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Working Paper No. 2/89

January 1989

DEPARTMENT OF ECONOMETRICS

ISSN 1032-3813

ISBN 0 86746 775 4

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TRANSFORMATIONS FOR AN EXACT GOODNESS-OF-FIT TEST OF STRUCTURAL CHANGE IN THE LINEAR REGRESSION MODEL[1]

By Maxwell L. King and Phillip M. Edwards[2]

Abstract: This paper considers testing for structural change of unknown form in the linear regression model as a problem of testing for goodness-of-fit. Transformations of recursive (or other LUS) residuals that reduce the problem to one of testing independently distributed uniform variables are presented. Exact empirical distribution function tests can then be applied without having to estimate unknown parameters. The tests are illustrated by their application to a money demand model.

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1. This research was supported by a grant from the Australian Research Council. It was also supported by the ESRC under grant HR8323 while the first author was visiting the Department of Economics at the University of Southampton. The authors wish to thank Simone Grose for research assistance and Walter Krämer for his helpful comments.

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1. Introduction

In many applications, the standard assumptions required for the classical linear regression model are somewhat questionable. This is particularly true in econometric applications, where for example, it is often difficult to find convincing arguments as to why the regression relationship is constant over time. In fact, the main point of the Lucas (1976) critique of quantitative economic policy analysis is that policy changes can cause parameter changes in economic relationships over time. Of course, if these changes are of a minor nature, then it may well be that the standard linear regression model provides a useful and meaningful approximation. It would be silly to build a complicated model when a simple one will do. It is therefore important to be able to test the adequacy of a fitted linear regression model. Typically, little may be known about how and when the regression relationship might change so that the test will need to cast a wide net. One possible approach is to apply a goodness-of-fit test to the linear regression.

The first such test that usually springs to mind is the well-known χ^2 test. This is less than ideal for, as Stephens (1974) observed, it has long been known that for goodness-of-fit problems in which the distribution function is continuous and completely specified, tests based on the empirical distribution function (EDF) are more powerful than the χ^2 test. A disadvantage of EDF based tests is that when unknown parameters in the distribution function are replaced by their estimates, the distributions of the test statistics under the null hypothesis change. Stephens gives some approximate critical values of various statistics for a random sample from the normal distribution with zero mean and unknown variance as well as unknown mean and variance.

The Cusum of squares test for structural change proposed by Brown, Durbin and Evans (1975) can be viewed as an approximate Kolmogorov-Smirnov EDF test applied to recursive residuals that have undergone a secondary nonlinear transformation. To see this, let

$$y = X\beta + u, \qquad u \sim N(0,\sigma^2 I_n) \tag{1}$$

denote the standard linear regression model where y is $n \times 1$, X is an $n \times k$ nonstochastic matrix of rank $k < n$, β is a $k \times 1$ vector of unknown parameters and σ is an unknown scale parameter. Also let $\hat{u}_j$, $j = k+1, \ldots, n$ denote the recursive residuals from (1). (For a definition of recursive residuals see, for example, Phillips and Harvey (1974), Brown, Durbin and Evans (1975) or Farebrother (1976b)). The Cusum of squares test is based on whether, for $r = k+1, \ldots, n$,

$$s_r = \left[\sum_{j=k+1}^{r} \hat{u}_j^2 \right] \Big/ \left[\sum_{j=k+1}^{n} \hat{u}_j^2 \right]$$

is always in the range

$$\pm c_o + (r-k)/m , \tag{2}$$

where c_o is an appropriately chosen value and $m=n-k$. Because $s_n = 1$, this acceptance region is equivalent to

$$\max_{i=1,\ldots,m-1} \left\{ s_{k+i} - i/m \right\} < c_o$$

and

$$\max_{i=1,\ldots,m-1} \left\{ i/m - s_{k+i} \right\} < c_o$$

which is of the form of the modified Kolmogorov-Smirnov test provided $s_{k+1}, \ldots, s_n$ is an ordered sample of independent observations from the uniform (0,1) distribution.

