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LOCALLY OPTIMAL ONE-SIDED TESTS FOR MULTIPARAMETER HYPOTHESES[1]

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Headnote

Recently, there has been an increased awareness of the one-sided nature of many econometric testing problems. For testing $H_0 : \theta = 0$ against $H_a : \theta \neq 0$ where θ is $p \times 1$, SenGupta and Vermeire (1986) introduced the class of locally most mean powerful (LMMP) unbiased tests. They are constructed to maximize the mean curvature of the power hypersurface in the neighbourhood of $\theta = 0$. Our interest is in testing H_0 against $H_a^+ : \theta_1 \geq 0, \ldots, \theta_p \geq 0$ with at least one strict inequality. We show how LMMP critical regions can be constructed and note that they suggest a new form for the Lagrange multiplier test in one-sided testing problems. Applications considered in the context of the linear regression model include joint one-sided testing for non-zero regression coefficients, autoregressive disturbances, heteroscedastic disturbances, random regression coefficients and variance components.

KEYWORDS: Autocorrelation, heteroscedasticity, invariance, Lagrange multiplier test, linear regression, locally most mean powerful test.

1. INTRODUCTION

In recent years, there has been an increased awareness of the potential one-sided nature of many econometric testing problems. It is recognised that economic theory frequently can provide information regarding the signs of various parameters and that this information should be used to improve the power of testing procedures. Also, functional considerations such as variances always being positive can give rise to natural one-sided testing problems. A good example of this increased awareness is the development of tests for inequality restrictions on regression coefficients; see for example Gouriéroux, Holly and Monfort (1982), Farebrother (1986), Hillier (1986), Judge and Yancey (1986), King and Smith (1986), Kodde and Palm (1986), Rogers (1986) and Wolak (1987). Much of this literature is concerned with one-sided versions of the likelihood ratio, Wald and Lagrange multiplier (LM) tests, the latter also being known as the Kuhn-Tucker test. Unfortunately, the asymptotic distributions of these test statistics under the null hypothesis are weighted sums of chi-squared distributions and the degenerate distribution at zero which make them difficult to apply. Also little is known of their small-sample performance - their main justification being asymptotic. In the case of testing two and three coefficient restrictions, Hillier (1986) has derived the exact distribution of the likelihood ratio statistic. However, it is clear that the complexity of deriving these distributions increases sharply with the number of restrictions.

Many testing problems involving the linear regression model can be reduced, either by conditioning on sufficient statistics or by invariance, to one of testing a simple null hypothesis against a composite alternative.[2] Ideally we would then like to use a uniformly most powerful (UMP) test but unfortunately such tests rarely exist. Cox and Hinkley (1974, p.102) discuss three possible approaches to constructing tests against composite alternatives when no UMP test exists. They are: (i) using a test which maximizes power at a "somewhat arbitrary 'typical' point" in the alternative parameter space, (ii) removing this arbitrariness by choosing the point to be close to the null hypothesis which leads to the locally best (LB) test and (iii) choosing the test which maximizes some weighted average of powers. Option (i) is sometimes known as the point optimal solution and is reviewed in some detail in King (1987b).

The LB solution has found some acceptance in the literature. Suppose we wish to test $H_0 : \theta = 0$ based on x which is an $n \times 1$ random vector whose distribution has probability density function $f(x|\theta)$ where θ is a $p \times 1$ vector of unknown parameters. When $p = 1$, the LB test of $H_0 : \theta = 0$ against $H_a^+ : \theta > 0$ is that with critical regions of the form

$$\left.\frac{\partial \ln f(x|\theta)}{\partial \theta}\right|_{\theta=0} > c_1 \tag{1}$$

where c_1 is a suitably chosen constant (Ferguson, 1967, p.235). It should also be noted that this test is equivalent to the one-sided LM test based on the square-root of the standard LM test statistic. Against the two-sided alternative $H_a : \theta \neq 0$ when $p = 1$, critical regions of the form

$$\left.\frac{\partial^2 f(x|\theta)}{\partial \theta^2}\right|_{\theta=0} > c_2 f(x|\theta)\Big|_{\theta=0} + c_3 \left.\frac{\partial f(x|\theta)}{\partial \theta}\right|_{\theta=0}$$

yield unbiased (LBU) tests when the constants c_2 and c_3 are chosen so that the critical region has the nominated size and is locally unbiased. Regions of this kind were labelled type A regions by Neyman and Pearson (1936).

For $p \geq 2$, the main emphasis in the literature has been on constructing LBU tests of $H_0 : \theta = 0$ against $H_a : \theta \neq 0$. Neyman and Pearson (1938) suggested the use of type C LBU regions when $k = 2$. These regions have constant power in the neighbourhood of $\theta = 0$ along a given family of concentric ellipses. They require knowledge of the relative importance of power in various directions in the neighbourhood of $\theta = 0$. Isaacson (1951) introduced type D regions to rectify this objection (also see Lehmann, 1959, p.342). Type D regions are obtained by maximizing the Gaussian curvature of the power function at $\theta = 0$. Recently, SenGupta and Vermeire (1986) introduced the class of locally most mean powerful unbiased (LMMPU) tests which maximize the mean curvature of the power hypersurface in the neighbourhood of $\theta = 0$ within the class of unbiased tests. The locally most mean powerful (LMMP) criterion is appealing because it incorporates both (ii) and (iii) of Cox and Hinkley's suggestions. SenGupta and Vermeire show that LMMPU tests have critical regions of the form

