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A VARIANT ON THE ARGUMENTS FOR THE

INVARIANCE OF ESTIMATORS IN A SINGULAR

SYSTEM OF EQUATIONS

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Working Paper No. 8/89

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ABSTRACT

Allocation models such as consumer demand systems typicall imply a degenerate error structure. The usual approach in estimation is to delete one equation, and to appeal to the results of Barten(1969) or Powell(1969) that parameter estimates are invariant to the equation deleted. However such proofs of invariance are not straightforward. This paper demonstrates that such systems are observationally equivalent to structures common in the simultaneous equations literature, for which invariance is obvious, and hence provides a more transparent demonstration of conditions for invariance

1.   INTRODUCTION

Consider the following two scenarios:

Scenario 1:

An econometrician wishes to estimate a macroeconometric system by
the use of maximum likelihood or other system methods.   Identities in
the system are divided into two categories.   First, they may define
endogenous variables which appear nowhere else in the system.   Such
identities can safely be disregarded as "redundant".   Second, they may
define endogenous variables which do occur elsewhere.   Then such
variables are "substituted out", effectively transforming the system so
that such variables are moved into category one, and their role taken by
constraints on parameters.   No-one ever questions whether the resulting
parameter estimates are invariant to alternative such transformations.

Scenario 2:

An econometrician wishes to estimate a demand system where the
system of equations is consistent with the optimization of some
criterion subject to one or more constraints.   Because of the
constraints, the error structure of the resultant system is degenerate,
and maximum likelihood estimation requires the deletion of one or more
equations.   The question of the invariance of maximum likelihood
estimates to the particular equation deleted then arises, and appeal is
made to Barten's result (Barten (1969)) (or to Powell's result for GLS,
Powell (1969)).   But what if the equation system differs from Barten's?
Should the econometrician provide a special proof for each particular
specification?   Such models abound in the literature in various forms.
A representative selection would include the consumer allocation models

(such as the LES, Translog, AIDS and Rotterdam models); their extensions to incorporate savings (such as the ELES of Lluch, Powell and Williams (1977) and extensions such as Cooper and McLaren (1983)); portfolio models (as originated by Brainard and Tobin (1968) and Parkin (1970)); integrated consumption and portfolio models (such as in Owen (1986)); and the models of firm behaviour based on cost functions (such as Fuss (1977)). Many examples are given in the book by Bewley (1986).

The purpose of this paper is to demonstrate that Scenario 1 and Scenario 2 are simply two aspects of the same situation, being considered separately simply because of the way they are usually written down. In fact, the "invariance problem" is not a problem at all, but merely an alternative way of stating an obvious result that most would take for granted (as in Scenario 1), if the problem had been written down in a slightly different form.

2.   THE GENERAL SPECIFICATION

The general simultaneous equation model may be written as:

$$Ay + Bx = u$$

where y is an n vector of endogenous variables, x is an m vector of exogenous variables, and u is an n-vector of disturbances. The coefficient matrices, A and B, are assumed to be uniquely defined as functions of an underlying vector of parameters, θ. Reparameterizations of the form $\theta = f(\mu)$ are allowed, provided f is one-to-one. The matrix A is assumed to be non-singular.

Consider now a number of particular specifications of the general form. The typical macroeconometric system is of the form:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} x = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \qquad (2.1)$$

where y_1 is of dimension n_1, y_2 is of dimension n_2, $n = n_1 + n_2$ and $E(u_1 u_1') = \Sigma_{u_1}$ is of rank n_1. Thus

$$E(uu') = \Sigma_u = \begin{bmatrix} \Sigma_{u_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

By definition, A_{21}, A_{22} and E_2 contain known coefficients, and do not depend on θ. Since A_{22} is non-singular, y_2 may be "substituted out" to give the form:

$$\begin{bmatrix} B_{11} 0 \\ B_{21} B_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} x = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \qquad (2.2)$$

in which B_{21}, B_{22} and F_2 are not functions of θ.