For the case when m is even, Brown, Durbin and Evans noted that the joint distribution of

$$s_{k+2},\ s_{k+4},\ \ldots,\ s_{n-2} \tag{3}$$

is identical to that of an ordered sample of independent observations from the uniform (0,1) distribution. If the test is based on these (m/2)-1 statistics then Durbin's (1969) table of significance points for the modified Kolmogorov-Smirnov EDF test can be used to determine c_o. Brown, Durbin and Evans suggested using this value, or a linearly interpolated value if m is odd, for c_o in (2). They reported that Monte Carlo evidence indicated that this choice of c_o value yields true significance levels slightly above nominal levels. An exact EDF test when m is even, could have been based on the (m/2)-1 statistics given by (3) with an obvious reduction in power.

In this paper we propose alternative transformations of recursive and other residuals which allow exact EDF tests to be applied to a full set of observations. Invariance arguments are used to reduce the goodness-of-fit testing problem to one of testing independent variables from the uniform (0,1) distribution so that the standard EDF tests such as the Kolmogorov-Smirnov, Cramer-von Mises, Kuiper, Watson and Anderson-Darling tests can be used. A similar approach has been suggested by Csörgö, Seshadri and Yalovsky (1973) (also see Mardia, 1980) for the special case of a random sample from the normal distribution with unknown mean and variance.

The proposed transformations are discussed in the next section and the results of an application of the proposed testing procedure to an annual model of the demand for money in the U.S.A. are presented in Section 3.

2. The Transformation

Our goodness-of-fit problem is one of testing

$$H_0 : y \sim N(X\beta, \sigma^2 I_n)$$

against

$$H_a : y \nsim N(X\beta, \sigma^2 I_n)$$

where both β and σ^2 are unknown. Observe that if H_a is true then at least one of either

(i) $E(y) \neq X\beta$,

(ii) $Var(y) \neq \sigma^2 I_n$,

(iii) y is non-normal ,

is true so we are indeed casting a wide net. While it is obvious how a structural change might result in (i) or (ii) being true, note that (iii) will occur in a regression whose errors switch distribution at some point in time.

This testing problem is invariant to transformations of the form

$$y^* = \gamma_0 y + X\gamma , \qquad (4)$$

where γ_0 is a scalar and γ is a $k \times 1$ vector. This is because if H_0 holds then

$$y^* \sim N(X(\gamma_0\beta + \gamma) , \gamma_0^2\sigma^2 I_n)$$

which means that H_0 also holds for y^*. Furthermore, if H_a is true because of at least one of (i), (ii) or (iii) holding then the same will also be true of y^* given the form of (4).

As King (1980) notes, the $m \times 1$ vector

$$v = P_1 z/(z' P_1' P_1 z)^{1/2}$$

is a maximal invariant under the group of transformations defined by (4) where $z = My$ is the vector of ordinary least squares residuals, $M = I_n - X(X'X)^{-1}X'$, and P_1 is an $m \times n$ matrix such that $M = P_1'P_1$ and $P_1P_1' = I_m$.

Under H_0, v is uniformly distributed over the surface of the unit m-sphere. Because of this, when v is transformed to polar coordinates, $\theta_j \in [0,\pi]$, $j = 1, 2, \ldots, m-2$, $\theta_{m-1} \in [0,2\pi]$, via

$$v_1 = \cos \theta_1 ,$$

$$v_j = \left[\prod_{i=1}^{j-1} \sin \theta_i\right] \cos \theta_j \qquad 2 \le j \le m-1 ,$$

$$v_m = \prod_{i=1}^{m-1} \sin \theta_i ,$$

it follows (see Goldman, 1976) that $\theta_1, \ldots, \theta_{m-1}$ are independent random variables under H_0 with probability density functions:

$$P_{\theta_j}(\theta_j) = \Gamma\{(m-j+1)/2\}\pi^{-1/2}\left[\Gamma\{(m-j)/2\}\right]^{-1}\sin^{m-1-j}\theta_j ,$$

$$\theta_j \in [0,\pi], \quad j = 1, 2, \ldots, m-2,$$

$$P_{\theta_{m-1}}(\theta_{m-1}) = 1/(2\pi) , \qquad \theta_{m-1} \in [0,2\pi] .$$

Observe that if e_i is the $m \times 1$ vector of zeros with the i^{th} element being unity, then θ_1 is the angle between e_1 and v, and θ_j is the angle between e_j and the projection of v onto the manifold spanned by $e_j, e_{j+1}, \ldots, e_m$ for $j = 2, \ldots, m-1$.