$$\sum_{i=1}^{p} \left. \frac{\partial^2 f(x|\theta)}{\partial \theta_i^2} \right|_{\theta=0} > c_0 \left. f(x|\theta) \right|_{\theta=0} + \sum_{i=1}^{p} c_i \left. \frac{\partial f(x|\theta)}{\partial \theta_i} \right|_{\theta=0} \qquad (2)$$

where the constants $c_0, c_1, \ldots, c_p$ are chosen so that the critical region is locally unbiased and of the nominated size. While finding these constants may not be particularly easy, at least we have a method of test construction. In contrast, type D regions have to be guessed and then verified.

Our interest in this paper is in the multivariate analogue of the one-sided testing problem, i.e. $H_0 : \theta = 0$ against $H_a^+ : \theta_1 \geq 0, \ldots, \theta_p \geq 0$ with at least one strict inequality.[3] There is always the possibility that one may be able to find a test that is LB in all directions from H_0 in the p-dimensional parameter space. Neyman and Scott (1967) call this property "robustness of optimality" while King and Evans (1988) call such tests uniformly LB (ULB). Like UMP tests, ULB tests may not always exist. In such cases, one may wish to consider a weaker optimality criterion. Our aim in this paper is to explore the option of constructing LMMP critical regions.

The plan of this paper is as follows. The next section gives the general method of constructing a LMMP test of H_0 against H_a^+ and a proof of this central result is given in the appendix. The test suggests a new form for the LM test which takes into account the one-sided nature of the alternative hypothesis. Because it has a N(0,1) asymptotic distribution under H_0, the new LM-type test can be readily applied in contrast to the Kuhn-Tucker test. It has the additional attraction of having a small-sample optimal power property when no nuisance parameters are present. In section 3, the method is used to construct a LMMP invariant (LMMPI) test for inequality restrictions on linear regression coefficients. We find that the LMMPI test is identical to King and Smith's (1986) additive t-test. In section 4, the method is used to construct a general LMMPI test of the error covariance matrix in the linear regression model. Applications of this test considered in section 5 include testing for joint first-order and fourth-order autoregressive (AR) disturbances, quarter-dependent AR(4) disturbances, heteroscedastic disturbances including heteroscedasticity arising from random regression coefficients and autocorrelation generated by variance components. Some concluding remarks are made in the final section.

2. THEORY

Assume we have observed the $n \times 1$ random vector x which has probability density function $f(x|\theta)$ where θ is a $p \times 1$ vector of unknown non-negative parameters. We are interested in testing

$$H_0 : \theta = 0$$

against

$$H_a : \theta > 0 .$$

A proof of the following result is given in the appendix.

Theorem: Suppose $f(x|\theta)$ is a density function which has at least one non-zero first-order derivative at $\theta = 0$ with respect to θ_i, $i = 1, \ldots, p$, and whose second-order derivatives at $\theta = 0$ all exist and are continuous. Then a LMMP test of H_0 against H_a is given by the critical region

$$s = \sum_{i=1}^{p} \left. \frac{\partial \ell n f(x|\theta)}{\partial \theta_i} \right|_{\theta=0} > c \qquad (3)$$

where c is an appropriate constant chosen to give the test the required significance level.

Observe that when $p = 1$, (3) reduces simply to (1). Also (3) is a much simpler critical region to construct than (2) given that it requires only one constant (critical value) to be determined while (2) requires the simultaneous determination of p+1 constants. Furthermore, s is the sum of the LB test statistics which each test H_0 against

$$H_a^i : \theta_i > 0 \quad \text{and} \quad \theta_j = 0, \quad j \neq i \ , \quad j = 1, \ldots, p \ ,$$

for i = 1, ..., p. This is advantageous in that if a calculated test statistic suggests H_0 should be rejected, we can look at the individual components of the statistic to see which θ_i are contributing most to this outcome. Unfortunately, under H_a it is extremely unlikely that the component statistics will be mutually independent and this should be taken into account when interpreting the results. The form of (3) also implies that a LMMP test is a LB test in the direction given by

$$\theta_1 = \theta_2 = \ldots = \theta_p > 0. \tag{4}$$

In other words, a LMMP test is the test with the steepest sloping power function at H_0 in the direction (4). This can be verified by replacing θ by $\tau\ell$, where ℓ is the $p \times 1$ vector of ones and τ is a nonnegative scalar. By using (1) to construct a LB test of $\tau = 0$ against $\tau > 0$, one gets (3). Also note that if a ULB test exists, then the LMMP test is also ULB. This follows from the LMMP test being a LB test in the direction (4) and therefore being LB in all directions due to the existence of a ULB test. As SenGupta and Vermeire (1986) note, LMMP tests are not, in general, invariant under reparameterization. This is a weakness which in some circumstances can be a strength. For example, it may be desirable to have good power in certain directions from H_0.

