Cattle - Feeding

UNIVERSITY COLLEGE OF WALES

ABERYSTWYTH

DEPARTMENT OF AGRICULTURAL ECONOMICS

THE JOINT PRODUCTION FUNCTION OF STARCH EQUIVALENT AND PROTEIN EQUIVALENT IN FEEDING DAIRY COWS

BY

MICHAEL B. JAWETZ

PRICE: 10S. 0D.

THE JOINT PRODUCTION FUNCTION
OF
STARCH EQUIVALENT AND PROTEIN EQUIVALENT
IN
FEEDING DAIRY COWS

BY

MICHAEL B. JAWETZ

1961

CONTENTS

INDEX OF TABLES AND FIGURES

ACKNOWLEDGMENTS

The author is deeply indebted to Dr. A. Durrant, Lecturer in Agricultural Botany at the University College of Wales, Aberystwyth, for his most valuable advice on the statistical problems of this study, for reading the chapters on statistics and for a critical discussion of them.

He would also like to thank Professor E. F. Nash of the Department of Agricultural Economics, Graham Hallett, Lecturer in this Department, and Principal K. P. Riley of the East Riding Institute of Agriculture, Beverley, East Yorks., for their comments on several points; and Miss E. Johnson, the Department's librarian, for her careful editing of the original draft—a laborious operation without which it would hardly have been readable—and for correcting the proofs.

Last, but not least, he would like to express his gratitude to two other members of this Department: Mr. B. C. Jones for his assistance with the layout of a number of tables and for supervising the technical details of production, and Miss Gwenda Jones for making the final drawings of the diagrams and their lettering.

Aberystwyth, July, 1960

Part One

INTRODUCTION

The present concepts underlying the study of input-output relationships in feeding dairy cows were outlined by the writer in an article[1] in which the literature on the subject was critically reviewed. They are mostly concerned with the energy requirements of milk output. It was shown in that article that the production requirements of starch equivalent (SE) were proportionately greater at high levels of yield than at low levels, and that the difference in SE requirements per gallon as between low and high yields was of the order of 60—100 per cent.

Much less is known about the production requirements of protein, and what little evidence exists appears to be contradictory. Mackintosh[2] conducted experiments on farms, for 2 years, in which several hundred cows each were fed rations containing either 0.6 lb. or 0.4 lb. PE per gallon of milk produced. He concluded that the lower protein ration was as effective as the higher. Apart, however, from the tendency on some farms to feed in excess of maintenance requirements, variations owing to breed of cow might have biased the two groups differently and it also appears that the inputs of SE were low: about 2.1 lb. SE per gallon were fed in the first year and less than 2.3 lb. in the second. This might have been the factor which limited the output of the cows which were fed the higher protein rations. A few years ago the writer presented estimates of protein requirements at various levels of yield[3] which showed a similar trend to those of SE; but the difference per gallon as between low and high yields appeared to be only 40 per cent. Later, Holmes and others carried out experiments at the Hannah Dairy Research Institute[4] and concluded that, with a given quantity of SE in the concentrate ration, an increase from 10 to 15 and 17 per cent of the digestible crude protein (DCP) content in the concentrates "led to no statistically significant differences in milk yield". But with a given content of DCP—16 per cent—and with SE in the concentrate increasing from 59 to 67 and 75, the two rations with higher SE content "resulted in significantly more milk than the low SE ration". Moreover, in farming practice there is an apparent contradiction between some protagonists of self-sufficiency, who succeed in obtaining relatively high yields from home-grown foods alone, and those who depend heavily on purchased concentrates. This problem is partly linked with economies

(1) Jawetz, M. B. *Dairy Science Abstracts*, 18, 1 (Jan., 1956).

(2) Mackintosh, J., Paper read at Belfast, July, 1939. National Institute for Research in Dairying, University of Reading.

(3) Jawetz, M. B., *Agriculture* 60, 56—61 (1953).

(4) Holmes, W., Waite, R., MacLusky, S., and Watson, J. N. *The Journal of Dairy Research*, 23, 1 (Feb., 1956).

of scale,[1] but it hinges on the rate of substitution—if indeed such substitution is possible at all—between SE and PE, of which nothing has been known so far.

The present study was well advanced when some more work on the subject came to the writer's notice. Burt[2] compared experiments giving estimates of responses to feed. He also reported on his own experiments. These have been further discussed by Orton,[3] who studied the problem both by examining experimental results and by analysing records from the National Milk Cost Investigation together with some Danish cost account records. He obtained some conflicting answers, owing in the writer's opinion to the fact that records for "6 winter months" were used, including considerable inputs of nutrients from grazing which were ignored. Gardner[4] analysed the published results from 5 bull progeny tests. He concluded that "the current standard of feeding (SE) should be regarded as a minimum which may need raising by at least one-third to obtain potential yield from certain cows". But these progeny tests had one great disadvantage: they were conducted on first-calf heifers and such immature animals bring in yet another unknown element, namely the nutrient requirements for growth of bone and muscle, as distinct from growth of fat tissues. Since this cannot be measured by any known method, such material is not suitable for the definition of input-output relationships. Actually some of Gardner's functions showed increasing returns from inputs. It is likely that this was due to the fact that high-fed animals could divert a larger proportion of the input to the production of milk—after the needs for growth and perhaps some fat production had been satisfied—whereas low-fed animals could "spare" less for their main task.

Broster and others[5] investigated the influence of steaming up on the subsequent input-output relationship. His experiments, too, were conducted on first-calf heifers, a fact which obscures some of his quantitative measurements. Nevertheless he was able to show that relatively small quantities of concentrates, when used for steaming up, had similar effects on yields to those of much larger amounts fed during the earlier stages of lactation.

Another factor obscuring the relationship of input and output is the difference between the food-converting capacities of individual dairy cows. Conventional feeding standards recognise variations in production requirements as between various breeds of cow, in so far as they are associated with the fat content of their milk. Jensen *et al.*[6] found that such differences exist between the TDN requirements of "good" cows and "poor" ones, and the writer found confirmation of this trend (on the basis of SE) by

(1) Substitution is possible within a fairly wide range between the number of cows stocked on a farm, home-grown foods produced, concentrates bought and the level of yield per cow.

(2) Burt, A. W. A., *Dairy Science Abstracts*, 19, 435, 1957.

(3) Orton, J. F., *The Farm Economist*, IX, 1 and 2, 1958.

(4) Gardner, T. W., *Journal of Agricultural Economics*, XIII, 4, Jan., 1960.

(5) Broster, W. H., Ridler, B., and Foot, A. S. *The Journal of Dairy Research*, 25, 3, Oct., 1958.

(6) Jensen, E., Klein, J. W., Ranchenstein, E., Woodward, T. E., and Smith, R. M. U.S. Department of Agriculture Tech. Bull. 815 (1942).

re-calculating some of the data presented by Yates *et al.*[1] The writer also drew attention to the fact that, for comparable outputs, heavier cows appear to have lower production requirements of SE than lighter ones.[2]

Before this study could be completed, Blaxter[3] penetrated a number of the problems involved—at least in terms of SE alone. For 5—6 years he conducted experiments which covered whole lactations. He found that 'low-yielding cows gave small responses of less than 0.5 lb. milk per lb. SE where high yielders gave over 2.2 lb. milk per lb. SE. Extension of these results to experiments within lactations showed that what applied to the lactation as a whole applied also at different stages of lactation. This relationship entails a relationship between food intake and milk yield which is curvilinear, the constants of the equations varying with the productive capacity of the animal . . . . The response to food was greater for the higher yielders . . . . Further investigation showed that the law of diminishing returns applied to these responses . . . ." Unfortunately the full details of his work are not yet available, but the little that is already known of it discloses an understanding of the subject far in advance of that shown by any other writer so far.

Some of the contradictions and obscurities mentioned above largely explain why advisory workers and nutrition experts still adhere to the conventional feeding standards, in spite of the existing evidence in favour of varying scales. But there is also another good reason for this, namely that the available input-output information applies to long periods— e.g. one full lactation in the American work and 2 to 3 months in the U.K. and Danish data analysed by Yates *et al.*[4] Its practical application is therefore very limited. The information would be adequate for the budgeting of the feed requirements of a herd for a whole season, in order to obtain the most economic input-output relationship when costs and prices were taken into account. But it cannot disclose *how* the feed should be distributed among the cows in that herd during the period concerned, since no information is given as to the actual distribution which resulted in the tabulated averages at different levels of lactation yield or as to the variation in the lactation yield which might have resulted if the recorded total feed input had been differently distributed through the lactation. The functional relationships between yield and input become extremely complex when the daily results are taken into account, since the yield obtained on any one day must be assumed to be affected not only by the current input of feed but by the level of feeding which has been given over the period of the lcatation which has already elapsed; conversely the feed input given on any one day plays some part in determining future as well as current yields. Some assumption is necessary, however, as to the distribution of the feed input over time, and there is presumably a certain optimal distribution for which a serviceable definition would perhaps be that it is a

(1) Jawetz (from Yates' Danish experiment B 136), *Dairy Science Abstracts* (Jan., 1956).

(2) Jawetz, ibidem.

(3) Blaxter, K. L., "Scientific Principles of Feeding Livestock." Proceedings of a Conference held at Brighton 11—13 Nov., 1958. *Dairy Cows*, Discussion, pp. 40, 41.

(4) Yates, F., Boyd, D. A., and Pettit, S. H. N., *Journal of Agricultural Science*, 32, 428 (1942).

distribution which maximises the aggregate lactation yield from the total quantity of feed given over the whole of the lactation to a particular cow. This optimal distribution is likely to result in a daily yield/input ratio which varies from one period of the lactation to another. Thus a cow yielding 1,000 gallons over 300 days would have an average yield of 3.33 gallons obtained at a daily average input of perhaps 12.3 lb. SE, i.e. 3.7 lb. SE per gallon. This point is illustrated in Figure I.

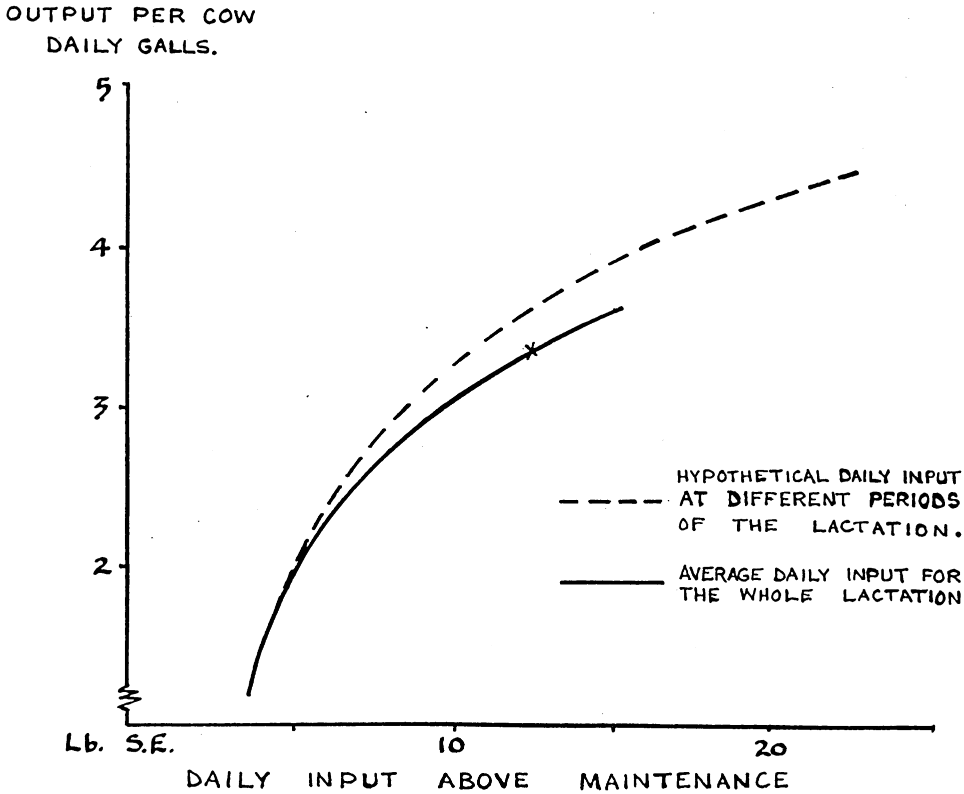

FIGURE I

Hypothetical Break-down of Average Lactation Input of SE above Maintenance into "Actual" Daily Input

The continuous curve represents average yields of 2.33 to 3.33 gallons a day for the whole lactation.[1] The interrupted curve is a hypothetical day-to-day output and input curve for cows with assumed input yields of 1,000 gallons, i.e. an average of 3.33 gallons a day and an average consumption over the whole period of 12.3 lb. SE per day. Each point on the interrupted curve relates to a different day and records the input and output

(1) The continuous curve is that of low yielding cows from the Jensen-Woodward project, recalculated by the author in terms of SE. See op. cit. in *Dairy Science Abstracts*, pp. 9 and 10 (Table IV).

of that day. Although this curve is, as it were, a break-down of the single point (marked X) on the continuous curve which represents the performance of the cow in question over the lactation, it does not pass through this point. At the stage of the lactation at which an actual daily yield of 3.33 gallons is obtained, the input of SE is only 11 lb. The process of constructing the interrupted curve could of course be repeated for any other point on the continuous curve.

Only recordings of daily, or at least weekly, inputs and outputs in a special experiment would produce data from which such actual daily curves could be obtained with full statistical accuracy. Unfortunately such experiments are among the most costly that can be made. Nevertheless before feeding standards which would take account of the diminishing biological productivity of dairy cows can become a practical proposition, a solution must be found to this problem.

The Purpose of This Study

It was originally intended to discover whether or not the findings of the Jensen-Woodward project and of the work of Yates and others would be confirmed by an analysis of the feeding records of British commercial dairy herds; and what variations in management, for example in size of herd, proportion of winter milk, types and proportions of bulky foods, might influence the input-output relationships.

As the analysis progressed a set of experimental data could be included in the study. Several lines of approach were abandoned or altered in view of the findings obtained. Finally it became possible to put forward two new hypotheses, namely:

(*a*) that cows of different breeds tend to have different conversion ratios of feed into milk, and

(*b*) that there is an interrelationship between the inputs of SE and PE and yield and that the ratio of SE to PE is different at various levels of yield.

In spite of the defects of the available data and the fact that the number of observations was uncomfortably small, the above hypotheses appeared to be sufficiently proven to form the basis of a complete theory of feeding dairy cows for production. Confirmation by especially designed and properly analysed experiments is, however, needed; and these experiments could also result in refinements of the hypotheses.

True and Hybrid Input-Output Functions

Apart from the greater accuracy of recording, there is an important difference between experimental data and those obtained in an investigation conducted in the field. Experiments are conducted under conditions of "constant" management and represent the situation prevailing in one firm, farm or herd. From them can be derived "true functions", from which "true" marginal curves can be determined. Although their aura of determinancy cannot fail in its appeal to the scientific worker, the applied value of such functions is diminished by at least two factors: in the first place, no exact repetition is possible in biological or economic experiments, owing to the variability of the human or animal element; in the second, practical conditions are never quite the same as those in which experiments are conducted, and experimental results—*ceteris paribus*—are hardly ever expected to be achieved in practice. On the other hand, field data implicitly represent varying management and conditions. Even if entrepreneurs co-operating in a survey consented to alter their inputs from year to year in order to provide intra-firm data the resulting time series would not represent true functions, since the cows would not be the same, and the inputs influenced by one season (in this context it might be the quality of grass or hay) would condition the input-output relationship in the next season to an extent which is beyond the reach of managerial intervention, and not measurable.

The only input-output functions which can be derived from experimental data are not true functions—i.e. those of a single firm, representing various points on the same curve—but "hybrid" functions, representing the positions of many firms each operating at one given point on their own individual curve. The underlying assumption is that if these firms applied similar inputs (or aimed at similar outputs) then their position on their input-output curve would be similar to that of other firms (farms) with comparable inputs or outputs. If they operated plants (herds) whose units (cows) had comparable productive capacity, and used sufficient managerial and technical skill, this assumption would be correct and the hybrid function as good as a true function. In practice, however, these factors vary from farm to farm. Hybrid functions, therefore, are often decried as unreliable and strongly biased in favour of well managed farms. This need not be the case at all; but even if a hybrid function is known to have some bias, related to the quality of the management, it can still provide a worthwhile approximation on condition that the nature and order of magnitude of the bias is taken into account.

In the context of this (a physical production function) study, the economic elements of herd management are irrelevant; only the technical elements need be considered. These can be dissected into the following components: (*a*) the choice of cows (units of "plant"), (*b*) the tending of the herd, (*c*) the degree of skill displayed in producing high-quality foods on the farm, (*d*) the input of feed in the preceding lactation and (*e*) the balancing and rationing of the current input of feed.

If all or most of the above five factors could be rated highly for high-output herds and low for low output ones, the resulting hybrid function would indeed be heavily biased. (Even such a function would be nearer the truth than the conventional feeding standard and more useful in application. It would permit a statement like this: "you could increase your output to X gallons by feeding Y feed (in any case more than indicated by conventional standards) *if* you improved the quality of your cows and that of your home-grown foods; and, if you are somewhat disappointed in the first lactation, you are likely to achieve the target in the second year of such management". Advice in this form is in fact given on conventional standards and it is sound, although the results tend to fall short of the target). But the existing evidence shows that only the current input of foods tends to be directly associated with output. High outputs are often forced from "poor" cows and still more often good cows are fed far short of their potential capacity. Good herd managers at some given time often have indifferent cows, owing to shortage of capital or erroneous breeding policies in the past; and the converse is frequently the case with indifferent herd managers. Similarly, high-output herds may be tended indifferently (poor housing is often the main handicap) and *vice versa*. The skill of producing high-quality foods is not necessarily associated with that of choosing the best cows and/or feeding for high yields, whereas poor home-grown foods are often skilfully balanced to yield high outputs. On this point also the element of luck in the plans of mice and men needs to be considered. It would seem that the only factor with an implicit bias is the level of feeding in the preceding lactation, which is apt to be positively associated with the current level of feeding. Experience tends to show that current yields may be affected by previous feeding, but nothing is known as to whether higher yields, when they are due to high-plane feeding in the past lactation, are associated with proportionately lower inputs in the current lactation. If they are, the function will be biased.