Note that this amounts to pre-multiplying the system (2.1) by the (non-singular) transformation matrix

$$T_1 = \begin{bmatrix} I & -A_{12} A_{22}^{-1} \\ 0 & I \end{bmatrix}.$$

Such a transformation is usually carried out explicitly prior to estimation, or implicitly if the estimation package used allows for identities to be retained in the system (such as TSP (1986)).

Also of interest is the reduced form,

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad (2.3)$$

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which can be derived either from (2.1) by premultiplying by A^{-1}, or from (2.2) by premultiplying by B^{-1} (where B is defined as the matrix on the left hand side in (2.2)). (Note that while $v_2 \neq 0$, Σ_v is of rank n_1, and G_2 is a function of θ.) Systems (2.1), (2.2) and (2.3) are each observationally equivalent, since each can be derived from any of the others by premultiplying by a non-singular nxn matrix. Of course there are many other observationally equivalent structures, but these three are of main interest.

Macroeconometric systems are usually specified in the form of (2.1) (although estimated in the form (2.2)) and demand systems (or other allocation models) are normally specified in the form (2.3). System (2.2) is the unifying form. Since all these systems are observationally equivalent, consider estimating (2.2). Clearly the last n_2 equations are redundant, and can be safely deleted. But the remaining sub-system is equivalent to the first n_1 equations of (2.3) (simply pre-multiply this sub-system by the non-singular matrix B_{11}^{-1}). Hence if (2.3) has been derived from (2.2) by premultiplying by a matrix B^{-1} it is clearly legitimate to delete the last n_2 equations from (2.3).

But what of invariance? Consider now an arbitrary system of the form (2.3). Assume only that Σ_v is of rank n_1.

<u>Theorem</u>   Given a reduced form such as (2.3) of dimension n, a necessary and sufficient condition for the validity of the deletion of the last $n_2 = n - n_1$ equations is that both Σ_v and Σ_{v_1} be of rank n_1.

<u>Proof</u>   (Sufficiency) Let K be the $n \times n_2$ matrix of characteristic vectors of Σ_v corresponding to the n_2 zero eigenvalues, ie. $\Sigma_v K = 0$. Form the transformation matrix

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$$T_2 = \begin{bmatrix} I & 0 \\ K' & \end{bmatrix}$$

and pre-multiply through (2.3), to give a representation of the form (2.2).

(Necessity) If after deleting the last n_2 rows Σ_{v_1} is of less than full rank, there can be no non-singular transformation between the first n_1 rows of (2.2) and (2.3).

In this notation, deletion of an alternative set of n_2 equations merely corresponds to a reordering of (2.3), and its validity requires that the resulting Σ_{v_1} be of full rank.

Define an invariant systems estimation method to be one which generates identical parameter estimates under alternative observationally equivalent specifications of a system. Define "invariance to deleted equations" to mean that estimates of all n dimensional subsystems of (2.3) for which the variance-covariance matrix is of rank n_1 provide identical parameter estimates under invariant system estimation methods.

Corollary. Given a reduced form such as (2.3) of dimension $\dot{n}$, with Σ_v of rank n_1, a necessary and sufficient condition for the invariance of parameter estimates to the deletion of any $n-n_1$ equations is that the rank of the variance-covariance matrix of the remaining system be n_1.

Proof. All such reorderings are observationally equivalent, and observational equivalence is transitive.

Thus, for example, invariance to deleted equations will hold under maximum likelihood or Aitken estimation, but not under trace minimization.

3.   EXAMPLES

The ideas of the previous section can be illustrated by the familiar linear expenditure system, (LES), which can be derived from the maximization of a direct utility function of the form

$$U(q) = \sum_{j=1}^{n} \alpha_j \ln (q_j - \gamma_j)$$

subject to the budget constraint $p'q = m$. The solution to this problem is usually written in the form (2.3) as

$$p_1 q_1 = p_1 \gamma_1 + \beta_1 (m - p'\gamma) + v_1$$
$$\vdots$$
$$p_n q_n = p_n \gamma_n + \beta_n (m - p'\gamma) + v_{n'}$$