Given the independence of $\theta_1, \ldots, \theta_{m-1}$ under H_0, the transformations

$$w_j = \int_0^{\theta_j} P_{\theta_j}(x)dx , \qquad j=1, \ldots, m-1 ,$$

result in independently distributed uniform variables on the interval (0,1) under H_0. These transformations can be performed using the following formulae:

$$w_{m-1} = \theta_{m-1}/(2\pi) .$$

For $1 \le j \le m-2$ and $m-j$ odd, let $q = (m-1-j)/2$. Then

$$w_j = \Gamma(q+1)\pi^{-1/2}\left[\Gamma(q+\tfrac{1}{2})\right]^{-1}\left[2^{-2q}\binom{2q}{q}\theta_j + (-1)^q\, 2^{-(2q-1)} \sum_{k=0}^{q-1} (-1)^k \binom{2q}{k}\left\{\sin(2q-2k)\theta_j\right\} / \left\{2q-2k\right\}\right] .$$

For $1 \le j \le m-2$ and $m-j$ even, let $q = (m-2-j)/2$. Then

$$w_j = \Gamma(q+3/2)\pi^{-1/2}\left\{\Gamma(q+1)\right\}^{-1}$$

$$\left[2^{-2q}(-1)^{q+1} \sum_{k=0}^{q} (-1)^k \binom{2q+1}{k}\left[\cos\left\{(2q+1-2k)\theta_j\right\}-1\right] / (2q+1-2k)\right]$$

The resultant w_j, $j=1, \ldots, m-1$, after having been sorted into ascending order

$$w_j^{(1)} \le w_j^{(2)} \le \ldots \le w_j^{(m-1)} ,$$

can be used to calculate standard test statistics based on the EDF as follows:

(i) The Kolmogorov-Smirnov statistics D, D^+, D^-:

$$D^+ = \max_{1 \le i \le m-1}\left\{i/(m-1)-w_j^{(i)}\right\} , \quad D^- = \max_{1 \le i \le m-1}\left[w_j^{(i)}-\left\{(i-1)/(m-1)\right\}\right]$$

and $D = \max\,(D^+, D^-)$.

(ii) The Cramer-von Mises statistic W^2:

$$W^2 = \sum_{i=1}^{m-1}\left[w_j^{(i)}-\left\{(2i-1)/(2m-2)\right\}\right]^2 + 1/\left\{12(m-1)\right\} .$$

(iii) The Kuiper statistic V:

$$V = D^{+} + D^{-} .$$

(iv) The Watson statistic U^2:

$$U^2 = W^2 - (m - 1)(\bar{w} - 0.5)^2$$

$$\text{where} \quad \bar{w} = \left[\sum_{i=1}^{m-1} w_i\right] / (m-1) .$$

(v) The Anderson-Darling statistic A^2:

$$A^2 = -\sum_{i=1}^{m-1}\left[(2i-1)\left\{\log w_j^{(i)} + \log\left[1-w_j^{(m-i)}\right]\right\} / (m-1)\right] - (m-1) .$$

Stephens (1974) presents tables for finding the critical values of each of the statistics. (Also see Pearson and Hartley, 1972.)

How should one compute v? Observe that $Var(P_1z) = \sigma^2 P_1MP_1' = \sigma^2 I_m$ so that $P_1z \sim N(0,\sigma^2 I_m)$. This implies that v can be regarded as a linear unbiased with scalar covariance matrix (LUS) residual vector divided by its norm. For any given regression model there are an infinite number of LUS residual vectors. Some of the best known are Theil's (1965, 1968) BLUS residuals and recursive residuals. These and other LUS residuals are reviewed by King (1987).

When testing for structural change, we recommend the use of recursive residuals. They can be calculated recursively either forwards in time or backwards in time. If one suspects that a change may have occurred late in the estimation period then tests based on backward recursive residuals are likely to have better power. Because BLUS residuals are "best" estimates of m of the unknown disturbances they may be preferable when testing specifically for non-normality. Algorithms for computing BLUS and recursive residuals may be found in Farebrother (1976a, 1976b).