If the current yield effect of past feeding is 5 per cent and the current inputs increase proportionately less, the bias at the top range of the input-output curve (or surface) is not likely to exceed 2—3 per cent, and it cannot be greater than 5 per cent. On the other hand, an element of bias is present throughout the whole range, since some lower output herds might have had the benefit of abundant grazing in the previous season. This is likely to lessen somewhat the effect of the bias in the top range of the function. Furthermore, at least in linear functions, any bias in the top range would be diminished by the pull exerted by the more numerous observations in the middle and lower part of the scatter.

It would therefore appear that in the physical input-output relationships studied here the hybrid functions may have had some slight bias due to management, but that the order of its magnitude would be insignificant.

The available data—both investigational and experimental—had more serious defects which will be discussed in the following pages.

The Data

In a previous study[1] the writer analysed in the aggregate a set of Milk Cost Investigation data of the Bristol I Province. In order to confine the observations to a period of hand feeding, the months January, February and March (1949) were selected. The dissected material from this study has been used for the present analysis. At a later stage an additional source of data was found in a paper prepared by Holmes and others on input-output experiments at the Hannah Dairy Research Institute.[2]

The main reason for the choice of the Bristol data was the fact that a great deal of work on extraction could be saved by their use. The dissected data for each farm comprised the number of cows in milk, suckling and dry,

(1) Jawetz, M. B., "The Winter Feeding of Dairy Cows", *University of Bristol*, Nov., 1950.

(2) Holmes, W., *et al.*, *op. cit.*

the output of milk—sold, consumed and fed to livestock—, the proportion of winter milk produced during the whole year, and the detailed quantities of every food fed to all the cows in the herd during the three months. But there were yet other reasons in favour of this material. In 1949 purchased concentrates were still rationed in this country and gross overfeeding with protein was therefore much less likely than under free market conditions. Furthermore, the author did the field work on over one-half of this sample and had some knowledge of the conditions prevailing, at the time at least, on those farms visited by him.

Originally the material consisted of records of 81 dairy herds. The records of four Jersey and five Guernsey herds were, however, discarded at the very outset of this study. Conventional feeding standards assume that the Channel Island breeds have higher production requirements than "ordinary" dairy breeds and their inclusion might have biased the results.

One of the defects of the data is the fact that only very rarely was there any detailed information about the quality of the hay and silage fed or the variety of kale consumed by the cows. In the great majority of cases, therefore, the average nutritive values of these foods had to be taken from the Ministry of Agriculture Bulletin Number 48.(1) A weighted average of these values for seeds and meadow hay has been taken. (The nutritive values applied in the analysis are given in Appendix Table I.) There was no reason to assume that the errors implicit in accepting average nutritive values for hay, silage and kale would be correlated ones.(2) The only possible correlation of such errors might have been one with the level of output, namely an association of nutritive values higher than average with high output and lower than average with low output. There was no indication of this at any stage of the analysis.

Another defect was soon revealed as the study progressed. A number of observations appeared to show gross over- or under-feeding. A proportion of them undoubtedly represented genuine cases, or were due to the first-mentioned defect; but a number could only come from errors in recording. On the other hand, the other main variable—output—was practically free from such errors. The quantities of milk sold wholesale were recorded from the Milk Marketing Board slips; and quantities sold or given to workers, consumed in the household or fed to livestock, though estimated by farmers, are subject to relatively insignificant error and in any case are only a small proportion of total output. Of the other possible variables size of herd, percentage of cows in milk and suckling, and proportion of winter milk produced were reasonably free from errors. But variables which would involve sub-stratification of the food input, e.g. according to the proportion of roughage in the ration or home-grown corn in the concentrates, or of protein derived from concentrates, would be biased by the errors of recording. On further analysis the errors of recording of food inputs were found to be practically uncorrelated; but they complicated the analysis, since they necessitated additional stratification.

(1) Woodman, H. E., "Rations for Livestock".

(2) Errors in a variable are correlated if the faulty observations do not fall equally above and below the "true" values of the variable throughout the range of the regression.

The Explanatory Analysis

The nutritive values of all the foods fed during the investigated period were calculated separately for each herd. Conventional maintenance requirements according to breed were calculated, and pregnancy requirements of 40 lb. SE and 45 lb. PE per annum per cow of 11 cwt.[1]—proportionately less for the Ayrshires—were added to maintenance. Subtraction of the above requirements from the total nutrients fed to each herd gave the quantities remaining for production, from which the inputs per cow and per gallon were calculated.

It was at this stage that cases of excessive "over-and under-feeding" became apparent. Some of them were of a magnitude which could only be explained by gross errors in recording. With the exception of two herds, which turned out to be Ayrshires, these cases were not associated with high levels of yield. On a scatter diagram they were grouped symmetrically along the regression near and below the mean output. The author knew from personal inspection one of the two Ayrshire herds mentioned above and in this herd the cows were distinctly larger than is typical for the breed. It seemed likely that the apparent over-feeding might have been due to the fact that the breed maintenance standards—6 lb. SE with 0.63 lb. PE—were too low for these heavy cows. When higher maintenance requirements were assumed for this herd its input conformed with the relevant section of the scatter. This implied that the production-inputs of some of the other Ayrshire herds might have been over-estimated as well.[2] Nevertheless, in the preliminary stages of the work the bias due to the correlated error in the input of (what appeared to be) one Ayrshire herd was ignored, although later it led to the exclusion of all the 11 Ayrshire herds from the statistical analysis.

The fact that input was subject to error precluded its use as an independent variable, since to employ it in this way would alter the slope of the regression from the true one.[3] In the exploratory stage, when inputs of SE and PE were analysed separately, this did not create serious problems. Even had input been free of error there would still have been two reasons for treating it as the dependent variable. In the first place, in input-output analyses based on other than experimental data there tends to be a distinctly higher degree of inter-relationship between output and good management than between input and such management. Since the latter is a "soft" variable, which defies measurement, it confounds a production function (i.e. one in which output is dependent on input) more than an input function (where output is the independent variable).[4]

(1) As estimated by the author in "The Winter Feeding of Dairy Cows", pp. 216—217.

(2) In the South-West of Britain there are strains of Ayrshires which are distinctly larger than the pure strain of the breed.

(3) Errors in the dependent variable lower the correlation coefficient (or index); but unless they are correlated they will not change the slope of the regression significantly from the true slope. Errors in the independent variable preclude valid results of both correlation and regression.

(4) This aspect of input-output functions has been discussed by the author in "Farm Size, Farming Intensity and the Input-Output Relationships of Some Welsh and West of England Dairy Farms", pp. 8—10. University College of Wales, Aberystywth, 1957.

Furthermore, all conventional feeding standards are expressed as input functions—lb. of SE, PE, TDN, etc., per gallon, 10 lb. or kg. of milk—and not the other way round.

The exploratory analysis, however, was made by classification and cross-tabulation and in order to exclude any strong bias when the classes were broken down into small groups (since the errors of input in the sample were concentrated in the lower part of the regression) the largest errors of recording, at least, had to be eliminated. A rough stratification by error was carried out as follows: all herds with average daily inputs of over 38 lb. of dry matter (DM) per cow, *or* over 4 lb. SE *or* 1 lb. PE, above maintenance (and pregnancy) requirements per gallon were rejected, and also all herds which averaged less than 20 lb. DM per cow daily, *or* under 1.9 lb. SE *or* 0.4 lb. PE, per gallon over maintenance.

The previous elimination of the Channel Island herds and the above stratification left a sample of 50 records(1) which, at first, were classified by yield into 5 groups of 10 herds each. The regression curves of inputs of SE on output were roughly similar to those in the Jensen-Woodward report. But the scatters fanned out in the region of higher yields and, as may be seen from Table 1, the mean input of the highest yield class was lower than that of the preceding class, thus indicating the influence of another variable.

TABLE 1

Daily Output and Input of Dry Matter per Cow in Herd and Nutrients above Maintenance Requirements per Gallon (3 Monthly Means)

Daily Yield per Cow in Milk	DM per Cow in Herd Daily	Input per Gallon	
		SE	PE
gallons	lb.	lb.	lb.
1.7	21.5	2.3	0.50
2.0	23.8	2.6	0.58
2.4	25.3	2.8	0.60
2.6	27.0	3.3	0.65
3.2	26.8	3.1	0.64

As a next step the herds were classified by the proportion of winter milk produced during the whole year and sub-classified by the mean daily yield per cow (in milk). In order to obtain roughly comparable yields in the higher and lower yield groups sub-stratification was carried out which eliminated 11 records. The results are shown in Table 2.

(1) Referred to by the author as "50 West of England Herds" in "Does It Pay to Feed Concentrates to Dairy Cows ? " in which the inputs (Table 2) were those calculated from the regression curve. (*Agriculture*, 60, 1953.)

TABLE 2

Output and Input of Nutrients above Maintenance Requirements per Gallon (3 Monthly Means)

According to Proportion of Winter Milk

Yield per Cow in Milk	Herds	Winter Milk	Daily Yield per Cow in Milk	Input per Gallon	
				SE	PE
	number	per cent	gallons	lb.	lb.
Higher	8	44.1	2.8	3.0	0.61
	11	54.8	2.9	3.2	0.65
Lower	12	44.2	2.1	2.6	0.55
	8	53.8	2.0	2.4	0.60

The small difference between the inputs can be explained by the slight variations between the yields, barring the trend of PE in the lower yield groups (of which more will be said later).

In their turn, variations owing to size of herd were investigated. The records were classified in three herd-size groups and sub-classified by higher and lower yields. The results are given in Table 3.

TABLE 3

Output and Input of Nutrients above Maintenance Requirements (3 Monthly Means)

According to Size of Herd

Higher Yield Group					Lower Yield Group				
Mean Herd Size and No. of Herds		Yield per Cow in Milk	Input per Gallon		Mean Herd Size and No. of Herds		Yield per Cow in Milk	Input per Gallon	
			SE	PE				SE	PE
Cows	Herds	gallons	lb.	lb.	Cows	Herds	gallons	lb.	lb.
16.2	9	3.0	3.1	0.67	17.3	9	1.8	2.5	0.54
27.5	8	2.8	3.1	0.62	30.0	9	2.1	2.7	0.58
49.5	7	2.6	3.1	0.64	53.7	8	1.9	2.5	0.58

In the higher yield group output per cow in milk declined with increasing herd size, but input of SE per gallon remained unchanged.

A likely explanation of this trend was sought in the composition of the rations; it is generally assumed that larger herds depend on home-grown bulky foods to a greater extent than do smaller ones. An analysis was attempted in which herds were sub-classified in two groups according to the proportion of SE obtained by them from bulk foods, the dividing line being fixed at 50 per cent. (The proportions of nutrients from the different categories of food fed to the herds as given in Table 1, though irrelevant in this context, may be of some interest and are presented in Appendix Table II.) In this classification, some groups dwindled to a few records or petered out, and the results, for what they were worth, showed no trend or consistency. But it did appear that groups with comparable outputs which had high inputs contained a majority of Shorthorn herds, while those with lower inputs had a majority of Friesians.

When the breeds of the 10 highest output herds were ascertained it was found that 5 of them were Friesian, 2 Ayrshire and 3 Shorthorn. The next output class contained one Friesian herd, 3 Ayrshire, 5 Shorthorn and one Shorthorn and Guernsey. Here, then, was the explanation of the fan-shaped scatter and the input-output peculiarity of these two groups. It appeared that breed was an important factor in the input-output relationship.

When the herds were grouped according to breed—Friesian, Shorthorn and Ayrshire—and sub-classified into higher and lower yield levels, a marked difference in the input-output ratio at higher yields was found as between the Friesians and the Shorthorns. The Ayrshire herds appeared to have higher inputs of SE per gallon than did the other two breeds, both in the higher and lower yield groups, and lower inputs of PE; but in view of the correlated errors of these inputs their relationship with outputs was biased. Consequently no significance can be attached to those figures which are given for the Ayrshire herds in the following Table 4. They are presented in order to demonstrate the effects of correlated errors in a dependent variable, and the order of magnitude of the bias exerted by the inclusion of Ayrshires in the preceding analysis. Since the Ayrshire herds were rather evenly distributed over the range of output, their effect on the over-estimation of inputs could only be small. But these Ayrshire herds could not be used for further analysis.

TABLE 4

Breed Variations in Inputs of Nutrients per Gallon (3 Monthly Means)

Inputs of Ayrshire Herds over-estimated owing to correlated errors

Yield Group	Breed of Cows	Number of Herds	Daily Yield per Cow in Milk	Input per Gallon	
				SE	PE
			gallons	lb.	lb.
Higher	Friesian	6	3.1	2.6	0.59
	Shorthorn	9	2.6	3.1	0.68
	Ayrshire	5	2.8	3.2	0.53
Lower	Friesian	6*	1.9	2.5	0.56
	Shorthorn	10	1.9	2.5	0.57
	Ayrshire	3†	1.7	2.8	0.51

*Including 3 Friesian and Shorthorn
†Including 1 herd of Ayrshire crosses

If the differences in input per gallon between the lower and higher output Friesians were significant at all, they were slight; whereas those differences appeared to be much larger with the other two breeds.

The above results were the more surprising because they were unexpected. Confirmation was sought and obtained from a sample of Welsh records of the Milk Cost Investigation for the three months January to April, 1951. The herds used for analysis were Friesian, Shorthorn and Ayrshire, in which the mean daily yields per cow in milk and suckling exceeded 2 gallons. Stratification for errors of recording in inputs was carried out in the manner previously described. This left 7 records each for Friesian and Shorthorn herds (originally 13 and 12) and 2 (out of 3) Ayrshire records. The results of the input-output analysis were as follows:

TABLE 5
Welsh Herds with Outputs Exceeding 2 Gallons Daily per Cow in Milk (3 Monthly Means)*
Breed Variations in Inputs of Nutrients per Gallon

Breed of Cows	Number of Herds	Daily Yield per Cow in Milk	Input per Gallon	
			SE	PE
	No.	gallons	lb.	lb.
Friesian	7	2.8	2.8	0.64
Shorthorn	7	2.7	3.5	0.62
Ayrshire	2	3.0	2.9	0.60

*January to April, 1951

The two Ayrshire observations cannot have much significance—although they look more sensible than those in Table 4. But the results obtained for the Welsh Friesian and Shorthorn herds confirmed those found for these breeds in the original sample. From this point it appeared that breed is a main explanatory variable in any analysis of input-output relationships of dairy cows. Before taking account of this in the mathematical analysis, however, it seemed necessary to re-trace some of the preliminary ones separately for Shorthorns and Friesians.

The number of Friesian herds proved to be too small for this, but the sample of Shorthorn herds was just adequate for further sub-classification.

TABLE 6
Shorthorn Herds Only
Influence of Seasonality of Production
(3 Monthly Means)

Yield per Cow in Milk	Number of Herds	Winter Milk	Daily Yield per Cow in Milk	Input per Gallon	
				SE	PE
	No.	per cent	gallons	lb.	lb.
Higher	4	44.7	2.5	3.2	0.68
	5	56.1	2.6	3.3	0.71
Lower	5	45.4	2.0	2.4	0.54
	5	52.3	2.1	2.5	0.58

The trends revealed by this table are similar to those given in Table 2 for all the herds although, in the higher output group, yields are lower and inputs higher than in Table 2.

TABLE 7

Shorthorn Herds Only
Input-Output of Nutrients above Maintenance Requirements
(3 Monthly Means)
According to Size of Herd

Higher Yield Group					Lower Yield Group				
Mean Herd Size and No. of Herds		Yield per Cow	Input per Gallon		Mean Herd Size and No. of Herds		Yield per Cow	Input per Gallon	
			SE	PE				SE	PE
Cows	Herds	gallons	lb.	lb.	Cows	Herds	gallons	lb.	lb.
22.6	7	2.6	3.26	0.73	25.0	9	1.9	2.67	0.55
50.8	5	2.6	3.33	0.70	64.2	4	2.0	2.34	0.56

The above results tend to show that, in this sample, herd size did not influence the input-output relationship—at least in the higher output classes when the breed of cow was taken into account. The difference between the inputs of SE in the lower yield groups is due to the crude method by which the errors of recording have been stratified at this stage. The errors are concentrated in the lower yield strata and, even though not correlated, impede sub-classification in a small sample: an over-estimated observation could be included in one class and the balancing under-estimated one in the other class, as happened in this case.

It will be noted that at comparable outputs lower inputs of SE appear to be associated with higher inputs of PE, although the differences are only fractional.

Attempts have been made to sub-classify the Shorthorn and Friesian herds of comparable yields according to the proportions of nutrients which they obtained from bulky foods. These proportions tended to vary at different yield levels (as can be seen in the Appendix Table II), a fact which necessitated further grouping. The resulting classes were very small and, in view of the "unbalancing" of the errors of input in such small groups, the results were not fit for presentation in any but the highest yield groups, whose inputs were free of gross error. The latter groups of Friesian and Shorthorn herds could be classified according to whether they obtained over or under 50 per cent of SE from bulky foods. The results are presented in Table 8.