(3.1)

where $\beta_i = \alpha_i / \iota'\alpha$ (ι is the unit column vector). Hence $\iota'\beta = 1$ and $p'q = m$ imply $\iota'v = 0$, and hence the system is singular. A minimal set of parameters is provided by $\theta = (\gamma_1, \ldots, \gamma_n, \beta_1, \ldots, \beta_{n-1})$. Since $\Sigma_v \cdot \iota = 0$, the transformation matrix T_2 of Section 2 is of the form

$$T = \begin{bmatrix} I & 0 \\ & \iota' & \end{bmatrix}$$

which converts (3.1) to the form correspondng to (2.2):

$$p_1 q_1 = p_1 \gamma_1 + \beta_1 (m - p'\gamma) + v_1$$
$$\vdots$$
$$p_{n-1} q_{n-1} = p_{n-1} \gamma_{n-1} + \beta_{n-1} (m - p'\gamma) + v_{n-1}$$

(3.2)

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$$p_n q_n = m - p_1 q_1 - \cdots - p_{n-1} q_{n-1}.$$

While there may be some question about deleting the last equation from (3.1) (even if β_n were written as $1 - \beta_1 - \cdots - \beta_{n-1}$), there is no question that the last equation can be safely deleted from (3.2). Such systems are normally written in the form of (3.1) on the grounds of "symmetry", but the form (3.2) is far more instructive. In general, it is the constraints of the optimization model that lead to degeneracy of the error structure when substituted out. A preferred procedure would be to always explicitly incorporate the constraints as identities in the estimation model. Note that deleting say the ith equation instead of the nth merely corresponds to the mapping from θ to μ that results from replacing β_i by β_n.

In this example the transformation from the form (2.3) to (2.2) was trivial, since it merely amounted to "reversing" the solution of the model for one equation. This will be true for any models which are derived explicitly as solutions to optimization problems subject to constraints, or from the equivalent dual specifications. But it will not be so straightforward if the solution is implicit, as for example in the Rotterdam models or in the "Pitfalls" type models.

To illustrate, consider an arbitrary three equation specification analogous to (3.2):

$$p_1 q_1 = p_1 \gamma_{11} + p_2 \gamma_{12} + p_3 \gamma_{13} + \beta_1 m + v_1$$
$$p_2 q_2 = p_1 \gamma_{21} + p_2 \gamma_{22} + p_3 \gamma_{23} + \beta_2 m + v_2$$
$$p_3 q_3 = p_1 \gamma_{31} + p_2 \gamma_{32} + p_3 \gamma_{33} + \beta_3 m + v_3$$

where $p_1 q_1 + p_2 q_2 + p_3 q_3 \equiv m$.

With no constraints other than adding up, this system may be

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estimated equation by equation by OLS, and the parameter estimates will obey the constraints imposed by adding up ($\sum_i \gamma_{ij} = 0$, $\sum_i \beta_i = 1$) and invariance is not an issue. But once other types of constraints are imposed, such as symmetry or exclusion restrictions, the issue is not so clear. To be concrete, consider the restrictions $\gamma_{12} = \gamma_{21}$, $\gamma_{13} = 0$. Now the first pair of equations may be estimated as a system, and the parameter estimates of the third equation derived by the adding up conditions. The third equation is a residual equation in Bewley's terminology, and its exclusion is valid. Can the first equation be deleted? If the second and third equations were estimated as a system without explicit consideration of the equality and exclusion restrictions, these restrictions would not be satisfied and hence invariance violated. The important point is that the adding up restrictions interact in this case with the other restrictions, and the conditions of the theorem are valid (i.e. rank (Σ_v) = 2) when we choose to identify θ with the parameters of the second and third equations, if and only if the constraints $\gamma_{23} + \gamma_{33} = 0$, $\gamma_{21} + \gamma_{22} + \gamma_{32} = 0$ are imposed. With this specification, the first equation may indeed be legitimately deleted.

