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MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA.

SMALL-DISTURBANCE ASYMPTOTICS AND THE DURBIN-WATSON AND RELATED TESTS IN THE DYNAMIC REGRESSION MODEL

Maxwell L. King and Ping X. Wu*
Department of Econometrics
Monash University
Clayton, Victoria 3168
Australia.

Abstract

Until recently, it was thought inappropriate to apply the Durbin-Watson (DW) test to a dynamic linear regression model because of the lack of appropriate critical values. Recently, Inder (1986) used a modified small-disturbance distribution (SDD) to find approximate critical values. This paper studies the exact SDD of statistics of the same general form as the DW statistic and suggests some changes to Inder's result. We show how to calculate true small-disturbance critical values and bounds for these critical values that take into account the exogenous regressors. Our results give a justification for the use of the familiar tables of bounds when the DW test is applied to a dynamic regression model.

1. Introduction

Until recently, it was thought inappropriate to apply the Durbin-Watson (DW) test to a dynamic regression model. As Durbin (1970) observed, appropriate critical values could not be computed. He suggested an adjusted DW statistic which, asymptotically, has a standard normal null distribution and which has become known as Durbin's h test. The adjustment is based on ordinary least squares (OLS) estimates and can break down because of the need to take a square root of a negative number. He also proposed an alternative test known as Durbin's t test. Monte Carlo studies of the relative power properties of the DW, h and t tests due to Kenkel (1974, 1975, 1976) and Park (1975, 1976) led to conflicting claims concerning the relative merits of the tests with Kenkel recommending the use of the upper bound as a critical value for the DW test. Inder (1984, 1985) addressed this problem by empirically determining critical values which allowed power comparisons of tests with the same exact, rather than nominal, size. His results support the use of the DW test if appropriate critical values can be determined.

Inder (1985, 1986) suggested using the exact critical value from the regression with the lagged dependent variables omitted. This has an asymptotic justification because these critical values are approximately those from the small-disturbance distribution (SDD) of the DW statistic. The SDD, which is the limit of the statistic's distributions as the disturbance variance tends to zero, has considerable appeal. In contrast to the large-sample distribution, the SDD of the DW statistic in the static model is the exact small-sample distribution because of the statistic's invariance to the disturbance variance. In a Monte Carlo comparison, Inder found that his approach generally yields sizes closer to the nominal size than do Durbin's h and t tests.

Nankervis and Savin (1987) report a parallel finding for testing linear coefficient restrictions in the dynamic regression model. They find the SDD of the F statistic is the same as the small-sample distribution of the statistic from the regression with the lagged dependent variable replaced by its mean. This paper reconsiders Inder's suggestion in the light of Nankervis and Savin's finding. In the next section we derive the SDD of a statistic of the same general form as the DW statistic, applied to the linear regression model with any number of lagged dependent variables. This distribution is the same as the exact distribution of the statistic applied to the analogous static model with the lagged dependent variables replaced by their means. Implications are explored in section 3. The final section observes that, contrary to text-book advice, there is a justification for using the familiar tables of bounds when applying the DW test to the dynamic model.

2. Theory

Consider the dynamic regression model

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \dots + \alpha_p y_{t-p} + x_t'\beta + u_t , \qquad (1)$$

$t = 1, \dots, n$, where y_t is the dependent variable, x_t is a $k \times 1$ vector of exogenous variables, $\alpha = (\alpha_1, \dots, \alpha_p)'$ and β are parameters and u_t is a disturbance term. If there are n observations available on each variable, the parameters are estimated using the last n-p observations. The model for these observations can be written as

$$y = Y_{-1}\alpha + X\beta + u , \qquad (2)$$

where y and u are (n-p)×1 vectors and Y_{-1} and X are (n-p)×p and (n-p)×k matrices, respectively.