TABLE 8

Influence of the Proportion of Nutrients from Bulky Foods
Highest Yielding Friesian and Shorthorn Herds
(3 Monthly Means, proportion of all nutrients including maintenance)

Breed of Cows	No. of Herds	Bulk as Source of		No. of Cows in Herd	Yield per Cow in Milk	Input per Gallon	
		SE	PE			SE	PE
	No.	per cent	per cent	No.	gallons	lb.	lb.
Friesian	3	62.9	49.4	17.4	3.0	2.58	0.57
	3	45.1	30.8	29.5	3.2	2.55	0.60
Shorthorn	3	64.1	53.5	43.8	2.8	3.36	0.70
	4	46.5	33.1	30.1	2.9	3.45	0.73

When the small size of the classes is taken into consideration, the differences in the input-output ratios as between higher and lower bulk-fed herds of the same breed appear to be inconsiderable. The fact that lower-bulk Friesian herds had a somewhat higher output, but a fractionally lower input of SE per gallon, than the higher-bulk ones *may* be due to the reverse position of PE. The results of Table 8 cannot be generalised, but they indicate that in this sample the input-output relationship within herds of the same breed tended to be similar for comparatively high yields, irrespective of the proportion of nutrients derived from bulky foods. Since it can be assumed that, if differences due to variations in nutrients derived from bulky foods did exist, they would be more marked with higher yielding herds than with low output ones, it follows that the results in Table 8 are also likely to be valid for the lower output herds.

Conclusion of Explanatory Analysis

Although the preceding analysis would not completely answer the problems under investigation it prepared the way for the main regression analysis. It appeared that in the input-output relationship breed is a classifying variable; and there were elusive but persistent indications of an interrelationship between inputs of SE and PE which seemed to warrant dissection by a sharper tool. On the other hand, there was no indication that the relationship might be influenced by the stage of lactation reached by the majority of the cows in a herd (as indicated by the proportion of winter milk produced during the whole year), or by the size of herd, or by the proportion of nutrients derived from bulky foods. This is no conclusive proof that these factors might not exert some influence on the "universe". But the evidence suggests that, in this sample, at least, very little, if anything at all, could be gained by using them as variables in further analysis.

Part Two

THE STATISTICAL ANALYSIS OF THE DATA

This analysis has been based on the 34 Shorthorn and the 21 Friesian herds of the original sample, and on the published data pertaining to 6 groups of Ayrshires in a Kirkhill experiment carried out by Holmes *et al.* Although the latter have serious defects for regression analysis (which will be discussed in due course), their use and the presentation of the results obtained was deemed worth while, since they supply a corroborative check on the Milk Cost data. Each breed has been analysed separately as follows:

(*a*) Two-variable regression of input of SE on output.

(*b*) Two-variable regression of input of PE on output.

(*c*) Tests of significance of the differences between the breeds in the slopes of regression of SE and PE on output.

(*d*) Stratification of the Shorthorn and Friesian sample for gross errors of observation in order to find out whether the regression curves are true ones, and if this is the case to make them fit for partial regression and joint regression analysis.

(*e*) Repetition of (*a*), (*b*) and (*c*) with samples stratified for error of observations.

(*f*) Linear partial regression of input of SE and PE on output.

(*g*) Curvilinear partial regression (Shorthorns only) of input of SE and PE on output.

(*h*) Linear joint regression of input of SE and PE on output.

(*i*) Linear joint regression with additional term.

(*j*) Tests of fit of joint regression curves.

(*k*) Production surfaces (iso-product curves) calculated from the best fitting equations. From them a theory of feeding dairy cows for production has been evolved in a further section of this study.

THE TWO-VARIABLE REGRESSION ANALYSIS

The variables were: x = output in gallons of milk per day per cow in milk (Mean of 91 days).

and y = input of SE in lb. per day per cow in milk, above maintenance and pregnancy requirements (Mean of 91 days).

or y_1 = input of PE in lb. per day as above.

The regressions were those of input on output owing to the errors of recording of the inputs.

The hypothesis was made that there is a significant difference between the slopes of regression of Shorthorn and Friesian herds.

The regression statistics were summarised as follows:

34 Shorthorn Herds

Linear Regression Statistics of SE on Output

	Output x	Input y	x^2	y^2	xy	
						$n = 34$
Sums	74.74	209.21	170.22	1697.91	494.52	$r_{xy} = 0.70274$***
Means	2.19823	6.1532				$s_{yx} = 2.51$

$$y = -6.71854 + 5.8555x$$

Linear Regression Statistics of PE on Output

	x	y_1	x^2	${y_1}^2$	xy_1	
						$n = 34$
Sums	74.74	30.24	114.53	50.42	75.00	$r_{xy_1} = 0.8074$***
Means	2.19823	1.35438				$s_{y_1x} = 0.33$

$$y_1 = -1.02213 + 1.0811x$$

21 Friesian Herds

Linear Regression Statistics of SE on Output

	x	y	x^2	y^2	xy	
						$n = 21$
Sums	51.01	138.49	132.154	1146.114	362.441	$r_{xy} = 0.59434$**
Means	2.42904	6.5947				$s_{yx} = 2.74$

$$y = -1.07475 + 3.1574x$$

Linear Regression Statistics of PE on Output

	x	y_1	x^2	${y_1}^2$	xy	
						$n = 21$
Sums	51.01	30.353	132.154	56.70	78.425	$r_{xy_1} = 0.4565$*
Means	2.42904	1.445381				$s_{y_1x} = 0.71$

$$y_1 = -0.062577 + 0.56928x$$

Owing to the errors of recording (and possibly to gross under- or over-feeding in some cases) correlation was lowered below the true level of that prevailing in the universe. Nevertheless, the correlation coefficients were highly significant, with the exception of that of PE in the Friesian group, which was significant at the 0.05 level. The standard errors of estimate were high, since they included elements of deviation due to errors of recording; they reflected on the defects of the samples, rather than on the order of magnitude of the errors of the true relationships as expressed by the regression curves derived from these samples. According to theory these curves should be true ones, and some proof of this will be sought later.

The Friesian group had lower correlation coefficients of both SE and PE on output, and higher standard errors of estimate than the group of Shorthorns. This indicated that it included relatively more bad records than the Shorthorn group.

The regression coefficients of the Shorthorn group were considerably higher than those of the Friesian group, i.e. the slopes of the regression curves varied considerably as between these two breeds. These differences were found to be significant at the 0.1 level only, t being 1.8803 for the difference between regression coefficients of SE on output and 1.7662 for that of PE on output. One full test has been presented in Appendix III.

The apparently low significance of the differences is an effect of the errors of recording; it will be seen later that the significance increased when the gross errors were eliminated. Moreover, the significance of the difference between the breeds in the regression slope of each nutrient is corroborated by a difference in the same direction of the other nutrient.

Elimination of Gross Errors of Recording by Stratification of the Samples

Although two-variable regression analysis can give correct results in spite of errors in one variable (if the errors are non-correlated and the logic of the function permits its use as the independent variable), such errors make it difficult to test the significance of curvilinearity—if indeed curvilinearity can be found at all in grossly inflated scatters. In input-output analysis this alone would be sufficient reason for the elimination of the "bad" observations. An even more compelling reason exists, however, when multi-variable analysis is contemplated on data in which more than one variable is subject to error. In such cases at least one of the unreliable variables is bound to be an independent one, and the resulting regression will not be the true one, unless a method is found which will isolate those reliable observations on which further analysis could be based.

It was clear that the method of stratification which had been applied in the exploratory analysis was not satisfactory. The following method was therefore devised: on the separate scatter diagrams of SE on output of the 34 Shorthorn and 21 Friesian herds were traced the linear regression curves of SE on output. The arbitrary decision was taken to regard as badly recorded those inputs of SE which were 25 per cent above and below the regression curve at any level of output. This is illustrated in Figures II (Shorthorn) and III (Friesians), which show the scatters and regression curves of SE on output and the 25 per cent "border" lines. It will be noted

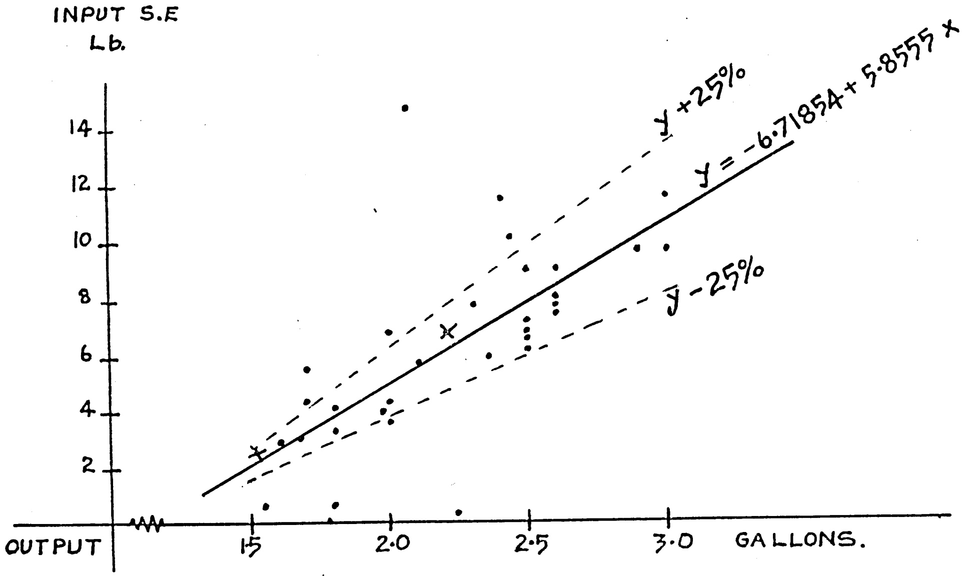

FIGURE II

34 SHORTHORN HERDS

Observations of Inputs of SE on Output, Linear Regression Curve and "Borderlines" 25% above and below the Regression

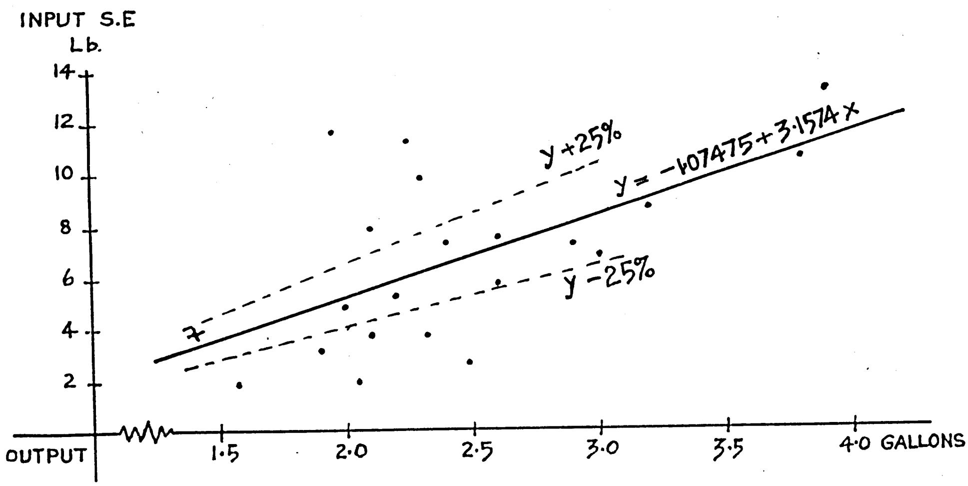

FIGURE III

21 FRIESIAN HERDS

Observations of Inputs of SE on Output, Linear Regression Curve and "Borderlines" 25% above and below the Regression that the observations beyond the "border" lines are connected with average yields of less than 2.5 gallons per cow and also that the errors do not appear to be correlated. This stratification brought down to 23 the sample of Shorthorn herds, while the Friesian herds dwindled to 11. In turn errors in the inputs of PE were similarly treated. Linear regression curves of PE on output were calculated separately for the remaining 23 and 11 herds, and "border" lines 25 per cent above and below these regression curves were drawn. Observations beyond these border lines—two Shorthorn and one Friesian—were then rejected as subject to error. They have been marked with crosses on Figures II and III.

The above method of stratification cut down the sample of Shorthorn herds to 21, i.e. by over one-third, and more than halved that of Friesian herds, only 10 of which remained. But these diminished samples could be regarded as reasonably accurate for three-variable analysis.

Final Two-Variable Regression Analysis of Samples Stratified for Error and Comparison with Original Samples

It was now possible to verify the law that non-correlated errors in the dependent variable will not change the slope of the regression significantly from the true slope, and also the assumption that the apparently low significance of the differences between the slopes of regression of inputs on output of Shorthorn and Friesian herds was due to errors of recording.

The regression statistics of the samples stratified for error are given below:

21 Shorthorn Herds

SE on Output

	Output x	Input y	x^2	y^2	xy	
						$n = 21$
Sums	49.39	141.21	119.63	1070.99	351.69	$r_{xy} = 0.95382$***
Means	2.3519	6.7242				$s_{yx} = 0.7449$

$$y = -6.5476 + 5.643x$$

PE on Output

	x	y_1	x^2	y_1^2	xy_1	
						$n = 21$
Sums	49.39	30.746	119.63	49.804	76.16	$r_{xy_1} = 0.94338$***
Means	2.3519	1.46409				$s_{y_1x} = 0.1623$

$$y_1 = -1.44684 + 1.10922x$$

10 Friesian Herds

SE on Output

	x	y	x^2	y^2	xy	
						$n = 10$
Sums	28.6	77.23	85.42	656.32	234.42	$r_{xy} = 0.919348$***
Means	2.86	7.723				$s_{yx} = 0.7125$

$$y = -2.9641 + 3.73675x$$

PE on Output

	x	y_1	x^2	x_1^2	xy_1	
						$n = 10$
Sums	28.6	16.23	85.42	28.72	49.14	$r_{xy_1} = 0.92679$***
Means	2.86	1.623				$s_{y_1x} = 0.19$

$$y_1 = -0.5251 + 0.7511x$$

TABLE 9

Comparison of Coefficients of Regression and Correlation and Standard Errors of Estimate between Original Samples and Those Samples Stratified for Errors

	SE on Output		PE on Output	
	34 Shorthorn	21 Shorthorn	34 Shorthorn	21 Shorthorn
b r_{xy} s_{yx}	5.8555 0.70274** 2.51	5.643 0.95382*** 0.74	1.0811 0.80746** 0.33	1.10922 0.94338*** 0.16
	21 Friesian	10 Friesian	21 Friesian	10 Friesian
b r_{xy} s_{yx}	3.1574 0.59434** 2.74	3.73675 0.919348*** 0.32	0.56928 0.4565* 0.71	0.7511 0.92679*** 0.19

Significance : * = 0.5 level, ** = 0.1 level, *** = 0.01 level.

In accordance with theory the regression co-efficients changed very little as between the original and the stratified samples, although the correlations became considerably better. The standard errors of input on output diminished in proportion with the increase in correlations.

The significance of the differences between the breeds in the slopes of regression of SE and PE on output has been tested for the stratified samples. Although the differences between the regression co-efficients were somewhat smaller than those in the unstratified samples, and in spite of the fall in the degrees of freedom from 51 to 27, the significance of these differences between Shorthorn and Friesian herds increased considerably. For the difference of SE on output $t = 2.734$—significant at the 0.01 level, and for that of PE on output $t = 2.568$—significant at the 0.05 level.

Before the analysis was carried further it was thought useful at this stage to bring in the Ayrshire data from the Hannah experiments at Kirkhill.

Ayrshire Cows

Data of Input-Output Experiments by W. Holmes and Others at the Hannah Research Institute, Kirkhill[1]

Ayrshire cows of the Institute herd were used in experiments lasting twelve weeks, from January to April, in 1953 and 1954. The cows calved mainly in November and December and the average lactation number was five. Each experiment was carried out with twelve cows, forming four 3 x 3 complementary latin squares.

In the first experiment (1953) the SE was kept (nearly) constant at 110 per cent of Woodman's standards, and digestible crude protein (DCP) was fed at three levels at 98, 113 and 123 per cent of these standards. The cows' average weight was practically identical on the three rations, being 1098 lb. on the average; and it remained stable throughout the experiments.

The second experiment (1954) was carried out with practically the same group of cows at the same stage of lactation. It was planned to compare the effect of three different levels of energy feeding (112, 117 and 125 per cent of Woodman's standards) whilst maintaining a relatively constant intake of DPC (129, 131 and 132 per cent of those standards). This high protein input was given in order "to avoid any possible lack of protein". The average weight of the cows was 1121 lb. Changes in weight were insignificant.

The detailed composition of the rations was given for each experiment. The mean inputs of SE and DCP and the mean outputs of milk per cow/day are shown below. They have been compiled from Tables 3, 6 (levels of feeding), 4 and 7 (average yield) of the report.[1]

Ration	A	B	C	X	Y	Z
per Cow/day	lb.	lb.	lb.	lb.	lb.	lb.
SE	18.3	18.0	17.7	18.7	20.3	21.3
DCP	3.6	3.9	2.9	4.2	4.3	4.2
Milk	38.8	39.1	36.9	38.9	40.6	40.9

It was concluded from the first experiment that no statistically significant difference between yields resulted from the different inputs of DCP, while the varying of SE in the second experiment had a small but significant effect on the yield (and the chemical composition) of milk. The experiments were not designed for regression analysis. But they had a serious defect even for analysis of variance: while PE in the second experiment was held constant at about 130 per cent of the conventional standard "to avoid any possible lack of protein", SE in the first experiment was only 110 per cent of standard, i.e. too low to avoid any possible lack of this nutrient. The only valid conclusion from experiment 1 is, therefore, that no significant differences between yields resulted from different inputs of DCP *when inputs of SE were relatively low*; and the conclusion from the second experiment should have the rider: *when inputs of DCP were relatively high.*

(1) W. Holmes *et al. Op. cit.*

In this study the average data of each group of cows in the above two experiments have been submitted to regression analysis. The observations of protein inputs are strongly biased, since three of them were kept high in the second experiment while the first experiment gave another relatively high and only one medium and one low observation. Consequently the resulting protein curve would be flatter than the "true" one and the estimates of protein requirements too high. The analysis of the full experimental data is therefore given here for the sake of check and comparison only. In order to obtain a more nearly true picture the author "re-designed" the experiments by omitting the observations with the protein inputs which were highest in relation to output, i.e. those of group X. These "improved" data have been analysed here as "5 groups of Ayrshires".

