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Dynamic Analysis of a 'Solow-Romer' Model of Endogenous Economic Growth

by

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Abstract

The model of endogenous economic growth developed by Paul Romer (1990a) is briefly reviewed and modified by substituting a Solow type consumption function in place of the utility maximising behaviour of consumers. The dynamic system and steady-state growth path of this *Solow-Romer* model are then derived. Such modification allows the dynamics of the model, in response to certain economic shocks, to be examined in terms of phase diagrams; and illustrates the instructional power of this approach. The impacts of the same economic shocks are also analysed more directly by numerical integration of the differential equations and boundary conditions describing the dynamic system of the model.

Adjustment processes are found to be relatively lengthy; and to be characterised by significant initial jumps or discontinuities in certain variables. Furthermore, in some cases these initial jumps can be in the opposite direction to that of the subsequent adjustment. Such results emphasise the importance of explicit analysis of the dynamics of the adjustment paths of growth models and their relevance for economic policy.

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Dynamic Analysis of a 'Solow-Romer' Model of Endogenous Economic Growth

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1. Introduction

Romer (1987 and 1990a) develops a model in which economic growth is endogenous. In the long-run the economy expands along a balanced growth-path where all the main variables exhibit constant and identical asymptotic rates of growth. The impact of various possible shocks to the economy (reflected through changes to the model parameters) are examined in terms of their effects on this long-run balanced growth rate. Such an approach is one of *comparative statics*, and as such it ignores the dynamics of the economy's adjustment from its pre-shock steady-state growth path to its ultimate post-shock one.

However, these dynamics are important for a number of reasons. The most obvious questions begged by the comparative statics approach concern timing:How long will it be until the new steady-state is (all but) reached? Will the paths of adjustment be gradual or precipitate? How will the costs and benefits of change vary over the adjustment path and how will they be shared between different economic agents? Clearly, a comparative statics analysis of the steady-state equilibria can shed no light on such matters. More specific issues of the dynamics arise from the fact that the adjustments of certain variables (usually prices and flow variables such as consumption) are characterised by initial discontinuous jumps. Furthermore, such jumps can be of the opposite sign to that of the subsequent (smooth) adjustment path, with obvious welfare implications in terms of timing and the gains and losses of different economic agents. Finally, as illustrated later in this article (also see Barro & Sala-i-Martin, 1995), adjustment periods tend to be lengthy. Thus,given the persistent and innumerable variety of shocks to which real economies are subject, it seems likely that they may more often be in a process of adjustment than on a steady-state growth path!

The purpose of this paper is to illustrate the dynamic behaviour of a'Romer-like' model. In particular, the technique of representing a dynamic system by its *phase-space* will be employed to demonstrate diagrammatically the adjustment responses to a variety of economic shocks. An analytical difficulty with the full Romer (1990a) model is its dimension. In its fundamental, balanced growth, form it comprises four coupled differential equations in four variables. Clearly this could not be visualised diagrammatically at all. The system can be transformed to a stationary one of three differential equations in three variables, but it remains a difficult *boundary value* problem. An initial value for one of the transformed variables is known, but only the asymptotic steady-state values are known for the other two. Also, while the system can then be represented pictorially with a three dimensional phase-space diagram, its analysis is messy and difficult to visualise.

Instead, by taking the apparently retrograde step of specifying a simple Solow type consumption function in place of the consumer optimising behaviour specified in the Romer system, a modified model is developed with a stationary dynamic system of only two variables, thereby allowing the phase-space to be constructed and examined in only two dimensions. This is termed the *Solow-Romer model*, and it differs from its progenitor in only a single aspect.[1] Its supply side is **exactly** the same as that of the Romer model, while on the demand side its consumption propensity is exogenous.

The structure of the paper is as follows: First, some background and a description of Romer's model are presented in Section 2. Next, in Section 3 the dynamic system and its steady-state are derived; and the phase-space of the system is constructed. Section 4 then uses this phase-space to examine the adjustment response of the system to a variety of economic shocks. In Section 5 the dynamics of the model are obtained by numerical analysis and the results compared with those obtained via the examination of the phase-space. Some concluding remarks are offered in the final section.

2. Background and description

2.1 Background

The fundamentally different nature of technological innovations compared to that of most economic goods plays a central role in Romer's (1990a) model. Technology, or knowledge, has long been taken to exhibit public good characteristics. As an input to production it is largely non-rival. A new design, a set of instructions, a computer program can all be used without diminution by an indefinitely large number of agents as often as desired, and at little additional cost once the (probably high) initial costs of development have been met.

[1] In fact, the S-R model is a particular parameterisation of the Romer model (See Kurz, 1968).

Non-rivalry also carries a strong implication of non-excludability through externalities. However, while external benefits and spillovers of knowledge are undoubtedly important outcomes from innovative activity, the fact remains that much, perhaps most, innovative activity is undertaken by private agents with the expectation of economic gain. The total benefits from technological improvements and the generation of knowledge must therefore be at least partially excludable. Consistent with this, growth in Romer's model is driven by the intentional and endogenous accumulation of knowledge, which is **non-rival and partially excludable**.

Non-rivalry in the technological input necessarily introduces a so called non-convexity into the production function, which must show **increasing** returns in respect of all inputs together. This can be readily demonstrated by a simple replication argument, noting that it is not necessary to replicate non-rival inputs, (Romer, 1990b). This means that a price taking equilibrium cannot hold unless technology is regarded as a public good, both non-rival and **completely non-excludable.** Market power is necessary to achieve an equilibrium in which technology is (at least) partially excludable (Schumpeter, 1942). Following this, in Romer's model equilibrium is supported by monopolistic competition among the producers of capital.

2.2. Description

2.2.1 General

Romer's (1990a) model comprises four factors: capital, labour, human capital and technology. Capital (K) is represented by a large variety of durable inputs available for final goods production, with the extent of the variety (specifically, the number of different types of capital) depending on the level of technology. The usual representation of ordinary (unskilled) labour (L) is adopted. Knowledge is separated into a rival component embodied in people – human capital (H) – and a non-rival technological component (A), which is independent of individuals and can be accumulated without bound on a per capita basis. Technology is represented as a stock of non-rival designs for the producer durables, which grows over time with research effort. The designs are excludable in terms of their **direct use** in the production of durables: For example they are patentable so that a durable can only be produced by a firm which owns the design for it. However, in adding to the general stock of design knowledge and contributing to subsequent designs, each design makes an **indirect** contribution to production that is not excludable.[2] Overall, technology is only partially excludable.

Formally the model comprises three sectors.[3] A research sector employs human capital and the existing stock of knowledge to producenew knowledge in the form of designs for new producer durables. A durable goods producingsector purchases the designs and uses them with foregone consumption to produce a wide variety of capital goods. These are then rented (or purchased) by a final goods sector, which uses them in conjunction with labour and human capital, to produce final output which can either be consumed or saved[4] (Figure 2.1).

2.2.2 Research sector

Aggregate production of designs is taken to be a deterministic function of the research inputs of human capital and the existing total stock of design knowledge. Specifically, the rate of increase of designs is:

$$\dot{A}(t) = \delta \, H_A(t)A(t) \qquad (2.1)$$

where δ is a productivity parameter, and $H_A(t)$ is total human capital employed in research. Note that the productivity of human capital in research is an increasing function of accumulated knowledge.

