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ISSN 1032-3813

ISBN 0 86746 944 7

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# A 'GORMANESQUE' APPROACH TO THE SOLUTION OF INTERTEMPORAL CONSUMPTION MODELS[1]

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## ABSTRACT

In this paper we compare the standard dynamic programming method for the derivation of closed form solutions to stochastic control problems with three alternatives. Our first alternative is a minor variant of dynamic programming which requires solution of an alternative partial differential equation. In the context of a basic stochastic intertemporal utility maximizing model, this alternative solution procedure is particularly useful when preferences are represented by an instantaneous profit function. Our second alternative is based on intertemporal duality theory, and is most useful when preferences are represented by an indirect intertemporal expected utility (or value) function. Our third alternative is based on matching restrictions across inverse marginal instantaneous and indirect intertemporal expected utility functions - Gorman "expenditure" and "wealth" functions. Each of the approaches is suitable to particular model specifications and preference representations. As a collection of methods based to varying degrees on the exploitation of separability and duality, the methodology of this paper may be termed 'Gormanesque' (in the spirit of Gorman (1976)). Taken together, these methods expand considerably the range of models for which closed form analytical solutions may be obtained.

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1 Earlier versions of this paper were presented to the Seventh Analytic Economics Workshop, Australian National University, Canberra, February 15-16, 1989, and to the 1989 Australasian Meetings of the Econometric Society, University of New England, Armidale, July 12-15, 1989.

## 1. INTRODUCTION

Since the seminal work of Hall (1978) a vast literature on Euler equation estimation has developed in applied econometric studies of the consumption function. While the undoubted simplicity of this approach is attractive, this achievement is not without cost. In particular, if the effects of unanticipated events on an economic agent's decision making are to be measured, a synthesised solution which links the agent's choice variables to fundamental exogenous variables must be analysed. Thus, from the point of view both of the economic theorist and the applied econometrician, there is a gap in the literature which needs to be filled.

In this paper we compare the standard dynamic programming approach to the derivation of synthesised or optimal feedback solutions to a consumer's stochastic intertemporal utility maximizing problem with three alternative solution procedures. The analytical derivation of closed form consumption functions seems to have begun and ended with the contribution of Merton (1971). The popularity of Euler equation estimation (as distinct from solution) seems to be due to a view that the extension of closed form techniques beyond the HARA class of utility function specifications considered by Merton is not analytically tractable. In the current paper we hope to show that this is not the case and that, by judicious use of separability and duality, considerable advances which are both of theoretical and empirical value may be made. The three methods which we propose to examine as alternatives to the standard dynamic programming approach rely upon the use of separability and duality (i.e. choice of the 'natural independent variables' in Gorman's (1976) methodology) to varying degrees. It is for this reason that we view our methodology as 'Gormanesque'.

## 2. A PROTOTYPE STOCHASTIC INTERTEMPORAL UTILITY MAXIMIZING MODEL

In this section we specify a prototype model in which the utility function is left as general as possible in order to illustrate the applicability of each of several approaches to the derivation of optimal feedback solutions.

Let c represent real total expenditure, w real wealth, and r the real rate of return on a risk free asset. We allow one risky asset with risk premium exogenously given as h. The expected real return on the risky asset is therefore (r + h)dt. The actual rate of return is affected by a diffusion $\sigma$ db, where b is standard Brownian motion. Let a represent the proportion of the consumer's portfolio which is held in the risky asset.

The consumer's decision making process is characterized by the problem:

(2.1) $$V(w,r,h,\sigma,\rho) = \max_{\{c,a\}} \ E_t \int_t^{\infty} e^{-\rho(\tau-t)}U(c(\tau))d\tau$$

subject to:

(2.2a) $$dw(\tau) = \{[a(\tau)h + r]w(\tau) - c(\tau)\}d\tau + a(\tau)\sigma w(\tau)db,$$

(2.2b) $$w(t) = w,$$

(2.2c) $$w(\tau) \geq 0 \quad \text{w.p.1.}$$

In the prototype model we treat $r, h, \sigma$ and $\rho$ as parameters of the consumer's decision making process. None of these simplifications are vital. We could equally have written r, h and $\sigma$ as variables, hence have written V as a function of "deeper" parameters, and sought a solution for c as a function of the deeper parameters. This may well be appropriate empirically, especially in view of (anticipated) time variation in $\sigma$, but since our primary interest is in synthesis in terms of fundamentally exogenous factors there is little point in introducing additional levels of nesting into the prototype.

The instantaneous utility function $U(c)$, where $c$ is real total expenditure, may be interpreted as an indirect utility function from static consumer demand theory, where $c = M/P$ with $M$ total money expenditure and $P$ a price index. The structure of the model admits two stage budgeting as optimal and we are concentrating on the first stage, the determination of aggregate real expenditure.

We seek a feedback solution of the form $c = f(w,r,h,\sigma,\rho)$ or, more generally, some parametric representation of this relationship capable of indicating the responsiveness of $c$ to shocks in $w$, $r$, $h$ or $\sigma$. For the purpose of derivation of the optimal response function, since $r$, $h$, $\sigma$ and $\rho$ are to be treated as parameters it will generally be convenient to suppress the dependence of $V$ on $r$, $h$, $\sigma$ and $\rho$, and to write the value function as $V(w)$, or simply $V$. Time dependence of $w,c,a$ is also suppressed wherever convenient to aid notational clarity.

## 3. PRELIMINARY RESULTS

In this section, our point of departure from the "Euler equation" approach is clarified. We use stochastic control theory/dynamic programming notation to facilitate transition to the synthesised approaches of the next section.