In order to bring them into line with the writer's own data, DCP as given in Tables 2 and 5 of the Kirkhill report was converted into PE, and milk yields were converted from pounds to gallons. Conventional maintenance requirements for 10 cwt. Ayrshires were deducted. The converted data are presented in Table 10.

TABLE 10

Input-Output Data of 6 Groups of Ayrshires Re-calculated from Tables 2—7 of Report by Holmes and Others

Ration	Per Cow/Day					
	A	B	C	X	Y	Z
SE:	lb.	lb.	lb.	lb.	lb.	lb.
Total	18.3	18.0	17.7	18.7	20.3	21.3
Maintenance ..	6.5	6.5	6.5	6.5	6.5	6.5
Production ..	11.8	11.5	11.2	12.2	13.8	14.8
PE :						
Total	3.20	3.47	2.58	3.91	4.0	3.91
Maintenance ..	0.7	0.7	0.7	0.7	0.7	0.7
Production ..	2.5	2.77	1.88	3.21	3.3	3.21
	gallons	gallons	gallons	gallons	gallons	gallons
Yield	3.77	3.80	3.58	3.78	3.94	3.97

The linear regression statistics are as follows:

6 Groups of Ayrshires

SE on Output

	Output x	Input y	x^2	y^2	xy	
						$n = 6$
Sums	22.84	75.3	87.042	955.25	287.526	$r_{xy} = 0.88029$*
Means	3.8066	12.55				$s_{yx} = 0.679$

$$y = -21.182 + 8.8614x$$

PE on Output

	x	y_1	x^2	y_1^2	xy_1	
Sums	22.84	15.65	87.042	41.7953	59.8496	$n = 6$
						$r_{xy_1} = 0.877448*$
Means	3.8066	2.60833				$s_{y_1x} = 0.265$

$$y_1 = -10.153385 + 3.40594x$$

5 Groups of Ayrshires

SE on Output

	x	y	x^2	y^2	xy	
Sums	19.06	63.1	72.754	806.41	241.41	$n = 5$
						$r_{xy} = 0.882336*$
Means	3.812	12.62				$s_{yx} = 0.747$

$$y = -21.68 + 8.9979x$$

PE on Output

	x	y_1	x^2	y_1^2	xy_1	
Sums	19.06	13.66	72.754	38.652	52.427	$n = 5$
						$r_{xy_1} = 0.987477**$
Means	3.812	2.732				$s_{y_1x} = 0.091$

$$y_1 = -11.2222 + 3.6606x$$

The slope of the regression of SE on output was practically identical as between the 6 and 5 groups, though a fraction steeper for the latter. The slope of PE on output was somewhat steeper for the 5 groups than for the six. All the correlation coefficients of both samples were significant. But whereas the elimination of the "unsuitable" set of observations improved the correlation of SE on output only fractionally it raised the significance of the correlation of PE on output to the 0.01 level.

Since the Ayrshire experiments were made on cows which were in the early stages of lactation, whereas the Shorthorn and Friesian groups studied here were, "on the average," in mid-lactation, no outright comparison was possible between the differences in the slopes of their linear regressions. The latter represent different sectors of regression curves. Curvilinear regression curves could be comparable, if they covered a sufficient range of observations; but the Ayrshire experiments covered only a very limited sector at the highest extremity of observations. Nevertheless, significance tests of the differences between the regression slopes of Ayrshire and the other two breeds were carried out which permit some valid conclusions.

With the Ayrshires (both 6 and 5 groups) and Shorthorns the difference between the slopes of PE on output was significant at the 0.01 level, whereas there was no significant difference between the slopes of SE on output.

This means that the Ayrshires (which averaged considerably higher yields) were significantly better converters of SE than the Shorthorns, provided only that they had more PE; they may also have been more efficient converters of PE, but no inference can be drawn in this respect.

The difference between the slopes of PE on output of the Ayrshires and Friesians was significant at the 0.01 level, too. But that between the slopes of SE on output was significant at the 0.1 level only. Although this means that the Friesians were significantly better converters at *lower yields*, it does not follow that there would be a difference at comparable outputs. From the available data it is not possible to conclude whether or not there is a significant difference in the input-output relationship as between Ayrshires and Friesians.

Curvilinearity of Regressions of SE and PE on Output

Most of the scatter diagrams indicated varying degrees of curvilinearity. Although the solution of the problem does not lie with two-variable analysis it is interesting to explore it to the limits, if only to obtain a familiarity with the data which could be useful in further analysis.

In Table 11 are given the correlation coefficients and indices of SE and PE on output for the three breeds, together with the degree of significance of curvilinearity where it could be found.

TABLE 11

Three Breeds of Cow

Correlation Coefficients and Indices and Significance of Curvilinear Regression

Breed of Cows and Groups	SE on Output		PE on Output	
	Correlation	Significance of Curved Regression	Correlation	Significance of Curved Regression
		at per cent level		at per cent level
21 Shorthorns	r_{xy} 0.95382*** p_{yx} 0.95450...	non sign.	r_{xy} 0.94334*** p_{yx} 0.95713***	0.05
10 Friesian ..	r_{xy} 0.919348*** p_{yx} 0.95480***	0.1	r_{xy} 0.92679*** p_{yx} 0.993076***	0.01
6 Ayrshire ..	r_{xy} 0.88029* p_{yx} 0.985698***	0.01	r_{xy} 0.877448* no curvilinearity	—
5 Ayrshire ..	r_{xy} 0.882336* p_{yx} 0.993378**	0.01	r_{xy} 0.987477** no curvilinearity	—

The correlation indices of SE on output were higher than the coefficients, and markedly so in the Ayrshire groups; but curvilinearity was not or only slightly significant in the Milk Cost data, whereas it was highly significant in the Ayrshire experimental ones. In the Shorthorn and Friesian groups the correlation indices of PE on output were also higher, but curvilinearity was significant at five or one per cent levels; whereas the Ayrshire groups showed no curvilinearity at all.

A rough pattern appears of a relationship between curvilinearity in the input-output function of one nutrient and linearity in that of the other; linear (or near linear) regression of SE on output would seem to be associated with curved regression of PE on output (Shorthorns and Friesians) and *vice versa* (Ayrshires). It is conceivable that the Hannah experiments could have been designed in such a way as to show degrees of curvilinearity similar to the pattern displayed by the Milk Cost data. On the other hand the Shorthorn and Friesian herds, if fed SE and PE in proportions approaching those given in the experiment, might have shown curved relationships of SE and linear ones of PE on outputs. The imprecise nature (stochasticity) of the two-variable regressions can only disclose that the problem exists. Its solution must be sought in multiple regression analysis.

Multiple Regression Analysis

In the following analysis yield (X_1) is the dependent variable, while input of SE (X_2) and of PE (X_3) are the independent ones.

Since the inputs can only be applied jointly, it was conceivable that in the two non-experimental samples the level of application of one of them might have depended on that of the other. In view of the fact that the feeding was influenced to a large extent by the theoretical requirements of the conventional feeding standards, a high degree of intercorrelation between inputs of SE and PE could be expected. If this intercorrelation ($r_{x_2x_3}$) were higher than the correlation of output either on SE ($r_{x_1x_2}$) or on PE ($r_{x_1x_3}$), this could distort the regression surfaces from their "true" slopes.[1]

The relevant correlation coefficients are set out for comparison as follows:

TABLE 12

Three Breeds of Cow

Intercorrelation of SE and PE

$r_{x_2x_3}$ in comparison with $r_{x_1x_2}$ and $r_{x_1x_3}$

Sample	$r_{x_2x_3}$	$r_{x_1x_2}$	$r_{x_1x_3}$
21 Shorthorn	0.9409	0.9538	0.9434
10 Friesian ..	0.8067	0.9193	0.9268
6 Ayrshire ..	0.7353	0.8803	0.8774
5 Ayrshire ..	0.8365	0.8823	0.9875

In all groups r_{23} was lower than both r_{12} and r_{13}, although the difference was very small in the Shorthorn group. The coefficients of the Ayrshire groups give some insight into the nature of the inter-relationships as found in experimental data: the improvement brought about by the rejection of the distorting observations increased the correlation of SE on PE from 0.74 in the 6 Ayrshire groups to 0.84 in the 5 groups. It is unlikely that in an especially designed experiment, with no bias caused by the relative numbers of "high" or "low" observations, r_{23} would be higher still. Apart from any speculation on the values of r in the above table and on their

(1) Where highly intercorrelated variables occur they tend to lower the accuracy of estimates of partial regression.

relative magnitude, it is apparent that the intercorrelation of SE and PE of the Shorthorn and Friesian herds was roughly comparable with that encountered in the Kirkhill experiments, even though some slight bias might have existed in the Shorthorn group. It follows that the accuracy of the estimates of partial regression cannot be biased to any considerable extent by the intercorrelation of the independent variables.

Although it is unlikely that linear multiple regression could express the relationships in the populations concerned, it was by no means clear what would be the nature of these relationships. They might, indeed, be linear; but it was more likely that they would be curved. Since the inputs operated jointly, joint equations might be necessary to express these relationships. Moreover, regressions which are curved over the total range of variables in a population may be linear over that portion of the range for which observations are available. In any case, it was uncertain whether the available scanty data for Friesians and Ayrshires would reveal any relationships which could stand significance tests. The step-by-step procedure already adopted has, therefore, been continued. As the first step, linear three-variable regression analysis has been applied.

Three-Variable Linear Regression

It should be borne in mind that the Ayrshire groups were not directly comparable with the Shorthorn and Friesian ones, since they represented different stages of lactation and therefore different average yields. The yields and input of the Shorthorn and Friesian herds under study were the three-monthly averages of herds ranging from about 45 to 55 per cent winter milk, whose cows were individually at various stages of lactation. Their average stage has been defined as "mid-lactation", on the basis of the quantity of milk which these two groups of herds produced during the three months, measured as a proportion of their total output in the full year. In fact the "mid-lactation" data for the Shorthorn and Friesian herds represented (roughly) averages for their whole lactations, whereas the Ayrshire data were averages for the peak of their lactations. On an average full-lactation curve or surface the first may be taken to represent the input-output relationships when yields and inputs are moderate and the second those when the output and inputs are highest.

The equations are set out below:

TABLE 13

Linear Regression Equations of Output on Inputs of SE and PE and Standard Errors of Estimate (3 monthly averages)

Stage of Lactation	Group of Herds	Equation	$S_{x_{1\cdot 23}}$
Middle ..	21 Shorthorn	$X_1 = 1.2016 + 0.09699X_2 + 0.34022X_3$	0.1089
,, ..	10 Friesian ..	$X_1 = 0.86399 + 0.120966X_2 + 0.654214X_3$	0.1433
Early ..	5 Ayrshire ..	$X_1 = 2.9679 + 0.01837X_2 + 0.224026X_3$	0.0167
,, ..	6 Ayrshire ..	$X_1 = 2.8054 + 0.050907X_2 + 0.128862X_3$	0.0423

It appears that, on the basis of an average figure for low yielding and relatively high yielding cows over a period of three months in mid-lactation, Shorthorn cows tended to increase their output by less than $\frac{1}{10}$ gallon of milk for each additional pound of SE and by over $\frac{1}{3}$ gallon for each pound of PE, while Friesians were inclined to produce over $\frac{1}{8}$ gallon—about 20 per cent more than Shorthorns—for each additional pound of SE, and about $\frac{2}{3}$ gallon—nearly twice as much as Shorthorns—for each additional pound of PE fed.

On the other hand, two groups of high yielding Ayrshires, on the average over a period of three months from the beginning of their lactations, tended to increase output by an insignificant amount—0.0184 gallon—for each additional pound of SE, and by less than $\frac{1}{4}$ gallon for each additional pound of PE.

The coefficients of multiple and partial correlation are given below, together with the standard errors of regression ($S_{1\cdot23}$) and the variance ratios (VR) by which the significance of the fit of the equations to the data has been measured. (The calculation of VR is given in Appendix IV.)

TABLE 14

Linear Three-Variable Regressions
Correlation Coefficients and Variance Ratios

Group of Herds	$R_{1\cdot23}$	$r_{12\cdot3}$	$r_{13\cdot2}$	VR
21 Shorthorn	0.9634***	0.589**	0.488*	108.45***
10 Friesian ..	0.9713***	0.774*	0.796*	58.34***
5 Ayrshire ..	0.9928**	0.651 n.s.	0.967*	56.235*
6 Ayrshire ..	0.9434*	0.723 n.s.	0.774 n.s.	15.568*

Significance : * = 0.5 level, ** = 0.1 level, *** = 0.01 level.

The degrees of significance which can be attributed to the above estimates can be gleaned from Table 14. Although the exclusion of one group of Ayrshires did not remove all the bias implicit in the Kirkhill experiment, it did increase the significance of R and made significant at least the coefficient of partial correlation of PE on output when SE was held constant;[1] moreover, the variance ratio improved, in spite of the loss of one degree of freedom, and it is likely that its significance would have been higher if there had been a few more observations.

(1) The conclusions on PE drawn by Holmes *et al.* from their Kirkhill experiments are diametrically opposite to those in this study—even for the 6 groups of Ayrshires. This is due to the fact (*a*) that those writers calculated protein in terms of digestible crude protein, while the composition of the roughages in their ABC groups was different from that in the XYZ groups—differences which could only be accounted for in terms of PE; and (*b*) that they adjusted maintenance requirements to the increases in weight caused by the various levels of feeding during the experiment, whilst in the context of feed input-milk output relationships such increases are simply waste and should be ignored.

The Linear Production Surfaces and Product Contours and Average (Constant) Rates of Nutrient Substitution

These have been calculated from the multiple regression equations.

The production surfaces are presented in tabular form in Appendix V. The top part of each table gives the estimated yields per cow resulting from various combinations of inputs of SE and PE within the range of observations for each group of herds. The bottom part shows the various inputs of SE and PE per gallon corresponding to the yields as estimated in the top half of the tables.

The product contours (iso-product curves) are presented in Table 15. This table shows the lowest and highest estimates of combinations of SE and PE, within the range of observations, which will tend to produce any given output.

The distortion caused by the bias of high PE observations in the Kirkhill experiments (6 Ayrshires) is particularly evident in the combination of high SE and low PE input estimates for an output of 3.6 gallons: the lowest input of PE in the experiment was 1.88 lb. per cow, equivalent to 0.523 lb. per gallon. Although this was coupled with only 11.2 lb. SE, it is not at all probable that, for an output of 3.6 gallons, an increase of 1.8 lb. SE to 13 lb. could bring down the PE requirement to 0.29 lb. per gallon.

From a comparison of the regression coefficients and the production surfaces in Appendices III and V it appears that, notwithstanding the good fit of the equations, the relationships cannot in fact be linear over the whole lactation. Although the regression surface of the Shorthorns is flatter than that of the Friesians, the difference between their slopes is small. It may express a genuine difference as between the nutrient substitution rates of these two breeds. But it is obvious that, irrespective of any peculiarity in the nutrient substitution rate that may be hidden in the Ayrshire data, the flat slope of this group's regression surface must be mainly due to the higher levels of its average outputs and inputs connected with the early stage of lactation. The results of the Kirkhill experiments with Ayrshires force one to the conclusion that, near their peak output, Shorthorn and Friesian cows also would have flattish production surfaces. It follows that, for the whole lactation range of a cow, the relationship is curved and cannot be expressed adequately by linear multiple regression equations.

The relationship is further illustrated by a graphic presentation of the product contours in Figure IV, in which both the Ayrshire groups are presented.