2.2.3 Manufacture of capital goods

Given the designs, capital (in the form of durable producer goods) is produced directly out of foregone consumption:

$$\dot{K}(t) = Y(t) - C(t) \qquad (2.2)$$

where $Y(t)$ and $C(t)$ are aggregate output and consumption respectively.

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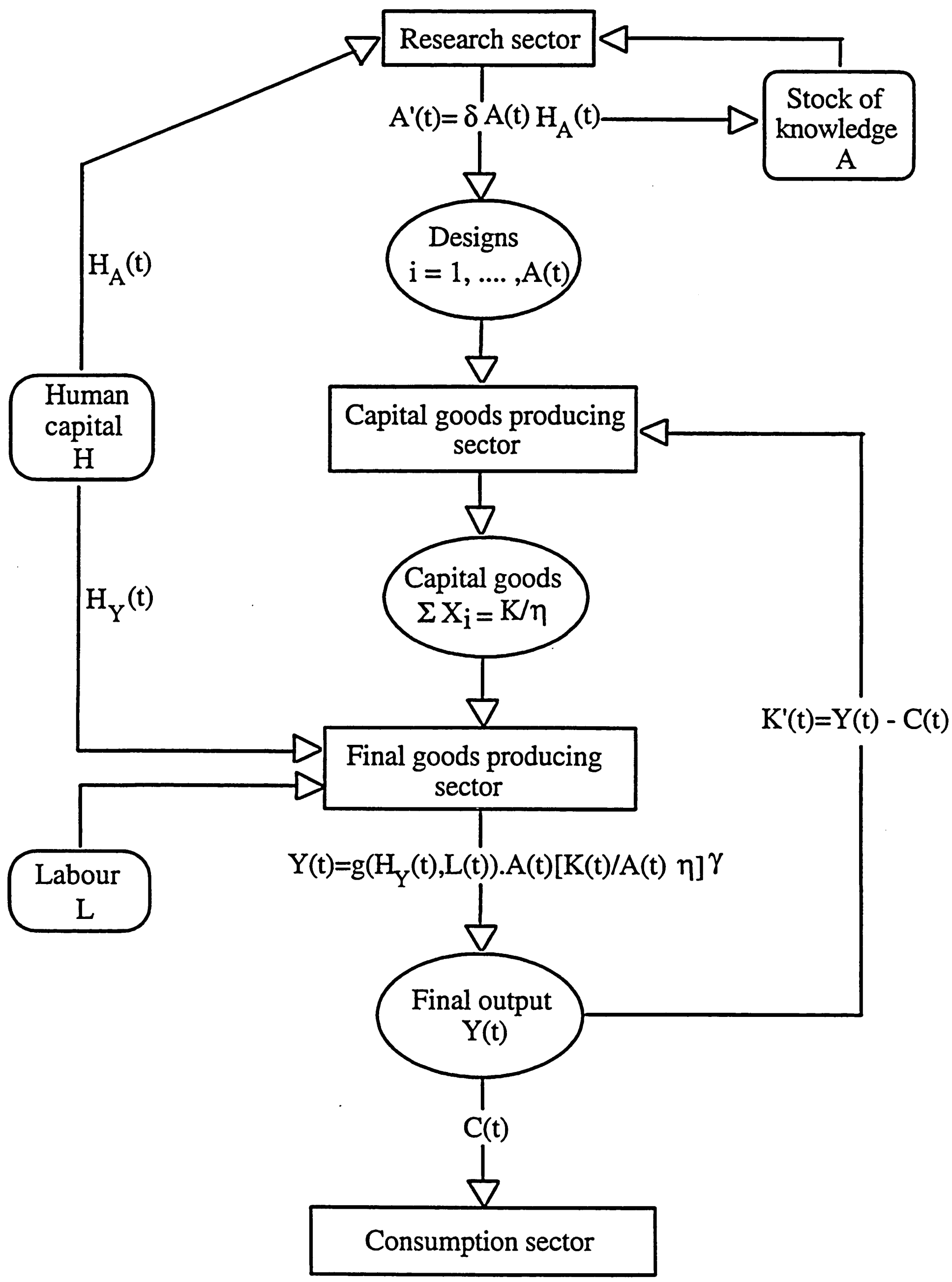
[2]While these externalities are undoubtedly highly relevant to the real world, they are **not necessary** to generate growth endogenously (Romer, 1987).

[3]Actually, this is merely a convenience to facilitate understanding the flows and transfer prices involved. A variety of institutional arrangements could apply.

[4]'Foregone consumption' is said to be used to produce capital. But such output is not actually produced; rather, the resources which would have been necessary to manufacture it are devoted instead to the production of capital. Given the existence of designs, the production technology for capital goods is identical with that of final output.

Figure 2.1 Diagrammatic representation of the Romer model



The number of different types of durables increases asnew designs are developed. At any time t there are $i = 1,2,.......,A(t)$ different types in existence, and there are $X_i(t)$ units of type i. Since the capital goods sector employs the same technology as that of final output, it is possible to 'exchange' consumption goods for capital goods. If it requires η units of output to produce one unit of any of the different durable goods types, the aggregate capital stock at time t is given by:

$$K(t)=\eta\sum_{i=0}^{A(t)}X_i(\mathrm{t}) \quad \text{or, ignoring indivisibilities} \quad K(t)=\eta\int_0^{A(t)}X(i,t)di \qquad (2.3)$$

The knowledge represented by new designs is excludable in terms of its direct use in the production of new durable goods (through the granting of infinitely lived patents for example). Each design is the property of only a single firm which produces the corresponding durable.

2.2.4 Production of final output

Labour (*L*), human capital (*H*), and physical capital (*K*) are the inputs to the production of final output (*Y*). Overall, the production technology is assumed to be linearly homogeneous. The unusual feature of the technology is that capital is disaggregated into the list of all the different types of producer durables available ($X_i(t)$, for i = 1,.......,$A(t)$).[5]

Aggregate (economy-wide) output is given by:

$$Y(t)=g(H_Y(t),L)\sum_{i}^{A(t)}X_i(t)^{\gamma} \quad \text{or} \quad Y(t)=g(H_Y(t),L)\int_0^{A(t)}X(i,t)^{\gamma}\,di \qquad (2.4)$$

where $H_Y(t)$ is the amount of human capital devoted to final goods production; and the function $g(.)$ is assumed to be homogeneous of degree $(1\text{-}\gamma)$. Specifying $g(.)$ as a 'Cobb-Douglas' function:

$$g(H_Y(t),L)=H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)} \qquad (2.5)$$

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[5]In contrast to the usual practice of employing a single aggregate for capital, where different types are implicitly assumed to be perfect substitutes, the approach adopted here means that different types of capital have additively separable effects on output. Thus, there are two distinct ways in which capital can grow: Extra units of already existing durable types can be added, and new types can be developed and brought into use. Diminishing returns apply to the former type of capital accumulation but not to the second. Even with aggregate capital (*K*) fixed, output can be increased by introducing new types; that is, by raising $A(t)$. However this is not costless: The range of capital types is constrained at any time by the fixed costs of their production. It is optimal for all the different types of capital goods to be used at the same level, so the productivity of every different type (including new ones as they are designed) is constant. Although each type of capital good is different, they all produce identical effects on output, and all exhibit the same diminishing returns.