So far, we have assumed the distribution of s is known, thus allowing an appropriate critical value to be found. For applications in which this is not the case, large-sample asymptotic critical values can be derived by the simple adaptation of the asymptotic theory of the LM test. Observe that s is the sum of the elements of the score vector evaluated under H_0. Under standard regularity conditions (see for example, Godfrey (1988, pp.6-7)) but without a requirement that the true parameter value be an interior point of the parameter space, it follows that

$$n^{-1/2}s \overset{a}{\sim} N(0,\ell' V\ell)$$

where $V = \lim_{n\to\infty} n^{-1}\mathcal{J}$ and $\mathcal{J}$ is the information matrix (minus the expected value of the Hessian matrix) evaluated at $\theta = 0$. Therefore, a LMMP critical region of approximately the desired size, can be obtained by rejecting H_0 for large values of

$$s^* = s \,/\, (n\ell' V\ell)^{1/2}$$

which is assumed to have a N(0,1) distribution under H_0.

Clearly, a test based on s* can also be viewed as an LM-type test which takes into account the joint one-sided nature of the alternative hypothesis and which maximizes the mean slope of the power function at H_0. This is assuming there are no nuisance parameters present. In the presence of nuisance parameters that cannot be eliminated by conditioning on sufficient statistics or by invariance arguments, the following analogous LM-type test suggests itself.

Suppose θ is partitioned as $\theta = (\theta_1', \theta_2')'$ where θ_1 is $p_1\times1$, θ_2 is $p_2\times1$ and $p_1+p_2 = p$. Our interest is in testing $H_0 : \theta_1 = 0$ against $H_a : \theta_1 > 0$ when θ_2 is unknown. If the value of θ_2 was known, then a LMMP test of H_0 is given by

$$s\dagger = \sum_{i=1}^{p_1} \frac{\partial \ell n f(x|\theta)}{\partial \theta_i}\Bigg|_{\theta = (0', \theta_2')'} > c .$$

The LM approach suggests that the unknown θ_2 should be replaced by its maximum likelihood estimate under H_0, denoted $\hat{\theta}_2$. Let $\mathcal{J}^{11}$ denote the upper $p_1 \times p_1$ block of the inverse of the information matrix and let $\hat{s}\dagger$ and $\hat{\mathcal{J}}^{11}$ denote $s\dagger$ and $\mathcal{J}^{11}$, respectively, evaluated at $\hat{\theta} = (0', \hat{\theta}_2')'$. The asymptotic theory of the LM test implies that rejecting H_0 for large values of

$$\hat{s}\dagger \;/\; (\ell'(\hat{\mathcal{J}}^{11})^{-1}\ell)^{1/2} , \qquad (5)$$

assuming a N(0,1) distribution under H_0, is an asymptotically valid test.[4] It can be viewed as an LM-type test which takes into account the joint one-sided nature of the alternative hypothesis. In contrast to the Kuhn-Tucker test, this new LM-type test can be readily applied because of its N(0,1) asymptotic distribution. It has the additional attraction of having a small-sample optimal power property when no nuisance parameters are present.

3. INEQUALITY TESTING OF RESTRICTIONS ON LINEAR REGRESSION COEFFICIENTS

Consider the classical linear regression model

$$y = X\beta + u \qquad (6)$$

where y is $n \times 1$, X is an $n \times k$ nonstochastic matrix of rank $k < n$, β is a $k \times 1$ vector of unknown parameters and u is an $n \times 1$ error vector. It is assumed that $u \sim N(0, \sigma^2 I_n)$ where σ^2 is unknown.

We are interested in testing

$$H_0 : R\beta = r$$

against

$$H_a : R\beta > r$$

where R is a known $p \times k$ matrix of rank $p < k$ and r is a known $p \times 1$ vector. Without loss of generality, we assume

$$R = [0 : I_p] \quad \text{and} \quad r = 0$$

because, as King and Smith (1986, p.369) show, if this is not the case, the testing problem can be reparameterized to one in which it is true.

Partition (6) as

$$y = X_1\beta_1 + X_2\beta_2 + u$$

where X_1 is $n \times q$, X_2 is $n \times p$ such that $q + p = k$, $X = [X_1 : X_2]$, β_1 is $q \times 1$ and β_2 is $p \times 1$ such that $\beta' = (\beta_1' : \beta_2')$. Our problem is now one of testing $H_0 : \beta_2 = 0$ against $H_a : \beta_2 > 0$.

This problem is invariant under the class of transformations

$$y \longrightarrow \gamma_0 y + X_1\gamma$$

where γ_0 is a positive scalar and γ is a $q \times 1$ vector. Let $M_1 = I_n - X_1(X_1'X_1)^{-1}X_1'$ and let P_1 be an $(n-q) \times n$ matrix such that $P_1P_1' = I_{n-q}$ and $P_1'P_1 = M_1$. Then

$$v = P_1y \,/\, (y'P_1'P_1y)^{1/2}$$

is a maximal invariant. Because $P_1y \sim N(P_1X_2\beta_2, \sigma^2 I_{n-q})$, v has density

$g(v)dv \;=$

$$\int_0^{\infty} (2\pi\sigma^2)^{-(n-q)/2} \exp\left\{\frac{-1}{2\sigma^2}\,(\lambda^2 - 2\lambda v'P_1X_2\beta_2 + \beta_2'X_2'M_1X_2\beta_2)\right\}\lambda^{n-q-1}d\lambda dv$$

where dv denotes the uniform measure on the surface of the unit m-sphere.