TABLE 15

Linear Product Contour Map

(Three monthly averages of daily inputs per cow and inputs per gallon and of output)

Output	lb. Inputs of Nutrients Above Maintenance															
	3 months mid lactation								3 months early lactation							
	21 Shorthorn				10 Friesian				5 Ayrshire				6 Ayrshire			
	SE		PE		SE		PE		SE		PE		SE		PE	
	per cow	per gall.	per cow	per gall.	per cow	per gall.	per cow	per gall.	per cow	per gall.	per cow	per gall.	per cow	per gall.	per cow	per gall.
2.0	4.0 5.5	2.0 2.75	1.20 0.78	0.6 0.39	4.0 5.5	2.0 2.75	1.0 0.72	0.5 0.36								
2.5	6.8 8.8	2.72 3.52	1.88 1.31	0.75 0.52	5.8 7.6	2.32 3.04	1.42 1.07	0.57 0.43								
3.0	9.6 12.0	3.2 4.0	2.54 1.87	0.85 0.62	7.7 9.8	2.57 3.27	1.84 1.45	0.61 0.48								
3.5					9.5 12.0	2.71 3.43	2.27 1.81	0.65 0.52								
3.6									11.0 13.0	3.06 3.61	1.92 1.75	0.53 0.49	11.0 13.0	3.06 3.61	1.82 1.03	0.51 0.29
3.9									13.0 15.0	3.33 3.85	3.09 2.93	0.79 0.75	13.0 15.0	3.33 3.85	3.36 2.57	0.86 0.66

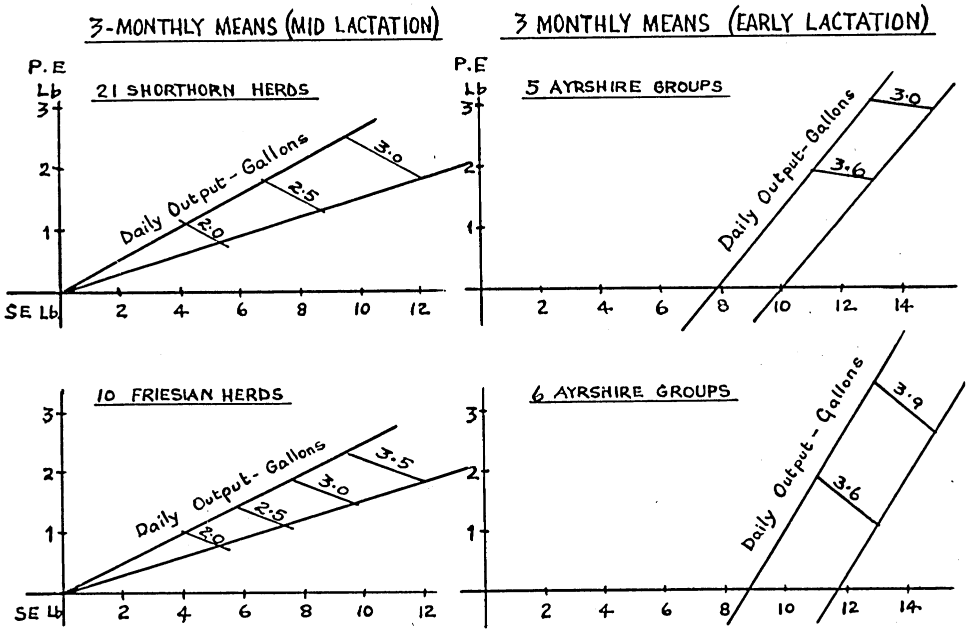

FIGURE IV

Linear Product Contour Maps (Iso-Product Curves) and Ridgelines. Daily Inputs per Cow above Maintenance and Daily Outputs

It is implicit that a production surface representing the relationship for the whole lactation should cut the origin (i.e. the bottom flat of the space in which it is placed): when a cow is dried off, zero inputs of SE and PE above maintenance are associated with zero output of milk. If a production surface begins (or, to be more correct, ends) at the bottom, then, on a product contour map which represents it on a plane, isoclines (ridge lines) connecting the limits of substitution between the contours must run through the origin. The isoclines may be linear or curved, in which case each would be concave to one axis; i.e. they would have the shape of a wedge or a cigar or an egg with the "point" touching the origin.

Although the limits of substitution cannot be defined from linear equations, they could approximately be determined by visual inspection of the scatters of the available observations. It may be seen in Figure IV that the contours of both the Shorthorn and the Friesian herds could roughly be contained between diagonal lines drawn through the origin, whereas the Ayrshire contours could only be contained by straight lines which would cross the horizontal axis considerably to the right of the origin, or by curved lines which could only pass through the origin if *both* were concave to the horizontal axis.

This suggests that average input-output relationships might be reasonably expressed by linear multiple regression over a range of outputs associated with all but the early stages—perhaps the first 13 to 15 weeks—

of a cow's lactation; but some other regression equations, which could express diminishing marginal productivity, would be needed in order to reveal the trend of the relationship over the whole range of outputs.

The range of substitution appears to be relatively narrow. Although the contours in Figure IV do not represent it in full, their extrapolation would soon express nonsensical combinations of nutrients. This would suggest that the "true" contours would be likely to curve upwards from some points near the isoclines, but would have only slight curvature between these points; i.e. that there would probably be all but perfect substitution along the product contours within the limits shown, but diminishing rates beyond these limits.

The rates of substitution are indicated by the slope of the product contours. They have also been calculated from the equations and are given in Table 16.

TABLE 16

Average (Constant) Rates of Nutrient Substitution

(3 months in mid-lactation)		(3 months in early lactation)	
21 Shorthorn	10 Friesian	5 Ayrshire	6 Ayrshire (strongly biased)
= 1 lb. SE will tend to substitute for :			
0.285 lb. PE	0.185 lb. PE	0.082 lb. PE	0.395 lb. PE
= 1 lb. PE will tend to substitute for :			
3.508 lb. SE	5.408 lb. SE	12.188 lb. SE	2.531 lb. SE

The average rates of substitution set out above can only be regarded as rough approximations, particularly those for the 5 groups of Ayrshires. (The rates for the original 6 Ayrshire groups are shown in order to demonstrate the extent and trend of their bias.) But they disclosed an important trend which provided a clue for subsequent analysis.

In discussing the slopes of the production surfaces it has been claimed that the flat slope of the (5) Ayrshires' production surface must be mainly due to this group's higher level of mean output, irrespective of any peculiarity in the rate of substitution which may be connected with the breed itself. But it is likely that the absolute level of yield also determines the rate of nutrient substitution within the function of any breed. The mean daily outputs were 2.35 gallons, 2.86 gallons and 3.81 gallons respectively for Shorthorns, Friesians and (5 groups of) Ayrshires. The slope of the product contours in Figure IV is steepest for the Shorthorns, somewhat flatter for the Friesians and a great deal flatter for the Ayrshires, and this picture is reflected in the rates of substitution given in Table 16. If the apparent connection between the mean yield of a group of herds and the rate of nutrient substitution is a genuine trend, then it should be possible to detect this trend within each breed group; substitution rates would widen with increasing yields. In that case some joint production function could be found which would fit the data better than an ordinary function and increase the accuracy of the estimates.(1)

CURVILINEAR MULTIPLE REGRESSION OF THE GROUP OF 21 SHORTHORN HERDS

At this stage it seemed hardly worth while to attempt curvilinear multiple regression analysis. The possibility could not be discarded, however, that a better fit might be obtained by using curvilinear equations. If this were the case the above conclusion, derived from the results of linear regression analysis, might have to be adjusted. The attempt has therefore been made.

A second-degree equation has been fitted to the data of the 21 Shorthorn herds by the method of least squares and simultaneous equations, as follows:

$$X_1 = 6.378 - 0.54416X_2 - 4.05876X_3 + 0.059985X_2^2 + 1.05541X_3^2$$

The variance ratio (VR) is 3.0174—only just significant at the 5 per cent level—while VR of the linear regression is 108.45, significant at the 0.01 level (see Table 14). The equation grossly over-estimates yields at the highest and lowest levels of input. In view of the very considerable difference between the variance ratios, there was no need to calculate the index of correlation and standard error of estimate of the quadratic equation. It could also be assumed that even a third-degree equation would not fit the data better than the linear one (though better than the quadratic equation). It was apparent that roughly similar results could be expected from an attempt to fit curvilinear regressions to the considerably smaller sample of Friesians, and the Ayrshires could not sustain an equation with five terms in any case. Curvilinear analysis was therefore abandoned at this stage.

JOINT REGRESSION ANALYSIS

Since the preceding analysis indicated that the relationship is practically linear within the range of each group of data, simple joint functions could be chosen. These could take three different forms, according to the nature of the joint relationship.

If the regression of X_1 (Output) on X_3 (PE) is substantially linear for any given value of X_2 (SE) and the slope of the regression $b_{13\cdot 2}$ changes at a constant rate with changes in X_2, the regression would have the form of equation (A):

$$X_1 = a + bX_3 + g(X_2X_3)$$

But the regression of X_1 on X_2 may be linear for any given value of X_3 and the slope of the regression $b_{12\cdot 3}$ may change with changes in X_3 when the equation (B) would be:

$$X_1 = a + bX_2 + g\,(X_2X_3)$$

(1) An "ordinary" three-variable production function determines the separate effect of each independent variable on the dependent one; a joint function determines the dependent variable resulting from any given combination of the independent variables.

Yield, however, may vary with inputs of SE or PE for other reasons than the effect of either (SE or PE) on the regression of yield on the other. If this is the case then the regression will need an additional term, as in equation (C):

$$X_1 = a + b\ X_2 + c\ X_3 + g\ (X_2X_3)$$

There was no indication as to the nature of the joint relationship. All three equations had therefore to be fitted and their fit compared with each other and with that of the multiple regression equation (D). Only the Shorthorn sample was large enough to permit precise deductions as to what relationships might be expected in other samples. But, even if the Friesian and Ayrshire samples were too small to provide by themselves any definition of a "law" of input-output, they could nevertheless supply bases for hypotheses with regard to these particular breeds, on two conditions, namely: that the "law" itself could be established on an adequate sample (i.e. the 21 Shorthorn herds) and that these hypotheses would agree with it.

Consequently two different criteria were applied for the final choice of the function. Its general validity was determined in the Shorthorn sample by the highest variance ratio with the resulting lowest standard error of estimate adjusted for the small number of observations ($\bar{S}_{1\cdot23}$). For the function thus chosen, hypothetical validity (best fit) could then be assumed for any of the smaller samples if it resulted in standard errors of estimate, *unadjusted* for the number of observations ($S_{1\cdot23}$)(1) which were lower than $S_{1\cdot23}$ of any other equation which had been fitted.

In Table 17 the three joint regression equations, as fitted to the data of the 21 Shorthorn herds, are represented together with measurements of the accuracy of the estimates. The measurements of the linear three-variable regression are also given for comparison.

All the above variance ratios are significant at the 0.1 per cent points, (A), (B) and (D) with 2 and 18 degrees of freedom and (C) with 3 and 17 d.f. The first two joint equations have VR's lower than the multiple regression equation, but the joint equation with an additional term has a VR considerably higher. The differences between the degrees of exactness in the fit of these equations are given diminishing emphasis by the other measurements of estimates, $\bar{S}_{1\cdot23}$ showing the larger and $S_{1\cdot23}$ the smaller difference.

It is evident that equation (C) fits the data considerably better than any of the other equations, while the joint equations without an additional term give poorer fits than the multiple regression.

Equation (A), which gave the poorest fit, has now been rejected from further analysis. The remaining two joint equations have been fitted to the data of the 10 Friesian herds, and are presented in Table 18. For purposes of comparison, the measurements of accuracy are also given.

(1) $S_{1\cdot23}$ is a useful measure of exactness of fit in a sample in absolute terms, whether or not it can be regarded as a precise guide to what may be expected in another sample or in the universe.

TABLE 17

Three Joint Regression Equations Fitted to Data of 21 Shorthorn Herds and Comparison with Multiple Regression

Equation	$S_{1\cdot 23}$	$\bar{S}_{1\cdot 23}$	VR
(A) $X_1 = 0.93133 + 1.1501X_3 - 0.0241X_2X_3$	0.1308	0.1412	78.009
(B) $X_1 = 1.29262 + 0.152312X_2 + 0.003213X_2X_3$	0.1219	0.1317	91.006
(C) $X_1 = 0.2075 + 0.178231X_2 + 1.387769X_3 - 0.099388X_2X_3$	0.0612	0.0655	264.054
(D) Multiple Regression	0.1014	0.1216	108.45

TABLE 18

Two Joint Regression Equations Fitted to Data of 10 Friesian Herds and Comparison wtih Multiple Regression

Equation	$S_{1\cdot 23}$	$\bar{S}_{1\cdot 23}$	VR
(B) $X_1 = 1.79241 + 0.02775X_2 + 0.06322X_2X_3$	0.1791	0.2141	36.2437
(C) $X_1 = -0.027 + 0.2354X_2 + 1.1895X_3 - 0.063834X_2X_3$	0.1315	0.1695	39.9260
(D) Multiple Regression	0.1432	0.1712	58.340

The VR's shown above are also significant at the 0.1 per cent points, (B) and (D) with 2 and 7 degrees of freedom and (C) with 3 and 6 d.f. Equation (B) has the poorest fit on all counts. From the standard errors of Equation (C), however, it is obvious that its VR would have been higher than that of the multiple regression if the sample had been somewhat larger.(1) The $\bar{S}_{1\cdot 23}$ of this equation is (fractionally) smaller than that of Equation (D), notwithstanding the arithmetical effect on such a small sample of the loss of one degree of freedom caused by the joint equation's additional term.

Finally, the comparative exactness in the fit of the three equations can be studied in Table 19, in which are presented the yields estimated from them, the observed yields and the residuals.

The best distribution of the estimates around the regression, with the resulting greatest reduction of the squared residuals, is given by Equation (C). Since this equation fulfils both criteria previously set for the choice of the function, the assumption can be made that it provides the best available definition of the input-output relationship for the Friesian sample.

Equation (B) can now be finally rejected. In Table 20 is presented the joint equation (C), with an additional term as fitted to the data of the group of 5 Ayrshires, and its measures of accuracy are compared with those of the multiple regression.

It is not surprising that, with 3 and 1 degrees of freedom (1 d.f. for the smaller mean square), the VR of Equation (C) is not significant. The VR of the multiple regression with 2 and 2 d.f. is significant at the 5 per cent point. In this very small sample the additional constant has increased the adjusted standard errors; but the structural relation of the variables is defined by the $S_{1\cdot 23}$ and by this criterion the joint equation (C) with additional term still gives a better fit. This can be seen from Table 21, in which the estimates obtained from the two equations are compared.

(1) Owing to the re-distribution of the degrees of freedom between the regression sum of squares and the error sum of squares.

TABLE 19

10 Friesian Herds
Yields Estimated by Three Equations, and Residuals

Equation	Multiple Regression			Joint Regression (B)			Joint Regression (C)		
Observed X_1	Estimated X_1'	$X_1 - X_1'$ (Z')	$(Z')^2$	Estimated X_1''	$X_1 - X_1''$ (Z'')	$(Z'')^2$	Estimated X_1'''	$X_1 - X_1'''$ $(Z)'''$	$(Z''')^2$
3.9	3.87	— 0.03	0.0009	3.95	0.50	0.0025	3.83	— 0.07	0.0049
3.8	3.87	0.07	0.0049	3.85	0.50	0.0025	3.81	0.01	0.0001
3.2	3.31	0.11	0.0121	3.20	0	0	3.37	0.17	0.0289
3.0	2.83	— 0.17	0.0289	2.73	— 0.27	0.0729	2.89	— 0.11	0.0121
2.9	2.69	— 0.21	0.0441	2.66	— 0.24	0.0576	2.73	— 0.17	0.0289
2.6	2.42	— 0.18	0.0324	2.43	— 0.17	0.0289	2.41	— 0.19	0.0361
2.6	2.57	— 0.03	0.0009	2.58	— 0.02	0.0004	2.61	0.01	0.0001
2.4	2.61	0.21	0.0441	2.60	0.20	0.0400	2.65	0.25	0.0615
2.2	2.19	— 0.01	0.0001	2.29	0.09	0.0081	2.11	— 0.09	0.0081
2.0	2.24	0.24	0.0591	2.30	0.30	0.0900	2.18	0.18	0.0324
Sums 28.6	28.60	0	0.2275	28.59	— 0.01	0.3029	28.59	— 0.01	0.2131

TABLE 20

Joint Regression Equation with Additional Term Fitted to Data of 5 Ayrshire Groups of Cows and Comparison with Multiple Regression

Equation	$S_{1\cdot 23}$	$\bar{S}_{1\cdot 23}$	VR
(C) $X_1 = 1.574328 + 0.142511X_2 + 0.644398X_3 - 0\ 037653X_2X_3$	0.01571	0.03513	25.941
(D) Multiple Regression	0.01844	0.02916	56.235

TABLE 21

5 Ayrshire Groups

Yields Estimated by Two Equations and Residuals

Equation	Multiple Regression (D)			Joint Regression (C) with Add. Term		
Observed X_1	Estimated X_1'	$X_1 - X_1'$ (Z')	$(Z')^2$	Estimated X_1''	$X_1 - X_1''$ (Z'')	$(Z'')^2$
3.97	3.959	— 0.011	0.000121	3.963	— 0.007	0.000049
3.94	3.961	0.021	0.000441	3.953	0.013	0.000169
3.80	3.800	0	0	3.799	— 0.001	0.000001
3.77	3.745	— 0.025	0.000625	3.755	— 0.015	0.000225
3.58	3.595	0.015	0.000225	3.589	0.009	0.000081
Sums 19.06	19.06	0	0.001412	19.059	— 0.010	0.000525

Again, Equation (C) gives the greater reduction of the squared residuals.

Although the bias contained in the original sample of 6 Ayrshire groups has been sufficiently demonstrated in the analysis preceding this section, the joint regressions (B) and (C) have been fitted to its data and compared with the multiple regression as a matter of interest. (As will be seen in the following section, this "pedantic" step gave an unexpected clue to the solution of a baffling problem.)

With 3 and 2 degrees of freedom the VR of Equation C is not significant. The VR's of Equations (B) and (D) with 2 and 3 d.f. are significant at the 5 per cent level. Again, Equation (C) has the smallest $S_{1\cdot 23}$ and Equation (B) the largest.

It appears, therefore, that the relationship can be defined as a joint one of SE and PE on output and expressed by the joint regression (C) with an additional term. On the other hand, it should be borne in mind that the samples are inadequate to carry conclusions which would express the underlying relationship with the reliability of a law. Conclusions reached in this study can only be theoretical until they are confirmed by suitable experiments.

Peculiarities of Joint Function with Additional Term Revealed by Product Contours

The calculation of a product contour map for the sample of 21 Shorthorn herds from Equation (C) revealed some unexpected results. Higher inputs (above maintenance) of both nutrients in various combinations were associated with higher, though proportionately less high outputs, up to an output of PE of just over 1.75 lb. At this point output appeared to be 2.67 gallons *irrespective of inputs of SE.* Inputs of PE of above 1.75 lb. were accompanied by higher output only if inputs of SE were also higher. On the other hand, holding PE constant at any level above 1.75 lb., and at the same time increasing inputs of SE, appeared to depress yields. This would denote diminishing physical output—a textbook phenomenon implicit in very high levels of input which is rarely encountered in biological experiments and hardly ever in practice. In this sample even

TABLE 22

Original 6 Ayrshire Groups
Two Joint Regression Equations and Comparison with Multiple Regression

Equation	$S_{1\cdot23}$	$\bar{S}_{1\cdot23}$	VR
(B) $X_1 = 3.23779 + 0.01291X_2 + 0.011373X_2X_3$	0.04101	0.05731	13.324
(C) $X_1 = 0.26275 + 0.27599X_2 + 0.91205X_3 - 0.06945X_2X_3$	0.03667	0.06353	7.406
(D) Multiple Regression	0.03784	0.05348	15.568

highest average inputs were a good deal below stomach capacity; and visual inspection of the scatter of the observations around the product contours which are presented in Figure V dispelled the possibility of diminishing physical output. It seemed, therefore, that the chosen equation rotated the production surface a little too much.

