ISSN 0729-0683

ISBN 0 86746 703 7

MONASH UNIVERSITY

SOME RECENT DEVELOPMENTS IN NON-LINEAR TIME SERIES MODELLING

Kuldeep Kumar

Working Paper No. 7/88

July 1988

DEPARTMENT OF ECONOMETRICS

ISSN 0729-0683

ISBN 0 86746 703 7

SOME RECENT DEVELOPMENTS IN NON-LINEAR TIME SERIES MODELLING

Kuldeep Kumar

Working Paper No. 7/88

July, 1988

DEPARTMENT OF ECONOMETRICS, FACULTY OF ECONOMICS AND POLITICS

MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA.

SOME RECENT DEVELOPMENTS IN NON-LINEAR TIME SERIES MODELLING

by

Kuldeep Kumar*
Indian Institute of Management
Lucknow 226020, India.

Key words and phrases: ARMA Models, Bilinear Models, Threshold Models, Exponential Autoregressive Models, Random Coefficient Autoregressive Models, Exponential Moving Average Models, State Dependent Models, Model Specification, Stationarity and Invertibility.

May, 1988

* This paper was written when the author was visiting Monash University. The author is most grateful to the Department of Econometrics, particularly, to Professor Maxwell L. King for inviting him to visit the Department and for providing the research support.

ABSTRACT

Most of the recent work in time series analysis has been done on the assumption that the structure of the series can be described by linear models such as Autoregressive (AR), Moving Average (MA) or mixed Autoregressive-Moving Average (ARMA) models. However, there are occasions on which subject matter, theory or data suggests that linear models are unsatisfactory and hence it is desirable to look at non-linear time series models.

In the last decade several non-linear time series models have appeared in literature, specifically, bilinear time series models, threshold AR models, exponential AR models, random coefficient AR models, exponential moving average models and other related models. In this paper we have reviewed various non-linear time series models. We have also reviewed various tests of non-linearities developed by various authors. Since the model specification is the most important step in any time series model building, we have discussed the problem of model specification in the context of bilinear and threshold models in detail.

SOME RECENT DEVELOPMENTS IN NON-LINEAR TIME SERIES MODELLING

1. INTRODUCTION:

Since the appearance in 1970 of the book by Box and Jenkins (1970,1976); the use of Autoregressive-Moving average (ARMA) models has become widespread in many fields for the analysis and prediction of time series data including economic forecasting. These models are commonly referred to as Box-Jenkins models and the whole approach is usually referred as the Box-Jenkins approach. The details of these models can also be found in Chatfield (1980), Priestley (1981), Hannan (1962,70), Anderson (1976), Cryer (1986), Vandaele (1983) etc. Box-Jenkins models are linear in nature, but there are occasions on which subject matter, theory or data suggests that linear models are unsatisfactory and the time series in question cannot be adequately described by linear models, hence it is desirable to look at non-linear time series models. Non-linear time series models are usually non-Gaussian in nature and it has been suggested that they can be transferred to Gaussian by using certain transformations. However, it has been shown by Granger and Andersen (1978b) that forecasts are usually biased and lead to higher mean square errors if such transformations are made. In a study Davies and Petrucelli (1985) analysed 234 real data sets and observed that 15% of these series are non-linear in nature. In another study conducted by the same authors, they analysed 160 real data sets and 67 of these were detected as non-linear. Predictions were considerably improved when non-linear time series models were used instead of ARMA models.

Tong (1983) has summarised the following limitations on ARMA models:

(i) having symmetrical joint distributions; stationary Guassian ARMA models are not ideally suited for data exhibiting strong asymmetry e.g. hydrological data,

(ii) ARMA models are not ideally suited for data exhibiting sudden burst of very large amplitude at irregular time due to non-normality,

(iii) ARMA models are not ideally suited for data exhibiting time irreversibility.

Subba Rao and Gabr (1980), Hinich (1982), McLeod and Li (1983), Keenan (1985), Petruccelli and Davies (1986), Chan and Tong (1986a) provided some tests for non-linearity which also suggests that many time series found in practice (for example the "classical" sunspot data, Canadian Lynx data, IBM daily common stock closing price data, etc.) cannot be adequately described by linear models; hence it is desirable to look at non-linear time series models. A brief review of these tests has been given in this paper.

Keeping in view the importance of non-linear time series modelling quite a few non-linear time series models have appeared in the literature in the last decade. However, certain restrictions should be imposed on non-linear models in order to make them useful. The first restriction is that they should not have an explosive solution, i.e. they should be stationary. A second limitation is that they are capable of producing forecasts and for this the models should be invertible. Besides, Tong and Lim (1980) proposed following requirements of non-linear time series models:

(a) identification of the model should not entail excessive computation;

(b) they should be general enough to capture some of the non-linear phenomenon;

(c) one step ahead prediction should be easily obtained from the fitted model. If the model is non-linear, its overall prediction performance should be an improvement over the linear model.

(d) they should possess some degree of generality and be capable of generalisation in multivariate case,

(e) they should reflect the structure of the mechanism generating the data and have some intuitive appeal.

Although the era of non-linear time series models started with Wiener (1956) and was discussed by Nelson and Van Ness (1973) it is in the last decade that most progress has been made in the field of non-linear time series modelling. Granger and Andersen (1978b) and Subba Rao and Gabr (1984) studied bilinear time series models in detail. These models were originally developed by control engineers to describe input-output relationships for a deterministic non-linear system. Tong (1983) has discussed another class of non-linear time series models commonly known as Threshold Autoregressive models. These models are general enough to capture the notion of limit cycles which plays a key role in the modelling of cyclical data and in physical and biological sciences. Other non-linear time series models are exponential AR models introduced by Haggan and Ozaki (1980,81), non-linear threshold models, random coefficient AR models introduced by Nicholls and Quinn (1980), exponential moving average and other related models introduced by Lawrence and Lewis (1977, 1980, 1985) and some other non-Gussian time series models introduced by McKenzie (1980) and Raftery (1982).

It may be mentioned that although many non-linear time series models have appeared in the literature, bilinear and threshold time series models have been discussed in detail by many authors and the application on real data sets for these models have also been discussed.

In this paper we have reviewed various non-linear time series models especially the developments in Bilinear and Threshold time series models. Since the most important step in any time series model building is the specification of the correct model, the specification problem of these models has been discussed in detail. In section 2, we have reviewed the various tests of non-linearity. In sections 3 and 4, we have discussed bilinear and theshold models in detail. In section 5, we have discussed some other non-linear time series models and in section 6, we have discussed state dependent models introduced by Priestley (1980). Finally conclusion has been drawn in section 7.

2. TESTS FOR NON LINEARITY

Recently a number of tests for non-linearity has been proposed in the literature. Some of these tests are based on a frequency domain approach whereas others use the time domain approach. Chan and Tong (1986a) argued the following reasons for developing the test of non-linearity:

(i) the tests will throw some light on the incidence rate of non-linearity in real time series;

(ii) suggest use of non-linear predictor in preference to linear ones;

(iii) what kind of non-linearity are present in the data.

Subba Rao and Gabr (1980) and Hinich (1982) have based their tests on frequency domain whereas other tests are based on time domain. Subba Rao-Gabr have constructed two tests aimed at detecting:

(a) whether the process is Gaussian, in which case given that it is stationary it must necessarily conform to a linear model.

(b) if the process is non-Gaussian, whether it conforms to a linear model.

The test is based on spectral and bispectral analysis of stationary time series data. Hinich (1982) proposed an improved and robustified version of the test. The main drawbacks of the test can be listed as follows:

(i) series length needs to be large for the application which is the usual drawback in spectral analysis.

(ii) great skill is necessary in applying it because of the large number of parameters involved. The user has to decide choice of lag window (Tukey, Parzen or Daniell), the truncation points and placing of grids.

(iii) Hinich (1982) pointed out that Subba Rao-Gabr test can be sensitive to outliers and hence proposed an improved and robustified version of the test.

(iv) the test may not work for the class of non-linear time series models with symmetric joint distribution.