Making use of the fact that $g(v)\Big|_{\beta_2=0}$ is a constant, one can show that

$$\frac{\partial \ell n g(v)}{\partial \beta_2}\Bigg|_{\beta_2=0} = \left(\frac{1}{g(v)}\,\frac{\partial g(v)}{\partial \beta_2}\right)\Bigg|_{\beta_2=0} = aX_2'P_1'v$$

where a is a positive constant. The theorem of the previous section implies that rejecting H_0 for large values of

$$s = \frac{\ell' X_2'M_1y}{(y'M_1y)^{1/2}}$$

yields a LMMPI test where ℓ is a $p \times 1$ vector of ones.

This test is equivalent to the additive t-test proposed by King and Smith (1986). To see this, note that their test is applied as a one-sided t-test of $H_0 : \alpha = 0$ against $H_a : \alpha > 0$ in the regression

$$y = X_1\beta + \alpha z + u \qquad (7)$$

where $z = X_2\ell$. The ordinary least squares (OLS) estimator of α in (7) is

$$\hat{\alpha} = (z'M_1z)^{-1}z'M_1y$$

and the unbiased OLS estimator of the error variance is

$$\hat{\sigma}^2 = y'(M_1 - M_1 z(z'M_1 z)^{-1} z'M_1)y \,/\, (n-q-1).$$

Thus the test statistic is

$$t = \hat{\alpha} \,/\, \left\{\hat{\sigma}(z'M_1 z)^{-1/2}\right\}$$

$$= \left\{(n-q-1)z'M_1 z\right\}^{-1/2} z'M_1 y \,/\, \left\{y'M_1 y - y'M_1 z(z'M_1 z)^{-1} z'M_1 y\right\}^{1/2}$$

$$= \left\{(n-q-1)z'M_1 z\right\}^{-1/2} s \,/\, \left\{1 - (z'M_1 z)^{-1} s^2\right\}$$

which is clearly a monotonic increasing function of the test statistic s.

This is an interesting example of a LMMPI test because King and Smith's test was originally constructed as a point-optimal test which optimizes power along the ray $\beta_{q+1} = \beta_{q+2} = \dots = \beta_k > 0$. For a range of testing situations involving p = 2,3 and 4, King and Smith compared the power of their additive-t test with the power envelope, the power of the F test and, for p = 2 only, the exact likelihood ratio test. Under H_a, they found their test is typically more powerful than the F test and, more often than not, also the likelihood ratio test. Their study also indicated that the additive-t test can have power within five per cent of the power envelope over a wide range of the parameter space, especially when the X_2 regressors are multicollinear. In contrast, the power of the F test was almost never found to be within five per cent of the power envelope.

4. TESTING THE ERROR COVARIANCE MATRIX OF THE LINEAR REGRESSION MODEL

Consider the normal linear regression model (6) with non-spherical disturbances, i.e. $u \sim N(0,\sigma^2\Omega(\theta))$, where σ^2 is an unknown scalar and $\Omega(\theta)$ is a symmetric matrix that is positive definite for θ in a subset of $(\{0\}\cup R^+)^k$ which is of interest. Without loss of generality, it is assumed that $\Omega(0) = I_n$.

We are interested in testing $H_0 : \theta = 0$ against $H_a : \theta > 0$. This problem is invariant under the group of transformations

$$y \longrightarrow \gamma_0 y + X\gamma$$

where γ_0 is a positive scalar and γ is a $k \times 1$ vector. Let $M = I_n - X(X'X)^{-1}X'$ and $e = My$ be the OLS residual vector from (6). In the case of $p = 1$, King and Hillier (1985) showed that a LB invariant (LBI) test against H_a is to reject H_0 for small values of

$$e'A_0e \,/\, e'e ,$$

where

$$A_0 = -\partial\Omega(\theta)/\partial\theta\Big|_{\theta=0} = \partial\Omega^{-1}(\theta)/\partial\theta\Big|_{\theta=0} .$$

For $p > 1$, the theorem of section 2 can be applied to construct a LMMPI test. Let $m = n - k$ and let P be an $m \times n$ matrix such that $PP' = I_m$ and $M = P'P$. The vector

$$v = Pe \,/\, (e'P'Pe)^{1/2}$$

is a maximal invariant under the above group of transformations. As King and Hillier note, the density of v can be shown to be

$$f(v)dv = \frac{1}{2}\,\Gamma(m/2)\pi^{-m/2}\left|P\Omega(\theta)P'\right|^{-1/2}\left(v'(P\Omega(\theta)P')^{-1}v\right)^{-m/2}dv , \qquad (8)$$

where dv denotes the uniform measure on the surface of the unit m-sphere. Let

$$A_i = -\partial\Omega(\theta)/\partial\theta_i\Big|_{\theta=0} = \partial\Omega^{-1}(\theta)/\partial\theta_i\Big|_{\theta=0} .$$

From (3) and (8), the LMMPI test is based on critical regions of the form

$$\sum_{i=1}^{p} -\frac{1}{2}\frac{\partial \ell n|P\Omega(\theta)P'|}{\partial\theta_i}\Big|_{\theta=0} + \sum_{i=1}^{p} -\frac{m}{2}\frac{\partial \ell n(v'(P\Omega(\theta)P')^{-1}v)}{\partial\theta_i}\Big|_{\theta=0} > c .$$

The first summation on the left is simply a scalar constant. Evaluating the second summation gives

$$-\frac{1}{2}c_1 - \frac{m}{2}\sum_{i=1}^{p} v'PA_iP'v > c$$

or

$$s = \sum_{i=1}^{p} s_i = \sum_{i=1}^{p} e'A_ie/e'e = e'Ae/e'e < c_2 \qquad (9)$$

where $A = \sum_{i=1}^{p} A_i$ and c_2 is a suitably chosen constant. Note that (9) is also a LBI test of H_0 in the direction (4).