Define the operator $\mathcal{L}$ by:

$$(3.1) \qquad \mathcal{L} = [(ah + r)w - c]\, \partial/\partial w + \frac{1}{2} a^2\sigma^2 w^2\, \partial^2/\partial w^2 \,.$$

The Hamilton-Jacobi-Bellman (HJB) equation (see for example, Kushner (1967), Ch. IV, Theorem 7) for problem (2.1) is:

$$(3.2) \qquad 0 = \max_{c,a} \{U + \mathcal{L}V - \rho V\}$$

implying first order conditions for c:

$$(3.3) \qquad U_c = V_w$$

(where subscripts indicate partial derivatives), and for a:

$$(3.4) \qquad a = -(h/\sigma^2)V_w \,/\, (wV_{ww}) \;.$$

Of great importance to the synthesised approaches is the Hamilton-Jacobi (HJ) equation, that is (3.2) written as an identity for optimal c,a. We substitute out the "nuisance" variable, a, using (3.4) and write the condensed HJ equation as:

$$(3.5) \qquad 0 = U + (rw-c)V_w - \tfrac{1}{2}(h/\sigma)^2 V_w^2 \,/\, V_{ww} - \rho V$$

where c is now understood to be optimal.

Because of the importance of the HJ equation (3.5) in the sequel, some interpretation may be in order. The interpretation is most straightforwardly given for (3.2) written as an identity for optimal c,a. This equation simply states that, allowing for time discounting ($\rho V$), potential expected utility V should be used up optimally, being balanced by the acquisition of actual instantaneous utility U as the optimal path evolves. The usage of potential expected utility is measured by $\mathcal{L}V$, a measure of the expected change in V which allows for the second order effects relevant by Ito's Lemma.

Of even more importance in the sequel is the sensitivity of the HJ equation to w. Differentiating (3.5) wrt w and using (3.3) we obtain:

$$(3.6) \qquad 0 = \left[r - \rho - (h/\sigma)^2\right]V_w + (rw-c)V_{ww} + \frac{1}{2}(h/\sigma)^2 V_w^2 V_{w(3)} \, / \, V_{ww}^2$$

where $V_{w(3)} \equiv V_{www}$. Equations (3.3), (3.5) and (3.6) are key equations for the various solution procedures to be discussed. The three alternatives to the standard dynamic programming method all make use of (3.6) in some way. Because of the structure of the model (the fact that U does not depend on w), (3.6) is structurally simpler than (3.5). One unifying feature of our alternative methods, and a justification for the term 'Gormanesque', is their exploitation of this natural structure.

In the remainder of this section we illustrate the "Euler equation" approach in the above terminology. It is important to note that Euler equation estimation does not involve solution of the Euler equation but simply represents a rewriting of the equations given above.

As is well known, (3.3) is the Euler equation in integral form. We may write this equivalently as:

$$(3.7) \qquad c = C^G(V_w)$$

where $C^G$ is the inverse marginal instantaneous utility function. (Our notation $C^G$ arises from our intention to refer to this function as the "Gorman Cost Function", to be discussed further in the sequel.) By Ito's Lemma, the stochastic Euler equation is:

$$(3.8) \qquad dc = C^G_{(1)} dV_w + \frac{1}{2} C^G_{(2)} [dV_w]^2$$

where $C^G_{(1)}$ and $C^G_{(2)}$ denote first and second derivatives respectively of $C^G$, and

$$(3.9) \qquad dV_w = V_{ww}\, dw + \frac{1}{2} V_{w(3)} [dw]^2 .$$

From (2.2a):

$$(3.10) \qquad [dw]^2 = a^2 \sigma^2 w^2 dt.$$

Using (2.2a), (3.10), (3.4) and (3.6), we may write (3.9) as:

$$(3.11) \qquad dV_w = (\rho-r)V_w\,dt - (h/\sigma)V_w\,db.$$

This in turn implies:

$$(3.12) \qquad [dV_w]^2 = (h/\sigma)^2V_w^2\,dt$$

so that (3.8) becomes:

$$(3.13) \qquad dc = \left[(\rho-r)V_wC^G_{(1)} + \tfrac{1}{2}(h/\sigma)^2V_w^2C^G_{(2)}\right]dt - (h/\sigma)V_wC^G_{(1)}db\ .$$

By re-expressing $C^G_{(1)}$, $C^G_{(2)}$ and $V_w$ as functions of c, (3.13) becomes the standard stochastic Euler equation for problem (2.1). To make this transformation, observe that (3.3) and (3.7) imply $c \equiv C^G(U_c(c))$, so that successive differentiation gives:

$$(3.14a) \qquad C^G_{(1)} = 1 / U_{cc}$$

$$(3.14b) \qquad C^G_{(2)} = -\,U_{c(3)} / U_{cc}^3\ .$$

The stochastic Euler equation can then be written:

$$(3.15) \qquad dc = \left[(\rho-r)U_c / U_{cc} - \tfrac{1}{2}(h/\sigma)^2U_c^2U_{c(3)} / U_{cc}^3\right]dt - \left[(h/\sigma)U_c / U_{cc}\right]db\ .$$

Equation (3.15) is amenable to discrete time approximation and estimation subject to a conditionally heteroscedastic error structure. However, since it does not contain a solution for c as a function of the fundamentals w, r, h, $\sigma$ and $\rho$, it cannot be used to examine the responsiveness of c to unanticipated events. In addition, it is worth noting that a further source of information on the parameters of the model is the analogous equation for the evolution of wealth:

(3.16) $$dw = \left[rw - c - (h/\sigma)^2 U_c \,/\, (U_{cc} f_w)\right]dt - \left[(h/\sigma)U_c \,/\, (U_{cc} f_w)\right]db.$$

To ignore (3.16) in estimation involves an efficiency loss, but to apply (3.16) requires the closed loop solution $f(w, r, h, \sigma, \rho)$. We turn therefore to approaches which allow the derivation of the synthesised solution.