Suppose $u \sim N(0,\sigma^2\Omega(\theta))$, where σ^2 is an unknown scalar and $\Omega(\theta)$ is a symmetric matrix which is positive definite for a subset of q × 1 parameter vectors, θ, of interest and is such that $\Omega(0) = I_{n-p}$. We wish to test $H_0 : \theta = 0$. This parameterization covers a number of important testing problems such as testing for, either separately or jointly, autocorrelation, various forms of heteroscedasticity and various forms of stochastic coefficients of the exogenous variables. King and Wu (1989) have shown that, in the context of the static model,

$$y = X\beta + u , \qquad (3)$$

a locally most mean powerful invariant (LMMPI) test against $H_a : \theta_1 \geq 0, \ldots, \theta_q \geq 0, \theta \neq 0$, is to reject H_0 for small values of

$$s = z'Az \,/\, z'z \qquad (4)$$

where z is the OLS residual vector from (3) and $A = \sum_{i=1}^{q} A_i$ in which

$$A_i = -\partial\Omega(\theta) \,/\, \partial\theta_i \Big|_{\theta=0} , \qquad i = 1, \ldots, q.$$

When q = 1, this reduces to King and Hillier's (1985) locally best invariant (LBI) test.

For testing $H_0 : \theta = 0$ in (2), one obvious approach is to use s as the test statistic where z is the OLS residual vector from (2). A difficulty is that the null distribution of s is unknown. As Inder (1985, 1986) and Nankervis and Savin (1987) have suggested, a fruitful way to proceed is to compute critical values based on the SDD of s. We now consider this distribution under each of the following assumptions.

Assumption A: $y_1, y_2, \ldots, y_p$ can be treated as constants.

Assumption B: The unobserved y_i, $i \leq 0$, have constant mean equal to $E(y_1)$ and deviations from this mean following the stationary AR(p) process

$$v_t = \alpha_1 v_{t-1} + \alpha_2 v_{t-2} + \ldots + \alpha_p v_{t-p} + u_t \tag{5}$$

in which $u_t \sim IN(0,\sigma^2)$ under H_0.

Assumption A is appropriate if we wish to make an inference conditional on the values taken by $y_1, \ldots, y_p$ while assumption B is analogous to the assumption made by Inder (1985, 1986).

$m_t = E(y_t)$ can be determined recursively from

$$m_t = \alpha_1 m_{t-1} + \alpha_2 m_{t-2} + \ldots + \alpha_p m_{t-p} + x_t'\beta \ , \quad t = p+1,\ldots,n \ , \tag{6}$$

where under assumption A,

$$m_i = y_i \ , \qquad i = 1, \ldots, p \ , \tag{7}$$

while under assumption B and $\sum_{i=1}^{p} \alpha_i \neq 1$,

$$m_1 = x_1'\beta \Big/ \left(1 - \sum_{i=1}^{p} \alpha_i\right) \ , \tag{8}$$

$$m_2 = x_2'\beta + \sum_{i=1}^{p} \alpha_i m_1 \ , \tag{9}$$

$$m_i = x_i'\beta + \sum_{j=1}^{i-2} \alpha_j m_{i-j} + \sum_{j=i-1}^{p} \alpha_j m_1 , \quad i = 3, \ldots, p . \quad (10)$$

Furthermore, subtraction of (6) from (1) yields the stationary AR(p) process (5) under assumption B and the nonstationary AR(p) process given by (5) together with $v_i = 0$, $i = 1, \ldots, p$, under assumption A.

Because $y_t = m_t + v_t$, under both assumptions A and B we can write

$$Y_{-1} = M_{-1} + O(\sigma)$$

where M_{-1} is the mean matrix of Y_{-1}. For any a×b matrix B of rank $b < a$, let $P_B = I_a - B(B'B)^{-1}B'$. Then

$$Y'_{-1}P_X Y_{-1} = M'_{-1}P_X M_{-1} + O(\sigma)$$

$$(Y'_{-1}P_X Y_{-1})^{-1} = (M'_{-1}P_X M_{-1})^{-1} + O(\sigma)$$

so that

$$P_{P_X Y_{-1}} = P_{P_X M_{-1}} + O(\sigma) .$$

Also note that $P_X P_{P_X M_{-1}} = P_{[M_{-1}:X]}$ so that

$$s = u'P_{[Y_{-1}:X]} A P_{[Y_{-1}:X]} u \,/\, u'P_{[Y_{-1}:X]} u$$

$$= \frac{u'\left\{P_{[M_{-1}:X]} + O(\sigma)\right\} A \left\{P_{[M_{-1}:X]} + O(\sigma)\right\} u}{u'\left\{P_{[M_{-1}:X]} + O(\sigma)\right\} u}$$