In the time domain approach Keenan (1985) has proposed a test for linearity. He assumed that the series, having a Volterra expansion, may be adequately represented by a second order expression

$$Z_t = \mu + \sum_{u=-\infty}^{\infty} \theta_u a_{t-u} + \sum_{v=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} \theta_{uv} a_{t-u} a_{t-v}$$

The approximation will be linear if and only if the last term on the right is zero, i.e. he is testing:

$$H_o : Z_t = \mu + \sum_{u=o}^{M} a_u a_{t-u} \qquad (2.1)$$

vs

$$H_1 : Z_t = \mu + \sum_{u=o}^{M} a_u a_{t-u} + \sum_{u=o}^{M'} \sum_{v=o}^{M''} a_{uv} a_{t-u} a_{t-v} \qquad (2.2)$$

where $\{a_t\}$ is a sequence of i.i.d. random variables and M, M′, M″ are sufficiently high. The mechanics of the test is similar to Tukey's one degree of freedom test for non-additivity and runs as follows using the observations $(Z_1, Z_2, \ldots Z_n)$,

(a) Fit model (2.1) to the data and calculate the fitted value $\{\hat{Z}_t\}$ and the residuals $\{\hat{a}\}_t$ for t = M+1,...,n, and the residual sum of square $\langle\hat{a}\hat{a}\rangle = \Sigma\hat{a}_t^2$.

(b) Regress $\hat{Z}_t^2$ on $\{1, Z_{t-1}, \ldots, Z_{t-m}\}$. Let $\{\hat{\xi}_t\}$ be the residuals.

(c) Let $\hat{\eta} = \sum_{t=M+1}^{n} \hat{a}_t \hat{\xi}_t$.

(d) Calculate $\hat{F} = \dfrac{\hat{\eta}^2(n-2M-2)}{\langle \hat{a}\hat{a}\rangle - \hat{\eta}^2}$

under $H_o : \hat{F} \sim F_{1,n-2M-2}$.

The advantages of Keenan's test are that it is easy and quick to implement involving little subjective choice of parameters. It generally gives quite stable results. However, one major drawback of Keenan's test is that it is valid only for the non-linear series having Volterra expansion but all the non-linear time series models do not possess Volterra expansion. For other drawbacks see Davies and Petruccelli (1985).

McLeod and Li (1983) propose a portmanteau test for non-linearity based on squared autocorrelation

$$Q_{aa} = n(n+2) \sum_{k=1}^{m} \hat{\gamma}^2_{aa}(k)/(n-k)$$

where $\hat{\gamma}_{aa}(k) = \sum_{t=1}^{n-k} (\hat{a}^2_t - \hat{\sigma}^2_a)(\hat{\sigma}^2_{t+k} - \hat{\sigma}^2_a) / \sum_{t=1}^{n} (\hat{a}^2_t - \hat{\sigma}^2_a)^2$

where $\hat{\sigma}^2_a = \sum_{t=1}^{n} \hat{a}^2_t / n$

Davies and Petruccelli (1985) have compared the tests of non-linearity developed by McLeod and Li (1983) and Keenan (1985). They have checked the empirical significance level and reported power studies when the tests are applied to bilinear and threshold models. They have found that the performance of the test statistic developed by McLeod and Li is poor except for large sample size whilst for Keenan statistic is better both for bilinear and threshold models.

Petruccelli and Davies (1986) propose a new portmanteau test (CUSUM test) for threshold type non-linearity based on cumulative sum of standardised one step ahead forecasts from fitted AR models. The authors have shown by evaluating the performance of the test on simulated and real data sets, that it is a reasonable alternative to the McLeod and Li and Keenan tests.

Chan and Tong (1986a) have discussed the development of a test of

$$H_o : b_i^{(1)} = b_i^{(2)} \text{ for } i = 0, 1, \ldots p,$$

within the class of SETAR models (the details of these models are given in section 4).

$$Z_t = \begin{cases} b_o^{(1)} + \sum_{i=1}^{p} b_i^{(1)} Z_{t-i} + e_t & \text{if } Z_{t-d} \le \gamma \\ b_o^{(2)} + \sum_{i=1}^{p} b_i^{(2)} Z_{t-i} + e_t & \text{if } Z_{t-d} > \gamma \end{cases} \qquad (2.3)$$

Chan and Tong (1986a) have taken the conventional likelihood ratio approach and the test statistic λ is given by

$$\lambda = \{ \hat{\sigma}^2_{NL} / \hat{\sigma}^2_{L} \}^{N/2}$$

where N denotes the sample size, $\hat{\sigma}^2_{NL}$ is the usual mean residual sum of squares under 2.3 and $\hat{\sigma}^2_{L}$ is that under H_o. Chan and Tong (1986a) have done a comparative study of Hinich, Keenan, CUSUM and their own test.

It may be mentioned that none of the tests is a sure-fire test and none of the tests provides a basis for choosing a model among competing non-linear models.

Two ideas which look potentially useful, but have not been exploited yet for testing non-linearity may be mentioned here.

In his book on threshold models Tong (1983) has used the regression function at lag-j, i.e. $E[Z_t/Z_{t-j} = \alpha]$ for investigating non-normality and non-linearity. For a particular value of j (say j=1), $E[Z_t/Z_{t-1}= \alpha]$ is expected to be a linear function of Z_{t-1} if the process is linear but not so if the process is non-linear (say bilinear or threshold). If the process is linear this function will be linear for all j. Kumar (1986b) has done some simulation studies to investigate the applicability of this test and the results were found to be quite encouraging. In the case of linear AR, MA or mixed ARMA model, the function gives more or less a straight line pattern but quite different non-linear shapes of the function were obtained for bilinear and threshold MA models. It would be interesting to study the behaviour of the function theoretically for different models and to construct some test for distinguishing between linear and non-linear functions.

In the case of linear time series models with Gaussian noise all third order moments

$$\mu_{k,\ell} = E[(Z_t - \mu)(Z_{t-k} - \mu)(Z_{t-\ell} - \mu)]$$

are zero for all values of k and ℓ but they will not be zero in the case of many non-linear time series or for linear models having (non-symmetric) non-normal noise. It may be worth trying to develop some test based on third order moments in the time domain which is equivalent to the tests based on bispectrum analysis.

3. BILINEAR MODELS.

Recently Granger and Andersen (1978b) (subsequently referred to as GA) and Subba Rao (1981) proposed a special class of non-linear model known as Bilinear time series models. This type of model has been extensively discussed in control theory to describe input-output relationships for a deterministic non-linear system; see for example Mohler (1973). Bilinear models offer a class of model that are potentially capable of analysis and they may pick up part of any non-linearity in the data and thus could suggest improved methods of forecasting. It has been shown by Subba Rao and Gabr (1984) that bilinear models have some interesting properties which can match known properties of real data. The outstanding advantage of the bilinear model is that it involves only a finite number of parameters and hence makes it feasible to consider the problem of fitting such models to real data. In this respect bilinear models may be regarded as natural non-linear extension of the ARMA models. It has been shown by GA that a series which is white noise according to autocovariance (COVA) analysis may well be forecastable from its own past using bilinear models and hence they show the importance of bilinear models in practice.

3.1 *Definition:*

The general bilinear autoregressive moving average model of order (p,q,P,Q) (abbreviated as BL(p,q,P,Q)) as defined by Subba Rao (1981) is

$$Z_t = \sum_{j=1}^{p} \phi_j Z_{t-j} + \sum_{i=0}^{q} \theta_i a_{t-i} + \sum_{k=1}^{Q} \sum_{\ell=1}^{P} \beta_{k\ell}\, a_{t-k} Z_{t-\ell} \qquad (3.1.1)$$

where $\{a_t\}$ are independently, identically distributed normal random variables with mean zero and variance σ_a^2. These models are linear in the Z_t's and also in the a_t's separately but not in both. The completely bilinear model is given by (3.1.1) with p=q=0; so that

$$Z_t = \sum_{k=1}^{Q} \sum_{\ell=1}^{P} \beta_{k\ell}\, a_{t-k}\, Z_{t-\ell} + a_t \qquad (3.1.2)$$

If $\beta_{k\ell} = 0$ for all $k > \ell$, the model is called superdiagonal, and if $\beta_{k\ell} = 0$ for all $k \neq \ell$, the model is said to be diagonal. Finally if $k < \ell$, the model is called sub-diagonal. The simple superdiagonal, subdiagonal and diagonal bilinear models can be written as follows

$$Z_t = \beta Z_{t-2}\, a_{t-1} + a_t \qquad \text{(superdiagonal model)} \qquad (3.1.4)$$

$$Z_t = \beta Z_{t-1}\, a_{t-1} + a_t \qquad \text{(diagonal model)} \qquad (3.1.4)$$

$$Z_t = \beta Z_{t-1}\, a_{t-2} + a_t \qquad \text{(subdiagonal model)} \qquad (3.1.5)$$

The models (3.1.3, 3.1.4 and 3.1.5) along with their generalisations have been discussed in detail by Granger and Andersen (1978b). The model (3.1.1) has been discussed in detail by Subba Rao and Gabr (1984).