## 4. APPROACHES BASED ON SYNTHESIS

### 4.1 Dynamic Programming

This is the standard approach, which uses the HJ equation (3.5) and employs the inverted first order condition (3.7) to give a nonlinear PDE in V:

(4.1) $$0 = U(C^G(V_w)) + \left[rw - C^G(V_w)\right]V_w - \tfrac{1}{2}(h/\sigma)^2 V_w^2 \,/\, V_{ww} - \rho V .$$

As the lack of progress since Merton (1971) indicates, analytical solution of this PDE for V(w), when U(c) is more general than HARA, remains intractable.

### 4.2 Dynamic Programming Applied to the Marginal Function

A potentially more amenable application of the dynamic programming method may be derived by the change in "variable" from V to $V_w$, which

results in the use of (3.6) rather than (3.5) as the underlying PDE. Thus the "derivative" form of the fundamental PDE of dynamic programming is:

(4.2) $$0 = \left[r - \rho - (h/\sigma)^2\right]V_w + \left[rw - C^G(V_w)\right]V_{ww} + \tfrac{1}{2}(h/\sigma)^2 V_w^2 V_{w(3)} \,/\, V_{ww}^2.$$

For certain instantaneous utility functional forms U and associated Gorman cost functions $C^G$, (4.2) may be more tractable (solving for $V_w$) than would be (4.1) (which must be solved for V). Clearly, a solution for $V_w$ is all that is required to enable (3.7) to represent the synthesised solution. Additionally, the approach based on (4.2) is viable in cases where the Gorman cost function $C^G$ has a closed form analytical representation even though the utility function U may not have a closed form. Clearly, this includes a wider class of preference orderings than the usual closed form U representations.

For example, let preferences be represented by a profit function:

$$(4.3) \qquad \Pi^F(\mu) = \max_c \{\mu U(c) - c\}$$

where $\mu$ is the "price" of utility. (See Gorman (1976) for a discussion of the consumer's profit function). By Hotelling's Lemma,

$$(4.4) \qquad u = \Pi^F_\mu(\mu)$$

and hence costs as a function of $\mu$ (the "Frisch cost function") may be defined, on rearrangement of (4.3), as:

$$(4.5) \qquad C^F(\mu) = \mu\Pi^F_\mu(\mu) - \Pi^F(\mu) \ .$$

The Frisch cost function $C^F(\mu)$ is an important duality concept associated with the profit function approach to the representation of preferences. However, in an intertemporal model it is $1/\mu$ rather than $\mu$ which is the "natural variable" in Gorman's (1976) sense. We denote by "Gorman cost function" the cost function defined in terms of the "natural variable" for our problem.

The first order condition of (4.3) requires $\mu = 1/U_c = 1/V_w$, allowing the Gorman and Frisch cost functions to be related simply by

(4.6) $\quad C^G(V_w) \equiv C^F(1/V_w)$ .

Combining (4.6) and (4.2) allows the approach of this section to be applied to problems in which preferences are specified in terms of the profit function $\Pi^F(\mu)$ even if preferences cannot be explicitly represented in terms of a utility function U(c).

### 4.3 Stochastic Intertemporal Duality

An alternative way to use (3.6) is to rearrange it as:

(4.7)
$$c = f(w,r,h,\sigma,\rho) = rw + [r - \rho - (h/\sigma)^2]V_w \,/\, V_{ww} + \tfrac{1}{2}(h/\sigma)^2 V_w^2 V_{w(3)} \,/\, V_{ww}^3 \; .$$

Since V is the stochastic intertemporal indirect utility function, (4.7) may be thought of as the stochastic intertemporal analogue of Roy's Identity. Equation (4.7), together with the asset solution (3.4), reproduced here as:

(4.8)
$$a = g(w,r,h,\sigma,\rho) = -\,(h/\sigma^2)V_w \,/\, (wV_{ww}),$$

provide a complete solution of the optimization problem (2.1) conditional on knowledge of the functional form of V.

The importance of these results hinges on the possibility of characterizing a value function $V(w,r,h,\sigma,\rho)$, to which we now turn. Consider functions U(c) satisfying the following properties:

(U1) U is non-decreasing in c;

(U2) U is concave in c;

over some set $\mathcal{C} \in R_+$. For any function U(c) let $X = \{(w,r,h,\sigma,\rho)) \in R_+^5$ | Problem (2.1) has a solution}. Let $\bar{\mathcal{C}} = \{c | c = f((w,r,h,\sigma,\rho);$

$(w,r,h,\sigma,\rho)\ \varepsilon\ X\}$. The proofs of the following theorems are in Appendix 1.

**Theorem 1. (U → V)**

For any specification of U satisfying (U1)-(U2) for which X is non-null, V defined by Problem (2.1) has the following properties on X:

(V1) V is non-decreasing in w;

(V2) V is concave in w;

(V3) V is non-decreasing in r;

(V4) V is non-decreasing in h;

(V5) V is non-increasing in $\sigma$;

(V6) Let $H(w,r,h,\sigma,\rho,c) = \rho V - (rw-c)V_w + \frac{1}{2}(h/\sigma)^2 V_w^2 / V_{ww}$, then for $c\ \varepsilon\ \mathscr{C}$, H is locally convex in $(w,r,h,\sigma,\rho)$ about its stationary point.

To introduce the derivation of U from V, consider the problem:

$$U(c) = \min_{w,r,h,\sigma,\rho} H(w,r,h,\sigma,\rho,c) \tag{4.9}$$

with H defined in terms of V as in (V6) above.

For any function $V(w,r,h,\sigma,\rho)$ let $X^*(c) = \{(w,r,h,\sigma,\rho))|$ Problem (4.9) has a solution$\}$ and let $\mathscr{C}^*$ be the set of $c\ \varepsilon\ R^+$ such that $X^*(c)$ is non-null.