The form of s is analogous to the Durbin-Watson (DW) statistic. This means that the critical value, c_2, may be found by standard numerical techniques used to calculate critical values of the DW statistic.[5] Unfortunately, the distribution of s is a function of the design matrix X through M. However, bounds for c_2 that are independent of X but dependent on A can be calculated in an analogous way to the familiar DW bounds (see King, (1987a, pp.28-29)). Also, methods of approximating the critical values of the DW statistic can also be used to approximate c_2 (see King (1987a, pp.25-27) and Evans and King

(1985)). Given that under suitable regularity conditions, the test statistic has an asymptotic normal distribution under the null hypothesis, the normal approximation has some appeal. This involves rejecting H_0 for small values of $(s-\mu)/\nu$ assuming a N(0,1) distribution under H_0 where

$$\mu = tr(MA) / m ,$$

$$\nu^2 = 2\left[m\ tr(MAMA) - \left\{tr(MA)\right\}^2\right] / \left\{m^2(m+2)\right\}$$

and tr(·) denotes the trace of the matrix.

5. EXAMPLES OF TESTS OF THE ERROR COVARIANCE MATRIX

5.1 Joint First-Order and Fourth-Order Autocorrelation

Among other things, the error term is included in the regression model to capture the effects of omitted or unobservable regressors and functional approximations. These are often expected to lead to autocorrelated disturbances and for quarterly econometric models, there is a recognition that this autocorrelation may possess a fourth-order component because of seasonal effects. For example, one would expect the omission of relevant variables with seasonal components to lead to both first-order and fourth-order effects in the disturbances. Consequently, one may wish to test whether the errors, u_t, have been generated by the stationary seasonal-autoregressive process

$$(1 - \rho_1 L)(1 - \rho_4 L^4)u_t = \varepsilon_t \qquad (10)$$

where

$$0 \le \rho_1 < 1 \quad \text{and} \quad 0 \le \rho_4 < 1$$

are unknown parameters, L is the lag operator such that $Lu_t = u_{t-1}$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)' \sim N(0, \sigma^2 I_n)$. Note that (10) is the AR(5) process

$$u_t = \rho_1 u_{t-1} + \rho_4 u_{t-4} - \rho_1 \rho_4 u_{t-5} + \varepsilon_t .$$

We wish to test

$$H_0 : \rho_1 = \rho_4 = 0$$

against

$$H_a : \rho_1 \ge 0, \ \rho_4 \ge 0, \ \rho_1^2 + \rho_4^2 > 0.$$

For this problem, the LMMPI test is given by (9) where

$$-A = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & & & & & \vdots \\
0 & 1 & 0 & 1 & 0 & 0 & \ddots & & & & \vdots \\
0 & 0 & 1 & 0 & 1 & 0 & & \ddots & & & \vdots \\
1 & 0 & 0 & 1 & 0 & 1 & & & \ddots & & \vdots \\
0 & 1 & 0 & 0 & 1 & 0 & \ddots & & & \ddots & \vdots \\
\vdots & \ddots & & \ddots & & & \ddots & & & & \vdots \\
\vdots & & & & & & 0 & 1 & 0 & 0 & 1 \\
\vdots & & & & & & 1 & 0 & 1 & 0 & 0 \\
\vdots & & & & & & 0 & 1 & 0 & 1 & 0 \\
\vdots & & & & & & 0 & 0 & 1 & 0 & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 0 & 1 & 0
\end{bmatrix}$$

so that

$$s = -2 \sum_{i=2}^{n} e_i e_{i-1} \Big/ \sum_{i=1}^{n} e_i^2 \; - \; 2 \sum_{i=5}^{n} e_i e_{i-4} \Big/ \sum_{i=1}^{n} e_i^2$$

$$= -2 \, (r_1 + r_4) ,$$

where r_i is the i^{th} order residual autocorrelation coefficient. Hence H_0 should be rejected for large values of $r_1 + r_4$.

It follows that a test which rejects H_0 for small values of the sum of the DW statistic and Wallis's (1972) test statistic is approximately LMMPI.