GA and Subba Rao (1981) discuss the condition for stationarity and invertibility of some simple bilinear time series models. The question of invertibility which is required if the models are to be useful for forecasting purposes, is particularly interesting in this context. Let $\hat{\varepsilon}_t$ be an estimate of the innovation ε_t, then Granger and Andersen (1978a) call the model invertible if

$$\lim_{t \to \infty} E[(\varepsilon_t - \hat{\varepsilon}_t)^2] = 0$$

when this condition is satisfied, the model can be projected forward in an obvious way, with $\hat{\varepsilon}_t$ replacing ε_t, to obtain forecast of future x_t. Tong (1981) has discussed the condition of ergodicity of a specific bilinear model. Stationarity and invertibility of simple bilinear time series models has been discussed by Quinn (1982) and Tuan and Tran

(1981). Ren, Zhi and Tong (1983) have studied the distribution of simple stationary bilinear processes and Liu (1985) has studied theoretical properties of some more general bilinear models BL(p,q,γ,1). Priestley (1980) has shown that bilinear time series models as well as other non-linear time series models can be considered as special cases of his state dependent models.

In their book on Bispectrum analysis and Bilinear time series models, Subba Rao and Gabr (1984) have discussed the bispectrum analysis and its application in developing a test of non-linearity in detail. They have also discussed the general bilinear time series models and its properties in detail including the conditions of its stationarity and invertibility. To estimate the parameters of the bilinear model they have used the Newton-Raphson method which has been used by Box and Jenkins (1970) as well for estimating the parameters of the linear time series models. In particular they have considered the problem of fitting a BL(p,0; m,k) model

$$Z_t + \sum_{i=1}^{p} a_i Z_{t-i} = \alpha + \sum_{i=1}^{m} \sum_{j=1}^{k} b_{ij} Z_{t-i} e_{t-j} + e_t$$

and have outlined a procedure for obtaining the initial estimates of the parameters. Subba Rao and Gabr have studied the sampling properties of parameters estimate of some simple BL(1,0,1,1) model. Existence, strict stationarity and ergodicity of bilinear time series models for a given input white noise and paramater values are studied in detail in the recent paper by Akamanam, Rao and Subramanyam (1986). The use of ergodicity in the estimation of parameter is also hinted in the article. Maravell (1983) provides detailed discussion of an interesting application in which a bilinear model is fitted to an economic time series.

Subba Rao (1981) has pointed out that sometimes there may be more paramaters in the full bilinear time series models and it may be possible to look for subset bilinear time series models. Keeping this in view Gabr and Subba Rao (1981) defined subset bilinear models and then described an algorithm for the estimation of these models. The method is illustrated with real time series data and the optimal several steps ahead forecast of these time series models are calculated.

3.2 *Specification of Bilinear Time Series Models.*

The specification problem in Bilinear time series models can be classified as

(i) how to distinguish bilinear models from linear models.

(ii) if the series is bilinear how to distinguish between various classes.

(iii) how to distinguish between various lags once a particular class of bilinear model has been specified.

So far in the literature GA (1978b), Subba Rao (1981), Li (1984), and Kumar (1986a), have tackled the problem of identification of bilinear time series models. While the method proposed by Subba Rao is based on AIC (see Akaike (1977)) and is automatic in nature, the other methods are semi-subjective in nature.

3.2.1 *Subba Rao's method*

Subba Rao's method for the identification of the order p,m,k of the bilinear model BL(p,0; m,k) is to estimate the parameters of the model for different values of p,m,k and in each case calculate the residual variance $\hat{\sigma}_e^2$. The information criterion due to Akaike (1977) is defined as AIC = -2(max log likelihood) + 2 (number of independent paramaters)

$$= (N - \gamma)\log\hat{\sigma}_e^2 + 2 \text{ (number of independent paramaters)}$$

and the normalised AIC (NAIC) is defined as

$$\text{NAIC} = \text{AIC}/(N-\gamma)$$

where $\hat{\sigma}_e^2$ is the residual variance, $(N-\gamma)$ is the number of effective observations. The chosen order is one for which AIC value is minimum.

The algorithm for choosing the order of the bilinear model BL(p,0;m,k) consists of first choosing a fixed integer γ which should be greater than or equal to the order of the best AR model for the data. Then fitting the linear AR(p) model and let the corresponding residual variance be $\hat{\sigma}_e^2(AR)$. Take the coefficients obtained from above as initial estimates of the AR part of the BL(p,0;1,1) model and set $b_{11}=0$. Calculate the corresponding $\hat{\sigma}_e^2$ and AIC values for the fitted model. Using the coefficient so obtained fit the BL(p,0;2,1) and BL(p,0;1,2) model and calculate the corresponding $\hat{\sigma}_e^2$ and AIC values for both fitted models. Take the coefficients obtained from BL (p,0;1,2) or BL(p,0;2,1) whichever has the smaller residual variance, as the initial values for fitting the BL (p,0;2,2) model. The procedure is continued for all possible combinations (m,k) s.t $m,k < \gamma$. For other values of p = $1,2,\ldots\gamma$ we repeat all the steps and the procedure stops if the residual variance $\hat{\sigma}_e^2$ increases as m and k increases. Finally, we choose that model for which the AIC is minimum. The details of the algorithm and computer program is given in Subba Rao and Gabr (1984, pp.176-77).

This approach requires tedious calculations and is difficult to use. AIC has been criticised by Shibata (1976) and Kashyap (1980) for its inconsistent behaviour in the case of linear model but nothing is known about non-linear models. It may be mentioned that since the distribution of bilinear models is not known, the AIC used here is pseudo AIC.

3.2.2 *GA's Method*

GA have noted that in most cases, the autocorrelation for the models defined in (3.1.2) will be zero or their structure will be similar to MA(q) models and hence it will be difficult to distinguish them from complete white noise or MA(q) models. The obvious moments to be considered next are third order moments.

$$\mu_{k,\ell} = E[(Z_t - \mu)(Z_{t-k} - \mu)(Z_{t-\ell} - \mu)]$$

which will all be zero for complete white noise and also in the case of pure AR, pure MA and mixed ARMA models with normal noise. But GA did not use the third order moments on the grounds that for some bilinear models like

$$Z_t = \beta\, Z_{t-2}\, a_{t-1} + a_t \tag{3.2.1}$$

"all the third order moments will be zero" and hence they will be of no use in discriminating between true white noise and bilinear models. However, Kumar (1986a) has shown that GA's assertion regarding third order moments for the model of type (3.2.1) is not valid. GA therefore considered the fourth order moments or rather a small subset of them, namely the autocovariance for the series $\{Z_t^2\}$. These are all zero for the white noise but typically not for series generated by (3.1.2). The method given by GA is good enough to distinguish bilinear models from linear models but it is difficult to distinguish between various subclasses of bilinear models, or to identify appropriate lags using this method.