**Theorem 2 (V → U)**

For any specification of V satisfying (V1)-(V6) for which $\mathscr{C}^*$ is non-null, $U^*$ defined by Problem (4.9) satisfies (U1) and (U2) on $\mathscr{C}^*$.

**Theorem 3**

For any U satisfying (U1)-(U2), define V by Problem (2.1). For this V, define $U^*$ by Problem (4.9). Then on their common domain of definition $U^* = U$.

**Theorem 4**

For any V satisfying (V1)-(V6), define U by Problem (4.9). For this U, define $V^*$ by Problem 2.1. Then on their common domain of definition $V^* = V$.

If in Theorem 3, $\mathscr{C}^* = \mathscr{C}$ then $U^*$ and U will coincide. Similarly, if in Theorem 4, $X^* = X$ then $V^*$ and V will coincide. Thus there is a duality between functions U satisfying properties (U1)-(U2) and functions V satisfying properties (V1)-(V6). The empirical content of this duality is summarised in the following:

**Theorem 5**

Let V satisfy properties (V1)-(V6) and define the synthesised functions $c = f(w,r,h,\sigma,\rho)$ by (4.7) and $a = g(w,r,h,\sigma,\rho)$ by (4.8). Then these functions are the solutions of Problem (2.1) corresponding to a function U with properties (U1) and (U2) dual to V.

Use of Theorem 5 allows the approach of this section to be applied to problems in which preferences are specified in terms of the value function $V(w,r,h,\sigma,\rho)$ even if preferences cannot be explicitly represented in terms of a utility function U(c).

For a discussion of intertemporal duality in a deterministic context see Cooper and McLaren (1980).

### 4.4 Matched Inverse Marginal Functions

This approach uses the equality of the instantaneous marginal utility of money, $U_c$, with the full marginal utility of wealth, $V_w$, to set up a two-equation parametric relationship between c and w. Let $\lambda = U_c = V_w$. Then $c = C^G(\lambda)$ and $w = W^G(\lambda, r, h, \sigma, \rho)$, where $W^G$ is the inverse of the $V_w$ function. (We refer to $W^G$ as the "Gorman wealth function".) From the identity $w \equiv W^G(V_w(w))$ we have $1 = W^G_\lambda V_{ww}$ and $0 = W^G_{\lambda\lambda}V^2_{ww} + W^G_\lambda V_{w(3)}$. Thus (4.7) may be used to define functional restrictions relating $C^G$ to $W^G$. Specifically,

$$C^G = \mathcal{D}W^G \tag{4.10}$$

where $\mathcal{D} = rI + \left[r - \rho - (h/\sigma)^2\right]\lambda\ \partial/\partial\lambda - \frac{1}{2}(h/\sigma)^2\lambda^2\ \partial^2/\partial\lambda^2$ .

Given U(c) and hence $C^G(\lambda)$, the approach requires construction of a function $W^G(\lambda, r, h, \sigma, \rho)$ such that (4.10) holds. For these matched functions the $c = f(w, r, h, \sigma, \rho)$ relationship is then given parametrically by:

$$c = C^G(\lambda) \tag{4.11a}$$

$$w = W^G(\lambda, r, h, \sigma, \rho). \tag{4.11b}$$

These can be complemented by the asset demand equation, also parameterised on λ, as:

$$a\ w = -(h/\sigma)^2\lambda W^G_\lambda \tag{4.11}$$

In common with the approach of Section 4.2, this approach is viable in cases where $C^G$ (but not necessarily U) has a closed form analytical representation. Specification of $C^G$ may proceed from the specification of a profit function via the association with a Frisch cost function, as illustrated in Section 4.2.

A detailed exposition of this approach in a more general setting may be found in Cooper and Madan (1986).

## 5. EXAMPLES

### 5.1 Features of the Alternative Solution Methods for HARA Preferences

We sketch all four methods when preferences are HARA in order to highlight the basic features of each approach. HARA functional forms for U and V are known to be self-dual, and this is the most general case for which the standard dynamic programming method has been applied to date.

HARA preferences are represented by an origin-translated Box-Cox generalization of a logarithmic instantaneous utility function:

$$(5.1) \qquad U(c) = \left[(c-\gamma)^{\beta} - 1\right] / \beta , \qquad c > \gamma,\ \beta < 1 ,$$

where $\gamma$ is interpreted as subsistence consumption and utility is only defined for $c > \gamma$.

Using (4.3), an equivalent representation of HARA preferences is given by the profit function:

$$(5.2) \qquad \Pi^F(\mu) = \mu\left\{\left[\mu^{\beta/(1-\beta)} - 1\right] / \beta - \mu^{\beta/(1-\beta)}\right\} - \gamma .$$

From (4.5), this implies a Frisch cost function:

$$(5.3) \qquad C^F(\mu) = \gamma + \mu^{1/(1-\beta)} ,$$

and from either (5.1) by differentiation and inversion, or from (5.3) by identification of $\lambda$ with $1/\mu$, the Gorman cost function is immediately obtained as:

$$(5.4) \qquad c = C^G(\lambda) = \gamma + \lambda^{-1/(1-\beta)} .$$

Since utility is defined only for $c > \gamma$, we expect the optimal value function $V(w,r,h,\sigma,\rho)$ to be defined only for wealth above what would be required to meet committed expenditure $\gamma$ in perpetuity. Of course, for this example the form of V is known. However, at this point it is convenient to write down a conjectured generic form for V to illustrate the application of the various solution methods. Let us conjecture:

$$(5.5) \qquad V(w,r,h,\sigma,\rho) = \theta_0 + \theta_1\left[(w-\gamma/r)^{\beta} - 1\right] / \beta$$

where $\theta_0$ and $\theta_1$ are unspecified functions of $r,h,\sigma,\rho$.