5.2 Quarter-Dependent Fourth-Order Autocorrelation

Through the work of Thomas and Wallis (1971) and Wallis (1972) among others, there has been much interest in the simple fourth-order error process

$$u_t = \rho u_{t-4} + \varepsilon_t$$

as a model for seasonal autocorrelation in quarterly data. It can be viewed as separately generating March-quarter, June-quarter, September-quarter and December-quarter errors from their own AR(1) processes. Largely for parsimonious reasons, all four AR(1) processes are assumed to have the same coefficient. Clearly it may be more realistic to allow the coefficient to vary and to model the error process by

$$\begin{aligned} u_t &= \rho_1 u_{t-4} + \varepsilon_t , && \text{for } t \text{ from quarter 1 ,} \\ &= \rho_2 u_{t-4} + \varepsilon_t , && \text{for } t \text{ from quarter 2 ,} \\ &= \rho_3 u_{t-4} + \varepsilon_t , && \text{for } t \text{ from quarter 3 ,} \\ &= \rho_4 u_{t-4} + \varepsilon_t , && \text{for } t \text{ from quarter 4} \end{aligned}$$

where $0 \le \rho_i < 1$, $i = 1, \ldots, 4$. It is straightforward to show that a LMMPI test of

$$H_0 : \rho = (\rho_1, \ldots, \rho_4)' = 0$$

against $H_a : \rho > 0$, rejects H_0 for large values of

$$r_4 = \sum_{t=5}^{n} e_t e_{t-4} \Big/ \sum_{t=1}^{n} e_t^2 .$$

Consequently, Wallis's test for simple AR(4) disturbances is an approximately LMMPI test of H_0 against H_a.

5.3 Heteroscedasticity and Random Coefficient Variation

Suppose the regression errors, u_t, are normally and independently distributed with mean zero and variance

$$\sigma_t^2 = \sigma^2 h(\alpha' z_t) \qquad (11)$$

where h is a monotonic-increasing positive function whose second derivative at zero is continuous, α is a $p \times 1$ vector of nonnegative parameters and z_t is a $p \times 1$ vector of nonconstant observations on p exogenous variables. The exact form of h need not be known although its first derivative at zero is assumed to be positive.

A natural example comes from the Hildreth-Houck (1968) random coefficient model which assumes the regression coefficients at time t are generated as

$$\beta_t = \beta + v_t$$

where $v_{it} \sim IN(0,\tau_i^2)$, $i = 1, \ldots, k$, and v_t is independent of v_s, $t \neq s$. Also, assuming $x_{1t} = 1$, $t = 1, \ldots, n$, the error term u_t is now part of v_{1t}. Thus

$$y_t = x_t'\beta_t$$

$$= x_t'\beta + x_t'v_t$$

$$= x_t'\beta + w_t$$

where $x_t = (x_{1t}, \ldots, x_{kt})'$ is the $k \times 1$ vector of observations at time t on the k regressors. Thus we have a regression model whose error term is w_t with $E(w_t) = 0$ and

$$\mathrm{Var}(w_t) = \sum_{i=1}^{k} \tau_i^2 x_{it}^2$$

which can be written in the form of (11) in which the α_i, $i = 1, \ldots, p$, are ratios of variances and therefore nonnegative.

In general, we are hypothesising a heteroscedastic model in which the variances are a function of some known exogenous variables. This is similar to the approach of Breusch and Pagan (1979) except we are also assuming knowledge of the signs of the derivatives

$$\partial\sigma_t^2 / \partial z_{it}, \qquad i = 1, \ldots, p .$$

To simplify the analysis, we assume these derivatives are all non-negative. If for a particular i, the derivatives are non-positive, then replace z_{it} with $-z_{it}$, $t = 1, \ldots, n$, before proceeding.

When $H_0 : \alpha = 0$ holds, $\sigma_t^2 = \sigma^2 h(0)$ are constant. We therefore wish to test H_0 against $H_a : \alpha > 0$. Now

$$\left.\frac{\partial h(\alpha' z_t)}{\partial \alpha_i}\right|_{\alpha=0} = z_{it} \left.\frac{\partial h(x)}{\partial x}\right|_{x=0}$$

so that a LMMPI test for H_0 against H_a is to reject H_0 for large values of

$$\sum_{t=1}^{n}\left(\sum_{i=1}^{p} z_{it}\right)e_t^2 \Big/ \sum_{t=1}^{n} e_t^2 .$$

In the special case of the Hildreth-Houck random coefficient model, a LMMPI test is to reject H_0 for large values of

$$\sum_{t=1}^{n}\left(\sum_{i=2}^{k} x_{it}^2\right)e_t^2 \Big/ \sum_{t=1}^{n} e_t^2 .$$

5.4 Testing for Variance Components

A general representation of the linear model with variance components (see Searle and Henderson, (1979)) is as

$$y = X\beta + \sum_{i=0}^{p} Z_i v_i , \tag{12}$$

where y, X and β are as in (6), v_i is a vector of q_i random effects and Z_i is a known $n \times q_i$ matrix with $v_0 = u$ being the $n \times 1$ error vector as in (6) (i.e., $v_0 \sim N(0,\sigma^2 I_n)$) and $Z_0 = I_n$. The random effects are such that

$$v_i \sim N(0,\sigma_i^2 I_{q_i}) , \qquad i = 1, \ldots, p$$

and

$$E(v_i v_j') = 0 , \qquad i \neq j = 0 , \ldots, p .$$

Then (12) can be written as

$$y = X\beta + w$$

where w is an $n \times 1$ error term such that $E(w) = 0$ and

$$\begin{aligned} Var(w) &= \sigma^2 I_n + \sum_{i=1}^{p} \sigma_i^2 Z_i Z_i' \\ &= \sigma^2 \left(I_n + \sum_{i=1}^{p} \tau_i Z_i Z_i' \right) \end{aligned}$$

where $\tau_i = \sigma_i^2 / \sigma^2 \geq 0$. In many applications (see Searle and Henderson), each of the $Z_i Z_i'$ matrices will be capable of being expressed as Kronecker products of identity matrices and square matrices of ones.

