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An adverse social welfare consequence of a rich-to-poor income transfer: A relative deprivation approach

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Abstract

A transfer from a richer individual to a poorer one seems to be the most intuitive and straightforward way of reducing income inequality in a society. However, can such a transfer reduce the welfare of the society? We show that a rich-to-poor transfer can induce a response in the individuals' behaviors which actually exacerbates, rather than reduces, income inequality as measured by the Gini index. We use this result as an input in assessing the social welfare consequence of the transfer. Measuring social welfare by Sen's social welfare function, we show that the transfer reduces social welfare. These two results are possible even for individuals whose utility functions are relatively simple (namely, at most quadratic in all terms) and incorporate a distaste for low relative income. We first present the two results for a population of two individuals. We subsequently provide several generalizations. We show that our argument holds for a population of any size, and that the choice of utility functions which trigger this response is not singular - the results obtain for an open set of the space of admissible utility functions. In addition, we show that a rich-to-poor transfer can exacerbate inequality when we employ Lorenz-domination, and that it can decrease social welfare when we draw on any increasing, Schur-concave welfare function.

Keywords: A rich-to-poor transfer; Relative income; Sen's social welfare function

JEL Classification: D30; D31; D60; D63; H21; I38

1. Introduction

We study together two of the main determinants of the wellbeing of societies: social welfare, and income inequality. The standard approach in economics and philosophy has been to measure aggregate wellbeing by means of a social welfare function, and income inequality by means of an index. More than four decades ago, Sen (1973) argued forcefully that the latter should be made part of the former. Specifically, Sen (1973) proposed to measure social welfare as income per capita times one minus the Gini coefficient of income inequality, arguing that income per capita alone cannot serve as a helpful guide to wellbeing. This stance informs us that, for example, the welfare of a society in which two individuals have incomes 2 and 2 is higher by one third than the welfare of a society in which two individuals have incomes 1 and 3. We are aware that income inequality can be measured in a variety of ways. Because our focus in this chapter is on studying social welfare as defined by Sen, we use the Gini coefficient of income inequality as our measure of income inequality.

For a century now, the Pigou-Dalton transfer principle (Pigou, 1912, Dalton, 1920) has been a lightning rod and fundamental guide in the design and implementation of policies aimed at redressing inequality. Pigou (1912, p. 24) wrote as follows: "[E]conomic welfare is likely to be augmented by anything that, leaving other things unaltered, renders the distribution of the national dividend less unequal. If we assume all members of the community to be of similar temperament, and if these members are only two in number, it is easily shown that any transference from the richer to the poorer of the two, because it enables more intense wants to be satisfied at the expense of less intense wants, must increase the aggregate sum of satisfaction." Dalton (1920, p. 321) expressed the principle in the following way: "If there are only two income-receivers, and a transfer of income takes place from the richer to the poorer, inequality is diminished. There is, indeed, an obvious limiting condition: the transfer must not be so large as more than to reverse the relative positions of the two income-receivers, and it will produce its maximum result, that is to say, create equality, when it is equal to half the difference between the two incomes." Subsequently, the Pigou-Dalton principle became an axiom that any admissible measure of income inequality has to obey (see, for example, Weymark, 2006).

It appears that the principle serves well Sen's social welfare function. However, considerable research (for example, Mirrlees, 1971; Saez, 2002; Choné and Laroque, 2005) indicates that implementing the Pigou-Dalton transfer may weaken the incentive to work. This response may reduce per capita income, thereby outweighing the redistributive welfare gains. As a consequence, while income inequality is reduced, Sen's social welfare is lowered. Another reason why the Pigou-Dalton transfer could decrease Sen's social welfare is the "leaky bucket" argument (Okun, 1975): if only a fraction of the tax ends up being transferred, per capita income decreases.

Here, however, we argue that there is yet another channel for the apparently seamless chain (a Pigou-Dalton transfer => lowered Gini coefficient => raised Sen's social welfare) to break down. So much so that, in fact, enacting the transfer could achieve the exact opposites: increase the Gini coefficient and reduce Sen's social welfare. Why could this be so? To present our argument, we track the effects of the transfer on the responses of the individuals concerned. We find that their adjustment in behavior in the wake of the transfer can exacerbate rather than reduce income inequality. In order to present this possibility efficiently, we employ several simplifications. (Later on in the chapter, we step out of several of these simplifications.) We consider a stylized production economy (an "artisan economy") in which two individuals produce a single consumption good. The utility of each individual depends negatively on his work effort and on his distaste for low relative income, and positively on his consumption. In such a constellation, a marginal rank-preserving transfer from a richer individual to a poorer individual can lead simultaneously to exacerbated inequality as measured by the Gini index and to a decrease in Sen's measure of social welfare. This finding is presented in the next section of the chapter by means of a constructive example. In Section 3 we assess the robustness of the insight provided by the example with respect to the form of the utility functions, the size of the population, and the measures of inequality and social welfare employed. Additionally, in an Appendix we show that the results do not depend on the format of the transfer: a transfer financed by a proportional tax levied on the income of the richer individual yields the same results as does the lump-sum tax and transfer assumed in the main text.

What drives our results are the individuals' distaste for low relative income and their responses in terms of effort to any variation in that income. Related work (Stark, 2013) provides evidence that low relative income / relative deprivation) matters, and that an increase (decrease) in relative deprivation intensifies (reduces) work effort so as to mitigate the increase (take advantage of the decrease).[1]

A Pigou-Dalton transfer from a richer individual to a poorer individual weakens the latter's incentive to work hard because the income "deprivation" experienced by the poorer individual is reduced. This scaling back of effort arises because, fundamentally, the benefits in terms of income when the poorer individual exerts effort are of two types: income as a means of enabling consumption, and income as a means of escaping an excessively low relative income. Thus, we provide a new behavioral explanation based on relative deprivation for the reduction of labor supply of the poorer individual as a result of redistributive policies, which is considered in the received literature on optimal taxation.[2] The triple assumptions that individuals like higher income, dislike effort, and dislike low relative income are not exceptional. In combination, the adjustments in effort by the two individuals can increase inequality, as measured by the Gini index,[3] and decrease social welfare, as measured by Sen's social welfare function. As could be anticipated, though, the wellbeing of the poorer individual improves.

Why is it important to identify a new explanation for the labor supply response of the recipient(s) of the transfer? One reason is that different explanations give rise to different policy responses. Suppose that a policy maker wants to encourage the recipient of a transfer not to scale back his work effort and reduce his output. When relative deprivation is a factor, then revision of the reference group of the poorer individual in conjunction with the transfer could leave the labor supply of the poorer individual intact. As an illustration, consider a transfer of one income unit from a richer individual whose income is 14 to a poorer individual whose income is 2, and measure income relative deprivation as the aggregate of the income excesses divided by the size

[1] We use the index of relative deprivation as a measure of low relative income (an Appendix in Stark, 2013, is a brief foray into relative deprivation).

[2] A classical work on this subject is Mirrlees (1971). For more recent studies on the response of labor supply to redistributive tax regimes see, for example, Saez (2002), and Choné and Laroque (2005). A second strand of literature investigating this issue builds on testing the "negative income tax" proposed by Friedman (1962); see, for example, Burtless (1986) who reports on reduced work supply among the beneficiaries of negative income in tax experiments in the U.S.

[3] It is not the objective of this chapter to inquire into the properties of the Gini coefficient as such. Rather, we study the behavior of individuals, and we ask how this behavior influences the Gini coefficient. Specifically, we do not look at the Gini coefficient as a function of the post-transfer incomes, $G(x_1, x_2)$, but rather at the Gini coefficient as a function of the transfer τ, $G(x_1(\tau), x_2(\tau))$, where the behavior of the individuals, as epitomized by the functions $x_i(\tau)$, shapes the "reaction" of the coefficient.

of the comparison group. The pre-transfer relative deprivation of the poorer individual is $(1/2)(14-2)=6$, and his post transfer relative deprivation is $(1/2)(13-3)=5$. This individual's relative deprivation can be retained at 6 if, for example, an individual whose income is 11 is added to the reference group of the poorer individual, because then $(1/3)[(11-3)+(13-3)]=6$.

At this juncture, three comments are in order. First, what adds value to our result is that in and by itself, the desire of individuals not to experience relative deprivation seemingly provides a built-in force favoring a more equal income distribution. "Interference" in the prevailing distribution by means of a rich-to-poor transfer could be expected to reinforce the already prevalent "bias" in favor of equality. Nonetheless, we observe the opposite. In a sense, the sum of the responses of the individuals whose preference is for less inequality to an income-equalizing transfer is a higher level of aggregate inequality, as measured by the Gini index.

Second, in this chapter we broaden the meaning and scope of the Pigou-Dalton transfer in that we embed it in a social context. This perspective leads us to drop the implicit Pigou-Dalton transfer principle assumptions of one-dimensionality and no-externalities. The introduction of factors that put a wedge between income and utility can then render the effect of a Pigou-Dalton transfer perverse. We consider broadening the domain in which the transfer is assessed. After all, individuals exhibit social preferences, income is hardly ever derived in isolation and, as a large body of modern day evidence suggests, while individuals like having income, they dislike having low relative income.[4] Starting from a "utility equilibrium" in which an individual strikes a balance between these two factors, the tradeoff between them could alone imply that improvement in the sphere of low relative income is accompanied by a muted desire for income.

Third, although in the main we analyze a population that consists of two individuals, our inequality cum welfare results are not limited to a two-individual Gini coefficient as a measure of inequality, and to a two-individual Sen's social welfare function as a measure of societal wellbeing. We address the case of a rich-to-poor transfer in a population of any number of individuals where the transfer exacerbates inequality which, in turn, is evaluated by Lorenz-

[4] Empirical studies that marshal evidence regarding the role of interpersonal comparisons for people's behavior include Zizzo and Oswald (2001), Luttmer (2005), Fliessbach et al. (2007), Blanchflower and Oswald (2008), Takahashi et al. (2009), Card et al. (2012), and Cohn et al. (2014).

domination, and decreases wellbeing as measured by a general, Schur-concave class of social welfare functions. In addition, we consider more than one method of taxing the income of the rich individual in the population.

The detailed structure of the chapter is as follows. In Section 2 we present a constructive example, built on specific utility functions, of the possibility that in the wake of a rich-to-poor transfer, income inequality will increase, and social welfare will take a beating. In Section 3 we provide generalizations of this example. In Proposition 1, presented in Subsection 3.1, we show that our results are not contingent on the specific utility functions to which we resort in Section 2. In Claim 2, displayed in Subsection 3.2, we relax the assumption regarding the number of individuals in the population, and we provide a generalization in terms of both the inequality measure and the welfare measure. Usage of these two alternative measures yields, once again, an increase and a decline, respectively, in the wake of a marginal rich-to-poor transfer. Section 4 concludes. We provide two appendices. In Appendix A, we present several technical lemmas that are needed for the proof of Proposition 1. In Appendix B, we show that the after-effects of a Pigou-Dalton transfer reported in the main text arise also when the transfer is financed by a proportional tax levied on the income of the richer individual. In Appendix C, we present the proof of part 1 of Claim 2.

2. A constructive example

To begin with, we assume that individual i, $i \in \{1,2\}$, converts units of work (costly effort) $e_i \geq 0$ into units of pre-transfer income, y_i, at the rate of one-to-one, that is, $y_i = e_i$. To allow for the possibility of a transfer of income between the individuals, we denote by $c_i \geq 0$ i's consumption, which is equal to i's post-transfer income, x_i. We resort to the following specification of the utility function $U_1 : \mathbf{R}^3 \to \mathbf{R}$, $U_1 \in C^2(\mathbf{R}^3)$ for individual 1

$$U_1(c_1, e_1, r_1) = dc_1 - 4gr_1^2 - fe_1, \tag{1}$$

and to the following specification of the utility function $U_2 : \mathbf{R}^3 \to \mathbf{R}$, $U_2 \in C^2(\mathbf{R}^3)$ for individual 2

$$U_2(c_2, e_2, r_2) = ac_2 - hr_2 - \frac{b}{2}e_2^2, \tag{2}$$

where $a, b, d, f, g, h > 0$ and where, acknowledging the distaste for low relative income in the individuals' preferences, r_i is the extent of the concern for low relative income of individual *i*, who compares his (post-transfer) income with the income of the other individual *j*.[5, 6] We note that the utility functions defined in (1) and (2) are concave in consumption and in the level of exerted effort, thus they can be thought of as standard utility specifications, enriched by acknowledging the concern for low relative income, which we operationalize by the index of relative deprivation. This index is

$$r_i = RD(x_i, x_j) = \frac{1}{2}\max\{x_j - x_i, 0\}, \tag{3}$$

where $j = 3 - i$. From the equality $x_i = c_i$, it follows that $r_i = RD(x_i, x_j) = RD(c_i, c_j)$.