Recently Li (1984) looked further at the idea of considering the autocorrelation of the square process $\{Z_t^2\}$ defined by

$$\ell_j = \frac{\mathrm{Cov}(Z_t^2, Z_{t-j}^2)}{\mathrm{Var}(Z_t^2)}$$

and tried to distinguish between bilinear models using ℓ_j : j=1,2,3,... He has claimed that these autocorrelations are vital in the determination of k and ℓ for simple bilinear model of the type

$$Z_t = \beta Z_{t-k} a_{t-\ell} + a_t$$

3.2.3 *Kumar's Method*

Kumar (1986a) has shown that GAs assertion regarding third order moments for the model of type (3.2.1) is not valid. In fact, some of the third order moments do not vanish at all for non-diagonal and diagonal bilinear models and the pattern of non-zero moments can be used to discriminate between true white noise and bilinear models and also between different subclasses of bilinear models. It has been observed that lags k and ℓ of the particular bilinear models can also be identified by looking at the third order moments. If it is assumed that the error terms $\{a_t\}$ are normally distributed then the third order moments $\mu_{k,\ell}$ will be zero in the case of pure AR, pure MA and mixed ARMA models but not for bilinear models. Kumar (1986b) has also obtained third order moments for some general bilinear models and it has been found that third order moments are successful in identifying these models as well.

The main theoretical results given in Kumar (1986a) can be summarised in the following lemmas:

Lemma 1: For the bilinear model

$$Z_t = \beta\, Z_{t-k'}\, a_{t-\ell'} + a_t$$

when $k' \neq \ell'$, $\mu_{k\ell}$ will not be zero when $(k=k',\ \ell=\ell')$ or $(k=\ell',\ \ell=k')$ and it will be zero for all other points for which $k \geq 0$, $\ell \geq 0$.

Lemma 2: For the bilinear model

$$Z_t = \beta\, Z_{t-k'}\, a_{t-\ell'} + a_t$$

when $k'=\ell'$, $\mu_{k\ell}$ will take non-zero values when

(i) $k=0,\ \ell=0$

(ii) $k=k',\ \ell=\ell'$

(iii) $k=0,\ \ell=\ell'$ or $k=k',\ \ell=0$

(iv) $k=k'+1,\ \ell=\ell'$ or $(k=k',\ \ell=\ell'+1)$

(v) $k=0,\ \ell=2,3,4,5,\ldots$ or $\ell=0,\ k=2,3,4,5,\ldots$

Other terms for which $k \geq 0$, $\ell \geq 0$ will be zero.

Kumar (1986b) has also obtained third order moments for some more general bilinear models like

(i) BL(0,1,2,1) : $Z_t = \theta_1\, a_{t-1} + \beta\, z_{t-2}\, a_{t-1} + a_t$

(ii) Ren-Zhi and Tong (1983): $Z_t = \phi_1 Z_{t-1} + \beta Z_{t-1} a_t + a_t$

(iii) BL(1,0,2,1) : $Z_t = \phi_1 Z_{t-1} + \beta Z_{t-2}\, a_{t-1} + a_t$

The procedure of specification for more general bilinear models as given in Kumar (1986b) can be listed as

(i) Obtain the ACF, PACF and construct the C-table using the method given by Kumar (1987).

(ii) Identify the best linear model using the ACF, PACF and C-table. Also identify the degree of the MA, AR or mixed ARMA model using the ACF, PACF or C-table.

(iii) to identify the bilinear part obtain third order moments

$$\hat{\mu}_{k\ell} = \frac{1}{n} \sum_{t=1}^{n} (Z_t-\bar{Z})(Z_{t-k}-\bar{Z})(Z_{t-\ell}-\bar{Z})$$

for different values of k and ℓ, such that $0 \le k \le \ell$ and arrange them in a two-way table.

(iv) look at the pattern of third order moments in the resulting third order moment table and match the pattern.

Another idea for the specification of mixed bilinear models, which looks quite plausible theoretically is to identify the best linear model as discussed in step (ii) above. In the next step fit the linear part and obtain the residuals. We can then calculate third order moments of the residuals and can identify the bilinear model by matching it with the third order moment table of super-sub-diagonal bilinear model.

Kumar (1986b) has done a lot of simulation studies to justify his results and thus concludes that third order moments looks promising as they can

(i) discriminate between bilinear and linear AR, MA, or ARMA model with normal noise,

(ii) discriminate between various subclasses of bilinear models,

(iii) discriminate between various lags within a subclass,

(iv) third order moments also play an important role in discriminating between linear models having normal noise and linear models having non-normal (skewed) noise.

Hence as far as third order moments can serve the purpose of identification in bilinear models, there is no need to look at 4th order moments. Third order moments have an advantage, since higher order moments are more unstable than lower order moments.

The study of third order moments in bilinear time series models is not new. Recently Sesay and Subba Rao (1986) obtained difference equations for the third and fourth order moments when the time series $\{Z_t\}$ satisfies a bilinear model and is stationary up to fourth order. The equations are similar to the well known Yule-Walker equations available for linear time series models. They have also given an alternative way of deriving third order moments, which leads to the Yule-Walker type of difference equation in the cumulants. They have also used these third order moments for distinguishing between a linear ARMA(1,1) model and the bilinear model BL(1,0,1,1). It should be mentioned that as far as the specification problem is concerned the results obtained by Kumar (1986b) are more general than those obtained by Sesay and Subba Rao (1986). Hinich and Patterson (1985) used third order moments for identifying the structure of the quadratic model

$$x(t) = e(t) + \sum_{\gamma=1}^{R} \sum_{s=0}^{S} a(\gamma,s)\, e(t-s)\, e(t-\gamma-s)$$

where e(t) are independently, identically distributed random variables with mean zero. Guegan (1982) and in subsequent papers has used third and fourth order moments for distinguishing between various bilinear models. Recently De Gooijer and Heuts (1987) and Nirmalan and Singh (1986) have given some general results for specification of bilinear models using third order moments.

3.3 Bivariate and Multiple Bilinear Time Series Models

Most of the work in bivariate and multiple time series modelling has been done considering the relationships between the dependent variables Y_t and the explanatory variable X_t as linear. But it is quite possible that for some Y_t and X_t the relationship may not be linear, but it may follow some other type of non-linear or say bilinear relationship. It has been found by Subba Rao and Gabr (1984) that some time series like monthly unemployment figures in West Germany are non-linear and the forecasts obtained by bilinear models are better that the forecasts obtained from linear models. However, it has been observed by them that a series like unemployment is influenced by many other variables and it would be interesting to study the multivariate extension of bilinear time series models. Granger and Andersen (1978b) considered briefly the possibility of extension of bilinear models to bivariate bilinear models and have mentioned that"more extreme" examples of the deficiencies of covariance technique occurs when bivariate bilinear models are considered. Subba Rao (1986) has defined a multivariate extension of the bilinear time series models defined earlier for the univariate case by Subba Rao (1981) and studied some of the statistical properties of these processes in a particular case.

Kumar (1988) has defined and studied bivariate bilinear models in detail. To specify these models he has extended the concept of cross correlation function to third order cross moments which can be defined between variables x and y at lag (k,ℓ) as

$$\gamma_{xy}(k,\ell) = E[(x_t - \mu_x)(y_{t+k} - \mu_y)(y_{t+\ell} - \mu_y)]$$

$$(k,\ell = 0,\pm1,\pm2,\ldots)$$

and between variable y and x at lag (k,ℓ) as

$$\gamma_{yx}(k,\ell) = E[(y_t - \mu_y)(x_{t+k} - \mu_x)(x_{t+\ell} - \mu_x)]$$

It was found that these moments are capable of distinguishing between linear bivariate models and bilinear bivariate models.

Recently Stenshot and Tjøstheim (1987) defined multiple bilinear time series models. They have also obtained the sufficient condition for the existence of a strictly stationary solution conforming to the model, along with a brief description of the first and second order structure. It may be mentioned that multiple bilinear time series models defined by Stenshot and Tjøstheim are very similar to those defined by Subba Rao (1986).

4. THRESHOLD MODELS

In section 3, we have discussed an important class of non-linear time series models namely bilinear time series models. An alternative and equally important class of non-linear time series models has been discussed by Tong and Lim (1980). These models, known as threshold AR models postulate a finite set of possible AR models that a process may obey at any point of time with threshold for the the passage from one member of the set to another. If the passage is determined by the location of the past data values relative to the thresholds these models are said to be self exciting or SETAR models. The essential idea underlying the class of threshold models is the piecewise linearization of non-linear models over the state space by the introduction of thresholds. In fact the idea of using piecewise linear model in systematic way for the modelling of discrete time series data was first introduced by Tong (1977) and reported in Tong (1978, 1980). Tong and Lim (1980) have shown that TAR models are general enough to capture the notion of limit cycle which can only exist in non-linear systems and plays the key role in the modelling of cyclical data and in physical and biological sciences. Details of threshold models can be found in the book by Tong (1983). Somewhat related ideas are employed by Lawrence and Lewis (1977, 1980, 1985) and Wecker (1981) in building "asymmetric time series model" which we have discussed in this section. The latest review on the work done on threshold models can be found in Chan and Tong (1986b) and Tong (1986).