2.2.5 Market structures, prices and wages

The final output sector is characterised by competitive price taking and constant returns to scale. Producers take rental prices *p(i,t)* for each durable *i* as given, and choose the quantities X(i,t) to maximise profits at all times t. In aggregate the problem is:

$$Max. \int_0^{A(t)} [g(H_Y(t),L)X(i,t)^\gamma - p(i,t)X(i,t)]di$$

yielding the demand function:

$$p(i,t) = \gamma\, g(H_Y(t),L)X(i,t)^{\gamma-1}$$

Firms in the capital goods sector bid for the **sole** rights to manufacture durables according to each new design. Thus, any particular type of durable is made by only a single firm, which can charge a price greater than the (constant) marginal cost of producing the durables. But monopoly power is restricted by the free entry of the bidding process, and the rental market for durables is one of *monopolistic competition.* Having incurred the fixed cost of purchasing a design, these firms take the demand functions for their durables arising from the final goods sector as given, and set prices to maximise the excess of their rental income over variable cost. The monopolist sector problem is:

$$Max. \pi(i,t) = [p(i,t)X(i,t) - r(t)\eta X(i,t)]$$

where *r(t)* is the interest rate denominated in final goods[6], and so variable costs are $r(t)\eta X(i,t)$ – the interest cost on the $\eta X(i,t)$ units of output needed to produce *X(i,t)* durables. First order conditions yield the optimal monopoly levels of prices, quantities and profits as:

$$X(i,t) = X(t) = [\gamma^2 g(H_Y(t),L)/r(t)\eta]^{1/(1-\gamma)} \qquad (2.6)$$

$$p(i,t) = p(t) = r(t)\eta/\gamma \qquad (2.7)$$

$$\pi(i,t) = \pi(t) = (1-\gamma)p(t)X(t) \qquad (2.8)$$

[6]Again since the production technology in the two sectors is exactly the same and goods can be converted into capital one-for-one, r is also the rate of return on capital.

Thus, all of the different types of durable goods available at any time will be supplied at the same level (i.e. $X(i,t)=X(t)$). This is also apparent from the symmetry of the model specification.[7] It follows from equation (2.3) that this level is:

$$X(i,t)=X(t)=K(t)/\eta A(t) \qquad (2.9)$$

Competition among capital goods producing firms to obtain the rights to any new design means that all the monopoly rents will be bid away. Thus, the **research sector** will be able to extract prices for its designs equal to the present value of the monopoly rents associated with each corresponding new capital good.[8] At any time t the price of designs is given by:

$$p_A(t)=\int_t^{\infty} e^{-\int_t^{\tau} r(s)ds}\,\pi(\tau)d\tau$$

which can be solved by differentiating with respect to time to yield

$$\dot{p}_A(t)=r(t)p_A(t)-\pi(t) \qquad (2.10)$$

As for ordinary labour (*L*), total human capital (*H*) is taken to be constant in the model. It is also considered to be homogeneous. In particular, there is no distinction between that employed in research and that employed in the production of final output. Thus total human capital in the model's economy is simply:

$$H=H_A(t)+H_Y(t) \qquad (2.11)$$

Moreover, the returns to human capital will be the same in each sector. In research human capital earns all the income, and to make one extra design ($\dot{A}(t)=1$) requires an amount $H_A(t)=1/\delta A(t)$. Thus, its wage is:

$$w_H(t)=p_A(t)\delta A(t)$$

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[7]Furthermore, because of the usual diminishing returns in the production technology of the durables sector, it would otherwise be possible to increase profits by diverting resources from high to low output goods.

[8]The decision of whether to incur the costs of new research and development will therefore be based upon (the expectation of) future monopoly rents exceeding such costs.

Although some of the knowledge of designs is non-excludable, since every researcher is free to exploit **all** the knowledge *A(t)*, any benefits external to one researcher are captured by others. In aggregate researchers collect all the benefits of the sale of designs.

Since the final output sector is competitive, human capital employed there receives the value of its marginal product. Wages are thus:

$$w_H(t) = \frac{\partial Y(t)}{\partial H_Y(t)} = \frac{\partial g(H_Y(t),L)}{\partial H_Y(t)} \int_0^{A(t)} X(i,t)^\gamma \, di$$

Equating the two expressions for $w_H(t)$ gives:

$$p_A(t) = \delta^{-1} \frac{\partial g(H_Y(t),L)}{\partial H_Y(t)} A(t)^{-1} \int_0^{A(t)} X(i,t)^\gamma \, di \qquad (2.12)$$

2.2.6 Consumption

In the pure Romer model the pattern of consumption is derived from the maximisation of the discounted sum of all future aggregate utility, subject to an aggregate budget constraint.[9] Here, a simple *Solow* type consumption function is specified:

$$C(t) = (1-s)Y(t) \qquad (2.13)$$

where s is the (exogenously constant) propensity to save.

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[9]Aggregate utility, a function of the consumption stream alone, is taken to display constant elasticity of substitution time-preferences, and the solution to the problem yields a first order differential equation in consumption. Since the optimisation takes place over an infinite horizon, all future consumers are assumed to "always be governed by the same motives as regards accumulation" (Ramsey, 1928). Families or dynasties of overlapping generations, each of which take account of the well-being of their progeny, are usually postulated.

3. The dynamic system

3.1 Condensation of the model equations

The *Solow-Romer* model is fully specified by equations (2.1) to (2.13). It can be condensed into a dynamic system of three first order differential equations, in the variables $A(t)$, $K(t)$, and $p_A(t)$ as follows:

- substitute equation (2.11) into equation (2.1):

$$\dot{A}(t) = \delta[H(t) - H_Y(t)]A(t) \tag{3.1}$$

- substitute equations (2.4), (2.6), (2.9) and (2.13) into equation (2.2) to produce:

$$\dot{K}(t) = s\frac{r(t)}{\gamma^2}K(t) \tag{3.2}$$

- substitute equations (2.7), (2.8),and (2.9) into equation (2.10):

$$\dot{p}_A(t) = r(t)\left[p_A(t) - \frac{1-\gamma}{\gamma}\frac{K(t)}{A(t)}\right] \tag{3.3}$$

where, substituting (2.5), and (2.9) first into equation (2.12), and then into equation (2.6) also using the result from (2.12):

$$H_Y(t) = \left[\frac{\alpha(1-\gamma)}{\delta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}[K(t)/A(t)]^{\gamma} \cdot p_A(t)^{-1}\right]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{3.4}$$

and

$$r(t) = \frac{\delta\gamma^2}{\alpha(1-\gamma)}H_Y(t)p_A(t)[K(t)/A(t)]^{-1} \tag{3.5}$$

However, this system is only **'quasi-stationary'**: $p_A(t)$ is asymptotically constant, but in the limit $A(t)$ and $K(t)$ merely grow at constant proportional rates. It can be transformed to a properly stationary system by defining the variable $\Psi(t) = K(t)/A(t)$, taking derivatives and substituting. In this way the stationary "Solow-Romer model" is specified as a system of just two (coupled) first order differential equations in the variables $\Psi(t)$ and $p_A(t)$. as follows:

$$\dot{\Psi}(t) = [s\,r(t)/\gamma^2 - \delta H + \delta H_Y(t)]\Psi(t) \tag{3.6}$$

$$\dot{p}_A(t) = r(t)[p_A(t) - \frac{1-\gamma}{\gamma}\Psi(t)] \tag{3.7}$$

where:

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\delta\eta^{\gamma}} L^{(1-\alpha)(1-\gamma)}\Psi(t)^{\gamma} p_A(t)^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{3.8}$$

and

$$r(t) = \frac{\delta\gamma^2}{\alpha(1-\gamma)} H_Y(t) p_A(t) \Psi(t)^{-1} \tag{3.9}$$

3.2 The steady-state

For the system to reach a balanced growth steady-state, both technology (A) and capital (K) must grow at constant relative rates. From equations (3.1) and (3.2), the rates of growth of A and K will be constant only if H_Y and r respectively are both constant. Then, from (3.4) and (3.5) this requires p_A and the ratio K/A (= Ψ) also to be constant. Thus, setting each of (3.6) and (3.7) to zero and solving the stationary system generates the steady-state values of the growth rate (g) and the other variables in terms of the parameters and exogenous variables of the model:

$$H^S_{Yss} = \frac{H}{1+(s/\alpha\gamma)} \tag{3.10}$$

$$H^S_{Ass} = \frac{H}{1+(\alpha\gamma/s)} \quad \text{and} \quad g^S = \frac{\delta H}{1+(\alpha\gamma/s)} \tag{3.11}$$

$$r^S_{ss} = (\delta\gamma/\alpha) H^S_{Yss} \tag{3.12}$$

$$\Psi^S_{ss} = [\frac{\alpha\gamma}{\delta\eta^{\gamma}} L^{(1-\alpha)(1-\gamma)} H^{S\ \alpha(1-\gamma)-1}_{Yss}]^{\frac{1}{1-\gamma}} \tag{3.13}$$

$$p^S_{Ass} = \frac{1-\gamma}{\gamma}\Psi^S_{ss} \tag{3.14}$$

3.3 Phase-space of the Solow-Romer system

In general, the phase-space of any dynamic system is defined by the loci of points for which the first derivatives of each of the system's dependent variables (x_i) with respect to its independent variable (t) are zero: $dx_i/dt = 0$, $i = 1,....,n$. The intersections of the hypersurfaces generated by such loci define regions of the phase-space for which the 'direction of motion' of each of the system's dependent variables can be ascertained. Also, any point (in hyperspace) of common intersection of all the hypersurfaces defines a steady-state of the system.

Since the Solow-Romer model comprises only **two** dynamic variables (neither of which can be negative), its phase-space is the positive quadrant of the (Ψ,p_A)-plane. Equations (3.6) and (3.7) respectively generate the loci of points for which $\dot{\Psi}(t)=0$ and $\dot{p}_A(t)=0$. In particular:

$\dot{\Psi}(t) \gtreqless 0$ as:

$$C_1\Psi(t)^{\gamma}\,[C_2\frac{p_A(t)}{\Psi(t)}+1]^{1-\alpha(1-\gamma)}-p_A \gtreqless 0$$

where (3.15)

$$C_1=\frac{\alpha(1-\gamma)}{\eta^{\gamma}\delta}L^{(1-\alpha)(1-\gamma)}H^{\alpha(1-\gamma)-1}$$

$$C_2=\frac{s}{\alpha(1-\gamma)}$$

and

$$\dot{p}_A(t) \gtreqless 0 \text{ as: } p_A(t) \gtreqless \frac{1-\gamma}{\gamma}\Psi(t) \qquad (3.16)$$

These expressions define the phase-space (or in this two-dimensional case the *phase-plane*) of the Solow-Romer model. The *directions of motion* in the various regions of the phase-plane can be seen directly from (3.15) and (3.16). Also, they can be further clarified by partial differentiation of the two differential equations (3.6) and (3.7), from which it can be shown that:

$$\frac{\partial\dot{\Psi}}{\partial p_A}<0 \quad \text{and} \quad \frac{\partial\dot{p}_A}{\partial\Psi}<0$$

With this information the directions of change in both p_A and Ψ are known for all points in the phase space, and a phase diagram of the dynamic system may be drawn. This has been done for a *benchmark* data set[10] and with the savings rate set at $s = 0.2$. The results

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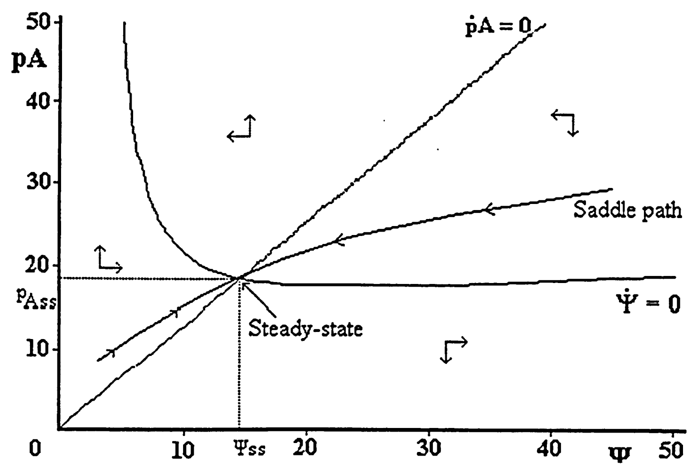
[10]The *benchmark* data set was intended to reflect the rudiments of the relevant magnitudes for the Australian economy: Australian National Accounts data over 1985-86 to 1994-95, indicate that the

are presented in Figure 3.1, where the directions of change in p_A and Ψ, in the (four) different regions of the phase-space, are indicated by "corner arrows". From these it is possible to draw streamlines[11] of the motion from any point in the phase space. In turn such streamlines indicate that the equilibrium (or steady-state) is one of *saddle path stability*. It is only from those (Ψ, p_A) couples which lie **on** the saddle path that the steady-state can be reached. Even very small discrepancies from such pointswill cause the system to diverge.

Figure 3.1 **Phase space of the Solow-Romer model. Calculated for a benchmark data-set and savings rate of *s*=0.2.**



shares of total wages and returns to capital in GDP were about 56 per cent and 44 per cent respectively. Similarly, Labour Force data indicate that of the nearly 8 million people employed, some 1 million are classified as "professionals". These data formed the basis for setting γ=0.44, H=1, and L=7. The 1/8 th share for the 'numbers of human capital workers' in total employment was adjusted up to 1/4 for their share of total labour income in order to reflect their likely higher productivity and wages. Thus the Cobb-Douglas income share parameter for human capital was set as α=0.25. The benchmark value for the discount rate was chosen as ρ=0.03 (partly in order to put the data set on an **annual** basis of measurement). Somewhat arbitrarily, the reciprocal of the elasticity of intertemporal substitution was set at σ=0.6, and the cost of a unit of one of the capital durables in terms of output was taken to be η=2.0. Finally, the value of the research productivity parameter, δ, (which is not dimensionless but rather, depends on the units in which H is counted) was adjusted so that the benchmark steady-state value of the interest rate, r_{SS} appeared a reasonable annual figure: With δ set at 0.1 the benchmark data set then returned a steady-state interest rate of r_{SS}= 6.25 per cent.