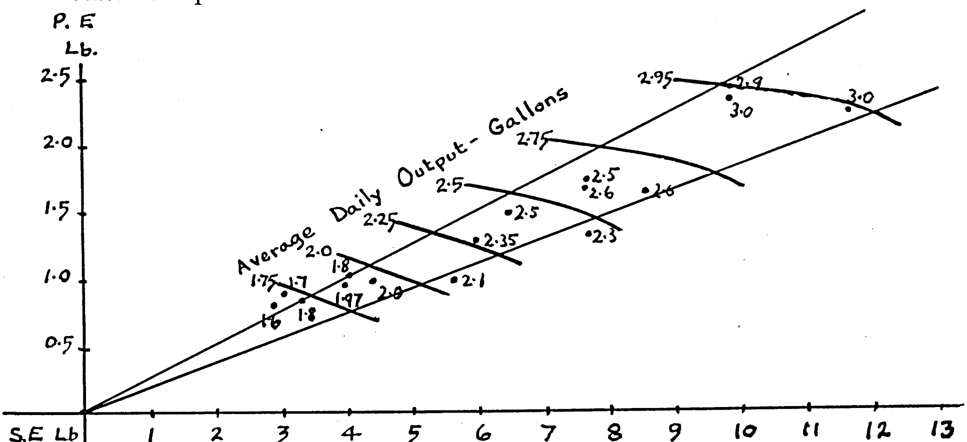

FIG. V. 21 Shorthorn Herds—*Scatter of Observations around (Joint) Product Contours*

Nevertheless, product contour maps calculated by this function for the samples of 10 Friesian herds and 5 Ayrshire groups revealed that the "exaggerated" rotation of the surface could not be a "built-in" peculiarity of the function when fitted to any set of data, since in these two samples higher inputs were associated with higher outputs (although outputs were proportionately less than inputs), but without reaching points beyond which outputs would actually appear to diminish. For these samples the product contours and the scatter of the data around them are presented in Figures VI and VII.

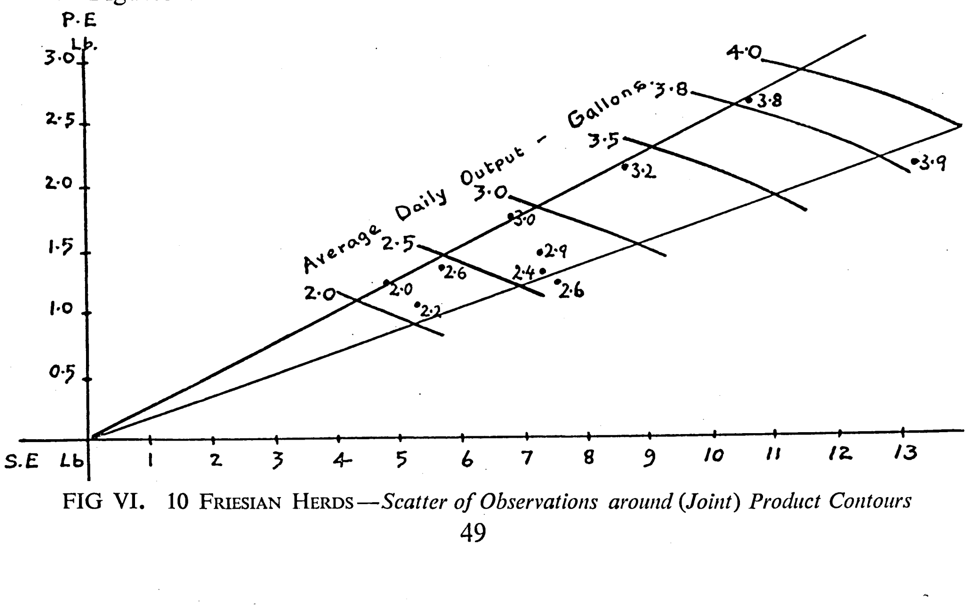

FIG VI. 10 Friesian Herds—*Scatter of Observations around (Joint) Product Contours*

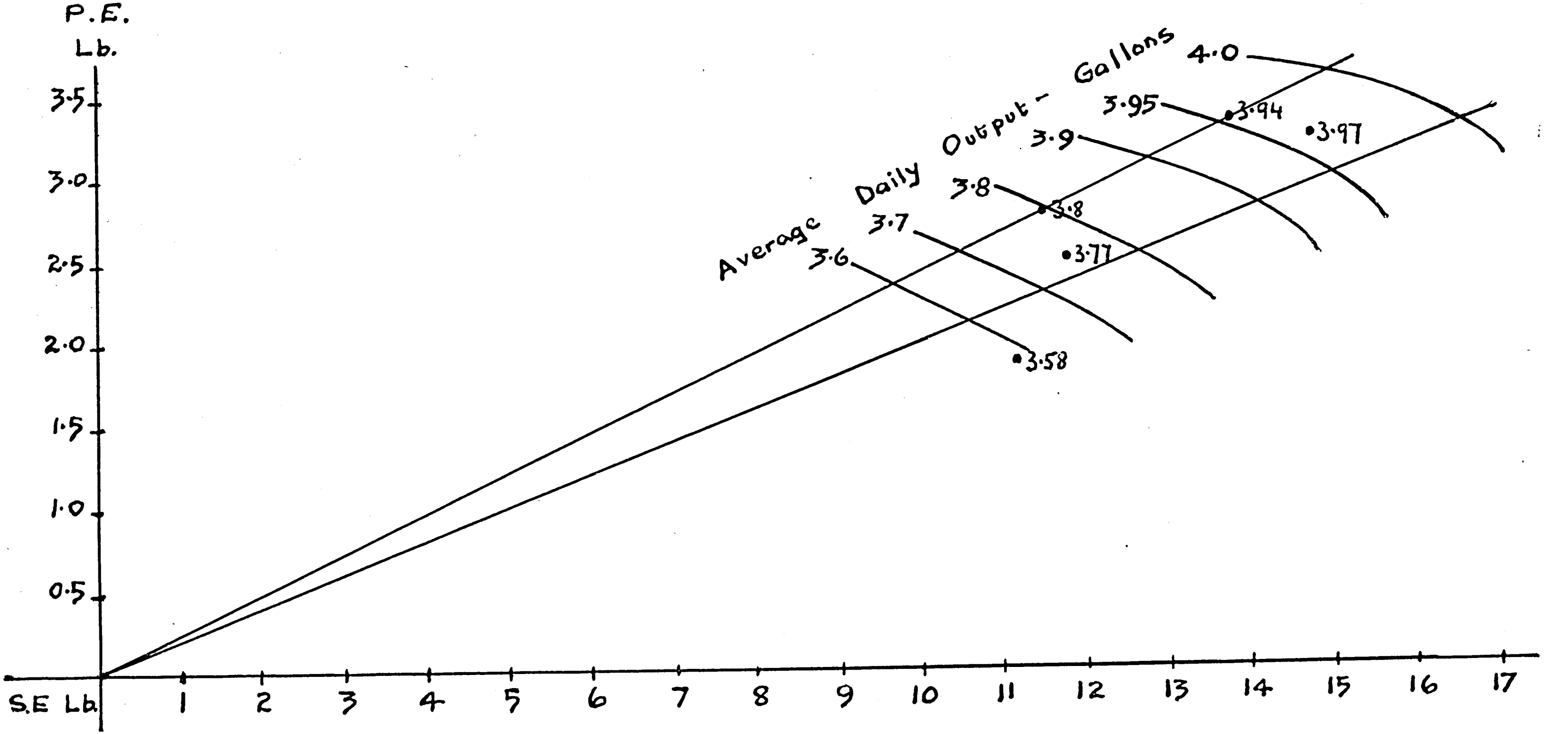

FIGURE VII

5 Groups of Ayrshires

Scatter of Observations around (Joint) Product Contours

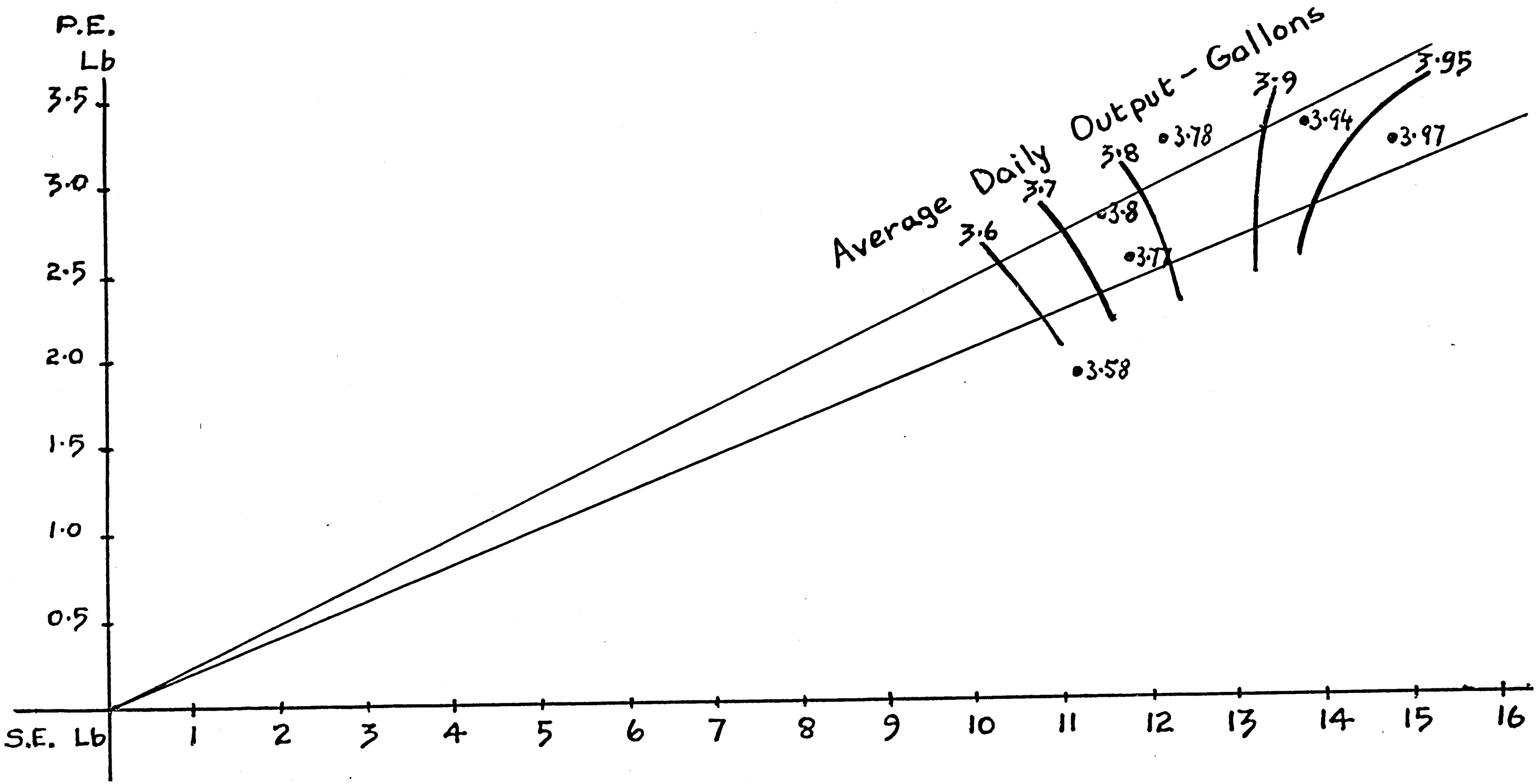

FIGURE VIII
6 Groups of Ayrshires
Scatter of Observations around (Joint) Product Contours

Finally, product contours were calculated for the original 6 Ayrshire groups. As can be seen from Figure VIII, they had much steeper slopes than those of the other samples. They also showed an association between input and output—the latter increasing at a diminished rate up to an input of just under 12.8 lb. SE, when output appeared to be 3.85 gallons *irrespective of inputs of PE.* Inputs of SE of above 12.8 lb. appeared to be connected with higher outputs only when inputs of PE were also higher. But holding SE constant at any level above 12.8 lb., and simultaneously increasing inputs of PE, caused outputs to diminish at an increasing rate.

The analogy with the Shorthorn product contours is apparent. Since the original sample of 6 Ayrshire groups had a known bias (PE) which appeared to be largely corrected by the rejection of the offending set of observations, it could be assumed that the excessive rotation of the production surface of the original 6 Ayrshires was connected with this bias. This gave the clue for an adjustment of the Shorthorn data which would provide a rational production surface. The adjustment will be discussed in the following section.

Another peculiarity of Equation (C) is that the product contours, while they tend to be linear at relatively low levels of input, show a slight and increasing curvature at high levels as they approach the contour around which the surface is rotated (i.e. the contour beyond which diminishing physical outputs appear to take place). The curvature is convex to the origin, thus implying *increasing* rates of marginal substitution as between SE and PE. In theory, marginal substitution for any level of output would take place at (nearly) constant rates or towards the limits of substitution at any level of input, at diminishing rates, and product contours would be straight lines or curves concave to the origin.

It is conceivable that the convex shape of the curves might be due to a slight under-estimate of PE in the roughages. For example, a small error might occur in the conversion of digestible crude protein into PE (according to Woodman's formula PE $= \frac{1}{2}x$ per cent dig. crude protein + dig. true protein); since for a given output the proportion of PE from roughages is implicitly higher in high SE rations than in low ones, protein in high SE rations, when expressed as PE, might be under-valued a little relatively to the protein in low SE rations. As a result, substitution of SE for PE might appear to occur at increasing marginal rates. While such a possibility cannot be disregarded entirely, it is not a likely one. The fact that the product contours nearer to the origin tend to be linear, and that the curvature increases in the area of the highest observations and finally swings out in the opposite direction (see Figures V and VIII), indicates that the convex shape of the curves is due to a peculiarity of the equation. Another aspect of this peculiarity is exemplified in Figure VII, in which the contours can be seen converging downwards towards a point below the horizontal axis, whereby an increasing degree of convexity is forced on them. Last, but not least, within the ranges of observation the slopes of the contours are practically flat; they are inclined to change drastically beyond the ridge lines. It is likely that the contours would be linear even at the highest rational levels of inputs or that they would show a curvature concave to the origin if more observations were available and if they had wider ranges.

In view of the insignificant (and in the writer's opinion irrational and accidental) curvature of the contours, the latter have been treated as if they were linear through the points of intersection with the ridge lines denoting the approximate range of the observations.

THE ADJUSTMENT OF THE DATA OF 21 SHORTHORN HERDS

If it is assumed that, in the 6 Ayrshire groups, the bias caused by the excessive number of observations with high PE inputs resulted in product contours for Equation (C) which ultimately became concave to the PE (vertical) axis, then the fact that those contours for the 21 Shorthorn herds ultimately became concave to the SE (horizontal) axis could imply a bias caused by an excessive number of observations with relatively high inputs of SE. Inspection of Figure V, however, supplied no evidence of this. But it did bring home the fact that, out of 21 observations with outputs ranging from 1.6 to 3 gallons, 9 had outputs of 2.5 and 2.6 gallons. It came to mind that the set of observations removed from the original sample of 6 Ayrshire groups had an output which fell at about the middle of that sample's range, and that 2 or more observations in that sample had practically identical outputs. Thus, out of 6 observations with outputs ranging from 3.58 to 3.97 gallons, three clustered between 3.77 and 3.8 gallons.

This could signify that Equation (C) is "allergic" to distributions in which a considerable proportion of the dependent variables are clustered near their mean.

In order to test the above proposition some method had to be found by which the distribution of the scatter of the 21 Shorthorn herds along the regression could be "improved".

Since this sample is not an experimental one, there was no possibility of breaking up the cluster of nine observations with outputs of 2.5 and 2.6 gallons by getting rid of the "offenders"—it was not even clear which, if any, of these observations were "offending". But it seemed acceptable to reduce the nine observations to four, by grouping their outputs in three pairs and one group of three and taking the means of the associated inputs.

Thus there would be two "observations", each of 2.5 and 2.6 gallons, instead of the original five and four observations respectively. Since there was a suspicion that some bias might be caused by observations with relatively high inputs of SE it was decided to group according to inputs of SE: three higher and two lower inputs at 2.5 gallons and two higher and two lower at 2.6 gallons.