We now use the relationships between consumption and post-transfer income and between effort exerted and pre-transfer income. Without a transfer, we have that $c_i = x_i = y_i = e_i$, namely

$$\begin{aligned} U_1(c_1, e_1, r_1) &= U_1(y_1, y_1, RD(y_1, y_2)) = dy_1 - 4g\left[RD(y_1, y_2)\right]^2 - fy_1, \\ U_2(c_2, e_2, r_2) &= U_2(y_2, y_2, RD(y_2, y_1)) = ay_2 - hRD(y_2, y_1) - \frac{b}{2}y_2^2, \end{aligned} \tag{4}$$

Throughout, we use the pre-transfer incomes $y_1, y_2 \geq 0$ (which are equal to the levels of work effort exerted by the individuals, e_1, e_2) as the decision variables in the utility maximization problem of the individuals.

We determine the ordering of the pre-transfer incomes and secure positivity of the exerted levels of effort at the optimum on assuming that

$$\frac{2ga}{b} > f - d > 0, \tag{5}$$

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5 By $C^2(\mathrm{X})$ we denote the set of functions from X to $\mathbf{R}$ that are twice continuously differentiable.

6 We defined the functions U_1 and U_2 on $\mathbf{R}^3$ in order to ensure traceability of their derivatives in the neighborhood of zero. Still, because the variables consumption, effort, and relative deprivation retain their intuitive interpretation only when they are non-negative, the domain of the maximization problems below will be constrained accordingly.

which ensures that $y_2^* > y_1^* > 0$; an asterisk denotes optimal values.[7] That is, under (5) and in the absence of any transfer, individual 1 is relatively poor, and individual 2 is relatively rich. Because $r_2 = RD(y_2^*, y_1^*) = 0$ (individual 2 is at the top of the income distribution and, therefore, he is not relatively deprived), the utility function of individual 2 for y_2, y_1 in the neighborhood of y_2^*, y_1^* reduces to

$$U_2(y_2, y_2, 0) = ay_2 - \frac{b}{2} y_2^2 . \tag{6}$$

Maximizing $U_2(y_2, y_2, 0)$ in (6) with respect to y_2, and maximizing $U_1(y_1, y_1, RD(y_1, y_2))$ in (4) with respect to y_1, yield

$$y_1^* = \frac{a}{b} - \frac{f-d}{2g},$$
$$y_2^* = \frac{a}{b}.$$

Because $y_2^* - y_1^* = \frac{f-d}{2g} > 0$ and $\frac{2ga}{b} > f - d \Rightarrow \frac{a}{b} - \frac{f-d}{2g} > 0$, (5) indeed ensures that $y_2^* > y_1^* > 0$.

Consider now a (marginal) transfer of income in the amount $\tau \geq 0$ from the richer individual 2 to the poorer individual 1.[8] The consumption levels, equal to the post-transfer incomes, of the two individuals become

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[7] We prove that (5) implies the ordering $y_2^* > y_1^*$ by contradiction. Suppose that $y_1^* \geq y_2^*$. Then, for $y_1, y_2 \geq 0$ in the neighborhood of y_1^*, y_2^*, we have that $r_1 = RD(y_1, y_2) = 0$ and $U_1(y_1, y_1, 0) = dy_1 - fy_1$. Because $f > d$, we have that the optimal effort exerted by individual 1 is $y_1^* = 0$. Because $y_1^* \geq y_2^* \geq 0$, we must have that $y_2^* = 0$. However, for $y_1 = 0$ we get that the right-hand derivative of the utility function of individual 2 at $y_2^* = 0$ is positive ($\lim_{y_2 \to 0^+} \frac{dU_2\left(y_2, y_2, RD(0, y_2)\right)}{dy_2} = a > 0$), yielding the inference that $y_2^* = 0$ cannot be optimal and, thus, contradicting the assumption $y_1^* \geq y_2^*$. That (5) also ensures positivity of y_1^*, y_2^* will be addressed momentarily.

[8] In Appendix B we study the possibility of a rich-to-poor transfer that is financed not by a lump-sum tax but, rather, by a proportional tax on the income of the richer individual.

$$c_1 = x_1 = y_1 + \tau,$$
$$c_2 = x_2 = y_2 - \tau.$$

Assuming that the transfer is small enough so as not to reverse the ordering of the incomes, that is, assuming that $y_1 + \tau < y_2 - \tau$, we have that $r_2 = RD(y_2 - \tau, y_1 + \tau) = 0$, and we can then consider the utility function of individual 1 as a function of the variables (y_1, y_2, τ), and the utility function of individual 2 as a function of the variables (y_2, τ). To avoid possible ambiguity, we denote these functions as

$$\begin{aligned} u_1(y_1, y_2, \tau) &= U_1(y_1 + \tau, y_1, RD(y_1 + \tau, y_2 - \tau)) = d(y_1 + \tau) - g(y_2 - y_1 - 2\tau)^2 - fy_1, \\ u_2(y_2, \tau) &= U_2(y_2 - \tau, y_2, 0) = a(y_2 - \tau) - \frac{b}{2} y_2^2. \end{aligned} \tag{7}$$

For a given $\tau \geq 0$, individual *i* maximizes u_i with respect to y_i. Solving simultaneously, the two first order conditions

$$\frac{\partial u_1}{\partial y_1} = 2g(y_2 - y_1 - 2\tau) - (f - d) = 0,$$
$$\frac{\partial u_2}{\partial y_2} = a - by_2 = 0,$$

allow us to express the optimal level of effort / pre-transfer income, $y_i^*(\tau)$, and the optimal level of consumption / post-transfer income, $x_i^*(\tau)$, as functions of τ.[9] We thus have that

$$\begin{aligned} y_1^*(\tau) &= \frac{a}{b} - \frac{f - d}{2g} - 2\tau, \\ y_2^*(\tau) &= \frac{a}{b}, \end{aligned} \tag{8}$$

and that

$$\begin{aligned} x_1^*(\tau) &= y_1^*(\tau) + \tau = \frac{a}{b} - \frac{f - d}{2g} - \tau, \\ x_2^*(\tau) &= y_2^*(\tau) - \tau = \frac{a}{b} - \tau. \end{aligned} \tag{9}$$

[9] Because $\frac{\partial^2 u_1}{\partial y_1^2} = -2g < 0$ and $\frac{\partial^2 u_2}{\partial y_2^2} = -b < 0$, the second order conditions for both maxima are satisfied.

We express the Gini index defined on the post-transfer incomes (which here constitute "net incomes") as a function of the transfer

$$G(\tau) = \frac{x_2^*(\tau) - x_1^*(\tau)}{2\left(x_1^*(\tau) + x_2^*(\tau)\right)}. \tag{10}$$

Let $W_S(\tau)$ be Sen's (1973) social welfare function - the product of per capita post-transfer income and one minus the Gini index - expressed as a function of the transfer, namely we let

$$W_S(\tau) = \frac{x_1^*(\tau) + x_2^*(\tau)}{2}\left(1 - G(\tau)\right). \tag{11}$$

We then have the following claim.

Claim 1. A marginal transfer from the richer individual to the poorer individual: (a) exacerbates income inequality as measured by the Gini index; (b) decreases Sen's measure of social welfare.

Proof.

(a) Combining (5), (9), and (10), we get that

$$G'(0) = \frac{d}{d\tau}\left[\frac{b(f-d)}{8ag - 2b(f-d+4g\tau)}\right]\Bigg|_{\tau=0} = \frac{2b^2 g(f-d)}{\left[4ag - b(f-d+4g\tau)\right]^2}\Bigg|_{\tau=0} = \frac{2b^2 g(f-d)}{\left[4ag - b(f-d)\right]^2} > 0. \tag{12}$$

Therefore, a marginal transfer τ increases the Gini index.

(b) Denoting by $X(\tau)$ the per capita post-transfer income as a function of the transfer, $X(\tau) = \left(x_1^*(\tau) + x_2^*(\tau)\right)/2$, we get that

$$X'(0) = \frac{d}{d\tau}\left[\frac{4ag - b(f-d+4g\tau)}{4bg}\right]\Bigg|_{\tau=0} = -1. \tag{13}$$

From (11) we get that

$$W_S(\tau) = X(\tau)\left(1 - G(\tau)\right). \tag{14}$$

From (12) and (13) we have that each of the two terms in (14) decreases after a marginal transfer and, therefore, $W_S(\tau)$ also decreases (because, obviously, $X(\tau) > 0$ and $1 - G(\tau) > 0$). Q.E.D.

The example presented in this section poses a puzzle: the income of the poorer individual and the change in Sen's measure of social welfare move in one direction, while the wellbeing of the poorer individual moves in the opposite direction. A Pigou-Dalton transfer could be expected to improve the wellbeing of the poorer individual, and in our case it indeed does: it is easy to confirm that following the transfer, the *utility* of the poorer individual increases.[10] However, at the same time, his post-transfer income (consumption) decreases;[11] in terms of *absolute* (*post-transfer*) *income*, he does not end up benefitting from the transfer. Yet, the difference between $x_2^*(\tau)$ and $x_1^*(\tau)$ is constant (equal to $\frac{f-d}{2g}$, refer to Figure 2 below) and, thus, it is independent of the size of the transfer. Consequently, the (extent of the) low relative income experienced by the poorer individual is also constant. On the other hand, the transfer allows the poorer individual to keep his low relative income stable while easing up his effort.[12] This suffices to increase his wellbeing, in spite of the drop in his consumption, because the combination of a constant low relative income and reduced effort is valued more highly than a higher consumption (recalling the relationship between f and d in the rightmost inequality in (5)).

This gain in the wellbeing of the poorer individual is not expressed when we look at Sen's measure of welfare. Apparently, Sen's social welfare function, built so as to take into account only a change in nominal (aggregate) income and a "synthetic" measure of inequality, does not embrace the satisfaction that the poorer individual derives from keeping his low relative income stable with a reduced work effort.

We illustrate our results graphically. To this end, we set the parameter values to $a=b=d=1/4$, and $f=g=h=1/2$. We note that these values satisfy assumption (5). Figures 1 through 5 display, respectively and as functions of the transfer τ: the optimal levels of the individuals' efforts / pre-transfer incomes $y_1^*(\tau), y_2^*(\tau)$; the optimal levels of the individuals'

[10] We have that marginal utility of the poorer individual with respect to the transfer is positive, that is

$$\frac{du_1\left(y_1^*(\tau), y_2^*(\tau), \tau\right)}{d\tau} = 2f - d > 0,$$

which follows from (5). Refer also to Figure 3.

[11] It is easy to see from (9) that $x_1^*(\tau)$ is a decreasing function of the transfer. See also Figure 2.