[11]The streamlines must cross the $\dot{\Psi} = 0$ locus with infinite slope and the $\dot{p}_A = 0$ locus with zero slope.

4. Dynamic response to economic shocks

Here the adjustment path of the Solow-Romer model in response to a variety of economic shocks is examined. The system is considered to be initially in equilibrium at the steady-state defined in Figure 3.1 (with the benchmark data set), and the effects of some sustained and unanticipated shock imposed at time zero is simulated. The following illustrative examples have been investigated:

- an increase in the savings rate, s, from 0.2 to 0.3;
- a rise in the productivity of researchers, as captured in the parameter δ, from 0.1 to 0.15; and
- an increase in the profit share of income, reflected through the parameter γ, from 44 per cent to 70 per cent.

4.1 Increase in the savings rate (s)

Since the savings rate is endogenous in the full Romer model any change in savings behaviour would have to be simulated there via shocks to other, more fundamental, parameters. Here it can be shocked exogenously, although it might still be thought of as arising indirectly from, say, some policy initiative such as a change in the Government's policy towards superannuation. In any event, an autonomous increase in the savings rate could be expected to stimulate investment thereby raising the capital-technology ratio (Ψ) and increasing the demand from the capital goods producing sector for new designs ($\dot{A}$). Such an increase in demand could be expected to push up the price of designs (p_A) until sufficient research activity were stimulated to re-establish equilibrium between the growth of capital and the growth of designs.

Thus, in terms of the model we would expect to observe increases in both $\Psi(t)$ and $p_A(t)$. Analysis of the phase space confirms this. The rise in the savings rate from 0.2 to 0.3 shifts the $\dot{\Psi}=0$ curve outwards and establishes both a new equilibrium and a new saddle path. The results are shown in Figure 4.1: In order for the system to converge to the new steady-state (SS2), it is necessary that it reach some state **on the saddle path.** However, since $\Psi(t)=K(t)/A(t)$ is a stock variable its evolution is (usually) continuous, with its immediate post-shock value equal to its immediate pre-shock value (Ψ1ss).

Conversely, prices readily jump discontinuously. Thus, the dynamic response of the Solow-Romer model to an increase in the savings rate is for the price of technology first to jump from its initial level to some intermediate value (pA1ss to pA12) while the capital-technology ratio remains constant (at Ψ1ss); and then for both variables to adjust smoothly along the saddle path towards the new steady-state of the system. The total adjustment path is along (SS1, P12, SS2) as indicated in Figure 4.1.

4.2 Increase in the productivity of researchers (δ)Increase in the savings rate (s)

This may be conceived of as arising from some government initiated measure of microeconomic reform. In any event, such an increase in productivity would be expected to raise the growth of research output and to lower its price. Thus, the capital-technology ratio could be expected to fall until the declining price of designs eventually restrains their growth and a new equilibrium is established. As before the adjustment path is composed of two parts; the first is an instantaneous drop in the price of technology (necessary to reach the saddle path of the dynamic system), and the second is a smooth transition towards the new steady-state along the saddle path. The dynamic adjustment path is indicated inFigure 4.2 as (SS1, P12, SS2).

Figure 4.1 **Phase plane analysis of the dynamic effects in the Solow-Romer model of a sustained rise in the savings rate (s) from 0.2 to 0.3 for a benchmark data-set.**

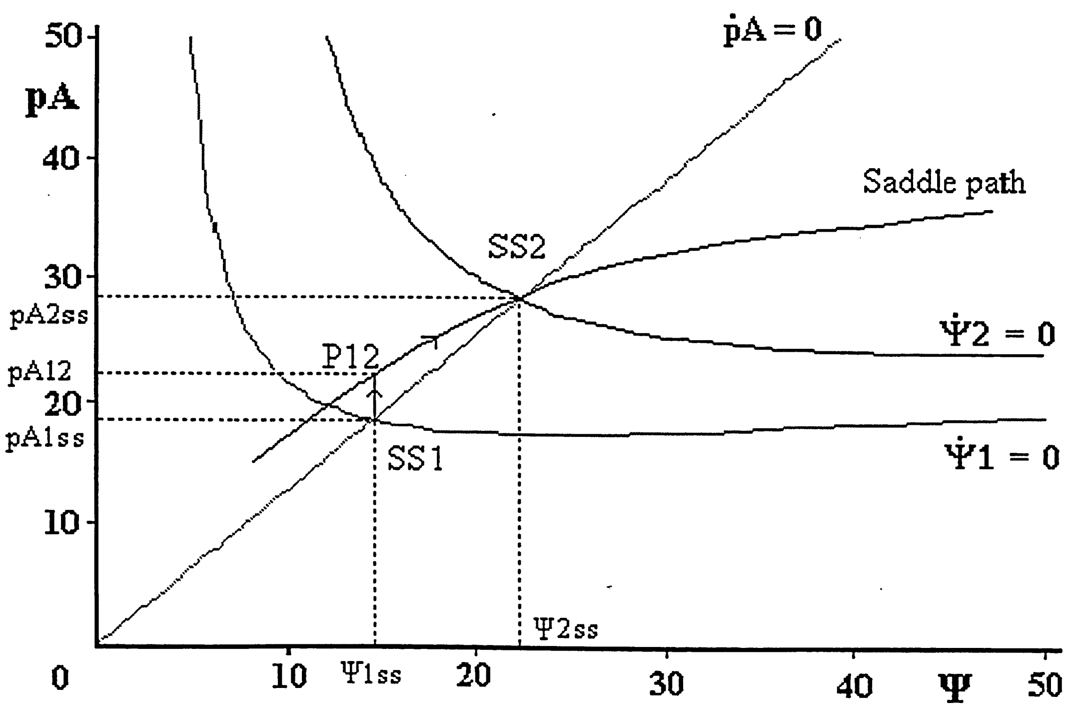
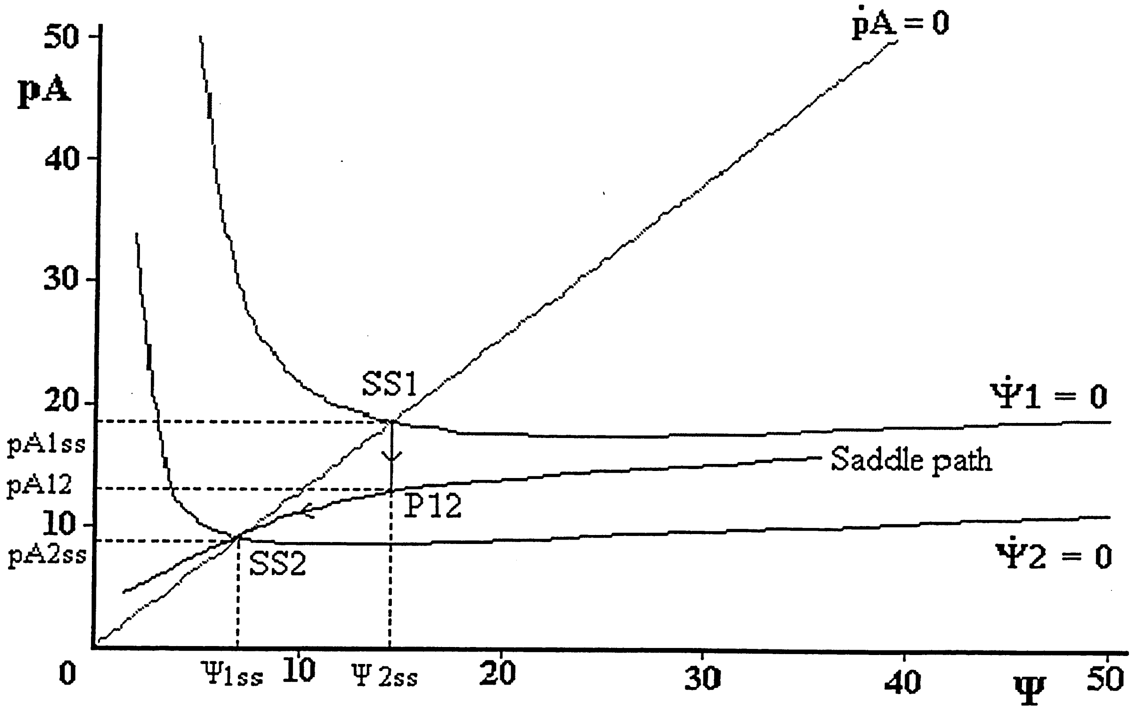