4.1 Definition

This class of non-linear models was introduced and developed by Tong (1980) and Tong and Lim (1980). This is the original threshold model and other models mentioned in this section including threshold moving average models are special cases and ramifications of the threshold models. Following them, let $\{ \gamma_o, \gamma_1, \ldots, \gamma_\ell \}$ denote an ordered subset of real numbers s.t. $\gamma_o < \gamma_1 \ldots < \gamma_\ell$ where γ_o and γ_ℓ are taken to be $-\infty$ and $+\infty$ respectively. Let $R_i = [\gamma_{i-1}, \gamma_i]$ then R_1, $R_2, \ldots R_\ell$ define a partition for the real line R, i.e.

$$R = R_1 U R_2 U \ldots U R_\ell$$

Then the process $\{X_t\}$ is called a self exciting threshold AR model of order $(\ell; k_1, k_2, \ldots k_\ell)$ to be written in short SETAR $(\ell; k_1, k_2, \ldots, k_\ell)$ if it satisfies

$$X_t = a_o^{(j)} + \sum_{i=1}^{k_j} a_i^{(j)} X_{t-i} + e_t^{(j)} \quad \text{if } X_{t-d} \in R_j \quad j=1,2,\ldots,\ell .$$

where $\{e_t^{(j)}\}$ is white noise and it is also assumed that $\{e_t^{(j)}\}$ and $\{e_t^{(j')}\}$ are independent. The numbers $\gamma_1, \gamma_2, \ldots, \gamma_{\ell-1}$ are called "thresholds". Since for $k \geq 2$ these models can give rise to limit cycle behaviour, they may be expected to be particularly applicable to series with a strong cyclical component. A simple SETAR $(2;k_1,k_2)$ model can be written as

$$Z_t + \phi_1^{(1)} Z_{t-1} + \ldots + \phi_{k1}^{(1)} Z_{t-k_1} = \mu_1 + a_t \text{ if } Z_{t-d} \leq C$$

$$Z_t + \phi_1^{(2)} Z_{t-1} + \ldots + \phi_{k2}^{(2)} Z_{t-k_2} = \mu_2 + a_t \text{ if } Z_{t-d} > C$$

Kumar (1986b) has developed threshold moving average models and discused the specification problem of such models in detail. Moving average models which were first introduced by Yule (1926) and studied in more detail by Wold (1938), play a key role in econometric modelling and may occur in economics in several ways. The essential idea underlying the threshold moving average model is similar to TAR models; i.e. different linear moving average models are used for different parts of the data. The necessity arises as some economic time series may not be represented by a single moving average model of order q (say) but may require different moving average models (may be also of different orders) for different parts of the data. A threshold moving average model of order $(2;q_1,q_2)$ can be defined as

$$Z_t = \mu_1 + a_t + \theta_1 a_{t-1} + \ldots + \theta_{q_1} a_{t-q_1} \text{ if } Z_{t-d} \leq C$$

$$Z_t = \mu_2 + a_t + \theta'_1 a_{t-1} + \ldots + \theta'_{q_2} a_{t-q_2} \text{ if } Z_{t-d} > C$$

where $\{a_t\}$ is a sequence of i.i.d. random variables.

Kumar (1986) has obtained the distribution of Z_t in the case of TMA(2;1,1) model

$$Z_t = a_t + \theta_1 a_{t-1} \quad \text{if} \quad Z_{t-1} \leq C$$
$$= a_t + \theta_1' a_{t-1} \quad \text{if} \quad Z_{t-1} > C$$

and it was found to follow a mixture of two normal distributions. ACF of TMA (2;1,1) model is found to be same as MA(1) model. It may be mentioned that for other TMA model when q_1 is different from q_2 it may not be possible to obtain the distribution and the ACF so easily.

Tong (1983) has illustrated the procedure for the estimation of the paramaters of the SETAR $(\ell; k_1, k_2, k_\ell)$ and has used the least squares method for it. He has also studied the sampling properties of these estimates and the details are given in Lim and Tong (1981). Tong (1983) has given various areas like Radio Engineering, Marine Engineering, Servo System, Oceanography, Biology, Economics, Medical Engineering etc. where the notion of threshold dominates and threshold models can be used successfully. Using threshold models Tong (1983) has done some case studies and analysed some real data sets including some ecological data, sunspot data, riverflow data and laboratory data. Tong and Wu (1982) and Tong (1982b) have studied the multistep ahead forecasting of cyclical data by threshold. Chan and Tong (1986) have discussed the problem of estimating the theshold paramaters, i.e. the change point, of a threshold AR model. The method is demonstrated by using the artificial and real data sets. Andel and Barton (1986) have investigated the marginal distribution and other properties of a threshold AR process of the first order with Cauchy innovation.

For the SETAR (2;1,1) model

$$Z_t = \theta_1 Z_{t-1} + a_{t(1)} \text{ if } Z_{t-1} \le \gamma$$

$$Z_t = \theta_2 Z_{t-1} + a_{t(2)} \text{ if } Z_{t-1} > \gamma$$

where $\{a_{t(i)}\}$ are i.i.d. random variables with mean 0 and variance $\sigma^2(i)$, i=1,2; Petruccelli (1986) has considered estimates of θ_1, θ_2 and γ which minimizes weighted sums of the sum of squares functions for $\sigma^2(1)$ and $\sigma^2(2)$. These include as a special case the usual least squares estimators.

4.2 Other Related Models

4.2.1 *Asymmetric Time Series Models*

A very similar and important model known as "asymmetric time series" model was introduced by Wecker (1981). Asymmetric time series respond to innovation with one of two different rules according to whether the innovation is positive or negative. It has been observed that when market condition changes, quoted prices are not revised immediately. The delay operates more strongly against reduction in price quotation than against increase. Asymmetric time series models are fitted to several economic time series by Wecker and it has been observed by him that they give particularly good fits to data on strong market return. The asymmetric moving average process of order one is given by

$$Z_t = u_t + \beta^+ u^+_{t-1} + \beta^- u^-_{t-1}.$$

where u_t is the sequence of i.i.d. random shocks;

$$u^+_t = \max(u_t, 0) \text{ the positive innovation}$$

$$u^-_t = \min(u_t, 0) \text{ the negative innovation}$$

β^{+} and β^{-} are fixed paramaters of the model. If $\beta^{+} = \beta^{-}$, the asymmetric model reduces to symmetric model

$$Z_t = u_t + \beta u_{t-1} .$$

Wecker has also mentioned the conditions of invertibility and a test for asymmetry. A generalisation of the asymmetric moving average model of order q is also given in the paper.

4.2.2 *Exponential Moving Average Models*

This class of models was introduced by Lawrence and Lewis (1977) for the reasons stated below:

(i) as an alternative to the normality theory of time series;

(ii) as a model for correlated positive random variables with exponential marginal distributions; and

(iii) as a simple point process model with which to analyse non-poisson series of events and to study the power of poisson tests particularly in situations where there is no physically motivated model.

The first order exponential moving average model is formed from an independent and identically distributed exponential sequence $\{\varepsilon_i\}$ according to the linear model

$$X_i = \begin{cases} \beta\varepsilon_i \text{ with prob. } \beta \\ \beta\varepsilon_i + \varepsilon_{i+1} \text{ with prob } 1-\beta \end{cases} \qquad (0 \le \beta \le 1,\ i=0,\pm1,\pm2,\dots)$$

It can be written as

$$X_i = \beta\varepsilon_i + I_i \ \varepsilon_{i+1}$$

where I_i are i.i.d. Bernoulli random variables which are 1 with probability $1-\beta$ and 0 probability β. In this model the stationary sequence of random variables $\{X_i\}$ has exponential marginal distributions and the X_i are random linear combinations of order one of an i.i.d. exponential sequence.