[12] From (8) we have that $y_1^*(\tau)$ is a decreasing function of the transfer; refer also to Figure 1.

consumptions / post-transfer incomes $x_1^*(\tau), x_2^*(\tau)$; the individuals' utility levels derived from the optimal efforts $u_1\left(y_1^*(\tau), y_2^*(\tau), \tau\right)$, $u_2\left(y_2^*(\tau), \tau\right)$; the Gini index $G(\tau)$; and Sen's social welfare $W_S(\tau)$. For $y_1^*(\tau)$ to be positive, it must hold that $\tau < \dfrac{a}{2b} - \dfrac{f-d}{4g} = 0.375$ and, thus, $\bar{\tau} = 0.375$ is chosen as the right end of the abscissa in each of the graphs.[13]

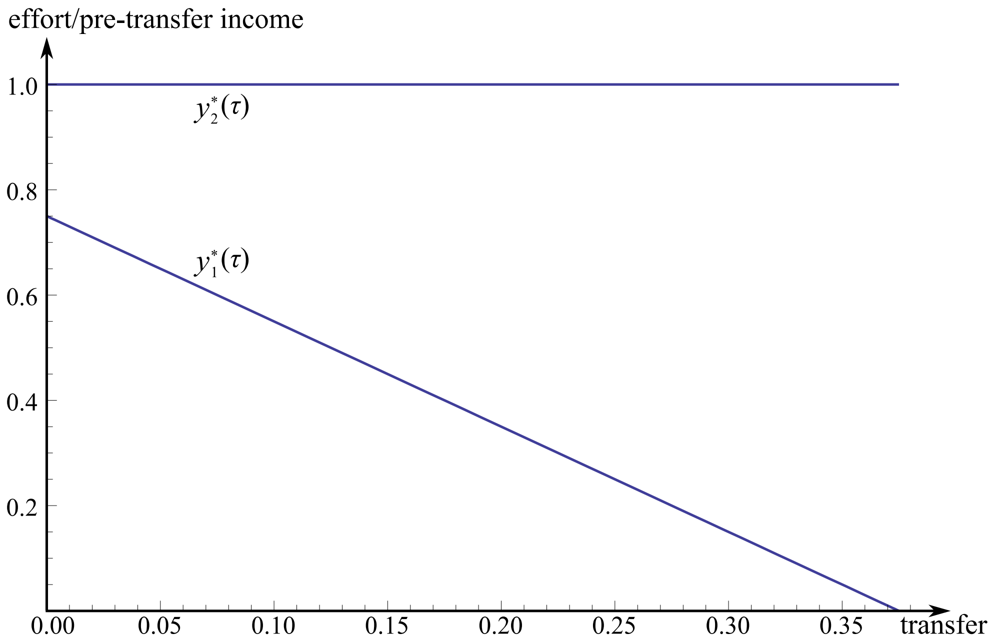

Figure 1. The optimal levels of the individuals' efforts / pre-transfer incomes, $y_1^*(\tau), y_2^*(\tau)$, as functions of the transfer τ.

[13] From (8) we get that $y_1^*(\tau) > 0$ if $\dfrac{a}{b} - \dfrac{f-d}{2g} - 2\tau > 0$, which is equivalent to $\tau < \dfrac{a}{2b} - \dfrac{f-d}{4g}$.

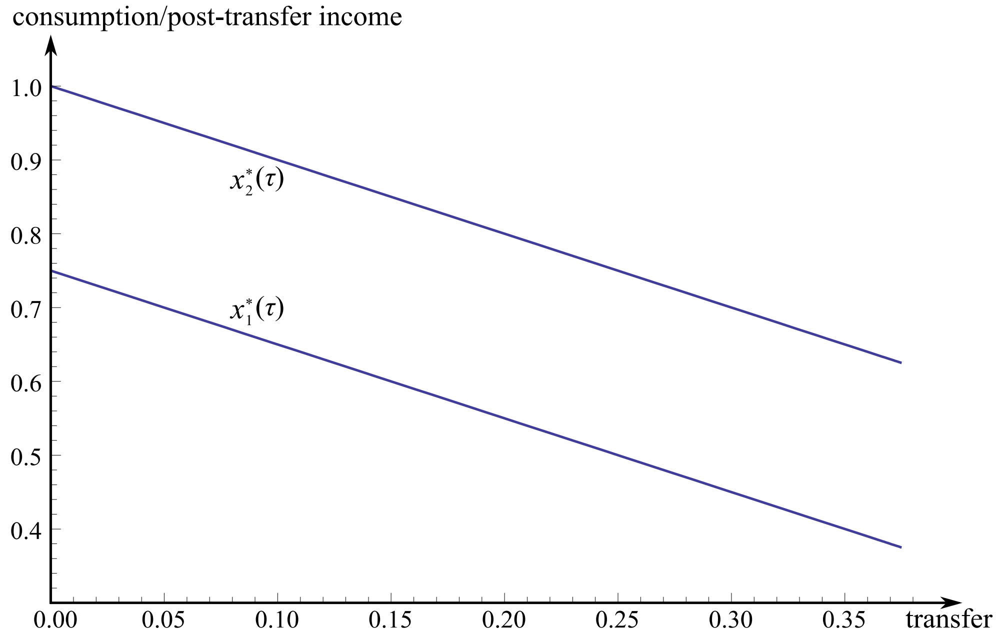

Figure 2. The optimal levels of the individuals' consumptions / post-transfer incomes, $x_1^*(\tau)$, $x_2^*(\tau)$, as functions of the transfer τ.

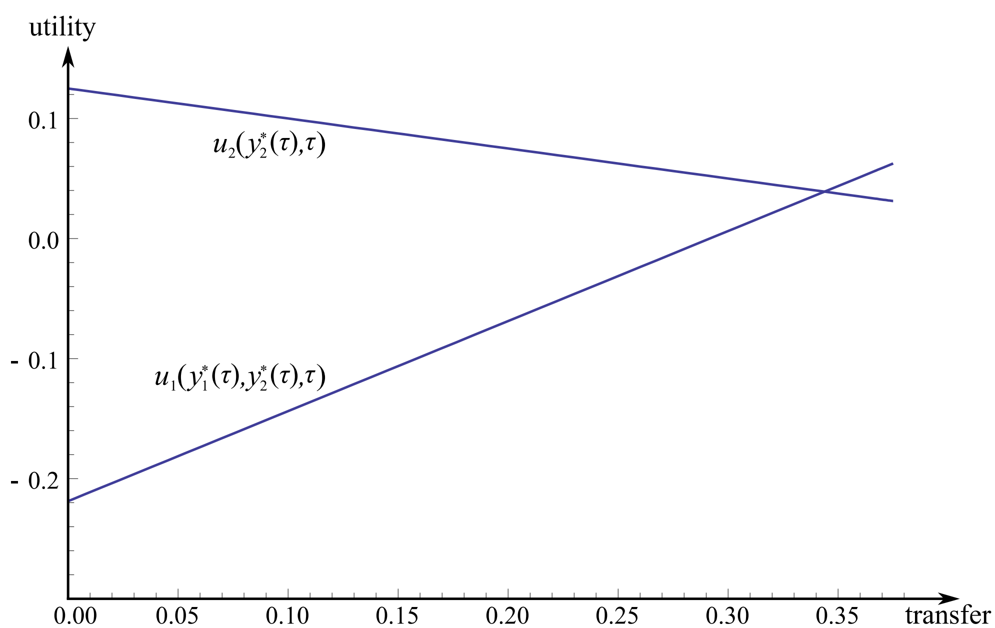

Figure 3. The individuals' utility levels derived from the optimal efforts, $u_1\left(y_1^*(\tau), y_2^*(\tau), \tau\right)$, $u_2\left(y_2^*(\tau), \tau\right)$, as functions of the transfer τ.

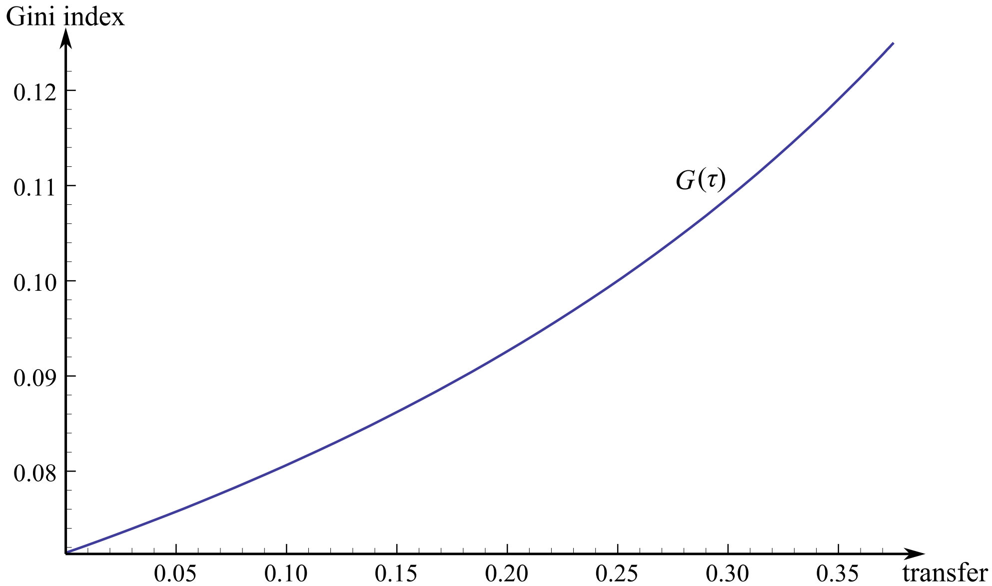

Figure 4. The Gini index, $G(\tau)$, as a function of the transfer τ.
Note: the scale of the vertical axis does not begin at zero.

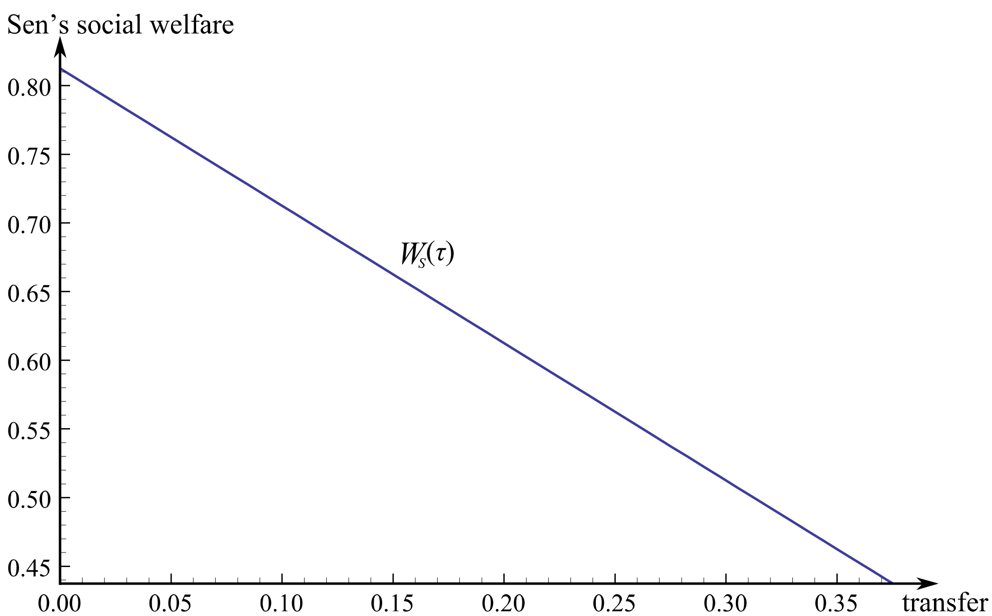

Figure 5. Sen's social welfare, $W_s(\tau)$, as a function of the transfer τ.

3. Generalizations

In this Section we strengthen our main argument in several respects. First, in Subsection 3.1 we show that the example of a rich-to-poor transfer that exacerbates income inequality, where inequality is measured by the Gini coefficient, and decreases wellbeing where social welfare is measured by Sen's welfare function, is not a singularity; it holds for all utility functions from an open set in the function space. In Subsection 3.2 we relax the assumption regarding the number of individuals in the population and we show that a Pigou-Dalton transfer can increase inequality and reduce social welfare under measures other than the Gini coefficient and Sen's social welfare function: specifically, a marginal rich-to-poor transfer can exacerbate inequality when the extent of inequality is assessed by Lorenz-domination, and it can decrease social wellbeing when social welfare is measured by any increasing, Schur-concave welfare function.

