\ell+(\nu+K_1+1)/2;\ q/2),$$

where

$$(75) \quad p_k(\lambda,\lambda_1) = \exp\{-(\lambda+\lambda_1)/2\} \sum_{j=0}^{k} \binom{k}{j} [((\nu+1)/2)_j/(\nu/2)_j](\lambda/2)^j(\lambda_1/2)^{k-j}.$$

THE GENERAL CASE

To evaluate $E(q) = \mathrm{MSE}(s)$ in the general case, first write (27) in the form

(76) $E(q) = K_1 + \beta' M_1' M_1 \beta + tr[A_1 E(rr')] - 2\beta' M_1' M_1 E(r).$

Now, using equations (46) and (48) from Hillier (1985a), the terms $E(rr')$ and $E(r)$ in (76) can be expressed as inverse Laplace transforms, and these can be evaluated using results from Davis (1979). Thus,

(77)
$$2\beta' M_1' M_1 E(r) = \Gamma_n(\nu/2) etr\{-M'M/2\}\, a_n \int_{Re(W)>0} etr(W)|W|^{-\nu/2}$$
$$ {}_1F_0(\nu/2;\ M'MW^{-1}/2)\ tr[\bar{M}(\beta\beta' M_1' M_1 + M_1' M_1 \beta\beta')\bar{M}' W^{-1}/2](dW)$$
$$= etr\{-M'M/2\} \sum_{\alpha,[1];\phi} \frac{(\nu/2)_\alpha}{j!(\nu/2)_\phi} \theta_\phi^{\alpha,[1]} C_\phi^{\alpha,[1]}(M'M/2,\ M'M(\beta\beta' M_1' M_1 + M_1' M_1 \beta\beta')/2),$$

where $a_n = [2^{n(n-1)/2}/(2\pi i)^{n(n+1)/2}]$ and, as before, $\bar{M}$ is any nxn matrix such that $\bar{M}'\bar{M} = M'M$. Also,

(78)
$$tr[A_1 E(rr')] = \Gamma_n(\nu/2) etr\{-M'M/2\}\, a_n \int_{Re(W)>0} etr(W)|W|^{-\nu/2}$$
$${}_1F_0(\nu/2;\ M'MW^{-1}/2)\Big\{tr[\bar{M}A_1\bar{M}'W^{-1}\bar{M}\beta\beta'\bar{M}'W^{-1}]/4$$
$$+ (1+\beta'\bar{M}'W^{-1}\bar{M}\beta/2)\,tr[A_1(I-\bar{M}'W^{-1}\bar{M}/2)]/(\nu-n-1)\Big\}\ (dW).$$

This integral can be resolved into the following three terms:

(79a)
$$\frac{tr(A_1)}{(\nu-n-1)}\Big\{1 + etr\{-M'M/2\} \sum_{\alpha,[1];\phi} \frac{(\nu/2)_\alpha}{j!(\nu/2)_\phi} \theta_\phi^{\alpha,[1]} C_\phi^{\alpha,[1]}(M'M/2,\ M'M\beta\beta'/2)\Big\},$$

(79b)
$$\frac{(\nu-n-2)}{(\nu-n-1)} etr\{-M'M/2\}\Big\{ \sum_{\alpha,[1];\phi} \frac{(\nu/2)_\alpha}{j!(\nu/2)_\phi} \theta_\phi^{\alpha,[1]} C_\phi^{\alpha,[1]}(M'M/2,\ M'MA_1/2)$$
$$+ \sum_{\alpha,[1],[1];\rho} \frac{(\nu/2)_\alpha}{j!(\nu/2)_\rho} \theta_\rho^{\alpha,[1],[1]} C_\rho^{\alpha,[1],[1]}(M'M/2,\ M'MA_1/2,\ M'M\beta\beta'/2)\Big\},$$

and

(79c)

$$- \operatorname{etr}\{-M'M(I+\beta\beta')/2\}\left\{\sum_{\alpha,[1],[\ell];\rho} \frac{(\nu/2)_\alpha (1)_\ell}{j!\,\ell!\,(\nu/2)_\rho} \theta_\rho^{\alpha,[1],[\ell]}\right.$$

$$\left. C_\rho^{\alpha,[1],[\ell]}(M'M(I+\beta\beta')/2,\ M'MA_1/2,\ M'M\beta\beta'/2)\right\}.$$