We may wish to test for variance components since their presence causes the OLS estimator and its associated predictor to be inefficient and standard OLS-based tests of the regression coefficients to be misleading. The null hypothesis is

$$H_0 : \tau = (\tau_1, \tau_2, \ldots, \tau_p)' = 0$$

and the alternative is $H_a : \tau > 0$. The LMMPI test for this problem is to reject H_0 for large values of

$$e' \left(\sum_{i=1}^{p} Z_i Z_i' \right) e \,/\, e'e \,.$$

6. CONCLUDING REMARKS

In this paper we have constructed a LMMP test of a simple null hypothesis against a multiparameter one-sided alternative. The resultant test statistic is the sum of the elements of the score vector evaluated at the null hypothesis. This makes it easy to apply and suggests a new form for the LM test which takes into account the one-sided nature of the alternative. In contrast to the Kuhn-Tucker test, this new LM-type test is readily applied because of its asymptotic normal distribution. It has the additional attraction of having a small-sample optimal power property when no nuisance parameters are present. The principle of invariance was used to eliminate nuisance parameters in multiparameter one-sided testing problems in the context of the linear regression model. This allowed the construction of LMMPI tests. For inequality testing of linear restrictions on regression coefficients, the LMMPI test is found to be equivalent to King and Smith's additive-t test which, as a result of an empirical power comparison, King and Smith (1986, p.382) conclude has "power very close to the power envelope over a wide range of parameter values". In the case of testing for spherical regression disturbances, the LMMPI test is of a form similar to the well-known DW test. Its critical values can therefore be calculated by standard numerical techniques used to calculate DW significance points, thus allowing it to be applied as an exact test. This is an option that currently popular tests such as the standard LM, likelihood ratio or Wald test generally not allow. Furthermore, such tests are typically applied ignoring information concerning the signs of the parameters under test.

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APPENDIX

Proof of Theorem

Consider a test of $H_0 : \theta = 0$ whose significance level is α, whose critical region is ω and whose power curve over the parameter space, $\theta \in (\{0\} \cup R^+)^p$, is denoted by $\beta(\theta)$ so that $\beta(0) = \alpha$ and $\beta(\theta) = \int_\omega f(x|\theta)dx$. Let

$$S_r^+ = \left\{ \theta : (\theta'\theta)^{1/2} \le r,\ \theta \in (\{0\} \cup R^+)^p \right\}$$

be the region in θ-space which is within a radius of r from $\theta = 0$. We are interested in the limit of the average volume between the power curve and $\beta(0) = \alpha$ above S_r^+ as r tends to zero. This average volume is given by

$$F(r) = \frac{\int_{S_r^+} (\beta(\theta) - \alpha)\, d\theta_1 \ldots d\theta_p}{\int_{S_r^+} d\theta_1 \ldots d\theta_p}$$

$$= 2^p V_p^{-1} \int_{S_r^+} (\beta(\theta) - \alpha)\, d\theta_1 \ldots d\theta_p \qquad (A.1)$$

where if p is even

$$V_p = \pi^{p/2} r^p \,/\, (p/2)!\ ,$$

while if p is odd,

$$V_p = \pi^{(p-1)/2} \{(p-1)/2\}!\ (2r)^p \,/\, p!\ ,$$

being the volume of a p-dimension hypersphere of radius r.

Consider the Taylor's series expansion of $\beta(\theta)$ about $\theta = 0$,

$$\beta(\theta) = \alpha + \theta' \frac{\partial \beta(z)}{\partial z}\bigg|_{z=0} + \frac{1}{2}\,\theta' \frac{\partial^2 \beta(z)}{\partial z \partial z'}\bigg|_{z=h(\theta)\theta} \theta .$$

where $0 < h(\theta) < 1$. We can write (A.1) as

$$F(r) = 2^p V_p^{-1} \int_{S_r^+} \theta' \frac{\partial \beta(z)}{\partial z}\bigg|_{z=0} d\theta_1 \ldots d\theta_p$$

$$+ 2^p V_p^{-1} \int_{S_r^+} \frac{1}{2}\,\theta' \frac{\partial^2 \beta(z)}{\partial z \partial z'}\bigg|_{z=h(\theta)\theta} \theta \, d\theta_1 \ldots d\theta_p$$

$$= F_1(r) + F_2(r) .$$

Transform θ to spherical coordinates in p-dimensional space as follows:

$$\theta_j = \rho \left(\prod_{i=1}^{j-1} \sin \phi_i \right) \cos \phi_j = \rho\mu_j(\phi) , \qquad 1 \le j \le p-1 ,$$

$$\theta_p = \rho \left(\prod_{i=1}^{p-1} \sin \phi_i \right) = \rho\mu_p(\phi) ,$$

subject to $0 \le \rho < \infty$, $0 \le \phi_j \le \pi/2$, $j = 1, \ldots, p-1$, where $\phi = (\phi_1, \ldots, \phi_{p-1})'$. Then $\theta = \rho\mu(\phi)$ where $\mu(\phi) = (\mu_1(\phi), \ldots, \mu_p(\phi))'$ and

$$d\theta_1 \ldots d\theta_p = \rho^{p-1} \prod_{i=1}^{p-2} \sin^{p-1-i} \phi_i \, d\rho d\phi_1 \ldots d\phi_{p-1} .$$