We now consider the relative strengths of the various methods in deriving the consumption function $c = f(w,r,h,\sigma,\rho)$ and the asset share function $a = g(w,r,h,\sigma,\rho)$.

**Method 1**

Given the $C^G$ function (5.4), the identification of $\lambda$ with $V_w$ allows the fundamental PDE of dynamic programming (4.1) to take the form:

$$(5.6) \qquad 0 = \left[V_w^{-\beta/(1-\beta)} - 1\right] / \beta + \left[rw - \gamma - V_w^{-1/(1-\beta)}\right]V_w$$
$$- \tfrac{1}{2}(h/\sigma)^2 V_w^2 / V_{ww} - \rho V .$$

The conjectured form for V does indeed solve this PDE (as is, of course, well known since Merton (1971)). The point we wish to make, however, is that to establish this it is necessary to determine suitable specifications for the parameters $\theta_0$, $\theta_1$. It can be seen that

$$(5.7a) \qquad \theta_0 = (1/\rho)\left[\rho\theta_1 - \theta_1^{-\beta/(1-\beta)}\right] / \beta$$

and

$$(5.7b) \qquad \theta_1 = \left[\rho/(1-\beta) - r\beta / (1-\beta) - \tfrac{1}{2}(h/\sigma)^2\beta / (1-\beta)^2\right]^{\beta-1}$$

justify (5.5) as a solution to (5.6). Although these parameters needed to be determined to examine the legitimacy of the V function, the information on $\theta_0$ (at least) is superfluous to requirements for determination of the closed loop consumption function. Having established the legitimacy of (5.5), we now use:

$$(5.8) \qquad V_w(w,r,h,\sigma,\rho) = \theta_1(w - \gamma/r)^{\beta-1}$$

together with (5.7b) in (5.4) to give the consumption function:

$$(5.9) \qquad f(w,r,h,\sigma,\rho) = \gamma + \left[\rho/(1-\beta) - r\beta/(1-\beta) - \tfrac{1}{2}(h/\sigma)^2\beta/(1-\beta)^2\right](w - \gamma/r) .$$

Similarly, we use (5.8) and its derivative in (3.4) to give the asset share function:

$$(5.10) \qquad g(w,r,h,\sigma,\rho) = (h/\sigma^2)\left[1 - \gamma/(rw)\right] / (1-\beta) .$$

**Method 2**

While Method 1 requires an explicit specification of U to employ in (4.1), the use of Method 2 allows us to begin at the point of specification of $C^G$ in (5.4). This may be thought of as corresponding to $\Pi^F(\mu)$ in (5.2) or to U(c) in (5.1), but U(c) need not be explicitly specified. Starting, for example, from the $\Pi^F$ function (5.2) we obtain (5.4), identify $\lambda$ with $V_w$, and hence employ the alternative PDE (4.2) which now takes the form:

$$(5.11) \qquad 0 = \left[r - \rho - (h/\sigma)^2\right]V_w + \left[rw - \gamma - V_w^{-1/(1-\beta)}\right]V_{ww} + \frac{1}{2}(h/\sigma)^2 V_w^2 V_{w(3)} \,/\, V_{ww}^2 \,.$$

As for Method 1, we could use the conjecture (5.5) for V. However, since V is not required, we may use the equivalent conjecture (5.8) for $V_w$. It can be seen that (5.8) solves (5.11) if $\theta_1$ satisfies

$$(5.12) \quad \theta_1 = \left\{\left[\rho - r + (h/\sigma)^2\right] / (1-\beta) - r - \frac{1}{2}(h/\sigma)^2(2-\beta) / (1-\beta)^2\right\}^{\beta-1}$$

(and this is, of course, equivalent to (5.7b)). The consumption and asset functions follow as for Method 1. Note that Method 2 does not require determination of the superfluous parameter $\theta_0$. More importantly, a specific form for U(c) is not required. This advantage will be developed in the example of Section 5.2.

**Method 3**

In this approach, we regard (5.5) not as a conjectured solution to a PDE but as an initial specification containing a representation of preferences. Application of the intertemporal analogue of Roy's Identity (4.7) leads immediately to the consumption function, and (4.8) gives the asset function. It is important to note that neither $\theta_0$ nor $\theta_1$ need be specified in this case. If V is thought of simply as an approximation to a regular optimal value function, this is the end of the matter. Many empirical applications of duality theory in static models follow this route, with the initially specified function representing a flexible (at some point) approximation to some unspecified regular underlying function. A similar methodological approach in stochastic intertemporal consumption models would suggest

Method 3 as the most straightforward procedure for derivation of estimable closed form consumption and asset equations.

If full regularity in the specification of V is desired, one may use (V6) of Theorem 1 to define $H(w,r,h,\sigma,\rho,c)$ and then check that (4.9) defines a regular U function. Except in simple cases, however, this appears to be impractical. An alternative approach is to impose regularity as a stochastic constraint on a flexible functional form.

**Method 4**

Given that (5.4) defines the Gorman cost function with $\gamma$ interpretable as committed expenditure, we conjecture a Gorman wealth function of the form:

$$(5.13)\qquad W^{G}(\lambda) = \gamma/r + \eta\lambda^{-1/(1-\beta)} .$$

Applying (4.10) we obtain:

$$(5.14)\qquad C^{G}(\lambda) = \gamma + \Big\{r - \Big[1/(1-\beta)\Big]\Big[r - \rho - (h/\sigma)^2\Big] - \tfrac{1}{2}(h/\sigma)^2\Big[-1/(1-\beta)][-1/(1-\beta) - 1\Big]\Big\}\eta\lambda^{-1/(1-\beta)}$$

and comparison with (5.4) implies:

$$(5.15)\qquad \eta = \Big\{\rho/(1-\beta) - r\beta(1-\beta) - \tfrac{1}{2}(h/\sigma)^2\beta/(1-\beta)^2\Big\}^{-1} .$$

Using (5.15), the consumption function is given parametrically by (5.4) and (5.13). The asset function follows immediately from (4.11).