Inserting equations (78) and (79a-c) into equation (76) yields an expression for E(q) in the general case. These equations simplify greatly, of course, when $n = 1$, and we find

(80) $$E(q) = K_1(1+\beta^2) + \beta^2(\nu-3)(\lambda_1+K_1)/2 +$$

$$\left[2\lambda_1\beta^2 + (\lambda_1+K_1)[1-\beta^2-\beta^2(\nu+\lambda)(\nu-3)/2]/(\nu-2)\right]e^{-\lambda/2}{}_1F_1((\nu-2)/2,\ \nu/2;\ \lambda/2).$$

This reduces to equation (71) when $\beta = 0$.

The density of q may be obtained by similar but much more tedious calculations and we omit this result.

7. THE LIML ESTIMATOR FOR THE EXOGENOUS COEFFICIENTS

The joint density $\mathrm{pdf}(r_1,\ s_1)$ is simply the product of equation (49) with the conditional normal density $\mathrm{pdf}(s_1|r_1)$ in equation (21), and this might well be the most useful result for these estimators from the point of view of hypothesis testing. However, to integrate out r_1 it is a simple matter to adapt the argument in Phillips (1984a) to this case, and hence to obtain an expression for $\mathrm{pdf}(s_1)$ in an operator form. We omit the details of this calculation, and give here instead the results for Cases A and B that provide the leading terms in the more general expression for the density.

CASE A: $(m_1, M_1) = 0$, $(\mu, M) = 0$

Since (omitting the subscripts on s and r for simplicity) $s|r \sim N(0, (1+r'r)I)$, and pdf(r) is as in (44), it is reasonably straightforward to obtain:

(81) $$\mathrm{pdf}(s) = (2\pi)^{-K_1/2}\exp\{-s's/2\}\ C(n,n)\ {}_1F_1(n/2,\ (n+K_1+1)/2;\ s's/2),$$

where $C(\nu,n)$ is defined in (59). Transforming to $q = s's$ then yields pdf(q) in exactly the form (58), but with ν replaced by n. Hence, in this case the densities of the OLS, TSLS, and LIML estimators have exactly the same form apart from a degrees of freedom parameter ν which takes the values $N-K_1$, K_2, and n respectively.

CASE B: $(\mu, M) = 0$, $(m\ , M\ ) \neq 0$.

Here (21) and (44) hold for pdf(s|r) and pdf(r) respectively. Using the same approach as in section 6 to integrate out r we find:

(82) $$\mathrm{pdf}(s) = (2\pi)^{-K_1/2}\exp\{-(s-m_1)'(s-m_1)/2\}\ C(n,n)$$

$$\sum_{j,k=0}^{\infty} \frac{(-1)^k(1/2)_{j+k}((K_1+1)/2)_j}{j!k!(1/2)_j((n+K_1+1)/2)_{2j+k}}\ C^{j,k}_{j+k}(M_1'(s-m_1)(s-m_1)'M_1/4,\ M_1'M_1/2)$$

$$_1F_1(j+k+n/2,\ 2j+k+(n+K_1+1)/2;\ (s-m_1)'(s-m_1)/2).$$

Transforming to $q = (s-m_1)'(s-m_1)$ yields pdf(q) exactly as in (62) except that ν is replaced by n throughout, so again the densities of the three estimators differ only in repect of the degrees of freedom parameter, ν.

Unfortunately, Case C ($\beta = 0$) is no simpler in the case of the LIML estimator than is the general case (see equation (43) above for evidence of this), so this special case must be treated by the methods described above for the general case. It does not seem possible to obtain expressions analogous to (81) and (82) for the general case.