To complete this section, consider the well-known textbook example of a macro-econometric system, writen in the form of (2.1) as

$$
\begin{aligned}
C &= \alpha + \beta Y + u \\
Y &= C + I \qquad ;
\end{aligned}
\tag{3.3}
$$

or in the form of (2.2) as

$$
\begin{aligned}
C &= \gamma_1 + \delta_1 I + v_1 \\
Y &= C + I
\end{aligned}
\tag{3.4}
$$

where $\gamma_1 = \alpha/(1-\alpha)$, $\delta_1 = \beta/(1-\alpha)$, $v_1 = u/(1-\alpha)$; or in the reduced form of (2.3) as

$$C = \gamma_1 + \delta_1 I + v_1$$
$$Y = \gamma_2 + \delta_2 I + v_2$$

(3.5)

where $\gamma_2 = \gamma_1$, $\delta_2 = 1/(1-\beta)$ and $v_2 = v_1$. Is there any substantive difference between system (3.5) and system (3.1)? Yet deletion of an equation from system (3.1) is usually considered to require justification, while deletion of an equation from (3.5) is not. For example, Wonnacott and Wonnacott (1979) state in a footnote that "Although we choose to work with (the first equation of (3.5)) it would be equally valid to work with the other reduced form equation. In fact, we would arrive at exactly the same estimators." (p. 263)

4. RELATIONSHIP TO OTHER INVARIANCE ARGUMENTS

Invariance is usually justified by appealing either to Barten's method (Barten (1969)) or the generalized inverse method (Powell (1969), Theil (1971)). Barten's method amounts to a demonstration that the likelihood of the sub-system can be identified with the likelihood of a full system in which the degenerate covariance matrix is replaced by one of full rank. The generalized inverse approach amounts to applying generalized least squares to the system by using a generalized inverse of the singular variance-covariance matrix. Dhrymes and Schwarz (1984a) show that the generalized inverse estimator does not exist if the equations of the system contain one or more explanatory variables in common (as is typically the case), but that it can be revived as a restricted estimator if the implied restrictions on parameters are explicitly incorporated. (This is similar to Bewley's proof that residual equations may be deleted - a residual equation contains all variables, and hence no zero restrictions are ignored). Dhrymes and Schwarz (1984b) refer to Barten's method as a "sleight of hand",

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suggesting that "such estimators will observe restrictions implied by adding up only when they are irrelevant, i.e. when all variables appear in all equations so that in fact they are identical to OLS procedures." They suggest that this method may also be revived as a restricted estimator, in which case it is algebraically equivalent to the restricted generalized least squares method.

In terms of the specifications of section 2 the Powell-Theil-Dhrymes-Schwarz proof of invariance demonstrates conditions under which estimates of a non-degenerate n_1 equation subset of (2.3) are equivalent to the estimate of (2.3) based on a generalized inverse of the variance-covariance matrix. The Barten proof of invariance amounts to a mapping of the likelihood of an n_1 equation subset of (2.3) into a "pseudo-likelihood" of system (2.3). The proof presented in this paper is based on a mapping from a specification of the form (2.3) to one of the form (2.2), for which invariance is obvious.

This paper is in the spirit of the approach of Dhrymes and Schwarz. To quote them again: "The heart of the problem is that the conditions on the parameters force the singularity of the covariance matrix - and to a certain degree the converse is true, i.e. the singularity of the covariance matrix implies certain restrictions." (Dhrymes and Schwarz (1984b), pp. 8-9). However, our approach is rather simpler. Instead of their approach of suggesting generalized inverse Aitken methods subject to all constraints applied to the full system, in most cases the degeneracy-inducing constraints can be easily substituted out to allow estimation of a non-degenerate system subject to any remaining constraints such as symmetry. In many systems such substitution amounts merely to the deletion of an equation.

5.   CONCLUSION

In the estimation of micro models there has been much confusion
about the "problem" of the degeneracy of the error variance-covariance
matrix.   But such models are observationally equivalent to structures
which have been common in macro models at least since the work of the
Cowles Commission, within which the solution is considered trivial.
Since most modern estimation packages (eg. TSP) explicitly allow for
identities, the simplest procedure is merely to explicitly carry the
identities in the model rather than substitute them out to create
degeneracies.

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