## 5.2 A Generalized Profit Function

Consider the profit function:

$$(5.16)\qquad \Pi^{F}(\mu) = \mu(\ell n\mu - \mu^{\alpha}) , \qquad \alpha < 1 .$$

An explicit analytical representation of U(c) cannot be recovered for this function (except in the trivial case $\alpha = 0$). Nevertheless, the stochastic intertemporal utility maximizing problem can be solved for this preference representation by either of Methods 2 or 4. We derive the consumption function. (The asset function follows trivially.) Observe that the Frisch cost function is:

$$(5.17) \qquad C^F(\mu) = \mu(1 - \alpha\mu^{\alpha}) .$$

To apply Method 2, we note firstly that the Gorman cost function is therefore:

$$(5.18) \qquad C^G(V_w) = V_w^{-1}\left[1 - \alpha V_w^{-\alpha}\right]$$

so that the PDE (4.2) becomes:

$$(5.19) \qquad 0 = \left[r - \rho - (h/\sigma)^2\right]V_w + \left[rw - \gamma - V_w^{-1}\left\{1 - \alpha V_w^{-\alpha}\right\}\right]V_{ww} + \tfrac{1}{2}(h/\sigma)^2 V_w^2 V_{w(3)} \,/\, V_{ww}^2 .$$

This PDE admits a solution for $V_w$ in implicit form:

$$(5.20) \qquad \theta_1 V_w^{-1} + \theta_2 V_w^{-(1+\alpha)} - w = 0$$

where:

$$(5.21a) \qquad \theta_1 = 1/\rho ,$$

$$(5.21b) \qquad \theta_2 = \alpha \Big/ \left\{\left[r - \rho - (h/\sigma)^2\right](1+\alpha) - r + \tfrac{1}{2}(h/\sigma)^2(1+\alpha)(2+\alpha)\right\} .$$

We consider only the case $\alpha \neq 0$. (The case $\alpha = 0$ gives the logarithmic form which may be solved easily as a special case of HARA.) Note that (5.20) may be written (in view of (5.21a)):

$$(5.22) \qquad \rho w = V_w^{-1}\left[1 + \rho\theta_2 V_w^{-\alpha}\right] .$$

Dividing (5.18) by (5.22) and rearranging we obtain:

$$(5.23) \qquad V_w^{-\alpha} = \left[1 - (1/\rho)(c/w)\right] / \left[\alpha + \theta_2(c/w)\right]$$

and use of (5.23) in (5.18) gives the nonlinear consumption function:

$$(5.24) \qquad c = \left[\frac{1-(1/\rho)(c/w)}{\alpha + \theta_2(c/w)}\right]^{1/\alpha} - \alpha \left[\frac{1-(1/\rho)(c/w)}{\alpha + \theta_2(c/w)}\right]^{(1+\alpha)/\alpha}$$

where $\theta_2$ is given by (5.21b)

This example is of considerable interest because the consumption function is obtained from an initial specification of the profit function $\Pi^F(\mu)$ rather than the instantaneous utility function U(c). Indeed, the function U(c) does not even have an analytical representation. The derivation above uses Method 2. Alternatively, using Method 4 we simply write the Gorman cost function (5.18) in terms of $\lambda$:

$$(5.25) \qquad c = \lambda^{-1}\left[1 - \alpha\lambda^{-\alpha}\right] .$$

Then, conjecturing a similar form for the Gorman wealth function $w = W^G(\lambda)$, say:

$$(5.26) \qquad w = \lambda^{-1}\left[\theta_1 + \theta_2\lambda^{-\alpha}\right] ,$$

we find on application of (4.10) that $\theta_1$ and $\theta_2$ must satisfy (5.21a) - (5.21b). Given these restrictions the consumption function is given parametrically by (5.25) - (5.26). For this example, Method 4 is much simpler to apply than Method 2, and we have found that this is generally the case.

## 5.3 A Generalized Optimal Value Function

In this example we illustrate Method 3 for the optimal value function

$$(5.27) \qquad V(w,r,h,\sigma,\rho) = \theta_0 + \theta_1 w^{\alpha} \ell n\, w \,, \qquad \alpha < 1 \,.$$

This function is non-decreasing and concave in w for $\theta_1 > 0$ and $\ell n\, w > 1/(1-\alpha) - 1/\alpha$. Although $\theta_0$ and $\theta_1$ are functions of r, h, $\sigma$ and $\rho$ in principle, these functions need not be explicitly specified for application of (4.7). The consumption function, although complex, may be derived trivially as:

$$(5.28) \qquad f(w,r,h,\sigma,\rho) = rw + \left[r - \rho - (h/\sigma)^2\right] \frac{w}{\alpha-1} \left\{\frac{\ell nw+1/\alpha}{\ell nw+1/\alpha+1/(\alpha-1)}\right\}$$

$$+ \frac{1}{2}(h/\sigma)^2 \, \frac{(\alpha-2)w}{(\alpha-1)^2} \left\{\frac{(\ell nw+1/\alpha)^2[\ell nw+1/\alpha+1/(\alpha-1)+1/(\alpha-2)]}{\ell nw + 1/\alpha + 1/(\alpha-1)}\right\}$$

while from (4.8) the asset function is:

$$(5.29) \qquad g(w,r,h,\sigma,\rho) = (h/\sigma)^2 \left\{\frac{\ell n\, w + 1/\alpha}{\ell n\, w + 1/\alpha + 1/(\alpha-1)}\right\} / (1-\alpha)$$

A major advantage of this approach is that the consumption function is necessarily linear in c (contrast (5.24)). A possible disadvantage is that neither U(c) nor $\Pi^F(\mu)$ is analytically derivable for even moderately complex examples such as (5.27). However, for many purposes an analytical representation of instantaneous preferences is irrelevant. In such cases Method 3 seems very attractive.