3.1. Non-singularity of the utility specification

We show that the results reported in the preceding section can be derived for utility functions that are more general than (1) and (2). We begin with a brief outline of the steps that we take to proceed. First, we look at the space of pairs of twice continuously differentiable functions, $C^2(\mathbf{R}^3)\times C^2(\mathbf{R}^3)$ as the space of possible utility functions of individual 1, V_1, and of individual 2, V_2. This space encompasses the utility functions U_1 and U_2 defined in Section 2. Second, constraining this space to the neighborhood of the fixed pair $U=(U_1,U_2)$, we simplify any pair of functions $V=(V_1,V_2)$ from this neighborhood into $v=(v_1,v_2)\in C^2(\mathbf{R}^3)\times C^2(\mathbf{R}^2)$, akin to what we did in Section 2 with respect to the simplification of $U=(U_1,U_2)$ into $u=(u_1,u_2)$. Third, we show that the results that we obtained in Section 2 hold for some open neighborhoods of u in the space $C^2(\mathbf{R}^3)\times C^2(\mathbf{R}^2)$, and thereby we (indirectly) show that the properties hold also for some open neighborhoods of U in the function space $C^2(\mathbf{R}^3)\times C^2(\mathbf{R}^3)$.

Let $\Omega_1 = C^2(\mathbf{R}^3)$ å $V_1(c_1,e_1,r_1)$ be the space of possible utility functions of individual 1, and let $\Omega_2 = C^2(\mathbf{R}^3)$ å $V_2(c_2,e_2,r_2)$ be the space of possible utility functions of individual 2. We endow the spaces Ω_1 and Ω_2 with the topology of uniform convergence with respect to second derivatives (or C^2 -uniform convergence topology), namely the topology where the open neighborhood basis for a function f consists of the sets:

$$\mathsf{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(f)=\left\{g\in C^2(\mathbf{R}^3):\forall_{x=(x_1,x_2,x_3)\in\mathbf{R}^3}\forall_{j,k=1,2,3}\left|g(x)-f(x)\right|<\varepsilon_1,\right.$$
$$\left.\left|\frac{\partial g}{\partial x_j}(x)-\frac{\partial f}{\partial x_j}(x)\right|<\varepsilon_2,\left|\frac{\partial^2 g}{\partial x_j\partial x_k}(x)-\frac{\partial^2 f}{\partial x_j\partial x_k}(x)\right|<\varepsilon_3\right\},$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, and with a product topology on the space $\Omega_1\times\Omega_2$.

Obviously, the utility functions U_1 and U_2 given by (1) and (2) belong, respectively, to Ω_1 and Ω_2.[14] We fix a pair of such functions, namely their coefficients a,b,d,f,g,h, which satisfy assumption (5). As already noted, we are interested in functions from the neighborhood of the pair $U=(U_1,U_2)$ in the space $\Omega_1\times\Omega_2$. From the analysis conducted in Section 2 we know that $U_1(y_1+\tau,y_1,RD(y_1+\tau,y_2-\tau))$ and $U_2(y_2-\tau,y_2,RD(y_2-\tau,y_1+\tau))$ have unique maxima, respectively, $y_1^*(\tau)$ and $y_2^*(\tau)$, such that $y_2^*(\tau)>y_1^*(\tau)\geq 0$ for a sufficiently small τ. Thus, a pair of functions $V=(V_1,V_2)\in\Omega_1\times\Omega_2$, which is close enough to U, taken as $V_1(y_1+\tau,y_1,RD(y_1+\tau,y_2-\tau))$ and $V_2(y_2-\tau,y_2,RD(y_2-\tau,y_1+\tau))$ will also have unique maxima denoted, respectively, by $y_{V,1}^*(\tau)$ and $y_{V,2}^*(\tau)$, which are in the neighborhoods of $y_1^*(\tau)$ and $y_2^*(\tau)$, respectively, so that $y_{V,2}^*(\tau)>y_{V,1}^*(\tau)\geq 0$ for a sufficiently small τ. Therefore, similar to Section 2, we can simplify the notation used in the definition of the pair of functions $V\in\Omega_1\times\Omega_2$ which are sufficiently close to U such that individual 1 is the poorer and individual 2 is the richer, and a marginal transfer τ does not reverse the ordering of the incomes so that the richer individual 2 does not experience relative deprivation. With these considerations in mind, it is convenient to change the variables in the pair of the utility functions $V\in\Omega_1\times\Omega_2$ in the following way:

[14] Here, while we take the functions U_1 and U_2 as given by formulas (1) and (2), we relax the ranges of their arguments beyond the sets in which these arguments retain their meaningful economic interpretation. In particular, in order to ensure differentiability of the utility functions in the neighborhood of zero, we allow the functions' third argument (relative deprivation) to take a negative value. This relaxation does not entail any ambiguity because soon thereafter, we confine our attention to small neighborhoods of income levels such that the relative deprivation of individual 1 is strictly positive, and the relative deprivation of individual 2 is equal to zero.

$$v_1(y_1, y_2, \tau) = V_1\left(y_1 + \tau, y_1, \frac{1}{2}(y_2 - y_1 - 2\tau)\right),$$
$$v_2(y_2, \tau) = V_2\left(y_2 - \tau, y_2, 0\right), \tag{15}$$

where τ is sufficiently small, $v_1 \in \omega_1 \equiv C^2(\mathbf{R}^3), v_2 \in \omega_2 \equiv C^2(\mathbf{R}^2)$,[15] and where we drew on the definition of low relative income, $r_1 = RD(c_1, c_2) = RD(y_1 + \tau, y_2 - \tau)$ and $r_2 = RD(c_2, c_1) = RD(y_2 - \tau, y_1 - \tau) = 0$.[16] These transformations of variables are smooth and do not change the value of the second argument in V_i, namely of $e_i = y_i$, so the optimal choices of effort levels / pre-transfer incomes of the individuals remain unchanged when their utility functions are transformed from $V = (V_1, V_2)$ to $v = (v_1, v_2)$, just as was the case upon "translating" $U = (U_1, U_2)$ into $u = (u_1, u_2)$ in Section 2.

We endow each of the spaces ω_1, ω_2 with the topology of uniform convergence with respect to second derivatives, and the space $\omega_1 \times \omega_2$ with a product topology. Obviously, $u_1 \in \omega_1$ and $u_2 \in \omega_2$.

For $U = (U_1, U_2)$ defined as in (1) and (2), and for the corresponding $u = (u_1, u_2)$ defined as in (7), we showed in Section 2 that under condition (5) a marginal transfer from the richer individual 2 to the poorer individual 1 exacerbates the Gini index, and decreases Sen's social welfare. In the following proposition we state and prove that there exists an open, non-empty neighborhood of u in the space $\omega_1 \times \omega_2$ (and, thus, also of U in $\Omega_1 \times \Omega_2$) consisting of utility functions for which the same results hold.

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[15] Actually, we only need to consider restriction of the function $V_1(c_1, e_1, r_1)$ to the neighborhood of the subspace $\{(c_1, e_1, r_1) \in \mathbf{R}^3_{>0} : c_1 = e_1\}$ in $\mathbf{R}^3_{>0}$, and of the function $V_2(c_2, e_2, r_2)$ to the neighborhood of the subspace $\{(c_2, e_2, 0) \in \mathbf{R}^2_{>0} \times \{0\} : c_2 = e_2\}$ in $\mathbf{R}^2_{>0} \times \{0\}$ and change their variables in these neighborhoods. However, the change of variables can be extended to the entire sets $\mathbf{R}^3$ and $\mathbf{R}^2 \times \{0\}$, so that the definitions of the functions v_1 and v_2 extend naturally to the entire $\mathbf{R}^3$ and $\mathbf{R}^2$.

[16] In the analysis provided in this section we formally allow $\tau < 0$ in order to ensure traceability of the derivatives with respect to τ of the optimal solutions to the individuals' maximization problems. However, only for $\tau \geq 0$ the economic interpretation of transfer from the "rich" to the "poor" individual is relevant.

Proposition 1. For any $u=(u_1,u_2)$ constructed as above, there exists an open neighborhood $N\subset\omega_1\times\omega_2$ of u (with respect to the C^2-uniform convergence topology) such that for any $v=(v_1,v_2)\in N$ a marginal transfer $\tau>0$ from the richer individual whose utility function is v_2 to the poorer individual whose utility function is v_1: (a) exacerbates income inequality as measured by the Gini index; and (b) decreases Sen's measure of social welfare.

Proof.

Let $u=(u_1,u_2)\in\omega_1\times\omega_2$ be associated with the given pair of utility functions $U=(U_1,U_2)\in\Omega_1\times\Omega_2$ defined by (1) and (2) and satisfying (5). For $\tau\in I_1$, where I_1 is a sufficiently small neighborhood of 0, $x_1^*(\tau)$, $x_2^*(\tau)$, $y_1^*(\tau)$, and $y_2^*(\tau)$ are all positive and $y_1^*(\tau)<y_2^*(\tau)$. Let $v=(v_1,v_2)\in\omega_1\times\omega_2$. If $\tau\in I_2$, where I_2 is a sufficiently small open neighborhood of 0, and if $v\in N_1(I_2)$, where $N_1(I_2)$ is a sufficiently small neighborhood of u, then the optimal effort levels of the individuals whose utility functions are v_1 and v_2, which we denote, respectively, by $y_{v,1}^*(\tau)$, and $y_{v,2}^*(\tau)$, exist, they are unique, and they observe $0<y_{v,1}^*(\tau)<y_{v,2}^*(\tau)$.

We show that following a marginal transfer from the richer individual 2 to the poorer individual 1 the Gini coefficient increases. The proof regarding the decrease of Sen's measure of social welfare follows analogously, and is therefore omitted.

We denote by $G_v(\tau)$ the Gini index defined on the optimal post-transfer incomes for the pair of utility functions $v=(v_1,v_2)$, namely $x_{v,1}^*(\tau)=y_{v,1}^*(\tau)+\tau$ and $x_{v,2}^*(\tau)=y_{v,2}^*(\tau)-\tau$. Mimicking (8), we have

$$G_v(\tau)=\frac{x_{v,2}^*(\tau)-x_{v,1}^*(\tau)}{2\left(x_{v,1}^*(\tau)+x_{v,2}^*(\tau)\right)}.$$

Thus,

$$G_v'(0)=\frac{x_{v,2}^{*}{}'(0)x_{v,1}^*(0)-x_{v,1}^{*}{}'(0)x_{v,2}^*(0)}{\left(x_{v,1}^*(0)+x_{v,2}^*(0)\right)^2}=\frac{(y_{v,2}^{*}{}'(0)-1)y_{v,1}^*(0)-(y_{v,1}^{*}{}'(0)+1)y_{v,2}^*(0)}{\left(y_{v,1}^*(0)+y_{v,2}^*(0)\right)^2},$$

so $G_v'(0)$ is a continuous function of $y_{v,1}^*(0), y_{v,2}^*(0), y_{v,1}^*{}'(0), y_{v,2}^*{}'(0)$. Therefore, it suffices to prove that $y_{v,1}^*(0), y_{v,2}^*(0), y_{v,1}^*{}'(0), y_{v,2}^*{}'(0)$ depend continuously on the initial choice of the pair of utility functions $v \in N_2 \subset N_1(I_2)$, where N_2 is a sufficiently small neighborhood of *u*. If this is true, then the function $v \mapsto G_v'(0)$ is continuous for $v \in N_2$, and the counter-image of $(0,\infty)$ of that function, which we denote by N_3 ($N_3 \subset N_2$), is an open neighborhood of *u* in $\omega_1 \times \omega_2$, such that $G_v'(0) > 0$ for any $v \in N_3$. Then, for each $v \in N_3$, G_v as a function of τ is increasing in an open neighborhood of 0.

To confirm continuity, we state and prove three lemmas in Appendix A. Lemma 1 shows that $y_{v,1}^*(\tau)$ and $y_{v,2}^*(\tau)$ are indeed differentiable functions of τ for any v from some neighborhood of *u*, which we denote by N_2. Lemmas 2 and 3 guarantee continuity (with respect to the utility functions) of the optimal effort level functions $y_{v,1}^*(\tau)$ and $y_{v,2}^*(\tau)$ and of their derivatives, respectively, at $\tau = 0$. Thus, Lemmas 2 and 3 imply that the function $v \mapsto \left(y_{v,1}^*(0), y_{v,2}^*(0), y_{v,1}^*{}'(0), y_{v,2}^*{}'(0)\right)$ is continuous and, therefore, $G_v'(0)$ is a continuous function of $v \in N_2$. Consequently $G_v'(0) > 0$ for $v \in N_3$. With $N = N_3$, the proposition follows. Q.E.D.