Later on (1980) this model was generalised by the same authors for AR and mixed ARMA models. Tong (1983) has shown that the class of EAR models and its extensions introduced by Lawrence and Lewis in a series of papers may be regarded as a subclass of the threshold models.

Lawrence and Lewis (1985) have recently proposed a newly developed type of second order AR process with random coefficients; called the NEAR(2) model

$$X_n = \begin{cases} \beta_1 x_{n-1} & \text{w.p } \alpha_1 \\ \beta_2 x_{n-2} & \text{w.p } \alpha_2 \\ 0 & \text{w.p } 1-\alpha_1-\alpha_2 \end{cases} + \varepsilon_n, \quad n=0,\pm1,\pm2$$

One of the drawbacks of this type of model is its lack of flexibility. There are also restrictions imposed by the relationships between the α's and β's. These restrictions are all presumably a consequence of the insistence on exponential marginal distribution. The behaviour of X_n seems unlikely to occur for real data. Jolliffe and Kumar (1985) have observed that TMA models with non-normal errors can give a wide range of shapes for the marginal distribution of X_n and will be more flexible.

4.2.3 *Relative Comparison between these Models*

The EMA model can be used as a basic model for positive time series, e.g. response times at a computer terminal. However, EMA models are a bit rare in practice and can be applied only in special circumstances. It may be more of theoretical interest as no application of EMA models has been cited by Lawrence and Lewis (1978,80). EMA models can be considered as a subclass of TMA models. It can be observed that while in ATS models the threshold is based on the innovations; in TMA models, the threshold is based on Z_{t-1} (previous observation), which may be considered more realistic. TMA models seem to be more flexible than EMA and ATS model as we can use different order of MA models for different parts of the time series.

4.3 Specification of Threshold Model

Tong (1980) has used the AIC for the identification of TAR models which already suffers from the drawbacks like excessive computation and inconsistency as mentioned earlier in section 3. The whole procedure is mentioned in detail in Tong (1983, Ch.4). Terasivarta and Leukonin (1985) investigated the performance of two model selection criterion AIC and SBIC for distinguishing between the linear and threshold models. They have shown through simulation studies that SBIC is clearly the better of these two alternatives. Kumar (1986b) has developed a very simple procedure named as "SPLIT METHOD" for distinguishing between TMA and MA models. The theoretical details can be seen in Kumar (1986b) but the procedure is outlined here.

In this method the observations are split accordiingly as $Z_{t-1} \leq \alpha$ or $Z_{t-1} > \alpha$ where α may be median, lower quartile or upper quartile and then we calculate the ACF and PACF of the two parts of the data. The features of the procedure can be listed as follows:

(i) One can distinguish between TMA and ordinary MA models by looking at the ACF (of various orders) for $Z_{t-1} \le \alpha$ and $Z_{t-1} > \alpha$.

(ii) If the model is MA, the magnitude of ACF for various orders should be the same in both parts, i.e. $Z_{t-1} \le \alpha$ or $Z_{t-1} > \alpha$ but not so in the case of TMA model.

(iii) For the TMA model if the order of MA is different according as $Z_{t-1} \lneq \alpha$ or $Z_{t-1} > \alpha$, the different cutting off is revealed in the ACF for two parts while for MA the cutting off is same in both parts.

(iv) In the case of TMA; if the order of MA is same in two parts, i.e. TMA (2;1,1) the cutting off will be same in both parts but magnitude of ACF will be different in two parts.

(v) The order of MA or the order/orders of TMA can be obtained by looking at the ACF in two parts.

(vi) TAR models can also be identified by looking at the PACF in two parts.

Hence the method can distinguish between

(a) ordinary MA or AR model with threshold model;

(b) TMA and TAR model;

(c) various orders of MA or AR models used for various parts.

However, there are certain limitations of this method. It is difficult to specify the correct cutting off point α using this method. Also when more than two models are involved it will be difficult to

Also when more than two models are involved it will be difficult to specify using this method.

5. SOME OTHER NON-LINEAR TIME SERIES MODELS

In this section we have briefly described two more non-linear time series models namely Random Coefficient Autoregressive model introduced by Nicholls and Quinn (1980) and Exponential AR models introduced by Haggan and Ozaki (1981).

5.1 Random Coefficient AR Model

Random Coefficient AR models as developed by Nicholls and Quinn (1980, 1981, 1982) are concerned with AR models in which the coefficients are assumed to be not constant but subject to random perturbations. The recent monograph by Nicholls and Quinn (1982) summarises the current state of knowledge concerning these models. The random coefficient AR model of order n generating a time series $\{x_t; t=0,\pm1,\pm2,...\}$ may be represented by

$$x_t = \sum_{i=1}^{n} (\beta_i + b_i(t))\, x_{t-i} + \varepsilon_t \qquad (5.1.1)$$

where the following assumptions are made

(i) $\{\varepsilon_t;\ t=0,\ \pm,1,\pm2...\}$ is a sequence of identically and independently distributed random variables with zero mean and variance σ^2.

(ii) β_i are constant

(iii) Let $b_t = (b_n(t),...,b_1(t))'$, $\{$then $b_t\ ;\ t=0,\pm1,\pm2,...\}$ is a sequence of i.i.d. random variables with zero means and $E(b_t b_t') = \Sigma$, furthermore $\{b_t\}$ is independent of $\{\varepsilon_t\}$.

(iv) The variance of σ^2 of ε_t is bounded below by $\bar{\delta}_1$, while the smallest eigenvalue of Σ is bounded below by $\bar{\delta}_2$, where $\bar{\delta}_1 > 0$ and $\bar{\delta}_2 > 0$ are both arbitrarily small.

(v) The parameters of β_i, i=1,2,...,n and Σ are such that there is a unique strict stationary solution to (5.1.1) which has finite second moment.

Nicholls and Quinn (1981) have observed the similarity between some bilinear models and random coefficient AR model. The important difference, however, is that the randomness in the coefficient in a bilinear model is produced by lags of the process $\{\varepsilon_t\}$ whereas in the RCA model the randomness occurs by means of a process $\{b_t\}$ which is independent of (ε_t). Nicholls and Quinn (1980) developed a two stage regression procedure for the estimate of the parameters of the unknown model. The estimates were found to be consistent. Nicholls and Quinn (1981) used these estimates as starting value in a Newton-Raphson algorithm which is employed to obtain the maximum likelihood estimate of a class of random coefficient autoregression. The problem of testing for randomness of the coefficient is also briefly discussed.

5.2 Exponential AR Model

Many observed stochastic processes display random vibration which is essentially non-Guassian in character. In most cases the analysis of this kind of data has been made using linear time series models, which can only provide an approximation to the true situation. In practice, it is found that many random vibrations display essentially non-linear behaviour. Haggan and Ozaki (1981) have introduced a discrete time series model namely "amplitude-dependent AR time series model" which has the properties similar to those of non-linear random vibrations. These models are of AR form with amplitude dependent coefficient. The exponential AR model of order p is defined as

$$Z_t = (\theta_1 + \pi_1 e^{-r} Z^2_{t-1})Z_{t-1} + (\theta_2 + \pi_2 e^{-r} Z^2_{t-1})Z_{t-2}$$

$$+\ldots+(\theta_p + \pi_p e^{-r} Z^2_{t-1})Z_{t-p} + e_t$$

where θ_1, θ_2, π_1, π_2 and r are constant.

Haggan and Ozaki (1980) have given the necessary condition for the existence of the above model. The order p of the fitted model is selected by use of the AIC criterion for non-linear time series models, (see Ozaki and Oda (1978)) and is given by

$$AIC(p) = (n-m) \log \hat{\sigma}^2_p + 2(2p+1)$$

where m is the maximum order of the model to be considered, n is the total number of observation and $\hat{\sigma}^2_p$ is the least square estimate of the residual variance of the model. The estimation procedure for the model and its application to Canadian Lynx data is given in the paper by Haggan and Ozaki (1981).