3. An Example

This section considers the application of the above exact EDF tests to an annual regression model of the demand for money in the U.S.A. suggested by Klein (1977). This model was used by Krämer and Sonnberger (1986) to illustrate the use of diagnostic testing in practice. Using Klein's notation, the model is

$$\log M = a_0 + a_1 \log y_p + a_2 r_S + a_3 r_L + a_4 r_M + a_5 \log S(\dot{P}/P) + u \qquad (5)$$

where M is the quantity of money (M2), y_p is real permanent income, r_S is a short term interest rate, r_M is the rate of return on money, $S(\dot{P}/P)$ is a measure of variability of the rate of price changes and u is the disturbance term. Annual observations of these variables for 1879-1974 are given by Krämer and Sonnberger (1986, Table A.1).

Farebrother's (1976b) algorithm was used to calculate recursive residuals forwards in time and backwards in time. Both sets of residuals, calculated using the full data set (1879-1974), were transformed as outlined above and the resultant w_j, j=1, ..., 89, were sorted into ascending order. The calculated values of each of the EDF test statistics are given in Table 1. With one exception, all tests reject H_0 at the one per cent significance level. The one exception is the D^+ test based on backwards recursive residuals which is significant at the five per cent level. There is ample evidence that the classical linear regression based on (5) does not fit the data well.

TABLE 1 : Values of the EDF test statistics for Klein's demand for money model; 1879-1974.

Test Statistics	Forward recursive residuals	Backward recursive residuals
D^+	0.2038	0.1328
D^-	0.2120	0.2469
D	0.2120	0.2469
W^2	1.6114	1.2790
V	0.4159	0.3797
U^2	1.6114	1.1222
A^2	9.3343	7.5476

References

Brown RL, Durbin J, Evans JM (1975) Techniques for testing the constancy of regression relationships over time. Journal of the Royal Statistical Society B 37: 149-163.

Csörgö M, Seshadri V, Yalovsky M (1973) Some exact tests for normality in the presence of unknown parameters. Journal of the Royal Statistical Society B 35: 507-522.

Durbin J (1969) Tests for serial correlation in regression analysis based on the periodogram of least squares residuals. Biometrika 56: 1-15.

Farebrother RW (1976a) Algorithm AS104: BLUS residuals. Applied Statistics 25: 317-322.

Farebrother RW (1976b) Recursive residuals - a remark on Algorithm AS75: Basic procedures for large, sparse or weighted linear least squares problems, Applied Statistics 25: 323-324.

Goldman J (1976) Detection in the presence of spherically symmetric random vectors, I.E.E.E. Transactions on Information Theory IT-22: 52-59.

King ML (1980) Robust tests for spherical symmetry and their application to least squares regression. The Annals of Statistics 8: 1265-1271.

King ML (1987) Testing for autocorrelation in linear regression models: A Survey. In: ML King and DEA Giles (eds) Specification Analysis in the Linear Model, Routledge and Kegan Paul, London

Klein B (1977) The demand for quality-adjusted cash balances: Price uncertainty in the U.S. demand for money function. Journal of Political Economy 85: 691-715.

Krämer W, Sonnberger H (1986) The Linear Regression Model Under Test, Physica-Verlag, Heidelberg

Lucas RE (1976) Econometric policy evaluation: A critique. Carnegie-Rochester Conferences in Public Policy 1: 19-46.

Mardia KV (1980) Tests for univariate and multivariate normality. In PR Krishnaiah (ed) Handbook of Statistics 1: Analysis of Variance, North-Holland, Amsterdam

Pearson ES, Hartley HO (1972) Biometrika Tables for Statisticians 2, Cambridge University Press, Cambridge

Phillips GDA, Harvey AC (1974) A simple test for serial correlation in regression analysis. Journal of the American Statistical Association 69: 935-939.

Stephens MA (1974) EDF statistics for goodness of fit and some comparisons. Journal of the American Statistical Association 69: 730-737.

Theil H (1965) The analysis of disturbances in regression analysis. Journal of the American Statistical Association 60: 1067-1079.

Theil H (1968) A simplification of the BLUS procedure for analyzing regression disturbances. Journal of the American Statistical Association 63: 242-251.

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