8. EXTENSIONS

As indicated in section 4, the result in equation (37) can be used to obtain, for instance, the joint density of the LIML and TSLS estimators, $pdf(r_1,r_2)$. By an obvious extension of the argument leading to equation (21) we also have the conditional result:

$$(83)\qquad (s_1,s_2)|(r_1,r_2) \sim N(\ (m_1,M_1)\begin{pmatrix} 1 & 1 \\ -r_1 & -r_2 \end{pmatrix},\ I_{K_1} \otimes \begin{pmatrix} 1+r_1'r_1, & 1+r_1'r_2 \\ 1+r_2'r_1, & 1+r_2'r_2 \end{pmatrix}),$$

so that the joint density $pdf(r_1,s_1,r_2,s_2)$ is easily obtained. Such results may be useful in comparing the properties of the two estimators.

More generally, Theorem 1 provides (after transforming $r \to r_1$ in equation (34)) the joint density $pdf(r_1,f_1,B,(x,X))$. Multiplying by $pdf(x_1,X_1)$ then gives $pdf(r_1,f_1,B,(x,X),(x_1,X_1))$, which is equivalent to the joint distribution of the (canonical) minimal sufficient statistic $t = (S,(x,X),(x_1,X_1))$. A variety of useful joint distribution results can then be obtained by transformation in this distribution. In particular, motivated by the curved exponential character of this model, one can consider the family of (one-to-one) transformations $t \to (\hat{\theta}(t),a(t))$ in which $\hat{\theta}(t)$ is the LIML estimate for all of the parameters in (2) (subject to (4) and (5)), and the $a(t)$ are statistics defined on the manifold (in t-space) defined by $\hat{\theta}(t)$ = const. that essentially capture the information lost in the reduction $t \to \hat{\theta}(t)$ (see Barndorff-Neilson (1980)). Of particular interest is the question of whether or not there exists an exactly ancillary statistic $a(t)$. These matters will be pursued in future work.

APPENDIX

Proof of Theorem 1 and its Corollaries

Proof of Theorem 1: The transformation $R \to (F,H)$ described in the Theorem is one-to-one, and the volume element transforms as

$$(A.1) \qquad (dR) = \prod_{i<j}(f_i-f_j)\prod_{i=1}^{n+1} df_i (H'dH)$$

Partitioning $H = (h,H_2)$, H_2 is an element of the Stiefel manifold orthogonal to h. That is, $H_2'H_2 = I_n$ and $H_2'h = 0$. This manifold may be generated by choosing, for each fixed h, a fixed matrix G, say, on the manifold (i.e., $G'G = I_n$, and $G'h = 0$, and so $GG' = I - hh'$), and then setting $H_2 = GH_1$, where H_1 ranges over the orthogonal group O(n). To ensure that the transformation $R \to (F,h,H_1)$ is one-to-one we can replace the condition that the elements in the first row of H_2 are positive by the condition that the elements in the first row of H_1 are positive. The volume element on O(n+1) transforms as

$$(A.2) \qquad (H'dH) = (h'dh)(H_1'dH_1),$$

where $(h'dh)$ denotes the invariant measure on the surface of the unit (n+1)-sphere, and $(H_1'dH_1)$ that on O(n) (see Muirhead (1982, Lemma 9.5.3, p. 397)).

Next, partition $h = (h_1,h_2')'$, with h_2 n×1, and define $r = -h_1^{-1}h_2$. The transformation $h \to r$ is one-to-one because h_1 is taken to be positive, and the transformation of the volume element is given in Phillips (1984b) as:

$$(h'dh) = (1+r'r)^{-(n+1)/2} \bigwedge_{i=1}^{n} dr_i \ .$$

The matrix G that defines the transformation $H \to (h,H_1)$ may be defined in terms of r by

(A.3) $$G = \begin{pmatrix} r' \\ I_n \end{pmatrix}(I+rr')^{-1/2},$$

since, under the transformation $r = -h_1^{-1}h_2$,

$$h = \begin{pmatrix} 1 \\ -r \end{pmatrix}(1+r'r)^{-1/2}.$$

Now define $\bar{F}_2$ by $F_2 = f_1\bar{F}_2$, so that $1 > \bar{f}_2 > \bar{f}_3 >\ldots> \bar{f}_{n+1} > 0$, and