The inference drawn from Proposition 1 is that the results obtained in Section 2 are not an outgrowth of a specific, singular choice of the pair of quadratic utility functions; they hold for an open set of utility functions. Specifically, for each pair of (at most quadratic) functions U_1, U_2 defined, respectively, in (1) and (2), such that the parameters a,b,d,f,g,h satisfy (5), there exists an open neighborhood of general utility functions for which the results of Claim 1 also hold. The sum of these neighborhoods constitutes an open subspace in the space of twice continuously differentiable functions.

3.2. Lorenz dominance, general social welfare functions, and a population of any size

In Section 2 we considered a population consisting of two individuals, and we assessed the effect of a Pigou-Dalton transfer from a richer individual to a poorer individual on the Gini coefficient as a measure of inequality, and on Sen's welfare function as a measure of social welfare. That a rich-to-poor transfer can increase inequality and reduce social welfare does not depend, however, on the population consisting of just two individuals, nor on the particular inequality and social welfare measures used in Section 2. Here we outline an example that confers robustness to the results of Section 2: the results continue to hold for a population of any number of individuals, and exacerbation of inequality can arise not only when the Gini coefficient is used as an index, but also when the Lorenz curve is the measurement rod. We show that the Lorenz curve defined on the post-transfer distribution of incomes is dominated by the Lorenz curve defined on the pre-transfer distribution of incomes. Furthermore, we provide a generalization of the previously reported result of a decrease of social welfare obtained when the social welfare function used was Sen's. Apparently, any increasing, Schur-concave social welfare function will register a decrease as a result of the transfer.

Consider a population of individuals $i \in \{1, 2, \ldots, n\}$, $n \in \mathbf{N}$, $n > 2$. We define utility functions $U_i : \mathbf{R}^3 \to \mathbf{R}$, $U_i \in C^2(\mathbf{R}^3)$, such that for individual 1 utility is

$$U_1(x_1, y_1, r_1) = dx_i - 4gr_1^2 - fy_1, \tag{16}$$

and for individuals $i \in \{2, ..., n\}$ utility is

$$U_i(x_i, y_i, r_i) = a_i x_i - h_i r_i - \frac{b_i}{2} y_i^2, \tag{17}$$

where x_i and y_i are, respectively, the post-transfer and the pre-transfer incomes,

$$r_i = RD_i(x_1, x_2, \ldots, x_n) \equiv \frac{1}{n} \sum_{k=1}^{n} \max\{x_k - x_i, 0\} \tag{18}$$

is the relative deprivation of individual *i*, and $d, g, f, a_i, h_i, b_i > 0$ for $i \in \{2, ..., n\}$.[17] We have the following claim.

Claim 2. For any $n > 2$ there exist parameters $d, g, f, a_i, h_i, b_i > 0$ for $i \in \{2, ..., n\}$ such that:

1. Without a transfer, the optimal levels of effort / income of individuals $i \in \{1, 2, \ldots, n\}$, x_i^*, are such that

[17] The relative deprivation measure in (18) is the index of relative deprivation defined for any population of size $n \geq 2$. In (3) the index was used for a population of two individuals.

$$x_1^* < x_2^* < ... < x_n^* .$$

2. For a marginal transfer $\tau > 0$, the pre-transfer distribution of incomes, $\left\{x_1^*, x_2^*, ..., x_n^*\right\}$, Lorenz-dominates the post-transfer distribution of incomes, $\left\{x_1^*(\tau), x_2^*(\tau), ..., x_n^*(\tau)\right\}$.

3. Following a marginal transfer $\tau > 0$, any increasing, strictly Schur-concave social welfare function defined as a function of the transfer τ, $W(\tau)$, will register a decrease, namely $W(0) > W(\tau)$.

Proof. For a given $n > 2$, let

$$d = 1,\ g = \frac{n^3}{2(n-1)\left[2^n(n+1) - n(n+3)\right]},\ \text{ and } f = 2, \tag{19}$$

and for $i \in \{2, ..., n\}$, let

$$a_i = 2^i,\ h_i = 2^i,\ b_i = 2. \tag{20}$$

The proof of part 1 of the claim replicates approximately the steps taken in Section 2, where we considered the case of two individuals. Because the proof is somewhat long, it is relegated to Appendix C. Here, we present a brief summary of the protocol of the proof. Proofs of parts 2 and 3 of the claim follow thereafter.

First, in the absence of a transfer, maximization of the utility functions (16) and (17) evaluated with the parameters in (19) and (20) yields the optimal efforts / incomes levels $x_1^* = 1$ and $x_i^* = 2^i - \frac{i}{n}2^{i-1}$ for $i \in \{2, ..., n\}$. Obviously, we have that $x_1^* < x_2^* < x_3^* < \ldots < x_n^*$, which is the essence of part 1 of the claim.

Next, we introduce a (marginal) transfer of income $\tau > 0$ from each of the "richer" individuals $2, 3, \ldots, n$ to the poorest individual 1. Analogously to the treatment in Section 2, we find that a marginal transfer does not change the optimal effort levels of these individuals, which yields the post-transfer optimal incomes

$$x_i^*(\tau) = x_i^* - \tau = 2^i - \frac{i}{n}2^{i-1} - \tau \tag{21}$$

for $i \in \{2,...,n\}$. In turn, the post-transfer income of individual 1 is [18]

$$x_1^*(\tau) = 1 - \tau . \tag{22}$$

To prove part 2 of the claim, we note that the Lorenz curve for an income distribution $(x_1,...,x_n)$ is defined as a piecewise linear curve connecting the points (F_i, L_i), $i = 1,...,n$, where $F_i = i/n$, and $L_i = X_i / X_n$ for $X_i = \sum_{j=1}^{i} x_j$. We rewrite the levels X_i and L_i as functions of the transfer τ, namely as

$$X_i(\tau) = \sum_{j=1}^{i} x_j^*(\tau) = \sum_{j=1}^{i} x_j^* - i\tau$$

and

[18] From steps that replicate the derivation presented in equations (C9)-(C11) in Appendix C, we get that the relative deprivation of individual 1 who receives from each of the other individuals, $i \in \{2,...,n\}$, a transfer of τ is

$$r_1 = \frac{1}{n}\sum_{i=2}^{n}\left\{x_i^*(\tau) - \left[y_1 + (n-1)\tau\right]\right\} = \frac{1}{n}\left\{\sum_{i=2}^{n}\left(2^i - i\, 2^{i-1}/n - \tau\right) - (n-1)\left[y_1 + (n-1)\tau\right]\right\}$$
$$= \frac{1}{n}\left[2^n(n+1)/n - 4 - (n-1)(y_1 + n\tau)\right].$$

Using (16) and (19), the utility function of individual 1 is

$$U_1[y_1 + (n-1)\tau, y_1, r_1] = \left[y_1 + (n-1)\tau\right] - \frac{n\left[2^n(n+1)/n - 4 - (n-1)(y_1 + n\tau)\right]^2}{2(n-1)\left[2^n(n+1) - n(n+3)\right]} - 2y_1.$$

Hence,

$$\frac{dU_1[y_1 + (n-1)\tau, y_1, r_1]}{dy_1} = -1 + \frac{2^n(n+1) - n\left[(n-1)(y_1 + n\tau) + 4\right]}{2^n(n+1) - n(n+3)} = (1 - y_1 - n\tau)\frac{(n-1)n}{2^n(n+1) - n(n+3)}.$$

By solving $dU_1[y_1 + (n-1)\tau, y_1, r_1]/dy_1 = 0$ we get that the optimal effort level / pre-transfer income of individual 1, expressed as a function of τ, is

$$y_1^*(\tau) = 1 - n\tau,$$

implying that the post-transfer income of this individual is

$$x_1^*(\tau) = y_1^*(\tau) + (n-1)\tau = 1 - \tau .$$

$$L_i(\tau) = \frac{X_i(\tau)}{X_n(\tau)} = \frac{\sum_{j=1}^{i} x_j^* - i\tau}{\sum_{j=1}^{n} x_j^* - n\tau}.$$

For both the pre-transfer distribution $\{x_1^*, x_2^*, ..., x_n^*\} = \{x_1^*(0), x_2^*(0), ..., x_n^*(0)\}$ and the post-transfer distribution $\{x_1^*(\tau), x_2^*(\tau), ..., x_n^*(\tau)\}$, the points F_i are the same, thus a Lorenz-domination of distribution $\{x_1^*(0), x_2^*(0), ..., x_n^*(0)\}$ over distribution $\{x_1^*(\tau), x_2^*(\tau), ..., x_n^*(\tau)\}$ with $\tau > 0$ is equivalent to $L_i(0) \geq L_i(\tau)$ for all $i = 1, ..., n-1$, and with $L_i(0) > L_i(\tau)$ for at least one $j \in \{1,..,.,n-1\}$. We next show that for any $i = 1, ..., n-1$, we have that $L_i'(0) < 0$, which is sufficient for $L_i(0) > L_i(\tau)$ to hold for a marginal transfer $\tau > 0$.

We have that

$$L_i'(0) = \frac{n\sum_{j=1}^{i} x_j^* - i\sum_{j=1}^{n} x_j^*}{\left(\sum_{j=1}^{n} x_j^*\right)^2}.$$

Using (21) and (22) for $x_j^* = x_j^*(0)$, $\sum_{j=1}^{i} x_j^*$ for any $i = 1, ..., n$ can be rewritten (drawing on the derivation presented in equations (C9)-(C11) in Appendix C) as

$$\sum_{j=1}^{i} x_j^* = 1 + \sum_{j=2}^{i}\left(2^j - \frac{j}{n}2^{j-1}\right) = 2^{1+i} - 3 - \frac{1}{n}2^i(i-1) = \frac{2^i(1+2n-i)}{n} - 3.$$

Therefore,

$$L_i'(0) = \frac{\left[2^i(2n+1-i) - 3n\right] - i\left[2^n(1+1/n) - 3\right]}{\left(\sum_{j=1}^{n} x_j^*\right)^2}$$

$$= \frac{2^i(2n+1-i) - i2^n(1+1/n) - 3(n-i)}{\left(\sum_{j=1}^{n} x_j^*\right)^2}.$$

In order to enable us to sign $L_i'(0)$, we define and investigate the following function:

$$f(x) \equiv 2^x(2n+1-x) - x2^n(1+1/n) - 3(n-x)$$

for $x \in [1,n] \subset \mathbf{R}$. This function is continuous for $x \in [1,n]$, and it is twice differentiable for $x \in (1,n)$. We have that

$$f'(x) = 2^x\left[(2n+1-x)\ln 2 - 1\right] - 2^n(1+1/n) + 3$$

and that

$$f''(x) = 2^x\left[(n+1-x)\ln 2 + (n\ln 2 - 2)\right]\ln 2 > 0$$

for any $n > 2$ and $x \in [1,n]$; that is, $f(x)$ is a convex function and although it does not have a local maximum for $x \in (1,n)$, it takes its maximum value either at $x = 1$, or at $x = n$. At these two points, respectively,

$$f(1) = n + 3 - 2^n(1+1/n) < 0,$$

$$f(n) = 0.$$

Thus, for $x \in [1,n)$, $f(x) < 0$.

For any $i = 1,...,n-1$ we can rewrite $L_i'(0)$ as

$$L_i'(0) = \frac{f(i)}{\left(\sum_{j=1}^{n} x_j^*\right)^2}.$$

From the preceding analysis of the properties of $f(x)$, we infer that for $i = 1,...,n-1$ and any $n > 2$, $f(i) < 0$ and, consequently, $L_i'(0) < 0$. This completes the proof of part 2 of the claim.