6.. STATE DEPENDENT MODELS

The scope of non-linearity in time series analysis is wide but Priestly (1980) developed a general class of non-linear time series models known as "state dependent models" (SDM) which includes as special cases bilinear models, threshold AR models, exponential AR models as well as standard linear time series models. These models are essentially autoregressive-moving average models in which the parameters are the functions of the past values of the time series. The model can be stated in a state-space form and estimated through an algorithm similar to the Kalman filter. Haggan, Haravi and Priestley (1984) extensively studied the application of state dependent models to a wide variety of non-linear time series data and have emphasised the ability of graphs of smoothed SDM parameter estimates to aid in identifying a variety of subclass of non-linear models (bilinear, threshold, exp AR etc.) which best fits the data.

If one considers the linear ARMA (k, ℓ) model

$$Z_t + \theta_1 Z_{t-1} + \ldots + \theta_k Z_{t-k} = \mu + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \ldots + \psi_\ell \varepsilon_{t-\ell}$$

at time $(t-1)$ the future development of the process $\{Z_t\}$ is determined by the values $\{\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-\ell}, Z_{t-1}, \ldots, \ldots, Z_{t-k}\}$ together with future values of ε_t. Hence the vector $Z_{t-1} = \{\varepsilon_{t-1}, \ldots, \varepsilon_{t-\ell}, Z_{t-1}, \ldots, Z_{t-k}\}$ may be regarded as the state vector of the process $\{Z_t\}$. The general non-linear SDM model may be expressed by allowing the coefficients of the above model to become a function of the state vector Z_{t-1}.

$$Z_t + \theta_1(Z_{t-1})Z_{t-1} + \ldots + \theta_k(Z_{t-1})Z_{t-k} = \mu(Z_{t-1}) + \varepsilon_t$$
$$+ \psi_1(Z_{t-1})\varepsilon_{t-1} + \ldots + \psi_\ell(Z_{t-1})\varepsilon_{t-\ell}$$

This is called state dependent model of order (k, ℓ). By choosing particular forms for the $\{\theta_u\}$ and $\{\psi_u\}$, it is easily seen that the SDM contained the linear ARMA models, the bilinear models, the threshold AR models and the exponential AR models. The details can be seen in Priestley (1980). Priestley has also discussed the problem of identification, estimation and forecasting using state dependent models.

7. CONCLUSIONS

In this paper we have reviewed almost all important non-linear time series models that have appeared in the literature. The literature is very vast and some sort of unification, similar to SDM of Priestley (1980) or more development in this area (SDM) is needed. It may be mentioned that there is no sure fire test of non-linearity and also there is no test or unified criterion to distinguish between various non-linear time series models.

REFERENCES

Akaike, H. (1977) On Entropy Maximisation Principle, Proc. Symp. on Application of Statistics, Dayton, Ohio, ed. P. R. Krishniah.

Akamanam, S.I., Rao, M.B. & Subramanyam, K. (1986) On the Ergodicity of Bilinear Time Series Models, Jour. Time Ser. Anal., 7, 157-163.

Anderson, O.D. (1976) Time Series Analysis and Forecasting, Butterworth.

Andel, J. & Berton, T. (1986) A Note on the Threshold AR(1) Model with Cauchy Innovation, Jour. Time Ser. Anal., 7, 1-5.

Box, G.E.P. and Jenkins, G.M. (1970,1976) Time Series Analysis: Forecasting and Control, San Fransisco, Holden Day.

Chan, W.S. & Tong, H. (1986a) On Tests for Non-linearity in Time Series Analysis, Technical Report, University of Kent.

Chan, W.S. & Tong, H. (1986b) A Survey of Statistical Analysis of Univariate Threshold Autoregressive Model. To appear in Advances in Statistical Analysis and Statistical Computing, Theory and Application, JAI Press Inc., U.S.A.

Chatfield, C. (1975, 1980, 1984) The Analysis of Time Series: An Introduction, Chapman and Hall, London.

Cryer, J.D. (1986) Time Series Analysis, Duxbury Press, Boston.

Davies, N. & Petruccelli, J.D. (1985) Experience with Detecting Non-linearity in Time Series: Identification and Diagnostic Checking, Research Report MSUR/8/85, Dept. of Maths, Trent Polytechnic, U.K..

De-Gooijer, J.G. & Heuts, R. (1987) Higher Order Moments of Bilinear Time Series Processes with Symmetrically Distributed Errors, Proc. of 2nd International Tampere Conference, ed. T. Pukkila and Puntanen, S.

Gabr, M.M. & Subba Rao, T. (1981) The Estimation and Prediction of Subset Bilinear time Series Models with Applications, Jour. Time Ser. Anal., 2, 153-171.

Granger, C.W.J. and Andersen, A.P. (1978a) On the Invertibility of Time Series Models, Stoch. Process Application, 8, 83-92.

Granger, C.W.J. and Andersen, A.P. (1978b) An Introduction to Bilinear Time Series Models, Vandenhoek and Ruprecht, Gottingen.

Guegan, D. (1982) Tests de Modèles Non Lineares, Alternative Approaches to Time Series Analysis, Proceedings of the 3rd France-Belgium Meeting of Statisticians, 45-65.

Haggan, V. & Ozaki, T. (1980) Amplitude Dependent Exponential AR Model Fitting for Non-linear Random Vibrations. Time Series, ed. O.D. Anderson, pp.57-71, Amsterdam, North Holland.

Haggan, V. & Ozaki, T. (1981) Modelling Non Linear Random Vibrations Using an Amplitude Dependent Autoregressive Time Series Model, Biometrika, 68, 189-96.

Haggan, V., Heravi, S.M. & Priestley, M.B. (1984) A Study of Applications of State-Dependent Models in Non-linear Time Series Analysis, Jour. Time Ser. Anal., 5, 69-102.

Hannan, E.J. (1962) Time Series Analysis, London, Methuen.

Hannan, E.J. (1970) Multiple Time Series, New York, Wiley.

Hinich, M.J. (1982) Testing for Gaussinity and Linearity of a Stationary Time Series, Jour. Time Ser. Anal., 3, 169-176.

Hinich, M.J. and Patterson, D.M. (1985) Identification of the Coefficients in a Non-linear Time Series of the Quadratic Type, Annals of Statistics, 30, 269-288.

Jolliffe, I.T. & Kumar, K. (1985) Discussion of the paper by Lawrence & Lewis, Jour. Roy. Stat. Soc., B, 47, 190-191.

Kashyap, R.L. (1980) Inconsistency of the AIC Rule for Estimating the Order of Autoregressive Models, IEEE Trans. Autom. Control, 25, 996-98.

Keenan, D.M. (1985) A Tukey Nonadditivity Type Test for Time Series Nonlinearity, Biometrika, 72, 39-44.

Kumar, K. (1986a) On the Identification of Some Bilinear Time Series Models, Jour. Time Ser. Anal., 117-122.

Kumar, K. (1986b) Some Topics in the Identification of Time Series Models, unpublished Ph.D. thesis, University of Kent at Canterbury, U.K..

Kumar, K. (1987) Identification of Autoregressive-Moving Average (ARMA) Models Using Padé Approximation, International Statistical Institute Bulletin, presented as invited paper 7.1 in 46th session of ISI, Tokyo, September, 1987.

Kumar, K. (1988) Bivariate Bilinear Models and their Specification, Non-linear Time Series and Signal Processing, ed. by R. Mohler, Springer-Verlag Berlin, Heidelberg.

Lawrence, A.J. and Lewis, P.A.W. (1977) An Exponential Moving Average Sequence and Point Processes, Jour. Appl. Prob., 14, 98-113.

Lawrence, A.J. and Lewis, P.A.W. (1980) The Exponential Autoregressive Moving Average EARMA(p,q) Process, Jour. Roy. Stat. Soc., B, 42, 150-161.

Lawrence, A.J. and Lewis, P.A.W. (1985) Modelling and Residual Analysis of Non-linear Autoregressive Time Series in Exponential Variables, Jour. Roy. Stat. Soc., B, 47, 165-202.

Lim, K.S. & Tong, H. (1981) On the Sampling Properties of Some Parameter Estimates of SETAR Models, Technical Report No. 148, UMIST.

Li, W.K. (1984) On the autocorrelation structure and identification of some bilinear time series, Jour. Time Series Anal., 5, 172-181.

Liu, S.I. (1985) Theory of Bilinear time Series Models, Comm. Statist. Theor. & Meth., 14 (10), 2549-2561.