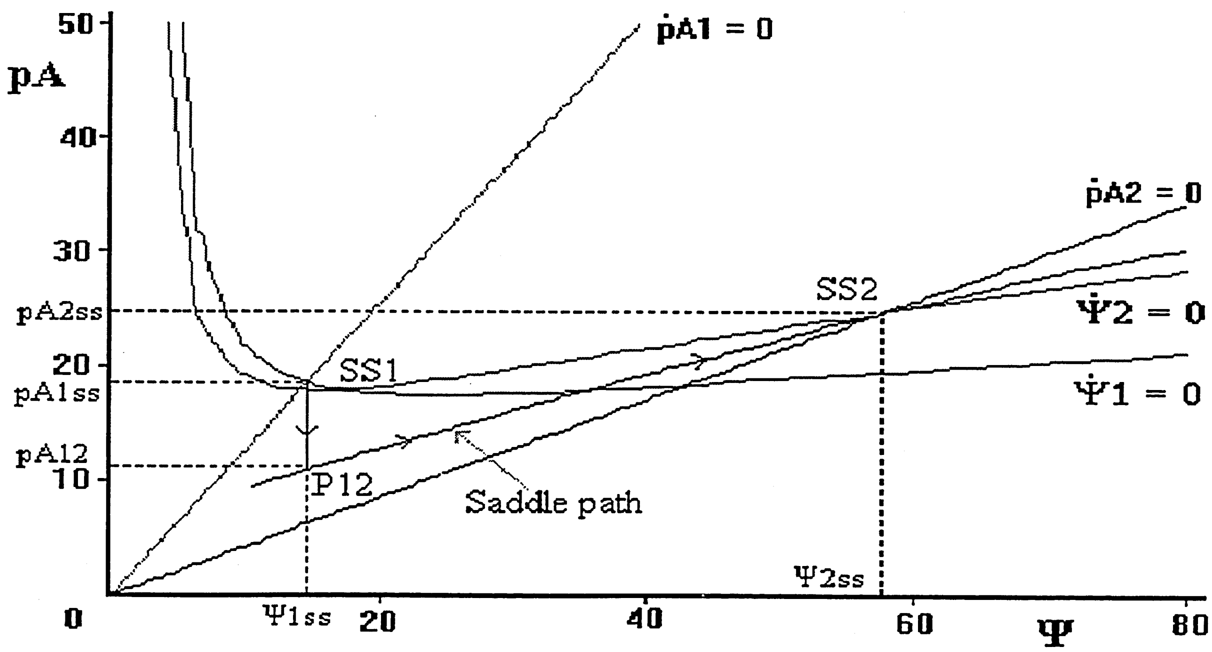
Figure 4.2 Phase plane analysis of the dynamic effects in the Solow-Romer model of a sustained rise in the productivity of researchers (δ) from 0.1 to 0.15 for a benchmark data-set.



4.3 Rise in the profit share of income (γ)

A rise in the profit share of income may be thought of, for example, as being due to some change in government tax policy. Since the marginal product of capital depends positively on the Cobb-Douglas parameter γ, it may also be readily thought of as arising from some form of microeconomic reform. An increase in the productivity of capital would then be expected to stimulate investment as producers moved around their transformation frontiers substituting capital for labour. As in the first example, increases in both the capital-technology ratio and the price of designs would follow. In terms of the model, the increase in γ lowers the $\dot{p}_A = 0$ locus and changes the shape of the $\dot{\Psi} = 0$ locus, raising it at the new point of intersection. The most interesting feature of the dynamics is that although the final equilibrium price of technology is greater than its original level, the initial response is a discontinuous fall (Figure 4.3).

Figure 4.3 **Phase plane analysis of the dynamic effects in the Solow-Romer model of a sustained rise in the profit share of income (γ) from 0.44 to 0.7, for a benchmark data-set.**



5. Numerical analysis of the model

Analysis of the phase space of a dynamic model, of which the saddle path is the key element, provides a highly instructive means of understanding the system's behaviour. However, while the saddle path represents a solution to the dynamic model, it is one for which time has been eliminated. Thus, phase space analysis cannot of itself provide answers to the sorts of timing questions that were raised in Section 1. For those purposes the model solution must take the form of providing explicit time paths for the model's variables.[12]

Since only the simplest of non-linear differential equation systems permit 'closed form', or complete analytical solutions (Roberts and Shipman, 1972), they must generally be solved by numerical methods instead. The Solow-Romer model is no exception. Now, a solution to the model is any set of time paths of the dynamic variables $\Psi(t)$ and $p_A(t)$ which satisfy

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[12]Determination of saddle paths and time paths for dynamic systems does not need to be independent. Clearly, if a system can be solved to obtain time paths for all its dynamic variables, then the *time variable* can simply be eliminated to generate the system's saddle path. Similarly, given saddle paths, initial values for all variables can be obtained and time paths derived as differential equation initial value problems (Mulligan and Sala-i-Martin, 1991)

both the differential equations and the boundary conditions. The latter comprise an'initial' value condition for the stock variable $\Psi(0)=K(0)/A(0)$ ($=\Psi_0$ say), and a 'final' value condition as described by the asymptotic or steady-state behaviour of the system![13] In particular, the 'final' boundary value condition can be specified as p^S_{Ass} (equations (3.10) to (3.14)).

The problem then, is to find an initial value for p_A (p_{A0}) such that the evolution of $p_A(t)$ and $\Psi(t)$ as prescribed by the differential equations, allow the steady-state to be reached. Such differential equation problems are known as *two-point boundary value problems* and a common approach to solving them is the numerical integration method of *shooting*. In this method an originally 'guessed' set of *free* initial values is iteratively updated according to how accurately the subsequent integration satisfies *the final value boundary conditions*, often transversality conditions (see Birkoff and Rota, 1969; Dixon et al., 1992: Keller, 1968; Press et al., 1986; and Roberts and Shipman, 1972).