In order to prove part 3 of the claim, we draw on (21) and (22), which state that a marginal transfer lowers the income of every individual. Thus, the mean income of the population is lowered. Consequently, we get a generalized-Lorenz-domination of the pre-transfer distribution of incomes over the post-transfer distribution of incomes, which (recalling Shorrocks, 1983) is equivalent to a decrease of any increasing, strictly Schur-concave social welfare function following a marginal rich-to-poor transfer. This completes the proof of part 3 of the claim. Q.E.D.

4. Discussion

By means of a constructive example, we presented a rationale as to why a rank-preserving transfer from a richer individual to a poorer individual might exacerbate (rather than reduce) income inequality (as measured by the Gini index or by the Lorenz-domination). This result was derived when the preference profiles of the individuals are quite natural and not overly restrictive, and it holds for a nonsingular set of possible utility functions. Specifically, we assumed that the individuals' utility functions exhibit distaste for low relative income. Demonstrating that a Pigou-Dalton transfer fails to decrease income inequality could imply that a more demanding transfer principle will be needed to secure reduced inequality. Moreover, when a Pigou-Dalton transfer increases inequality, then on such a transfer, the wellbeing of the population, as measured by a broad class of social welfare functions, registers a decline. Specifically, Sen's (1973) social welfare function indeed does that, in spite of the fact that after the transfer, the utility of the poorer individual increases.

Interestingly, Pigou (1920) himself provided two reasons why the "principle of transfers" might fail to reduce inequality. First, when the richer individual employs poorer individuals, the amount taken from the richer individual will hurt the poorer individuals because the former will not be able to create as many workplaces, or pay as much. Second, on receipt of the transfer, the poorer individual will agree to work for a lower wage because some of his needs will be catered for by the transfer.

The argument advanced in this chapter is related to this second reasoning if we consider that keeping low relative income in check is a "need" for the poorer individual, which is now catered for by the transfer. Additionally, our argument highlights important considerations that a social planner who is concerned about income inequalities should bear in mind. When acting in the seemingly simplest and most straightforward way to address inequality, namely making a Pigou-Dalton transfer, the social planner cannot be sure that the transfer and its consequences will increase the income of the poor, improve inequality measures, and raise welfare. Having said all that, we note that the wellbeing of the poor might still improve as a result of the transfer.

Appendix A: Continuity lemmas

As a preliminary, we note that $\frac{\partial^2 u_2}{\partial y_2^2}(y_2,\tau) = -b < 0$ and that $\frac{\partial^2 u_1}{\partial y_2^2}(y_1,y_2,\tau) = -2g < 0$ for any $(y_1,y_2,\tau) \in \Delta = \mathbf{R}^2_{>0} \times (I_1 \cap I_2)$. Therefore, there exist $N_2 \subset N_1(I_2)$ such that for any $v = (v_1,v_2) \in N_2$ it holds that $\frac{\partial^2 v_2}{\partial y_2^2}(y_2,\tau) < 0$, and that $\frac{\partial^2 v_1}{\partial y_2^2}(y_1,y_2,\tau) < 0$ for any $(y_1,y_2,\tau) \in \Delta = \mathbf{R}^2_{>0} \times (I_1 \cap I_2)$.

In the following lemmas we assume that $\tau \in I_1 \cap I_2$.

Lemma 1.

For any $v \in N_2$ there exist an open interval J such that $0 \in J \subset I_1 \cap I_2$, and there exist continuously differentiable functions $\varphi : J \to \mathbf{R}$, $\psi : J \to \mathbf{R}$ such that

(a) $\forall_{\tau \in J} \varphi(\tau) = y^*_{v,2}(\tau)$;

(b) $\forall_{\tau \in J} \psi(\tau) = y^*_{v,1}(\tau)$.

Proof.

To prove part (a), we note that because $v = (v_1,v_2) \in N_2$, then $y^*_{v,2}(\tau)$ is an optimal effort level for utility function v_2 and for $\tau \geq 0$ iff $\frac{\partial v_2}{\partial y_2}(y^*_{v,2}(\tau),\tau) = 0$. Also, from the assumptions on N_2 it follows that $\frac{\partial^2 v_2}{\partial y_2^2}(y_2,\tau) < 0$ (in particular, $\frac{\partial^2 v_2}{\partial y_2^2}(y_2,\tau) \neq 0$) for any $(y_1,y_2,\tau) \in \Delta$ and, hence, the implicit function theorem can be applied to complete the proof of part (a).

Analogously, for part (b), because $v = (v_1,v_2) \in N_2$, then $y^*_{v,1}(\tau)$ is an optimal effort level for utility function v_1 for $\tau \geq 0$ and for $y^*_{v,2}(\tau)$ iff $\frac{\partial v_1}{\partial y_1}(y^*_{v,1}(\tau), y^*_{v,2}(\tau),\tau) = 0$. From the assumptions on N_2 we have that $\frac{\partial^2 v_1}{\partial y_1^2}(y_1,y_2,\tau) < 0$ (in particular, $\frac{\partial^2 v_1}{\partial y_1^2}(y_1, y^*_{v,2}(\tau),\tau) \neq 0$) for any $(y_1,y_2,\tau) \in \Delta$, and from part (a) we have that $y^*_{v,2}(\tau)$ is a continuous and differentiable function of $\tau \in J$. Applying the implicit function theorem to the condition $\frac{\partial v_1}{\partial y_1}(y^*_{v,1}(\tau), y^*_{v,2}(\tau), \tau) = 0$ yields

$$y^{*}_{v,1}{}'(\tau) = -\frac{\frac{\partial^2 v_1}{\partial y_1 \partial y_2}\left(y^*_{v,1}(\tau), y^*_{v,2}(\tau), \tau\right) y^{*}_{v,2}{}'(\tau) + \frac{\partial^2 v_1}{\partial y_1 \partial \tau}\left(y^*_{v,1}(\tau), y^*_{v,2}(\tau), \tau\right)}{\frac{\partial^2 v_1}{\partial y_1^2}\left(y^*_{v,1}(\tau), y^*_{v,2}(\tau), \tau\right)}. \qquad \text{(A1)}$$

This completes the proof of part (b). Q.E.D.

In Lemmas 2 and 3 below, we apply Heine's definition of continuity. We consider a sequence $(v_n)_{n=0}^{\infty} = ((v_{1,n}, v_{2,n}))_{n=0}^{\infty}$ of functions from the space $\omega_1 \times \omega_2$, such that as n tends to infinity, the sequence tends (with respect to the product topology) to $v \in N_2$. Because we are only interested in the asymptotic behavior of that sequence, we can assume that $v_n \in N_2$ for each n . From Lemma 1 we know that for each pair $v_n = (v_{1,n}, v_{2,n}) \in N_2$ there exist a function $y^*_{v_{2,n},2}(\tau)$ (the solution to the maximization problem of the richer individual with a utility function $v_{2,n}(\cdot, \tau)$) and a function $y^*_{v_{1,n},1}(\tau)$ (the solution to the maximization problem of the poorer individual with a utility function $v_{1,n}(\cdot, y^*_{v_{2,n},2}(\tau), \tau)$).

Lemma 2.

For any $v \in N_2$ and any sequence $(v_n)_{n=0}^{\infty} = ((v_{1,n}, v_{2,n}))_{n=0}^{\infty}$ such that $v_n \xrightarrow[n\to\infty]{} v$, we have that

(a) $y^*_{v_{2,n},2}(0) \xrightarrow[n\to\infty]{} y^*_{v,2}(0)$;

(b) $y^*_{v_{1,n},1}(0) \xrightarrow[n\to\infty]{} y^*_{v,1}(0)$.

Proof.

As in Lemma 1, the proofs of parts (a) and (b) are analogous, albeit here part (b) requires additional attention because the behavior of the poorer individual (function $y^*_{v,1}(\tau)$) depends on $y^*_{v,2}(\tau)$, but not *vice versa*. Thus, we only provide here a proof for part (b), assuming that part (a) is true.

Let $\varepsilon > 0$. We show by contradiction that there exists n_1 such that for any $n > n_1$ we have $\left| y^*_{v_{1,n},1}(0) - y^*_{v,1}(0) \right| < \varepsilon$. Let $\frac{\partial^2 v_1}{\partial y_1^2}(y^*_{v,1}(0), y^*_{v,2}(0), 0) \equiv -\alpha < 0$, which follows from the fact that $v \in N_2$. From the continuity of the first and second derivatives of v_1, and from the fact that $\frac{\partial v_1}{\partial y_1}(y^*_{v,1}(0), y^*_{v,2}(0), 0) = 0$, we know that there exists $\delta > 0$ such that

$$\forall_{\substack{y_1 \in \left[y^*_{v,1}(0) - \delta,\, y^*_{v,1}(0) + \delta \right] \\ y_2 \in \left[y^*_{v,2}(0) - \delta,\, y^*_{v,2}(0) + \delta \right]}} \left| \frac{\partial^2 v_1}{\partial {y_1}^2}(y_1, y_2, 0) \right| > \frac{3\alpha}{4} \text{ and } \left| \frac{\partial v_1}{\partial y_1}(y_1, y_2, 0) \right| < \frac{\varepsilon\alpha}{4}. \tag{A2}$$

Let $\epsilon = \min\{\varepsilon, \delta\}$. We choose n_1 such that for any $n > n_1$ and for any $(y_1, y_2) \in \mathbf{R}^2_{>0}$, we have that $\left| \frac{\partial v_1}{\partial y_1}(y_1, y_2, 0) - \frac{\partial v_{1,n}}{\partial y_1}(y_1, y_2, 0) \right| < \frac{\epsilon\alpha}{2}$ (we can so write because $v_{1,n}$ tends to v_1 in our C^2-uniform convergence topology), and that $\left| y^*_{v_{2,n},2}(0) - y^*_{v,2}(0) \right| < \delta$ (we can so write because $y^*_{v_{2,n},2}(0)$ tends to $y^*_{v,2}(0)$, refer to part (a)). Thus, because $\frac{\partial v_{1,n}}{\partial y_1}(y^*_{v_{1,n},1}(0), y^*_{v_{2,n},2}(0), 0) = 0$,

$$\left| \frac{\partial v_1}{\partial y_1}(y^*_{v_{1,n},1}(0), y^*_{v_{2,n},2}(0), 0) \right| = \left| \frac{\partial v_1}{\partial y_1}(y^*_{v_{1,n},1}(0), y^*_{v_{2,n},2}(0), 0) - \frac{\partial v_{1,n}}{\partial y_1}(y^*_{v_{1,n},1}(0), y^*_{v_{2,n},2}(0), 0) \right| < \frac{\epsilon\alpha}{2} \le \frac{\varepsilon\alpha}{2}. \tag{A3}$$

Now, if we assume that $\left| y^*_{v_{1,n},1}(0) - y^*_{v,1}(0) \right| \ge \varepsilon$ for some $n > n_1$, then

$$\begin{aligned}
&\left| \frac{\partial v_1}{\partial y_1}(y^*_{v_{1,n},1}(0), y^*_{v_{2,n},2}(0), 0) \right| \\
&= \left| \frac{\partial v_1}{\partial y_1}(y^*_{v_{1,n},1}(0), y^*_{v_{2,n},2}(0), 0) - \frac{\partial v_1}{\partial y_1}(y^*_{v,1}(0), y^*_{v_{2,n},2}(0), 0) + \frac{\partial v_1}{\partial y_1}(y^*_{v,1}(0), y^*_{v_{2,n},2}(0), 0) \right| \\
&= \left| \int_{y^*_{v,1}(0)}^{y^*_{v_{1,n},1}(0)} \frac{\partial^2 v_1}{\partial y_1^2}(y_1, y^*_{v_{2,n},2}(0), 0) dy_1 + \frac{\partial v_1}{\partial y_1}(y^*_{v,1}(0), y^*_{v_{2,n},2}(0), 0) \right| \\
&\ge \left| \int_{y^*_{v,1}(0)}^{y^*_{v_{1,n},1}(0)} \frac{\partial^2 v_1}{\partial y_1^2}(y_1, y^*_{v_{2,n},2}(0), 0) dy_1 \right| - \left| \frac{\partial v_1}{\partial y_1}(y^*_{v,1}(0), y^*_{v_{2,n},2}(0), 0) \right|
\end{aligned}$$

$$\geq \min\left\{\left|\int_{y^*_{v,1}(0)-\varepsilon}^{y^*_{v,1}(0)} \frac{\partial^2 v_1}{\partial y_1^2}(y_1, y^*_{v_{2,n},2}(0),0)dy_1\right|, \left|\int_{y^*_{v,1}(0)}^{y^*_{v,1}(0)+\varepsilon} \frac{\partial^2 v_1}{\partial {y_1}^2}(y_1, y^*_{v_{2,n},2}(0),0)dy_1\right|\right\}$$

$$-\left|\frac{\partial v_1}{\partial y_1}(y^*_{v,1}(0), y^*_{v_{2,n},2}(0),0)\right| \geq \min\left\{\left|\int_{y^*_{v,1}(0)-\varepsilon}^{y^*_{v,1}(0)} \frac{3\alpha}{4}dy_1\right|, \left|\int_{y^*_{v,1}(0)}^{y^*_{v,1}(0)+\varepsilon} \frac{3\alpha}{4}dy_1\right|\right\} - \frac{\varepsilon\alpha}{4}$$

$$\geq \frac{3\varepsilon\alpha}{4} - \frac{\varepsilon\alpha}{4} = \frac{\varepsilon\alpha}{2},$$

where the second to last inequality holds due to (A2). This clearly contradicts (A3). Therefore, $\left|y^*_{v_{1,n},1}(0) - y^*_{v,1}(0)\right| < \varepsilon$ for any $n > n_1$. Q.E.D.