Maravell, A. (1983) An Application of Non-linear Time Series Forecasting. Jour. Bus. Econ. Statist. 1, 66-74.

McKenzie, E. (1980) Non-linear and Non-Gaussian Time Series Models, personal communication, University of Strathclyde, U.K..

McLeod, A.I. and Li, W.K. (1983) Diagnostic Checking ARMA Time Series Models Using Squared Residual Autocorrelations, Jour. of Time Ser. Anal., 4, 269-273.

Mohler, R.R. (1973) Bilinear Control Processes, Addison Wesley.

Nelson, J.Z. and Van Ness, J.W. (1973) Choosing a Nonlinear Predictor, Technometrics, 15, 219-231.

Nicholls, D.F. and Quinn, B.G. (1980) The Estimation of Autoregressive Model with Random Coefficients I, Jour. Time Ser. Anal., 1, 37-46.

Nicholls, D.F. and Quinn, B.G. (1981) The Estimation of Random Coefficient Autoregressive Model II, Jour. Time Ser. Anal., 2, 185-203.

Nicholls, D.F. and Quinn, B.G. (1982) Random Coefficient Autoregressive Models: An Introduction, Lecture Notes No.11, Springer-Verlag, Inc.

Nirmalan, T. and Singh, N. (1986) On the Problem of Model Selection in Bilinear Time Series Analysis, Comm. Stat. Theor. Meth., 15 (11), 3445-3457.

Ozaki, T. and Oda, H. (1978) Non Linear Time Series Model Odentification by Akaiki's Information Criterion. In Information and System, pp.83-91, Oxford.

Petruccelli, J.D. (1986) On the Consistency of Least Squares Estimate for a Threshold AR(1) Model, Jour. Time. Ser. Anal., 7, 269-278.

Petruccelli, J.D. & Davies, N. (1986) A Portmanteau Test for Self Exciting Threshold Autoregressive Type Nonlinearity in Time Series, Biometrika (to appear).

Priestly, M.B. (1980) State Dependent Models: A General Approach to Non-Linear Time Series Analysis, Jour. Time Ser. Anal., 1, 47-71.

Priestly, M.B. (1981) Spectral Analysis and Time Series, Vol.I, London, Academic Press.

Quinn, B.G. (1982) Stationarity and Invertibility of Bilinear Models, Stoch. Processes & Their Appl., 12, 225-30.

Raflėry, A.E. (1982) Generalised Non-Normal Time Series Models. Time Ser. Analysis, Theor. & Prac., I, O.D. Anderson, ed. North Holland.

Ren, W.S., Zhi, A.H. and Tong, H. (1983) On the Distribution of a Simple Stationary Bilinear Process, Jour. Time Ser. Anal., 4, 209-216.

Sesay, S.A.O. & Subba Rao, T. (1986) Yule Walker Type Difference Equations for Higher Order Moments of Bilinear Time Series Models, Technical Report, UMIST.

Shibata, R. (1976) Selection of the Orders of An AR Model by Akaike's Information Criterion, Biometrika, 63, 117-126.

Stenshot, B.K. and Tjostheim, D. (1987) Multiple Bilinear Time Series Models, Jour. Time Ser. Anal., 8, 2, 119-126.

Subba Rao, T. (1981) On the Theory of Bilinear Time Series Models, Jour. Roy. Stat. Soc., B, 43, 244-255.

Subba Rao, T. (1986) Statistical Analysis of Bivariate Bilinear Time Series Models, Technical Report No. 177, UMIST.

Subba Rao, T. & Gabr, M.M. (1980) A Test for Linearity of Stationary Time Series, Jour. Time Ser. Anal., 1, 145-158.

Subba Rao, T. & Gabr, M.M. (1984) An Introduction to Bispectral Analysis and Bilinear Time Series Models, Lecture Notes in Statistics, Springer-Verlag.

Terasvirta, T. & Leukkonen, R. (1985) Choosing Between Linear and Threshold AR Models. Time Ser. Analysis, Theor. & Prac., 7, O.D. Anderson, ed. North Holland.

Tong, H. (1977) Discussion of the paper by A.J. Lawrence & N.T. Kothegoda, Jour. Roy. Stat. Soc., A, 140, 34-35.

Tong, H. (1980) A View on Non-linear Time Series Building. Time Series ed. O.D. Anderson, Amsterdam, North Holland.

Tong, H. (1981) A Note on a Markov Bilinear Stochastic Process in Discrete Time, Jour. Time Series Anal., 2, 279-284.

Tong, H. (1982) A Note on Using Threshold Autoregressive Model for Multi Step Ahead Prediction of Cyclical Data, Jour. Time Series. Anal., 3, 137-140.

Tong, H. (1983) Threshold Models in Non-Linear Time Series Analysis, Lecture Notes in Statistics, Springer-Verlag.

Tong, H. (1986) Nonlinear Time Series Models of Regularly Sample Data: A review presented at the First World Congress of Bernoulli Society for Math. Stat. & Probability, Tashkent, USSR, Sept. 1986.

Tong, H. & Lim, K.S. (1980) Threshold Autoregression, Limit Cycles and Cyclical Data, Jour. Roy. Stat. Soc., B, 42, 245-292.

Tong, H. & Wu, Z.M. (1982) Multi-step-ahead Forecasting of Cyclical Data by Threshold Autoregression. Time Series Anal.: Theory & Practice I, ed. O.D. Anderson, 733-753. North Holland.

Tuan, P.D. & Tran, L.T. (1981) On the First Order Bilinear Time Series Model, J. Appl. Prob., 18, 617-627.

Vandaele, W. (1983) Applied Time Series and Box Jenkins Models, Academic Press, Inc.

Wecker, W.E. (1981) Asymmetric Time Series, Jour. Amer. Stat. Assoc., 76, 16-21.

Weiner, N. (1956) Nonlinear Problems in Random Theory, John Wiley & Sons.

Wold, H.O. (1938) A Study in the Anlysis of Stationary Time Series, Uppsala, Sweden.

Yule, G.U. (1926) Why do we sometimes get nonsense-correlation between time series? A study in sampling and the nature of time series, Jour. Roy. Stat. Soc., 89, 1, 1-64.

Recent Working Papers
(Available on Request)

1987

1/87 Maxwell L. King and Merran A. Evans, "Locally Optimal Properties of the Durbin-Watson Test".

2/87 R.D. Snyder, "Computational Aspects of Kalman Filtering with a Diffuse Prior Distribution".

3/87 Merran A. Evans, "The Twelfth Order Analogue of the Durbin-Watson Test".

4/87 Radhey S. Singh, "A Family of Improved Estimators in Linear Regression Models with Errors Having Multi-variate Student-t Distribution".

5/87 Radhey S. Singh, "Estimation of Error Variance in Linear Regression Models with Errors Having Multi-variate Student-t Distribution with Unknown Degrees of Freedom."

6/87 Radhey S. Singh, "Melo and Empirical Bayes Estimators for Reciprocals of Population Means and Regression Coefficients."

7/87 A.R. Tremayne, "A Synthesis of Econometrics and Time Series Methods to Model and Forecast UK Demands for Imports".

8/87 Merran A. Evans and Maxwell L. King, "A Further Class of Tests for Heteroscedasticity".

9/87 Asraul Hoque, "Size of Holding and efficiency in Bangladesh: An Application of a Random Coefficient Model".

10/87 Brett A. Inder, "Bias in the Ordinary Least Squares Estimator in the Dynamic Linear Regression Model with Autocorrelated Disturbances".

11/87 Russel J. Cooper and Keith R. McLaren, "Approaches to the Solution of Intertemporal Consumer demand Models".

12/87 Esme Preston, "On Choosing a Technical Wordprocessor".

1988

1/88 John Preston and Esme Preston, "Two Papers on Simple Word Processors for the PC".

2/88 Esme Preston, "Multilingual and Mathematical Text Processing".

3/88 Esme Preston and John Preston, "The Accountant's PC: Getting Started".

4/88 Maxwell L. King, "Towards a Theory of Point Optimal Testing".

5/88 Ralph D. Snyder, "Statistical Foundations of Exponential Smoothing".

6/88 Grant H. Hillier, "On the Interpretation of Exact Results for Structural Equation Estimators".