The model was solved in this way for the same illustrative simulations as were previously examined by the phase plane analysis of Section 4. Numerical solutions were obtained by fixing the initial value of Ψ at its steady-state level for the benchmark data set (Ψ_{1SS} in Figures 4.1, 4.2, and 4.3), making appropriate adjustments to parameters or exogenous variables to simulate the economic shock under consideration, and selecting initial values for p_A with which to *shoot* at the system's steady-state (or equivalently, its transversality condition) by numerical integration![14]

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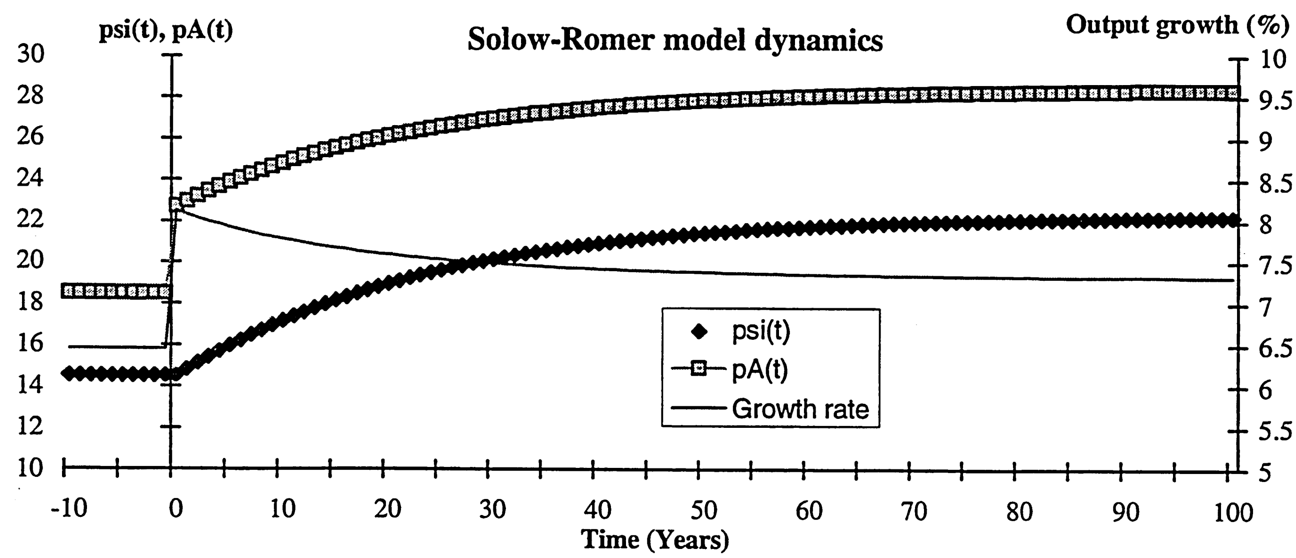
[13]More fundamentally, the final boundary value arises from what is called a *transversality condition*. This is a necessary condition for the dynamic optimisation problem (for producers) which underlies the model. It is this condition which forces the system to a steady-state and constrains the set of solutions to saddle-paths. Thus, it is of fundamental importance in establishing **both** the long-run equilibrium or steady-state behaviour of the model, and its transient dynamics.

[14]The fundamental approach of all numerical integration is that of using derivatives multiplied by step-sizes to add small increments to the functions to be integrated. Effectively, the differential equations are replaced by difference equation approximations (constructed from Taylor series expansions of the primitive functions), and the integration effected through iterative application of these difference equations. The simplest approach is "Euler's method" where all second order (and higher) *step-size* terms from the Taylor expansion are simply ignored. However, this is generally considered to be of more conceptual significance than of practical value. Superior and more sophisticated methods include the modified mid-point method (or Gragg's method); the Runge-Kutta method; Richardson extrapolations and the particular implementation of the Bulirsch-Stoer method; and predictor-corrector methods (see Press et al., 1987; and Pearson, 1991). Nevertheless, in the current application the Euler method (with a step-size of unity) was found to be highly accurate and was the method used. In practice the transversality condition was assumed to have been met when p_{At} had remained constant over (at least) t=250 to t=300. At that stage the other 'asymptotic variables' were also constant.

The results are reported in Figures 5.1, 5.2 and 5.3 and in Table 5.1. Dynamic adjustment paths for the rate of growth of output are provided in addition to those for the capital-technology ratio and the price of technology. Despite the fact that time is an explicit variable in the numerical integration solutions of the model while it is eliminated in the phase space analysis, the agreement between these two analytical methods can be clearly seen from a comparison of Figures 4.1, 4.2 and 4.3 with Figures 5.1, 5.2 and 5.3 respectively.

Figure 5.1 **Dynamic effects from the Solow-Romer model of a (sustained) rise in the savings rate(s) from 0.2 to 0.3 from time zero. 'Euler' method of numerical integration.**



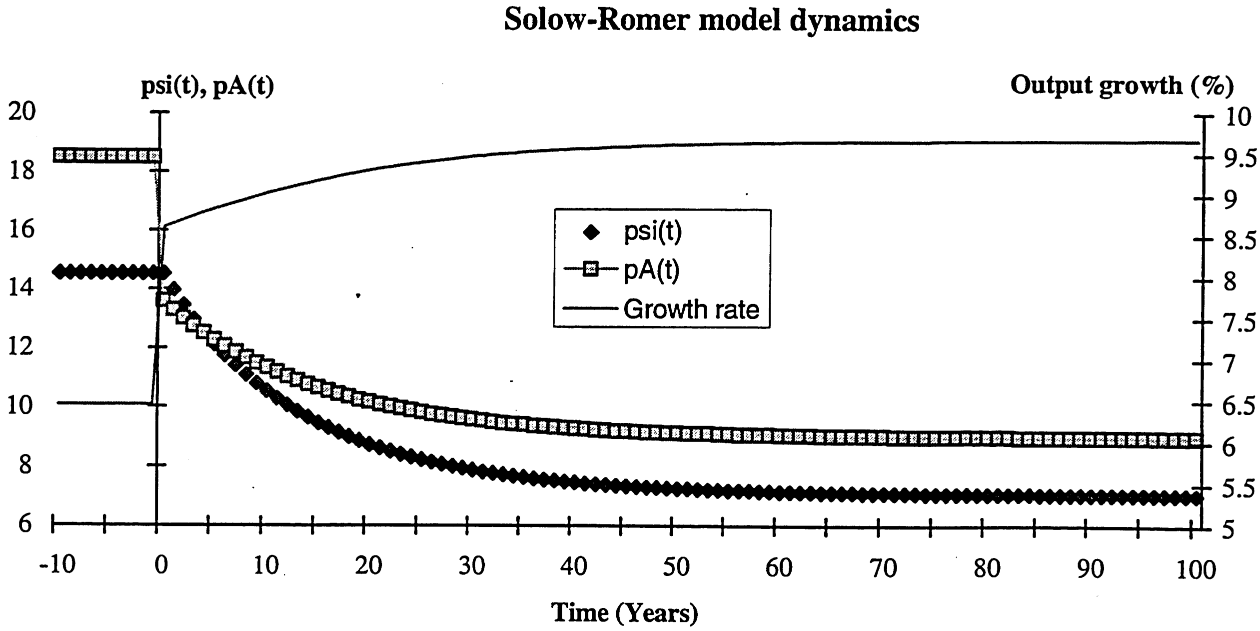
Numerical integration reveals an interesting feature of the adjustment dynamics of the model's growth rate. In the case of the simulated rise in the savings rate, while the growth rate of the economy ends up being higher, virtually all of its adjustment path is one of decline! The instantaneous response of the growth rate to the savings shock is a discontinuous rise, but following that the growth rate declines gradually to its new steady-state - a level which is between its pre-shock and immediate post-shock levels (Figure 5.1 and Table 5.1). In the 'increased profit share simulation', the growth rate also jumps upwards initially and subsequently declines gradually to its new steady-state.