Lemma 3.

For any $v \in N_2$ and any sequence $(v_n)_{n=0}^{\infty} = ((v_{1,n}, v_{2,n}))_{n=0}^{\infty}$ such that $v_n \xrightarrow[n\to\infty]{} v$, we have that

(a) $y^*_{v_{2,n},2}{}'(0) \xrightarrow[n\to\infty]{} y^*_{v,2}{}'(0)$;

(b) $y^*_{v_{1,n},1}{}'(0) \xrightarrow[n\to\infty]{} y^*_{v,1}{}'(0)$.

Proof.

As in the case of Lemma 2, we need to prove only part (b) because the proof of part (a) is analogous.

Let ò>0. We seek to show that there exists an M such that for any $n > M$ we have $\left|y^*_{v_{1,n},1}{}'(0) - y^*_{v,1}{}'(0)\right| <$ ò.

For the sake of the brevity of the notation, we define functions $v_1^{yy}(y_1,y_2) \equiv \dfrac{\frac{\partial^2 v_1}{\partial y_1 \partial y_2}(y_1,y_2,0)}{\frac{\partial^2 v_1}{\partial {y_1}^2}(y_1,y_2,0)}$, $v_1^{y\tau}(y_1,y_2) \equiv \dfrac{\frac{\partial^2 v_1}{\partial y_1 \partial \tau}(y_1,y_2,0)}{\frac{\partial^2 v_1}{\partial {y_1}^2}(y_1,y_2,0)}$ (and we do likewise for the function $v_{1,n}(y_1,y_2,0)$, thereby obtaining functions $v_{1,n}^{yy}(y_1,y_2)$ and $v_{1,n}^{y\tau}(y_1,y_2)$); and in $y^*_{v,1}(0)$, $y^*_{v,2}(0)$, $y^*_{v,2}{}'(0)$, $y^*_{v_{1,n},1}(0)$, $y^*_{v_{2,n},2}(0)$, and $y^*_{v_{2,n},2}{}'(0)$ we omit the zero as an argument. Because $\frac{\partial^2 v_1}{\partial y_1^2}(y_1,y_2,0) < 0$, which follows from the fact that $(v_1,v_2) \in N_2$, we have that

$\dfrac{\partial^2 v_{1,n}}{\partial y_1^2}(y_1, y_2, 0) < 0$ for any $(y_1, y_2) \in \mathbf{R}^2_{>0}$. Thus, the functions $v_1^{yy}(y_1, y_2)$, $v_1^{y\tau}(y_1, y_2)$, $v_{1,n}^{yy}(y_1, y_2)$, and $v_{1,n}^{y\tau}(y_1, y_2)$ are well-defined and continuous for $(y_1, y_2) \in \mathbf{R}^2_{>0}$ and, due to the convergence of $(v_n)_{n=1}^{+\infty}$, we have that $(v_{1,n}^{yy})_{n=1}^{+\infty}$ tends to v_1^{yy}, and that $(v_{1,n}^{y\tau})_{n=1}^{+\infty}$ tends to $v_1^{y\tau}$ (with respect to C^2-uniform convergence topology) as n tends to infinity.

The implicit function theorem implies (recall (A1)) that

$$
\begin{aligned}
y_{v,1}^{*}{}'(0) &= -\frac{\dfrac{\partial^2 v_1}{\partial y_1 \partial y_2}\left(y_{v,1}^*(0), y_{v,2}^*(0), 0\right) y_v^{*\prime}(0)}{\dfrac{\partial^2 v_1}{\partial y_1{}^2}\left(y_{v,1}^*(0), y_{v,2}^*(0), 0\right)} - \frac{\dfrac{\partial^2 v_1}{\partial y_1 \partial \tau}\left(y_{v,1}^*(0), y_{v,2}^*(0), 0\right)}{\dfrac{\partial^2 v_1}{\partial y_1{}^2}\left(y_{v,1}^*(0), y_{v,2}^*(0), 0\right)} \\
&= -v_1^{yy}(y_{v,1}^*, y_{v,2}^*) y_{v,2}^*{}' - v_1^{y\tau}(y_{v,1}^*, y_{v,2}^*).
\end{aligned}
\tag{A4}
$$

Let $v_1^{yy}(y_{v,1}^*, y_{v,2}^*) = \beta$. If $\beta \neq 0$, then due to the convergence of $(y_{v_{2,n},2}^*{}')_{n=1}^{\infty}$ (cf. part (a) of the lemma), there exists n_0 such that for any $n > n_0$ we have that $\left| y_{v,2}^*{}' - y_{v_{2,n},2}^*{}' \right| < \dfrac{\epsilon}{5|\beta|}$. If $\beta = 0$, then n_0 can be any.

Let $y_{v,2}^*{}' = \gamma$. If $\gamma \neq 0$ then, due to the convergence of $(y_{v_{2,n},2}^*{}')_{n=1}^{\infty}$ and $(v_{1,n}^{yy})_{n=1}^{+\infty}$, there exists n_1 such that for any $n > n_1$ we have that $\left| y_{v,2}^*{}' - y_{v_{2,n},2}^*{}' \right| < |\gamma|$ and, thus, that $\left| y_{v_{2,n},2}^*{}' \right| < 2|\gamma|$; that $\left| v_1^{yy}(y_1, y_2) - v_{1,n}^{yy}(y_1, y_2) \right| < \dfrac{\epsilon}{10|\gamma|}$ for any $(y_1, y_2) \in \mathbf{R}^2_{>0}$; and, due to the continuity of $v_1^{yy}(y_1, y_2)$, that there exists $\delta_0 > 0$ such that

$$
\forall_{\substack{y_1 \in \left[y_{v,1}^* - \delta_0,\, y_{v,1}^* + \delta_0 \right] \\ y_2 \in \left[y_{v,2}^* - \delta_0,\, y_{v,2}^* + \delta_0 \right]}} \left| v_1^{yy}(y_{v,1}^*, y_{v,2}^*) - v_1^{yy}(y_1, y_2) \right| < \frac{\epsilon}{10|\gamma|}.
$$

If $\gamma = 0$, there exists n_1 such that for any $n > n_1$ we have that $\left| y_{v_{2,n},2}^*{}' \right| < \sqrt{\epsilon}$ and $\left| v_1^{yy}(y_1, y_2) - v_{1,n}^{yy}(y_1, y_2) \right| < \dfrac{\sqrt{\epsilon}}{5}$ for any $(y_1, y_2) \in \mathbf{R}^2_{>0}$, and there exists $\delta_0 > 0$ such that

$$\forall_{\substack{y_1 \in \left[y^*_{v,1} - \delta_0 ,\, y^*_{v,1} + \delta_0 \right] \\ y_2 \in \left[y^*_{v,2} - \delta_0 ,\, y^*_{v,2} + \delta_0 \right]}} \left| v_1^{yy}(y^*_{v,1}, y^*_{v,2}) - v_1^{yy}(y_1, y_2) \right| < \frac{\sqrt{\varepsilon}}{5}.$$

Additionally, due to the continuity of $v_1^{y\tau}(y_1, y_2)$ there exists $\delta_1 > 0$ such that

$$\forall_{\substack{y_1 \in \left[y^*_{v,1} - \delta_1 ,\, y^*_{v,1} + \delta_1 \right] \\ y_2 \in \left[y^*_{v,2} - \delta_1 ,\, y^*_{v,2} + \delta_1 \right]}} \left| v_1^{y\tau}(y^*_{v,1}, y^*_{v,2}) - v_1^{y\tau}(y_1, y_2) \right| < \frac{\varepsilon}{5},$$

and due to the convergence of $(v_{1,n}^{y\tau})_{n=1}^{+\infty}$ there exists n_2 such that for any $n > n_2$ we have that $\left| v_1^{y\tau}(y_1, y_2) - v_{1,n}^{y\tau}(y_1, y_2) \right| < \frac{\varepsilon}{5}$ for any $(y_1, y_2) \in \mathbf{R}^2_{>0}$.

Let $\delta = \min\{\delta_0, \delta_1\}$. Due to the convergence of $(y^*_{v_{1,n},1})_{n=1}^{\infty}$ and $(y^*_{v_{2,n},2})_{n=1}^{\infty}$, there exists n_3 such that for any $n > n_3$ we have that $\left| y^*_{v,1} - y^*_{v_{1,n},1} \right| < \delta$ and that $\left| y^*_{v,2} - y^*_{v_{2,n},2} \right| < \delta$.

Finally, for $M = \max\{n_0, n_1, n_2, n_3\}$, applying the implicit function theorem to both $y^*_{v_{1,n},1}$ and $y^*_{v,1}$ (cf. (A4)), we have that

$$\begin{aligned}
&\left| {y^*_{v_{1,n},1}}' - {y^*_{v,1}}' \right| = \left| v_1^{yy}(y^*_{v,1}, y^*_{v,2}) {y^*_{v,2}}' + v_1^{y\tau}(y^*_{v,1}, y^*_{v,2}) - v_{1,n}^{yy}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) {y^*_{v_{2,n},2}}' - v_{1,n}^{y\tau}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) \right| \\
&\le \left| v_1^{yy}(y^*_{v,1}, y^*_{v,2}) {y^*_{v,2}}' - v_1^{yy}(y^*_{v,1}, y^*_{v,2}) {y^*_{v_{2,n},2}}' + v_1^{yy}(y^*_{v,1}, y^*_{v,2}) {y^*_{v_{2,n},2}}' - v_1^{yy}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) {y^*_{v_{2,n},2}}' \right. \\
&\left. + v_1^{yy}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) {y^*_{v_{2,n},2}}' - v_{1,n}^{yy}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) {y^*_{v_{2,n},2}}' \right| \\
&+ \left| v_1^{y\tau}(y^*_{v,1}, y^*_{v,2}) - v_1^{y\tau}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) + v_1^{y\tau}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) - v_{1,n}^{y\tau}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) \right| \\
&\le \left| {y^*_{v,2}}' - {y^*_{v_{2,n},2}}' \right| \left| v_1^{yy}(y^*_{v,1}, y^*_{v,2}) \right| + \left| {y^*_{v_{2,n},2}}' \right| \left| v_1^{yy}(y^*_{v,1}, y^*_{v,2}) - v_1^{yy}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) \right| \\
&+ \left| {y^*_{v_{2,n},2}}' \right| \left| v_1^{yy}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) - v_{1,n}^{yy}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) \right| \\
&+ \left| v_1^{y\tau}(y^*_{v,1}, y^*_{v,2}) - v_1^{y\tau}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) \right| + \left| v_1^{y\tau}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) - v_{1,n}^{y\tau}(y^*_{v_{1,n},1}, y^*_{v_{2,n},2}) \right| \\
&< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon. \text{ Q.E.D.}
\end{aligned}$$

Appendix B: A transfer financed by a proportional tax levied on the income of the richer individual

In Section 2, we considered a lump-sum transfer that did not reduce the richer individual's incentive to exert effort. It is natural to ponder whether the outcome of a rich-to-poor transfer exacerbating income inequality and decreasing Sen's social welfare arises also when the transfer is enacted via a proportional tax of the income of the richer individual.