## 6. THE DISCRETE TIME CASE

In this section we set out the consumer's decision making problem in a discrete time framework, demonstrate which of the previous results have close analogues in the discrete time setting, and how the other results necessary to employ the various solution approaches may be obtained by appropriate approximation procedures.

In discrete time, the consumer's decision making process may be characterized by the problem:

$$(6.1) \qquad V(w,r,h,\sigma,\rho) = \max_{\{c,a\}} E_t \sum_{\tau=t}^{\infty} (1+\rho)^{-(\tau-t)} U(c_\tau)$$

subject to:

$$(6.2a) \qquad w_{\tau+1} = [a_\tau h + r + 1]w_\tau - c_\tau + a_\tau \sigma w_\tau \varepsilon_\tau ,$$

$$(6.2b) \qquad w_t = w ,$$

$$(6.2c) \qquad w_\tau \geq 0 \qquad \text{w.p. } 1.$$

The definitions of these variables are essentially as in Section 2 except that $\varepsilon_\tau$ is a discrete time stochastic process with mean zero and variance unity. In order to enable the satisfaction of (6.2c) it is necessary that at any time $\tau$ the set $\Gamma$ of all pairs (c,a) satisfying non-negativity of next period's wealth for all possible values of $\varepsilon_\tau$ be non empty. This imposes some constraints on the class of stochastic processes to which $\varepsilon_\tau$ may belong. For example, $\varepsilon_\tau$ could not be modelled as a continuum over the whole line (e.g. Normal) but would be consistent with a Binomial or Multinomial approximation to the Normal.

The Bellman equation (the analogue of (3.2)) is:

$$(6.3) \qquad V(w,r,h,\sigma,\rho) = \max_{(a,c)\in\Gamma} \{U(c) + (1+\rho)^{-1} E\, V([ah+r+1]w - c + a\sigma w\varepsilon,\ r,h,\sigma,\rho)\}.$$

U(c) can be reconstructed from the optimization problem:

$$(6.4) \qquad U(c) = \min_{(w,r,h,\sigma,\rho,a)} E\{V(w,r,h,\sigma,\rho) - (1+\rho)^{-1} V([ah+r+1]w - c + a\sigma w\varepsilon, r,h,\sigma,\rho)\}$$

where the optimization is over the set

$$\psi(c) = \{(w,r,h,\sigma,\rho,a) \mid (ah+r+1)w - c + a\sigma w\varepsilon \geq 0 \ \forall \ \varepsilon\}$$

which ensures a non-negative value of next periods wealth.

This optimization problem is the analogue of (4.9), and in Appendix 2 the analogous properties of V and U so defined are demonstrated. Note, however, that this duality is not operational because it has not been possible to derive the analogues of Roy's Identity (corresponding to 4.7 - 4.8) which allow the closed form derivation of optimal c and a from the function V. This is due to the presence of the stochastic term $\varepsilon$ in (6.3). This precludes the derivation of exact solution methods analogous to the alternative approaches based on synthesis.

On the other hand, if we are willing to make an approximation to the discrete time Bellman equation (6.3), we may replicate all of the results previously derived for the continuous time model.

Dividing each unit time interval into sub-intervals of length $\Delta t$, and suppressing arguments where appropriate for clarity, (6.3) becomes:

$$(6.5) \qquad V(w_\tau) = \max_{(c_\tau, a_\tau)\in\Gamma_\tau} \{U(c_\tau)\Delta t + (1+\rho\Delta t)^{-1} E_\tau V(w_{\tau+\Delta t})\}$$

where:

$$(6.6) \qquad w_{\tau+\Delta t} = w_\tau + [(a_\tau h + r)w_\tau - c_\tau]\Delta t + a_\tau \sigma w_\tau \varepsilon_\tau (\Delta t)^{1/2} .$$

Taking a Taylor Series approximation of $V(w_{\tau+\Delta t})$ around $w_\tau$, taking expectations and dividing by $\Delta t$, we observe that if we drive $\Delta t$ to zero we obtain:

$$(6.7) \qquad 0 = \max_{(c,a)\varepsilon\Gamma} \{U(c) + [(ah + r)w - c]V_w + \frac{1}{2} a^2\sigma^2 w^2 V_{ww} - \rho V\} .$$

It is clear that the first order conditions for c and a from (6.7) are equivalent to (3.3) and (3.4). Hence (6.7) at the optimum is equivalent to (3.5). Thus the key result (3.6) applies, and all of the solution approaches discussed in Section 4 apply _mutatis mutandis_ to allow the derivation of approximate solutions to the discrete time problem (6.1). It also follows _á fortiori_ that all of the examples of Section 5 may serve to illustrate the application of the solution methods in a discrete time formulation.

## 7. CONCLUSION

In this paper we have considered procedures for the derivation of synthesised solutions to a typical stochastic intertemporal consumption model. We have demonstrated that, if the natural structure of these types of models is exploited and if preferences are specified by one of the natural dual representations (such as the profit function or the optimal value function), the range of solution procedures is considerably expanded. The relationship among the natural solution procedure and the natural preference representation and the associated function which is derived as part of the solution procedure can be summarised in the following table.

SOLUTION PROCEDURES

SUMMARY BASED ON SYNTHESIS

| Solution Procedure | Exploitation of Structure | Preference Representation | Associated Function |
|---|---|---|---|
| Dynamic Programming | No | $U(c)$ | $V(w,r,h,\sigma,\rho)$ |
| D.P. Applied To Marginal Function | Yes | $\Pi^F(\mu)$ | $V_w(w,r,h,\sigma,\rho)$ |
| Intertemporal Duality | Yes | $V(w,r,h,\sigma,\rho)$ | - |
| Matched Inverse Marginal Functions | Yes | $\Pi^F(\mu)$ | $W^G(\lambda,r,h,\sigma,\rho)$ |

As the examples in the paper illustrate, the use of those solution procedures which exploit the natural structure of the problem considerably expand the range of preference representations for which closed form consumption functions and asset demand equations may be derived.