Also let

$$\dot{\beta}_i = \frac{\partial \beta(z)}{\partial z_i}\bigg|_{z=0} \quad \text{and} \quad \dot{f}_i = \frac{\partial f(x|\theta)}{\partial \theta_i}\bigg|_{\theta=0} .$$

$$F_1(r) = 2^p V_p^{-1} \int_0^r \int_0^{\pi/2} \cdots \int_0^{\pi/2} \rho^p \left\{ \sum_{i=1}^{p} \mu_i(\phi) \dot{\beta}_i \right\} \prod_{i=1}^{p-2} \sin^{p-1-i} \phi_i \, d\rho d\phi_1 \ldots d\phi_{p-1}$$

$$= \begin{cases} 2^p V_p^{-1} \dfrac{r^{p+1}}{(p+1)} \left(\dfrac{\pi}{2} \right)^{\left(\frac{p-1}{2}\right)} \dfrac{1}{(p-1)!!} \displaystyle\sum_{i=1}^{p} \dot{\beta}_i & \text{if p is odd} \\ 2^p V_p^{-1} \dfrac{r^{p+1}}{(p+1)} \left(\dfrac{\pi}{2} \right)^{\left(\frac{p-2}{2}\right)} \dfrac{1}{(p-1)!!} \displaystyle\sum_{i=1}^{p} \dot{\beta}_i & \text{if p is even} \end{cases}$$

$$= r \, G_p \sum_{i=1}^{p} \dot{\beta}_i$$

where

$$G_p = \begin{cases} \dfrac{p!}{2^{p-1}\left[\left(\frac{p-1}{2}\right)!\right]^2 (p+1)} & \text{if p is odd} \\ \dfrac{2^{p+1}\left[\left(\frac{p}{2}\right)!\right]^2}{\pi(p+1)!} & \text{if p is even} \end{cases}$$

and

$$(p-1)!! = \begin{cases} (p-1)(p-3) \ldots 2 & \text{if p is odd} \\ (p-1)(p-3) \ldots 1 & \text{if p is even .} \end{cases}$$

$$F_2(r) = 2^p V_p^{-1} \int_0^r \int_0^{\pi/2} \cdots \int_0^{\pi/2} \frac{1}{2} \rho^2 \mu'(\phi) \left. \frac{\partial^2 \beta(z)}{\partial z \partial z'} \right|_{z=h(\theta)\theta} \mu(\phi)$$

$$\rho^{p-1} \prod_{i=1}^{p-2} \sin^{p-1-i} \phi_i \, d\rho d\phi_1 \ldots d\phi_{p-1}$$

$$= 2^{p-1} V_p^{-1} \frac{r^{p+2}}{(p+2)} \int_0^{\pi/2} \cdots \int_0^{\pi/2} \mu'(\phi) \left. \frac{\partial^2 \beta(z)}{\partial z \partial z'} \right|_{z=h(\theta)\theta} \mu(\phi)$$

$$\prod_{i=1}^{p-2} \sin^{p-1-i} \phi_i \, d\phi_1 \ldots d\phi_{p-1} .$$

There exists a sufficiently small δ such that for $0 < r < \delta$,

$$\left. \frac{\partial^2 \beta(z)}{\partial z \partial z'} \right|_{z=h(\theta)\theta} = \int_\omega \left. \frac{\partial^2 f(x|z)}{\partial z \partial z'} \right|_{z=h(\theta)\theta} dx$$

is bounded and

$$| F_2(r) | < r^2 A$$

where A is a sufficiently large constant. Then

$$\lim_{r \to 0} \frac{F(r)}{r} = \lim_{r \to 0} \frac{F_1(r)}{r} + \lim_{r \to 0} \frac{F_2(r)}{r}$$

$$= G_p \sum_{i=1}^{p} \dot{\beta}_i .$$

Thus the problem of finding a LMMP test reduces to one of choosing a critical region which maximizes $\sum_{i=1}^{p} \dot{\beta}_i$. Given that

$$\sum_{i=1}^{p} \dot{\beta}_i = \int_\omega \sum_{i=1}^{p} \dot{f}_i \, dx ,$$

the generalized Neyman-Pearson lemma implies that the critical region which rejects H_0 for

$$\sum_{i=1}^{p} \dot{f}_i > cf(x|0) \qquad (A.2)$$

where c is an appropriate constant, is a LMMP test. Alternatively, (A.2) can be written as (3).

Footnotes

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2. For examples of reduction by conditioning see Hillier (1986, 1987). King and Hillier (1985), King and Smith (1986) and King (1987b) provide examples of reduction by invariance.

3. If a and b are vectors of the same dimension then $a \geq b$ will denote $a_i \geq b_i$ for every i and $a > b$ will denote $a_i \geq b_i$ for every i with at least one strict inequality.

4. Just as for the standard LM test statistic (see for example Godfrey (1988, section 3.4)), alternative methods for computing $(\hat{\mathcal{J}}^{11})^{-1}$ can be used in (5).

5. See King (1987a, p.27) for a review and Shively, Ansley and Kohn (1989) for a recent development.

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