$$= s_0 + O(\sigma)$$

where

$$s_0 = \frac{u' P_{[M_{-1}:X]}\; A\; P_{[M_{-1}:X]}\; u}{u' P_{[M_{-1}:X]}\; u}$$

which is of the form of s applied to the static regression

$$y = M_{-1}\gamma + X\delta + u\ , \tag{11}$$

i.e., the original dynamic regression (2) with Y_{-1} replaced by its mean matrix M_{-1}. Thus the SDD of s applied to (2) is given by the small-sample distribution of s applied to the static regression[1] (11).

3. General Implications

The first obvious feature of the above result is that not all the regressors in (11) are known; the elements of M_{-1} are functions of the unknown parameters α and β through (6) and (7) under assumption A, and through (6), (8), (9) and (10) under assumption B. We write M_{-1} as $M_{-1}(\alpha,\beta)$ to highlight this point. Thus the SDD of s depends on the nuisance parameters, α and β, through $P_{[M_{-1}(\alpha,\beta):X]}$ and by continuity arguments we can infer that critical regions based on s are nonsimilar.[2] From Durbin and Watson's (1950) lemma it follows that

$$\Pr\left[s_o = z'Az \,/\, z'z < c \mid u \sim N(0,\sigma^2 I_{n-p})\right]$$

$$= \Pr\left[\sum_{i=1}^{n-k-2p} (\nu_i - c)\xi_i^2 < 0\right] \tag{12}$$

where z is the OLS residual vector from (11), $\nu_1, \nu_2, \ldots, \nu_{n-k-2p}$ are the eigenvalues of $P_{[M_{-1}(\alpha,\beta):X]}A$ other than k+p zeros and $\xi_i \sim IN(0,1)$. Standard numerical methods such as those described by King (1987, pp.27-28) can be used to calculate (12).

How then should we set a critical value based on the SDD? The standard approach in the case of nonsimilar tests (see for example Lehmann and Stein, 1948), is to control the maximum probability of a Type I error by ones choice of critical value. For our problem, this involves finding a critical value c such that

$$\sup_{\alpha,\beta} \Pr\left[\sum_{i=1}^{n-k-2p} (\nu_i - c)\xi_i^2 < 0 \right] = a \tag{13}$$

where a is the desired level of significance. We may wish to restrict α to that part of the parameter space which makes (5) a stationary AR(p) process. Even so, to solve for c is a complex numerical problem that may require excessive amounts of computer time.

An alternative and simpler approach, is to control the probability of a Type I error at a predetermined point in the nuisance parameter space. This requires choosing α and β values, say α^* and β^*, at which size is to be controlled and then finding the value of c that makes (12) equal to the desired significance level where ν_i, i = 1, ..., n-k-2p, are the non-zero eigenvalues of $P_{[M_{-1}(\alpha^*,\beta^*):X]}A$.

A third approach is to calculate bounds for the critical values of s_o that are independent of the unknown $M_{-1}(\alpha,\beta)$ regressors but which take into account the known regressors, X. Based on King's (1981) extension to the DW lemma, we have

$$s_{oL} \leq s_o \leq s_{oU} \tag{14}$$

where

$$s_{oL} = \sum_{i=1}^{n-k-2p} \lambda_i \xi_i^2 \Big/ \sum_{i=1}^{n-k-2p} \xi_i^2 , \quad s_{oU} = \sum_{i=1}^{n-k-2p} \lambda_{i+p} \xi_i^2 \Big/ \sum_{i=1}^{n-k-2p} \xi_i^2 ,$$

in which $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_{n-k-p}$ are the eigenvalues of $P_X A P_X$ other than k zero roots and under H_0, $\xi_i \sim IN(0,1)$, i=1, ..., n-k-2p. The first (second) inequality of (14) is an equality if the column space of $M_{-1}(\alpha,\beta)$ is spanned by the eigenvectors associated with the p largest (smallest) eigenvalues of $P_X A P_X$ (excluding k zero eigenvalues).