Table 5.1 Initial jump effects and ultimate changes in steady-state levels in the S-R model for three simulated shocks (percentage changes).

c variable of	Savings rate simulation (50% rise)		Research productivity simulation (50% rise)		Profit share simulation (59.1% rise)	
the S-R model	tial jump (%)	change in ss (%)	tial jump (%)	change in ss (%)	Initial jump (%	Final change in ss (%
Ψ	0	53.6	0	-51.5	0	298
pA	22.8	53.6	-26.5	-51.5	-59.0	34.0
Growth rate	25.9	13.4	33.5	50.0	7.3	-17.3

However in this case the new steady-state is less than the initial level. Thus here, although the system's growth rate eventually declines, it initially jumps to a higher level and only falls below its original (pre-shock) value after some time (Figure 5.3 and Table 5.1). Specifically, the simulation indicates that it would be eleven years before the economy's growth rate fell below its initial level.

Figure 5.2 Dynamic effects from the Solow-Romer model of a (sustained) rise in the productivity of researchers (δ) from 0.1 to 0.15 from time zero. 'Euler' method of numerical integration.



This brings us to an issue raised in Section 1; namely, that of the duration of the adjustment process. It would seem that the greater the speed of approach of the economy to a new steady-state, the less significance its transient dynamics would have for economic management.[15] However, the evidence here is that adjustment can be relatively slow. Based upon the capital/technology ratio the model exhibits a *half-life* of 17 years for the savings simulation, 10 years for the research productivity simulation, and 40 years for the profit share simulation.[16] The measure of adjustment speed is based upon the capital/technology ratio due to complications arising with the initial discontinuities in the adjustment of the other variables. When such a variable initially jumps towards its final equilibrium it will record an 'understated' half-life, while when the initial jump is away from its new equilibrium the measured half-life will be exaggerated. For example, in the research productivity simulation (Figure 5.2), with both the price of technology and the growth rate initially jumping more than half way towards their final steady-states, both would return half-lives of zero! Conversely, in the profit share simulation (Figure 5.3), where both these variables initially jump in the opposite direction to their new steady-state, they return half-lives of 82 and 33 years respectively!

6. Concluding remarks

The modification of Romer's (1990a) model to the Solow-Romer version presented here allowed the dynamic responses of the model to a variety of exogenous economic shocks to be studied in terms of phase diagrams. This approach amounts to a pictorial representation of the model and is a powerful analytical tool for understanding its operation (and thus how it represents the real world); and for examining the relationship between its variables as they adjust from one equilibrium towards another. However, since it involves the elimination of the time variable, phase-space analysis is not particularly helpful in answering the types of timing questions posed in Section 1. Numerical integration of the model's dynamic system was undertaken for that purpose.

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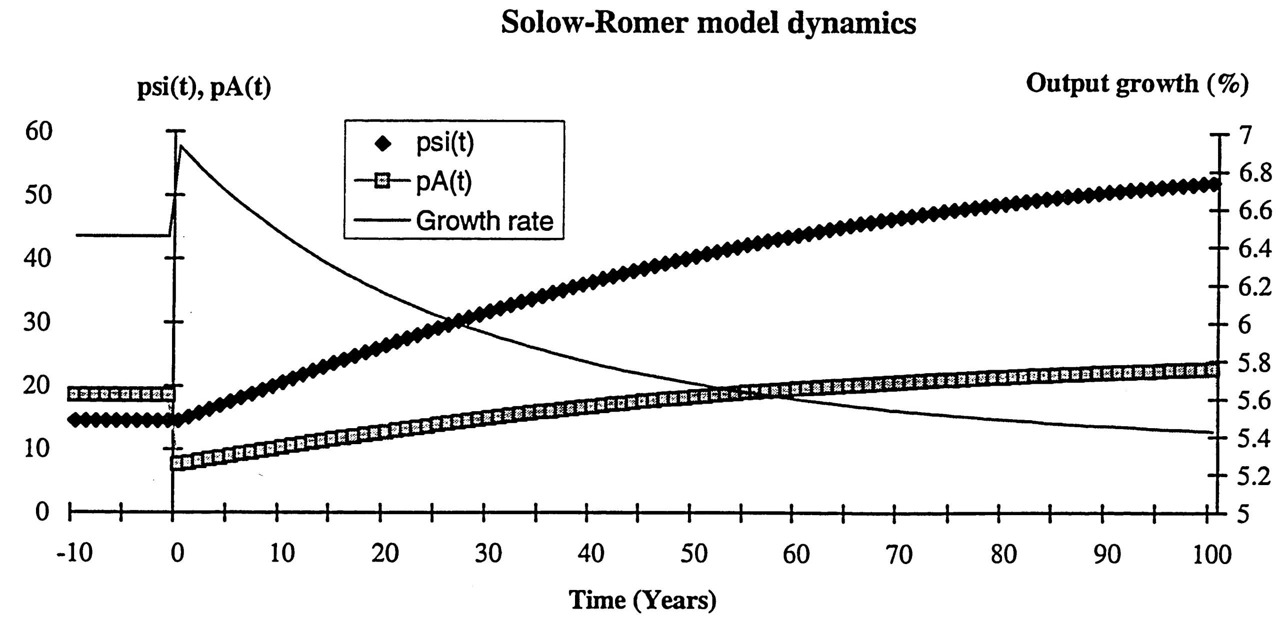
[15]This is leaving aside the whole issue of discontinuous jumps.

[16]The concept of a *half-life* refers to the time taken for the dynamic system to close half of the gap between its initial and final steady-states. In addition to the figures here, work with the full Romer model has indicated a capital/technology half-life of 33 years for a simulation in which the profit share of total income (γ), was raised by 10 per cent (for a slightly different benchmark data set to that used here).

Figure 5.3 **Dynamic effects from the Solow-Romer model of a (sustained) rise in the profit share of income (γ) from 0.44 to 0.7 from time zero. 'Euler' method of numerical integration.**



Adjustment processes were found to be relatively lengthy; and to be characterised by significant initial jumps or discontinuities in certain variables. Furthermore, it was found that in some cases these initial jumps could be in the opposite direction to that of the subsequent adjustment. Such results have important implications for economicpolicy: The longer the adjustment period the greater the relevance of adjustment issues for economic policy.[17] Also, the phenomena of jumps means that economic change can have quite precipitate effects, despite relatively protracted adjustment.

These matters emphasise the importance of explicit analysis of the dynamics of the adjustment paths of growth models. Dynamic analysis would seem to have a much greater potential for the provision of relevant policy advice than simply the comparative statics examination of alternative equilibria.

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[17]This is **not** to say that long term policy objectives should be compromised by short term adjustment costs.

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