We consider the individuals' utility specifications as in Section 2, albeit with one difference: the amount taken from the richer individual 2 and transferred to the poorer individual 1 is a proportion of the pre-transfer income of the richer individual 2, and is equal to ty_2, where $t \in [0,1)$ is small enough so that the taxing of the richer individual's income does not reverse the ordering of the incomes of the two individuals.

Using the utility functions (1) and (2) with parameters that satisfy (5), we know that without a transfer ($t = 0$), the optimal levels of effort and incomes are the same as those calculated in (7). When we apply a proportional (marginal) transfer $t > 0$, the post-transfer incomes of the two individuals are

$$\begin{aligned} x_1 &= y_1 + ty_2, \\ x_2 &= y_2 - ty_2. \end{aligned}$$

Reenacting the utility maximization procedure of Section 2, we obtain the following optimal levels of effort / pre-transfer incomes as functions of t

$$\begin{aligned} y_1^*(t) &= \frac{a(1-t)}{b}(1-2t) - \frac{f-d}{2g}, \\ y_2^*(t) &= \frac{a(1-t)}{b}, \end{aligned} \tag{B1}$$

which yield the following optimal levels of consumption / post-transfer incomes

$$\begin{aligned} x_1^*(t) &= y_1^*(t) + ty_2^*(t) = \frac{a(1-t)^2}{b} - \frac{f-d}{2g}, \\ x_2^*(t) &= y_2^*(t) - ty_2^*(t) = \frac{a(1-t)^2}{b}. \end{aligned} \tag{B2}$$

The Gini index defined on the post-transfer incomes and expressed as a function of the transfer t is

$$G(t)=\frac{x_2^*(t)-x_1^*(t)}{2(x_1^*(t)+x_2^*(t))}=\frac{\dfrac{a(1-t)^2}{b}+\dfrac{f-d}{2g}-\dfrac{a(1-t)^2}{b}}{2\left[2\dfrac{a(1-t)^2}{b}-\dfrac{f-d}{2g}\right]}=\frac{b(f-d)}{8ga(1-t)^2-2b(f-d)}. \tag{B3}$$

Combining (5) and (B3) we get that

$$G'(0)=\frac{d}{dt}\left[\frac{b(f-d)}{8ga(1-t)^2-2b(f-d)}\right]\Bigg|_{t=0}=\frac{4abg(f-d)(1-t)}{\left[4ga(1-t)^2-b(f-d)\right]^2}\Bigg|_{t=0}=\frac{4abg(f-d)}{\left[4ga-b(f-d)\right]^2}>0,$$

that is, a (marginal) proportional transfer t also increases the Gini index.

From the second line of (B1) we see that $y_2^*(t)$ is a decreasing function of t, that is, in this setting, the transfer indeed reduces the incentive of individual 2 to exert effort. Additionally, from (B2) we see that both $x_1^*(t)$ and $x_2^*(t)$ are decreasing functions of t. In conjunction with an increase of the Gini index, this implies that a (marginal) proportional transfer t decreases Sen's social welfare as well.

Appendix C: Proof of part 1 of Claim 2

For the parameters d, g, f, a_i, h_i, b_i defined in (19) and (20), the utility functions $U_i : \mathbf{R}^3 \to \mathbf{R}$, $U_i \in C^2(\mathbf{R}^3)$ of individuals $i \in \{2,3,\ldots,n\}$ are

$$U_i(x_i, y_i, r_i) = 2^i x_i - 2^i r_i - y_i^2,$$

and the utility function of the individual 1 is

$$U_1(x_1, y_1, r_1) = x_1 - \frac{n^3}{2(n-1)\left[2^n(n+1) - n(n+3)\right]} r_1^2 - 2y_1. \tag{C1}$$

We first derive the optimal effort levels / incomes of individuals in the absence of a transfer, that is, for $x_i = y_i$. Defining $S_i = \sum_{j: y_j > y_i} y_j$ for $i \in \{1,2,\ldots,n\}$, for individuals $i \in \{2,3,\ldots,n\}$ we have that

$$U_i(y_i, y_i, r_i) = 2^i y_i - 2^i \cdot \frac{S_i - (n-k_i) y_i}{n} - y_i^2, \tag{C2}$$

where k_i denotes the position of individual i in the pre-transfer ordering of incomes (such that $k_i = n$ if individual i is the richest, and $k_i = 1$ if individual i is the poorest). From the first order condition for maximum of the utility functions in (C2) we have that the optimal income / effort level y_i^*, $i \in \{2,3,\ldots,n\}$, is

$$y_i^* = 2^i - \frac{k_i}{n} 2^{i-1} \equiv \gamma_i(k_i). \tag{C3}$$

From (C3) it follows that for any $i \in \{2,3,\ldots,n\}$

$$\gamma_i(k_i) \in [2^{i-1}, 2^i), \tag{C4}$$

thus,

$$\gamma_n(k_n) > \gamma_{n-1}(k_{n-1}) > \ldots > \gamma_3(k_3) > \gamma_2(k_2)$$

and

$$n \geq k_n > k_{n-1} > \ldots > k_3 > k_2 \geq 1. \tag{C5}$$

In particular, when $i > k_1$, then $k_i = i$.

We next show that for individual 1 whose utility function is given in (C1), the optimal level of effort / income in the absence of a transfer is such that $y_1^* = 1$ and, thus, as follows from (C4), this individual will be the poorest in the population; that is $k_1 = 1$. We have that

$$U_1(y_1, y_1, r_1) = y_1 - \frac{n^3}{2(n-1)\left[2^n(n+1) - n(n+3)\right]} r_1^2 - 2y_1$$
$$= y_1 - \frac{n^3}{2(n-1)\left[2^n(n+1) - n(n+3)\right]} \left[\frac{S_1 - (n-k_1)y_1}{n}\right]^2 - 2y_1,$$

and, thus,

$$\frac{dU_1(y_1, y_1, r_1)}{dy_1} = -1 + \frac{n(n-k_1)\left[S_1 - (n-k_1)y_1\right]}{(n-1)\left[2^n(n+1) - n(n+3)\right]}. \tag{C6}$$

For $k_1 = n$, the derivative in (C6) is negative (in particular, it is not zero) so that for individual 1 to choose y_1 so high that $k_1 = n$ is not optimal. Then, surely, $k_1 < n$. By solving the first order condition $\frac{dU_1(y_1, y_1, r_1)}{dy_1} = 0$, we get that

$$y_1^* = \frac{n(n-k_1)S_1 - 2^n(n^2-1) + n(n-1)(n+3)}{(n-k_1)^2 n} \equiv \gamma_1(S_1, k_1) \tag{C7}$$

and, therefore,

$$\frac{\partial \gamma_1(S_1, k_1)}{\partial S_1} = \frac{1}{n-k_1} > 0,$$

namely $\gamma_1(S_1, k_1)$ is an increasing function of S_1. We can rewrite S_1 as $S_1 = \sum_{i=k_1+1}^{n} \gamma_i(i) = \sum_{i=k_1+1}^{n} \left(2^i - \frac{i}{n} 2^{i-1}\right)$ because, as already noted, for $i > k_1$ we have that $k_i = i$, and $\gamma_i(i) = 2^i - \frac{i}{n} 2^{i-1}$. On defining

$$S \equiv \sum_{i=2}^{n} \gamma_i(i) = \sum_{i=2}^{n} \left(2^i - \frac{i}{n} 2^{i-1}\right), \tag{C8}$$

we get that $S \geq S_1$ and, consequently, that $\gamma_1(S,k_1) \geq \gamma_1(S_1,k_1)$. In addition, we have that the two terms on the right hand side of (C8) can be expressed, respectively, as

$$\sum_{i=2}^{n} 2^i = 2^{n+1} - 4, \tag{C9}$$

and as

$$\begin{aligned} \sum_{i=2}^{n} \frac{i}{n} 2^{i-1} &= \frac{1}{n}\left(2\sum_{i=1}^{n-1} 2^i + \sum_{i=2}^{n-1} 2^i + \sum_{i=3}^{n-1} 2^i + \ldots + \sum_{i=n-2}^{n-1} 2^i + 2^{n-1} \right) \\ &= \frac{1}{n}\left[2(2^n - 2) + (2^n - 4) + (2^n - 8) + \ldots + (2^n - 2^{n-2}) + (2^n - 2^{n-1}) \right] \\ &= \frac{1}{n}\left(n \cdot 2^n - 4 - \sum_{i=2}^{n-1} 2^i \right) = \frac{1}{n}\left[n \cdot 2^n - 4 - (2^n - 4) \right] = \frac{n-1}{n} 2^n. \end{aligned} \tag{C10}$$

Insertion of (C9) and (C10) in (C8) yields

$$S = \sum_{i=2}^{n} \left(2^i - 2^{i-1} \frac{i}{n} \right) = 2^{n+1} - 4 - \frac{n-1}{n} 2^n = \frac{n+1}{n} 2^n - 4, \tag{C11}$$

and, thus, from the definition of $\gamma_1(S_1,k_1)$ in (C7),

$$\begin{aligned} \gamma_1(S,k_1) &= \frac{n(n-k_1)\left[2^n (n+1)/n - 4 \right] - 2^n (n^2 - 1) + n(n-1)(n+3)}{(n-k_1)^2 n} \\ &= \frac{n\left[(n+1)(n-3) + 4k_1 \right] - 2^n (n+1)(k_1 - 1)}{(n-k_1)^2 n}. \end{aligned} \tag{C12}$$

To help investigate the properties of the function $\gamma_1(S,k_1)$, we define an auxiliary function

$$f(z) = \frac{n\left[(n+1)(n-3) + 4z \right] - 2^n (n+1)(z-1)}{(n-z)^2 n} \tag{C13}$$

for $z \in [1,n] \subset \mathbf{R}$. Then, from comparing (C12) and (C13), we get that $\gamma_1(S,k_1) = f(k_1)$. We have that

$$
\begin{aligned}
f'(z) &= -\frac{2^n(n^2+nz+z-n-2)-2n(n^2+2z-3)}{(n-z)^3 n} \\
&= -\frac{2^n\left[n^2+2z-3+(n-1)(z-1)\right]-2n(n^2+2z-3)}{(n-z)^3 n} \\
&= -\frac{(2^n-2n)(n^2+2z-3)+2^n(n-1)(z-1)}{(n-z)^3 n} < 0
\end{aligned}
$$

for any $n>2$ and $z \in [1,n]$ and, thus,

$$y_1^* \le \gamma_1(S,k_1) = f(k_1) \le f(1) = \gamma_1(S,1) = 1 < \gamma_2(k_2)$$

for any $k_2 \in \{1,...,n\}$. Therefore, the optimal income of individual 1 is lower than the income of individual 2 and, thus, individual 1 is the poorest. By enlisting (C5) we have that $k_i = i$ for each $i \in \{1,2,...,n\}$, and that, without a transfer, the optimal effort levels / income in the population are

$$
\begin{aligned}
y_1^* &= \gamma_1(S,1) = 1, \\
y_i^* &= \gamma_i(i) = 2^i - \frac{i}{n}2^{i-1}
\end{aligned}
$$

for $i \in \{2,...,n\}$, such that $y_1^* < y_2^* < ... < y_n^*$. Because without a transfer we have that $x_i^* = y_i^*$, then

$$x_1^* < x_2^* < ... < x_n^*,$$

which completes the proof of part 1 of the claim. Q.E.D.

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