If s^*_{oL} and s^*_{oU} denote the significance points of s_{oL} and s_{oU}, respectively, then we reject (do not reject) H_0 if the calculated value of s is below s^*_{oL} (above s^*_{oU}). The test is inconclusive if s falls between s^*_{oL} and s^*_{oU}. The lower bound, s^*_{oL}, can be used as a critical value for a conservative test of H_0, i.e. one whose size, at least under the SDD, is below the desired significance level. Note that it is very unlikely that s^*_{oL} will be identical to the nonsimilar critical value found by solving (13). This is because the elements of $M_{-1}(\alpha,\beta)$ are constrained by (6) and (7) or (6), (8), (9) and (10) making equality in (14) improbable. Similarly, the upper bound, s^*_{oU}, can be used as a critical value for a liberal test of H_0, i.e. one whose size, at least under the SDD, is above the desired significance level. Tighter bounds could be found using (13) to compute the lower bound and

$$\inf_{\alpha,\beta} \Pr\left[\sum_{i=1}^{n-k-2p} (\nu_i - c^*_U)\xi_i^2 < 0 \right] = a$$

for the upper bound, c^*_U.

Inder's (1985, 1986) suggestion amounts to ignoring the $M_{-1}\gamma$ term in (11) when computing critical values and yields a critical value between s^*_{oL} and s^*_{oU}. His empirical evidence suggests that this works reasonably well when $p = 1$; its performance when $p > 1$ is less certain.

There are also implications for the power of the test. Under H_a, the SDD of s is the distribution of s applied to the static regression (11) in which $u \sim N(0,\Omega(\theta))$. This implies that approximate power can be calculated in an obvious way assuming $u \sim N(0,\Omega(\theta))$ in (11). Note that although s_o is of the form of a LBI test when $q = 1$ (or a LMMPI test when $q > 1$), we cannot claim that the original test applied to the dynamic model is a small-disturbance LBI (or LMMPI) test. This is because of the nonsimilar nature of the test. Its small-disturbance size is a function of the nuisance parameters through $P_{[M_{-1}(\alpha,\beta):X]}$ and so typically will differ from the desired significance level.

4. Implications for the Durbin-Watson Test

For the DW test applied to the dynamic model, (11) suggests a justification for the use of tables of bounds computed for the static model. Inder (1985, 1986) advocated the use of bounds which assume the exogenous regressors, X, are the only regressors. It is clear from (11) that account should also be taken of $M_{-1}(\alpha,\beta)$. Thus tabulated bounds of the DW test can be applied in the normal manner.

How then should the DW test be applied in practice to (2)? First the calculated statistic should be compared with the appropriate bounds from published tables for the static model. If this produces an inconclusive result, tighter bounds, s^*_{oL} and s^*_{oU}, can be computed using standard numerical methods. This may also produce an inconclusive result, in which case, following the spirit of Kenkel's (1974, 1975) suggestion and also noting that Inder's (1985, 1986) approach tends to yield a lower than nominal probability of a Type I error, one might use the upper bound, s^*_{oU}, as the critical value. Alternatively, one could decide on values of the nuisance parameters α, β and σ^2, at which a desired level significance is required and then use the Monte Carlo method, or the numerical methods outlined above based on small-disturbance asymptotics, to estimate the critical value.

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Footnotes

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1. Nankervis and Savin (1987) proved their result concerning the F test of linear restrictions on the coefficients of (2) for $p = 1$ and under assumption A. The above approach can be used to extend their result to $p \geq 1$ under either assumption A or B.

2. This confirms the findings of Monte Carlo studies of Inder (1985, 1986), among others, that the DW test is nonsimilar in the context of (1).

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