In order to obtain some information about any possible bias caused by SE, these observations were also grouped according to higher and lower inputs of PE and, finally, according to the order in which they appeared in the columns: the first three and second two at 2.5 gallons and the first and second pairs at 2.6 gallons. The means resulting from the three methods of grouping are given overleaf:

TABLE 23

21 Shorthorn Herds

9 Clustered Observations of Output Reduced to 4 by averaging inputs in 3 different ways

Output Gallons	Inputs of SE	Means SE	Means PE	Inputs of PE	Means SE	Means PE	Both Inputs	Means SE	Means PE
2.6	2 higher	8.58	1.635	2 lower	8.45	1.610	2 second	8.32	1.625
	2 lower	7.67	1.675	2 higher	7.80	1.700	2 first	7.93	1.685
2.5	3 higher	7.69	1.710	3 higher	7.68	1.717	3 first	7.69	1.710
	2 lower	6.50	1.465	2 lower	6.50	1.455	2 second	6.50	1.465

TABLE 24

Linear Three Variable Regression Equations (three monthly averages in mid-lactation) Fitted to Reduced Sample of 16 Shorthorns, and Comparison with Original Equation

Groups	Equation	$\bar{S}_{x_{1\cdot 23}}$	VR
16 Shorthorns	$X_1 = 1.80722 + 0.111907X_2 + 0.285929X_3$	0.1081	132.382***
21 Shorthorns	$X_1 = 1.2016 + 0.09699X_2 + 0.34022X_3$	0.1216	108.450***

TABLE 25

Joint Regression Equations with Additional Term fitted to Reduced Sample of 16 Groups of Shorthorn Herds and Comparison with Original Equation

Group	Equation	$\bar{S}_{x_{1 \cdot 23}}$	VR
16 Shorthorn	$X_1 = 0.504602 + 0.176979X_2 + 0.964188X_3 - 0.06838X_2X_3$	0.08103	160.750***
21 Shorthorn	$X_1 = 0.2075 + 0.178231X_2 + 1.387779X_3 - 0.099388X_2X_3$	0.0655	264.054***

The differences between both the higher and the lower "observations" averaged by the three different methods are very small. This indicates that there can be no bias in the data which it would be worth while to consider. (It also appears that there was a higher inter-correlation between inputs of SE and PE associated with outputs of 2.5 gallons than with those of 2.6 gallons, but this is irrelevant in this context.) Although grouping by PE gave fractionally better results, by minimising the difference between the means of the observations of SE, the original decision to group by SE was maintained when the number of observations was reduced from 21 to 16.

Reduced Sample of 16 Shorthorn Sets. Multiple and Joint Regression Analysis

The equations and measurements of accuracy for the 16 sets are shown on page 54. For the sake of comparison those for the original 21 Shorthorn herds are also shown.

Reducing the sample by the method adopted resulted in only fractional changes of the multiple regression coefficients: the new equation attributes a little more output to SE and a little less to PE than the original one did. This is due to the fact that the reduction of observations to 16 lowered the mean output and gave more weight to the cluster of observations with low outputs and inputs. The average rates of substitution are therefore biased towards lower yields, whereas those rates calculated from the original observations may be a little biased in the opposite direction. The comparison of those rates of substitution for the original and reduced samples is as follows:

16 Shorthorn:		1 lb. SE will substitute for (0.39 lb. PE)
21	,,	1 lb. SE will substitute for (0.29 lb. PE)
16	,,	1 lb. PE will substitute for (2.56 lb. SE)
21	,,	1 lb. PE will substitute for (3.51 lb. SE)

In Table 25 is given the joint regression equation with additional term (C) fitted to the "group" of 16 Shorthorns, that for the 21 herds being added for the sake of comparison.

The equation does not "fit" the reduced sample as well as did the original one derived from 21 sets of observations. But its VR is still significant at the 0.01 level and it gives a better fit than the other regressions shown in Tables 24 and 17.

In the joint regressions the coefficients are not directly comparable. The effect of the reduction of the number of observations on these regressions can be seen from a comparison of the product contour maps for the original 21 herds with the reduced Shorthorn sample presented in Figure IX.

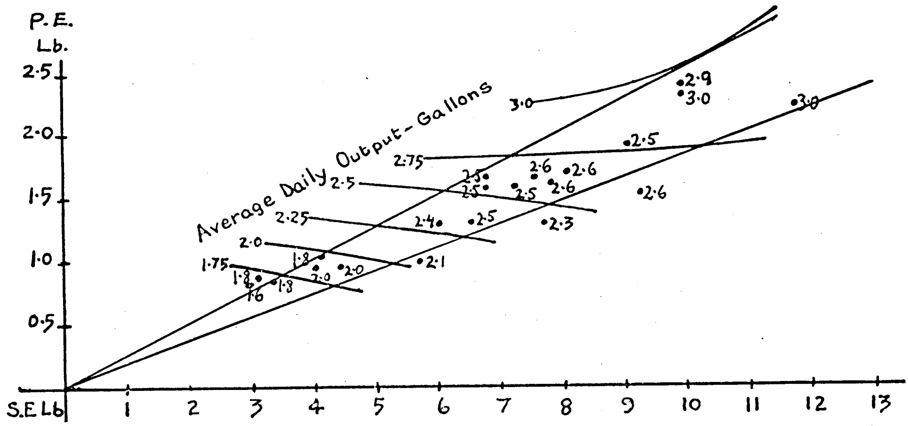

FIGURE IX. 16 Groups of Shorthorns—*Scatter of Observations around (Joint) Product Contours*

It can be seen from this figure that the iso-product curves, including the 2.95 gallon contour—the highest—now fit the relevant observations very well. Further calculations concerning Shorthorns have, therefore, been based on the reduced sample.

Joint Production Surfaces and the Marginal Productivity of the Nutrients

These have been calculated from the joint equations with additional term.

The production surfaces are presented in tabular form in Appendix VI. The lower part of that table shows the inputs of SE and PE per gallon (average inputs) which correspond to the yields given for total input in the higher part. At low input levels the total productivity of the nutrients is practically the same for Shorthorns as for Friesians, but with larger inputs it becomes nearly 29 per cent higher for the latter. The Ayrshires appear to have in early lactation a production surface roughly similar to that of the Friesians in mid-lactation.

The marginal productivity of the nutrient inputs for the three breeds is given in Table 26. On the left-hand side of this Table are shown the marginal outputs, for $\frac{1}{10}$ lb. increments of PE, which are associated with inputs of SE held constant at various total levels. Even at low input levels, the marginal productivity of PE is nearly one-third lower with Shorthorns than with Friesians; at higher levels of input it appears to be two-thirds lower with Shorthorns. The corresponding proportions for the Ayrshires at high levels of input are nearly one-half and over one-half less respectively than those for the Friesians. Again, the marginal productivity of PE diminishes at a rate which increases most for the Shorthorns and least for the Friesians.

On the right-hand side of Table 26 the marginal outputs for one lb. of SE are given, as associated with inputs of PE held constant at various total levels. The differences between the breeds in the marginal conversion rates of SE into milk appear to be roughly proportional to the differences in the conversion of PE. In each case, when there is a high input of one

TABLE 26

Marginal Productivity of Nutrients above Maintenance Computed from Production Surfaces in Appendix IV and within the input ranges given in that Table

When Average Daily Input of SE* Held Constant at lb.	Marginal Output per 0.1 lb. PE			When Average Daily Input of PE* Held Constant at lb.	Marginal Output per 0.1 lb. SE		
	16 Shorthorn gallon	10 Friesian gallon	5 Ayrshire gallon		Shorthorn gallon	Friesian gallon	Ayrshire gallon
4	0.070	—	—	—	—	—	—
5	0.063	0.088	—	0.9	0.115	—	—
6	0.057	0.081	—	1.2	0.095	0.160	—
7	0.050	0.074	—	1.5	0.075	0.138	—
8	0.042	0.067	—	1.8	0.055	0.120	0.080
9	0.035	0.060	—	2.1	0.034	0.102	0.060
10	0.028	0.054	—	2.4	0.013	0.085	0.050
11	0.021	0.048	0.025	2.7	—	—	0.040
12	0.014	0.043	0.020	3.0	—	—	0.030
13	—	0.037	0.016	3.3	—	—	0.020
14	—	—	0.011	—	—	—	—
15	—	—	0.007	—	—	—	—

*Above Maintenance Requirements

nutrient, the marginal productivity of the other is only a fraction of what it would be at a lower input of that nutrient.

Another conclusion to be drawn from Table 26 is that the highest yielding Shorthorns and Ayrshires tended to be quite near the maximum physical levels of output, whereas the marginal productivity (particularly of SE) of the Friesians was still some distance from that point. This implies that, in this investigation, the Friesians were a relatively better "herd" than the Shorthorns or even the Ayrshires, though the latter were an institutional (Hannah Research Institute) herd and probably above the average of their breed.

The Product Contours and the Marginal Rates of Nutrient Substitution

The product contours given in Figures VI, VII and IX have been tabulated in Table 27. This shows the highest and lowest estimates of combinations of SE and PE, within the range of observations, which tend to produce any given output.

TABLE 27

Product Contours within Isoclines which delimit the Observations (Ranges of Inputs of Nutrients for Given Outputs) 3 monthly averages per cow

Average Daily Output gallons	Daily Input of Nutrients above Maintenance, lb.					
	16 Shorthorns Mid-lactation		10 Friesians Mid-lactation		5 Ayrshires Early lactation	
	Daily Input of Nutrient, above Maintenance, between:					
	lb. SE with	lb. PE	lb. SE with	lb. PE	lb. SE with	lb. PE
1.75	3.48 4.05	0.86 0.77	— —	— —	— —	— —
2.00	4.36 5.11	1.08 0.94	4.35 5.26	1.09 0.90	— —	— —
2.25	5.36 6.30	1.32 1.16	— —	— —	— —	— —
2.50	6.45 7.76	1.60 1.42	5.75 6.92	1.43 1.19	— —	— —
2.75	7.88 9.60	1.97 1.77	— —	— —	— —	— —
3.00*	9.72 12.27	2.43 2.22	7.24 8.80	1.80 1.50	— —	— —
3.50	— —	— —	9.07 11.1	2.27 1.90	— —	— —
3.70	— —	— —	— —	— —	10.52 11.47	2.52 2.30
3.80	— —	— —	10.45 12.78	2.61 2.20	11.55 12.65	2.76 2.52
3.90	— —	— —	— —	— —	12.85 14.08	3.07 2.82
3.95†	— —	— —	11.47 14.08	2.89 2.44	13.69 15.00	3.27 3.00

*Shorthorn 2.95 gallons
†Friesians calculated for 4 gallons

TABLE 28

Marginal Rates of Substitution Between Inputs of SE and PE above Maintenance. Averages over 3 monthly periods. Shorthorns and Friesians in mid lactation, Ayrshires in early lactation

When Daily Output per cow gallons	21 Shorthorn	16 Shorthorn	10 Friesian	5 Ayrshire	21 Shorthorn	16 Shorthorn	10 Friesian	5 Ayrshire
	$\frac{\Delta X_3}{\Delta X_2}$ = 1 lb. SE substitutes for PE				$\frac{\Delta X_2}{\Delta X_3}$ = 1 lb. PE substitutes for SE			
	lb.	lb.	lb.	lb.	lb.	lb.	lb.	lb.
1.75	0.110	0.192	—	—	9.11	5.18	—	—
2.00	0.098	0.187	0.209	—	10.30	5.36	4.79	—
2.25	0.074	0.170	0.206	—	13.60	5.88	4.90	—
2.50	0.053	0.145	0.202	—	18.89	7.00	4.96	—
2.75	—	0.116	0.199	—	—	8.60	5.05	—
3.00*	—	0.082	0.192	—	—	12.14	5.20	—
3.50	—	—	0.181	—	—	—	5.51	—
3.70	—	—	0.180	0.231	—	—	5.57	4.32
3.80	—	—	0.176	0.225	—	—	4.68	4.44
3.90	—	—	0.174	0.215	—	—	5.75	4.64
3.95	—	—	0.173	0.206	—	—	5.77	4.85

*Shorthorn 2.95 gallons

The combinations of lower SE and higher PE have nutrient ratios of roughly 4 : 1, whereas those of higher SE and lower PE have ratios above 5.5 : 1. The nutrient ratios of the group of 5 Ayrshires work out at 4.2 : 1 and 5 : 1 respectively for low SE/high PE and high SE/low PE combinations. This may be partly due to the bias in PE implicit in the feeding of this group, but it is largely the result of the narrow range of observations in the experiment.

It should be borne in mind that for the other two breeds also the limits of these product contours are approximately those of the available observations. It is possible that in practice these limits could be extended somewhat in both directions. But it is likely that such extended contours would soon become increasingly curved, when the rates of nutrient substitution would change not only between the contours but also along them. The fitted equations permit only the computation of marginal rates of substitution which change as between various yields but are constant at any given yield. These rates of substitution are shown in Table 28. They are presented also for the group of 21 Shorthorns, alongside those for the reduced sample of 16, in order to demonstrate the more rational estimates achieved by using the sample of 16.

At low yields, i.e. at low levels of feeding, 1 lb. SE tends to substitute for roughly 1/5 lb. PE, the latter figure being somewhat less with Shorthorns and a little more with Friesians. At higher yields it would appear that 1 lb. SE will substitute for only 1/10 lb. PE with Shorthorns, but still for nearly 1/5 lb. PE in the case of Friesians. At the highest average yields the rate of substitution for 1 lb. SE will still tend to be 1/6 lb. PE with Friesians and, bearing in mind a possible bias, just over 1/5 lb. PE with Ayrshires. Alternatively 1 lb. PE would appear to substitute for roughly 5 lb. SE with Shorthorns or Friesians at low yields, rising as high as 12 lb. with Shorthorns at higher yields, while with Friesians the rate would tend to be less than 6 lb. even at the highest average yields. With Ayrshires it would appear to be less than 5 lb.

The above rates of substitution can only be regarded as gross approximations. Certainly they cannot be used as a yardstick for any precise inter-breed comparisons. But they reveal the trends underlying nutrient substitution, explain much about some feeding systems, and permit a rational, albeit rough, measurement of the quantities involved in them. The relatively high rate at which PE tends to substitute for SE at high levels of feeding (yield) reveals that a high PE content in home-grown bulky foods and pasture tends to save not only the protein that would have had to be bought if the content had been low, but also some of the total quantity of the food that needs to be fed, since substitution would effect a saving in SE.

The fact that there can be substitution of carbohydrates and protein for each other, and the ranges of nutrient substitution which have been established here, provide a theoretical explanation for the relative successes of the two broad systems of milk production which have been associated in this country with the names of Rex Patterson and Professor Boutflour. The first is based on the extensive use of grazing and bulky foods and on a sparing application of concentrates. It depends on the substitution of large quantities of SE for relatively small ones of PE, and sacrifices high yields for the sake of inexpensive feeding. On the iso-product curves in Table 27 the average nutrient requirements of such a system, for various levels of yield, are represented by the higher limits of SE with lower PE. The Boutflour system, which depends on an intensive use of concentrates and aims at high yields, substitutes relatively small amounts of PE for large quantities of SE. Its nutrient requirements appear to be near the lower limit of SE with higher PE. (The economic superiority of either system depends on a complicated combination of a number of factors, namely: prices of concentrates and milk, breed and yield capacity of the cows, quality of the grass swards, grassland management and preservation, skill in rationing for production and cowmanship, availability of capital for additional buildings and cows and, last but not least, the attraction of alternative uses of the available capital.)

Part Three

A NEW SYSTEM OF RATIONING (*a*)

THE INTER-LACTATION PRODUCTION FUNCTION

Conventional Balanced Cakes

From the product contours it is possible to calculate the total requirements of feed, over a period of time, for "average" cows of the three main breeds. In most cases the ration will be calculated from roughages or grazing for maintenance plus some production and, separately, from balanced concentrates for production above that quantity. Usually there will be a surplus of SE in the roughage, above the quantity necessary for maintenance and whatever output is expected from the roughage. The production of PE above this surplus SE can be calculated from the (if necessary, extended) product contours. But it is simpler to balance the surplus of SE with some high protein feed and to use balanced concentrates for the remainder of the ration. This remainder can be regarded as the variable input which changes at different rates at different levels of yield—although conventional feeding standards do not take account of it.

Balanced concentrates appear on the market in two main forms: "ordinary" dairy cake of which 4 lb. for each gallon over maintenance is usually recommended, and "high production" dairy cake, of which $3\frac{1}{2}$lb. per gallon is recommended. The former contains about 62.5 SE and 13.25 PE, whereas the latter contains approximately 72.5 SE and 15.4 PE. Both cakes have a nutrient ratio of about 4.7 : 1 and tend to cost practically the same per unit of nutrient. The only practical difference between them lies in their dry matter content and in the quantity of each required per unit of output, which is 12.5 per cent less for "$3\frac{1}{2}$ lb. per gallon cake" than for ordinary ("4 lb. per gallon") cake. If the rates of feeding with one of them are known then the necessary quantities of the other can be calculated from these rates.

Best production rations of "4 lb. cake" (thereafter referred to as ordinary cake or simply cake) have been compiled from the product contour maps by the graphical method. On the product contours combinations of SE and PE have been selected in which the quantity of each nutrient, divided by its content in the cake, had to be equal: $\frac{\text{lb. SE}}{62.5} = \frac{\text{lb. PE}}{13.25} = \text{lb.}$ ordinary cake. If correct, these combinations would all lie on a straight isocline going through the origin of the contour maps. In Table 29 the ordinary cake isoclines are given for the three breeds in tabular form, together with the total inputs (above maintenance requirements), calculated on each nutrient separately, and the marginal and average inputs of cake per gallon.