(A.4) $$\prod_{i<j}(f_i-f_j)\bigwedge_{j=1}^{n+1} df_i = (f_1^{n(n+3)/2}df_1)(\prod_{j=2}^{n+1}(1-\bar{f}_j)\prod_{\substack{i,j=2\\ i<j}}^{n+1}(\bar{f}_i-\bar{f}_j)\bigwedge_{j=2}^{n+1} d\bar{f}_j).$$

Finally, set $B = H_1\bar{F}_2H_1'$, $0 < B < I$. Since the elements in the first row of H_1 are positive, and the elements on the diagonal of $\bar{F}_2$ are ordered, the transformation $(\bar{F}_2,H_1) \to B$ is one-to-one and we have

(A.5) $$\prod_{\substack{i,j=2\\ i<j}}^{n+1}(\bar{f}_i-\bar{f}_j)\bigwedge_{j=2}^{n+1} d\bar{f}_j(H_1'dH_1) = (dB).$$

Combining (A.1) - (A.5), and noting that, in (A.4), $\Pi_{j=2}^{n+1}(1-\bar{f}_j) = |I-B|$, we have

(A.6) $$R = f_1\left[(1+r'r)^{-1}\begin{pmatrix} 1 \\ -r \end{pmatrix}\begin{pmatrix} 1 \\ -r \end{pmatrix}' + (r,I)'(I+rr')^{-1/2}B(I+rr')^{-1/2}(r,I)\right]$$

(A.7) $$(dR) = f_1^{n(n+3)/2}(1+r'r)^{-(n+1)/2}|I-B|df_1\bigwedge_{i=1}^{n} dr_i(dB).$$

Since $H_2 = GH_1$, $H_1 = G'H_2$ and we can express B in terms of r, $\bar{F}_2$, and H_2 as

$$B = (I+rr')^{-1/2}(r,I)H_2\bar{F}_2H_2'(r,I)'(I+rr')^{-1/2},$$

thus completing the proof of part (a).

Part (b) follows from equations (31), (A.6), and (A.7) on noting that $|R|^{(m-n-2)/2} = f_1^{(n+1)(m-n-2)/2}|B|^{(m-n-2)/2}$.

Proof of Corollary 1.1: The matrix B may be integrated out of $pdf(r,f_1,B)$ in equation (34) by using Muirhead (1982, Theorem 7.4.2). This gives

$$\text{(A.8)} \qquad pdf(r,f_1) = C_1^* f_1^{m(n+1)/2-1}(1+r'r)^{-(n+1)/2}|V|^{-m/2}$$

$$\exp\{-\tfrac{1}{2}f_1(1,-r')V^{-1}(1,-r')'/(1+r'r)\}\ {}_1F_1((m-1)/2, (m+n+2)/2, -\tfrac{1}{2}f_1 G'V^{-1}G)$$

with G as in (A.3) above. Equation (35) then follows on using the Kummer formula for the confluent hypergeometric function (James (1964, equation (51))): ${}_1F_1(a,\ c;\ S) = \operatorname{etr}\{S\}{}_1F_1(c-a,\ c;\ -S)$.

Proof of Corollary 1.2: To integrate out r in equation (35) we can invert the transformation $h \to r$ used above to define $h = \begin{pmatrix} 1 \\ -r \end{pmatrix}(1+r'r)^{-1/2}$. Then, only the confluent hypergeometric function in (35) involves h and its argument is, apart from the factor $(f_1/2)$, the rank-n matrix $(I-hh')V^{-1}$. Integrating term by term we have

$$\int_{h'h=1} C_\lambda((I-hh')V^{-1})(h'dh) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \int_{O(n+1)} C_\lambda\left(H\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}H'V^{-1}\right)(dH)$$

$$= \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} C_\lambda\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} C_\lambda(V^{-1})/C_\lambda(I_{n+1}),$$

which vanishes unless ℓ_{n+1}, the (n+1)-th part of the partition λ, is zero, and is equal to

$$\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \frac{(n/2)_\lambda}{((n+1)/2)_\lambda} C_\lambda(V^{-1})$$

when $\ell_{n+1} = 0$. Since $(n/2)_\lambda = 0$ unless $\ell_{n+1} = 0$, this yields equation (36).

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