## APPENDIX 1

### Proof of Theorem 1 ($U \to V$)

**V1.** For $w^2 > w^1$, let $c^1(\tau)$, $a^1(\tau)$ and $w^1(\tau)$ be the optimal solutions corresponding to $w^1$. The paths $c^1$ and $a^1$, with implied path $w(\tau)$, are feasible, but not necessarily optimal, for $w^2$. Hence $V(w^2) \geq V(w^1)$.

**V2.** For $0 \leq p \leq 1$, let $c^i(\tau)$, $a^i(\tau)$, $w^i(\tau)$ be the optimal paths corresponding to initial values $w(t) = w^1, w^2$ and $w^p = pw^1 + (1-p)w^2$, $i = 1, 2, p$. The paths $\bar{c}^p = pc^1 + (1-p)c^2$, $\bar{w}^p = pw^1 + (1-p)w^2$

and $\bar{a}^p = \dfrac{p\ a^1\ w + (1-p)\ a^2\ w^2}{p\ w + (1-p)\ w^2}$

are by construction feasible, but not necessarily optimal, for $w(t) = w^p$.

Hence $V(pw^1 + (1-p)w^2)$

$$= E_t \int_t^\infty e^{-\rho(\tau-t)} U(c^p(\tau))d\tau$$

$$\geq E_t \int_t^\infty e^{-\rho(\tau-t)} U(\bar{c}^p(\tau))d\tau$$

$$\geq E_t \int_t^\infty e^{-\rho(\tau-t)} \left[p\ U(c^1(\tau)) + (1-p)\ U(c^2(\tau))\right] d\tau$$

(by concavity of U( ))

$= pV(w^1) + (1-p)V(w^2)$

**V3.** For $r^2 > r^1$, let $c^1(\tau)$, $a^1(r)$ and $w^1(\tau)$ be the optimal solutions corresponding to $r = r^1$. The paths $c^1 + (r^2 - r^1)w^1$ and $a^1$, with implied path $w^1(\tau)$, are feasible, but not necessarily optimal, for $r = r^2$. Hence

$$V(w, r^2, h, \sigma, \rho) > V(w, r^1, h, \sigma, \rho)$$

**V4.** Analogous to V3. (Noting that $a \geq 0$ by (3.4) and V1 and V2.)

**V5.** Let $\sigma^2 = k\sigma^1$, $k > 1$. Let $c^2$, $a^2$ be optimal for $\sigma = \sigma^2$. Then $w^2$ satisfies

$$\begin{aligned} dw &= \{(a^2h + r)w - c^2\}dt + a^2\sigma^2 w\, db \\ &= \{(ka^2h + r)w - (c^2 + (k-1)a^2hw)\}dt + ka^2\sigma^1 w\, db \end{aligned}$$

which defines a feasible, but not necessarily optimal, path when $\sigma = \sigma^1$, with a higher value than the optimal path for $\sigma^2$. Hence

$$V(w, r, h, \sigma^2, \rho) < V(w, r, h, \sigma^1 \rho)$$

**V6.** For any point $(\hat{w}, \hat{r}, \hat{h}, \hat{\sigma}, \hat{\rho}) \varepsilon X \ \exists \ \hat{c} \ \varepsilon \ \mathscr{C}$ such that $U(\hat{c}) = H(\hat{w}, \hat{r}, \hat{\sigma}, \hat{\rho}, \hat{c})$. For any other point $(w, r, h, \sigma, \rho) \ \varepsilon X$,

$$U(\hat{c}) \leq H(w, r, h, \sigma, \rho, \hat{c})$$

since $\hat{c}$ is not necessarily optimal for $(w, r, h, \sigma, \rho)$.

Hence $H(w, r, h, \sigma, \rho, \hat{c}) \geq H(\hat{w}, \hat{r}, \hat{\sigma}, \hat{\rho}, \hat{c})$.

Proof of Theorem 2 ($V \rightarrow U$)

By the envelope theorem, $U_c = V_w$. That U is concave follows by standard duality arguments.

## Proofs Theorems 3, 4, 5

Follow directly from the condensed HJ equation (3.5).

## APPENDIX 2

### Stochastic Intertemporal Duality in the Discrete Time Case

1. Properties of V.

V can be defined directly by (6.1), and also satisfies (6.3). By (6.1), V is non-decreasing in w (V1), since a small increment in w leaves the original consumption plan feasible but not optimal. To prove the other results, the non-negativity of a is necessary. The partial of (6.3) with respect to a is

$$U_c + (1-\rho)^{-1}E\{V_w([ah+r+1]w - c + a\sigma w\varepsilon, r, h, \sigma, \rho)[hw + \sigma w\varepsilon]\}$$

When evaluating at a=0 the term $V_w$ is a constant independent of the random variable $\varepsilon$, and so comes out of the expectation operator. Since $E\varepsilon = 0$, the partial at a=0 is $U_c + (1+\rho)^{-1}hwV_w([1+r]w - c, r, h, \sigma, \rho) > 0$, hence the optimal a is non-negative. (Note that a is bounded from above by the possible negativity of $\varepsilon$.) Properties V2 to V6 then follow as for the continuous time case, with H defined as the minimand of (6.4).

2. Properties of U

U may be defined from V by (6.4). Since an increase in c reduces the set $\psi(c)$ and hence raises the minimum, U is non-decreasing in c (U1). Again, concavity (U2) follows by standard duality arguments.

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