TABLE 29

Total, Marginal and Average Input of Cake (6.25 SE with 13.25 PE) at Various Levels of Yield Above Maintenance Requirements

(Shorthorns and Friesians: 3 monthly averages in mid-lactation = lactation averages)
(Ayrshires: 3 monthly averages in early lactation)

Average daily output gallons	Nutrient Isocline for given combination SE lb.	PE lb.	Calculated on 62.5 SE cake lb.	13.25 PE cake lb.	Marginal Input of cake lb.	lb.	Average Input of cake per gallon lb.
16 Shorthorn							
					per ¼ gallon	per gallon	
1.75	3.78	0.81	6.05	6.08			3.46
2.00	4.76	1.01	7.62	7.62	1.57	6.28	3.81
2.25	5.85	1.24	9.36	9.36	1.74	6.96	4.16
2.50	7 20	1 52	11.52	11.47	2.16	8.64	4.61
2.75	8 88	1.88	14 13	14 14	2.61	10.44	5.14
2.95	11 02	2 34	17.63	17.66	3.50*	17.50	5.98
10 Friesian							
					per ½ gallon	per gallon	
2.00	4.75	1.01	7.60	7.60			3.80
2.50	6.30	1.32	10.08	9.96	2.48	4.96	4.03
3.00	7.94	1.68	12.70	12.68	2.62	5.24	4.22
3.50	9.98	2.12	15.97	16.00	3.27	6.64	4.56
4.00	12.70	2.71	20.32	20.45	4.45	8.90	5.08
5 Ayrshire							
					per $\frac{1}{10}$ gallon	per gallon	
3.70	11.17	2.38	17.87	17.96			4.83
3 80	12.26	2.60	19.62	19.62	1.75	17.50	5.16
3 90	13.68	2.91	21.89	21.92	2.27	22.70	5.61
4.00	16.09	3.42	25.74	25.77	3.85	38.50	6.44

*Per $\frac{1}{5}$ gallon

It appears that the average lactation input of cake, per gallon of output, varied from less than 3.5 lb. to about 6 lb. with Shorthorns and 5 lb. with Friesians—the latter's higher yields notwithstanding. The Kirkhill Ayrshires are not directly comparable with the other breeds, owing to a different period of lactation. For comparable outputs their input per gallon seemed to fall between those of the Shorthorns and those of the Friesians. It is, however, the marginal inputs that determine the limits of economic input at any given level of input cost and output price or value. For example, at an average price for milk amounting to 3s. per gallon and for cake amounting to £33 a ton (3.5d. per lb.), the optimum input-output position would be reached when marginal input per gallon was $\frac{36\text{d.}}{3.5\text{d.}} = 10.2$ lb. of ordinary cake. It will be seen from Table 29 that only the marginal inputs for the Friesians do not need to exceed 10.2 lb. of cake for any average level of output. Marginal input requirements of the Shorthorns for average yields in excess of about 2.7 gallons would tend to be higher than 10.2 lb. It also appears that, on the average, the 5 Ayrshire groups in the Kirkhill experiment were fed above the best economic level.

Balanced Concentrates with Higher than Conventional PE Content and Narrower Nutritive Ratio

It may be asked whether the present ratio of SE to PE contained in balanced dairy cakes is the best one from the economic point of view.

Since, within technical limits, one pound of PE can substitute for several pounds of SE, it would be economic to increase the proportion of PE in the ration within these limits until the price of one lb. SE equalled the price of $\frac{x}{n}$ lb. PE, n being the (marginal) rate of substitution; or, to put it differently, until the price ratio of SE/PE equalled the nutritive ratio of the concentrates mixture. The price ratio of PE to SE in concentrates seems to be around 3 : 1,[(1)] while the nutritive ratio tends to be considerably higher. This suggests that the proportion of PE to SE in concentrates should be higher than the conventional one of 4.5 : 1. The lowest nutritive ratios within the range of available observations are found along the isoclines representing the lowest inputs of SE in combination with the highest inputs of PE for any given output. In Table 30 these isoclines have been tabulated as read from the product contour maps. From them the concentrates curves have been calculated for mixtures of 62.5 SE, at which the best PE values ranged from 15 per cent, as calculated from the Ayrshire data, to 15.6 and 15.7 per cent, obtained respectively from the Shorthorn and Friesian data. Estimates of total input requirements of such non-conventional concentrates are given, as calculated on each nutrient separately; and the marginal and average inputs are also shown.

(1) See the author's "The Relative Costs and Values of Protein in Purchased Dried Grass, Dried Lucerne and Concentrates". *The Farm Economist*, VII, 7, 1954.

TABLE 30

Least Cost Nutrient Composition of Concentrates. Total, Marginal and Average Input of these Concentrates at Different Levels of Yields

(Shorthorns and Friesians: lactation averages; Ayrshires: 3 monthly averages in early lactation)

Average Daily Output gallons	Least Cost Isoclines lb.	lb.	Total Input of Concentrates Calculated on (*a*), (*b*) and (*c*) below SE lb.	PE lb.	Input of Least Cost Concentrates: Marginal lb.	lb.	Input of Least Cost Concentrates: Average per gallon lb.
	16 Shorthorn. Least Cost Concentrates: (*a*) 15.6% PE, 62.5 SE						
					per ¼ gallon	per gallon	
1.75	3.78	0.86	5.57	5.51			3.18
2.00	4.36	1.08	6.98	6.92	1.41	5.64	3.49
2.25	5.36	1.32	8.58	8.46	1.54	6.16	3.81
2.50	6.45	1.60	10.32	10.26	1.80	7.20	4.12
2.75	7.88	1.97	12.61	12.63	2.37	9.48	4.59
2.95	9.72	2.43	15.55	15.58	2.94	11.76	5.27
	10 Friesian. Least Cost Concentrates: (*b*) 15.7% PE, 62.5 SE						
					per ½ gallon	per gallon	
2.00	4.35	1.09	6.96	6.94			3.48
2.50	5.75	1.43	9.20	9.11	2.24	4.48	3.68
3.00	7.74	1.80	11.58	11.46	2.38	4.76	3.86
3.50	9.07	2.27	14.51	14.46	2.93	5.86	4.15
4.00	11.47	2.89	18.35	18.41	3.84	7.68	4.59
	5 Ayrshires. Least Cost Concentrates: (*c*) 15% PE, 62.5 SE						
					per 1/10 gallon	per gallon	
3.70	10.52	2.52	16.83	16.80			4.55
3.80	11.55	2.76	18.48	18.40	1.65	16.50	4.86
3.90	12.85	3.07	20.56	20.47	2.08	20.80	5.27
4.00	15.05	3.60	24.03	24.00	3.52	35.20	6.02

The following mixtures would provide the composition of nutrients given above:

			PE%
(*a*) Mixture ..	86% cake	=	11.40
	14% high protein cake (30% PE)	=	4.20
			15.60
(*b*) Mixture ..	85.5% cake	=	11.33
	14.5% high protein cake (30% PE)	=	4.35
			15.68
(*c*) Mixture ..	89% cake	=	11.79
	11% high protein cake (30% PE)	=	3.30
			15.09

If Table 30 is compared with Table 29, it appears that increasing the proportion of PE from the conventional 13.25 per cent to about 15.7 per cent would tend to diminish the necessary average inputs of 62.5 SE cake by roughly 0.3 lb. per gallon at low average levels of yield and 0.4 to 0.7 lb. per gallon at high levels. Marginal inputs would also diminish.

An increase of the PE content in balanced cake without a corresponding increase in SE would cost very little. If the new mixture contained ordinary cake and high protein cake of 30 PE, the necessary proportions in 100 lb. would be as follows:

85.5 lb. balanced cake at 13.25 PE	=	11.33 lb. PE	(at £33 per ton)
14.5 lb. high protein cake at 30.0 PE	=	4.35 lb. PE	(at £38 per ton)
100.0 lb. new type cake	=	15.68 lb. PE	(at £33.73 per ton)

With ordinary cake at £33 per ton, i.e. 3.54d. per lb., and H.P. cake at £38 per ton, one ton of the new mixture would cost £33.725, equivalent to 3.61d. per lb.

But the cost per gallon (or per cow) of the new concentrate would be less than that of the conventional cakes. This is shown in Table 31, in which the inputs of cake for the Shorthorns and Friesians, as given in Tables 29 and 30, have been calculated on the basis of the costs set out above.

Although the cost of cake with the suggested higher PE content would be nearly 15/- per ton more than that of conventional cake, total costs (per cow/day) and average costs (per gallon) would be over 6.5 per cent less at low average output and 8—10 per cent less at high output. Marginal costs would be still lower.

It would appear that Ayrshires would benefit slightly less from an increase of PE in cake to 15 per cent. It is, however, impossible to attach much significance to the exact proportion of PE in a "best nutrient combination" calculated from the small Ayrshire sample with its known bias. In view of the crudeness of the Shorthorn and Friesian samples, and their small sizes, neither of them separately would give a reliable basis for a "best nutrient combination" in balanced concentrates. On the other hand, in view of the differences between the samples and the different adjustments to which they have been subjected it is surprising that the results should have turned out so similar. It would appear that a narrowing of the nutritive ratio in the production ration would be economic. In balanced concentrates, with 62.5 SE, an increase of PE from 13.25 to about 15.5 (or, with 72.5 SE, from 14.9 PE to about 17.45 PE) might lower the cost per cow and per gallon by roughly 6.5 to 7.5 per cent and the marginal cost by perhaps 8 or 9 per cent. Precise quantities could be determined only by experiment.

TABLE 31

Total, Marginal and Average Costs above Maintenance

Conventional cake of 62.5 SE, 13.25 PE (column I) at £33.00 per ton
and suggested cake of 62.5 SE, 15.70 PE (column II) at £33.73 per ton
at various average outputs

Average Daily Output	16 Shorthorn						10 Friesian					
	Cost of Concentrates											
	Total		Marginal*		Average		Total		Marginal*		Average	
	I	II	I	II	I	II	I	II	I	II	I	II
gallons	d.	d.	d.	d.	d.	d.	d.	d.	d.	d.	d.	d.
2.0	26.97	25.20			13.49	12.60	26.90	25.13			13.45	12.56
			13.81	12.06					8.78	8.08		
2.5	40.78	37.26			16.32	14.87	35.68	33.21			14.27	13.28
			21.63	18.88					9.28	8.59		
3.0	62.41	56.14			21.17	19.02	44.96	41.80			14.97	13.93
			—	—					11.57	10.58		
3.5	—	—			—	—	56.53	52.38			16.14	14.98
			—	—					15.40	13.86		
4.0	—	—			—	—	71.93	66.24			17.98	16.57

*per 0.5 gallon

The Different Production Functions of Cows of Various Yield-Capacities

So far the analysis has resulted in the computation of average (inter-lactation) input-output curves for the three breeds with which this study has been concerned. On the average a Shorthorn or Friesian cow, fed a certain quantity of nutrients (over maintenance) during the three months in mid-lactation, will tend to produce a certain quantity of milk. If the quantities of nutrients are altered by given amounts, the milk yields will be altered by predictable quantities. It can be assumed that mid-lactation data can be raised to represent the total lactation, since the daily averages will be roughly similar in both cases. The average input-output curves of Shorthorn and Friesian cows in this study can, therefore, be regarded as those for the whole lactations.(1) On the other hand the Kirkhill data for Ayrshires refer to three months early in the lactation and so cannot be raised to represent full lactation. As a result, Ayrshires have had to be left out of the following computations.

If the yielding capacities of cows, even of one breed, differed only slightly, then the "average" input-output curves could serve as a useful guide in budgeting rations. Actually, however, the productive capacities of cows differ very widely. In theory each cow has her own production function which, moreover, is variable in time. In practice cows of one breed can be classified according to their yield capacity, with separate input-output curves computed for each class.

Very large samples of animals, in suitably designed feeding experiments or recording schemes, would be needed in order to obtain sufficient data for statistical determination of production functions varying with productive capacity. But speculative estimates of such functions, which appear to be adequate, can nevertheless be based even on the existing data.

Irrespective of the cows' breed and productive capacity, their input-output curves always have to run through the origin. The lower parts of curves may be taken to be similar up to average daily outputs of about 2 gallons, i.e. roughly 600 gallons per lactation; and they must fan out at higher outputs. Points on the curves which are above the 2-gallon one can be determined by using as standards for their computation the average input-output curves of the breeds, as found in the preceding analysis. The underlying logic is that, at more intensive inputs, a "better" cow will produce higher yields when fed at the same rate as a cow of lower capacity.

In Table 32 is shown the computation of 2 points above the 2-gallon one for 3 curves each of the Shorthorn and Friesian cows of higher than the average yield capacities. It seemed convenient to classify the capacities of the cows in 200-gallon yield intervals, since this was roughly the range of the yields above the average outputs in the data raised to 305 days' lactations. The actual average input and output have been rounded off to the nearest 100-gallon output points, along the "average" curves.

(1) E.g. it can be estimated that an "average" Friesian cow fed at an average daily rate of 10 lb. of ordinary (4 lb. per gallon) cake or equivalent above maintenance, i.e. a rate of 3,080 lb. over a lactation of 305 days, would produce a daily average of 2.5 gallons of milk or about 760 gallons in the lactation. The same cow or one of similar productive capacity could be forced to an average yield of 4 gallons daily—about 1,220 gallons per lactation—by increasing her average daily rate of intake of "4 lb. cake" or equivalent to 20½ lb., equal to 6,157 lb. per lactation.

TABLE 32

Computation of Some Points in the Higher Parts of Input-Output Curves (Cake above Maintenance)

Shorthorn and Friesian Cows of Different Yield Capacities. Averages of 305 days' Lactations

Yield Capacity (galls.)	700 raised to 900*		900 raised to 1100**		1100 raised to 1300		1300 raised to 1500		1500 raised to 1700	
	Shorthorn Cows									
Averages of:										
Daily Output	2.30 g.	2.95 g.	2.95 g.	3.61 g.	3.61 g.	4.26 g.	4.26 g.	4.91 g.	—	
Daily Input of Cake	9.80 lb.	17.63 lb.	12.58 lb.	21.59 lb.	15.38 lb.	25.48 lb.	18.15 lb.	29.36 lb.	—	
Cake per gallon	4.26 lb.	5.98 lb.	4.26 lb.	5.98 lb.	4.26 lb.	5.98 lb.	4.26 lb.	5.98 lb.	—	
	Friesian Cows									
Averages of:										
Daily Output	—		2.95 g.	3.61 g.	3.61 g.	4.26 g.	4.26 g.	4.91 g.	4.91 g.	5.57 g.
Daily Input of Cake	—		12.51 lb.	16.80 lb.	15.31 lb.	19.83 lb.	18.06 lb.	22.88 lb.	20.85 lb.	25.94 lb.
Cake per gallon	—		4.24 lb.	4.65 lb.	4.24 lb.	4.65 lb.	4.24 lb.	4.65 lb.	4.24 lb.	4.65 lb.

**Shorthorns:* Rounded-off figure. Averages of 16 herds: capacity 717 gallons = daily 2.35 gallons. Input 10.2 lb. cake equivalent daily per cow = 4.34 lb. per gallon.

***Friesians:* Rounded-off figure. Averages of 11 herds: capacity 872 gallons = daily 2.86 gallons. Input 12.50 lb. cake equivalent daily per cow = 4.20 lb. per gallon.

In the first column of this Table the average and the high output points on the input-output curves of Shorthorns and Friesians are used as standards, in order to compute these points for the curves of cows with lactation capacities 200 gallons higher. For example, the "average Friesian cow" producing 900 gallons, equivalent to an average of 2.95 gallons daily, will need on the average 12.51 lb. of ordinary cake over maintenance, i.e. an average of 4.24 lb. per gallon. The same cow can be pushed to 1,100 gallons, i.e. 3.61 gallons daily, if fed on the average a daily equivalent of 16.8 lb. ordinary cake, i.e. 4.65 lb. per gallon. But a better cow, with a productive capacity 200 gallons higher, may produce 1,100 gallons at an average input of 4.24 lb. per gallon (for which the poorer cow produced only 900 gallons), i.e. of 15.31 lb. cake daily; and when forced to 1,300 gallons, i.e. 4.26 gallons daily, will require 4.65 lb. cake per gallon on which the "standard" cow would produce only 3.61 gallons. In turn the average daily outputs assumed, and inputs computed, for the cow of 1,100-gallons' capacity forced to 1,300 gallons, have been used for the calculation of daily inputs for a 1,300-gallon cow forced to 1,500 gallons, and so on.

The above method was used to draw the average daily cake equivalent input-output curves, for Shorthorn and Friesian cows of various yield capacities, which are presented in Figure X.

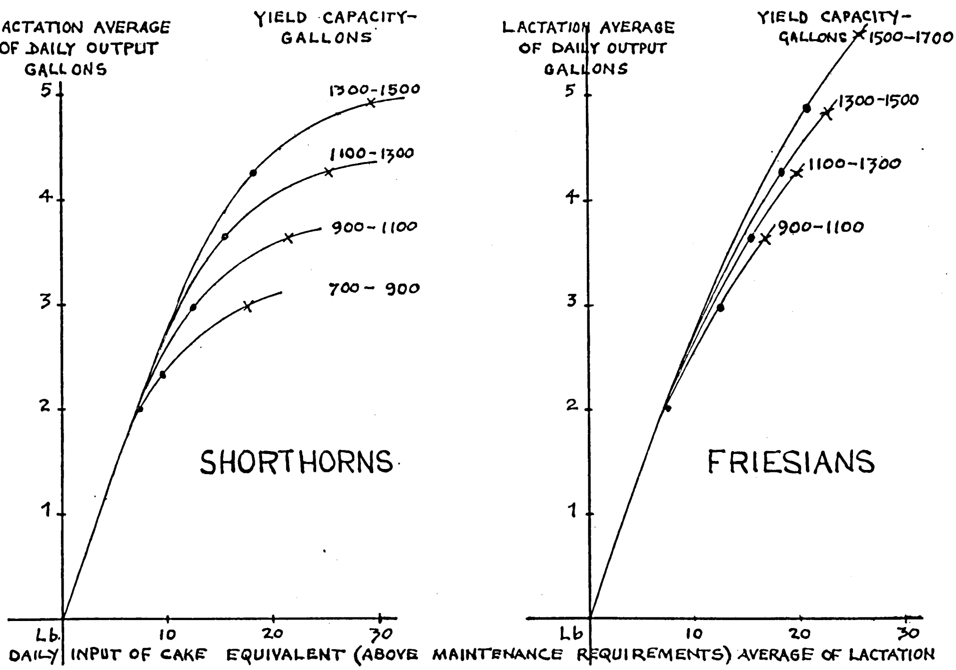

FIGURE X

Average Daily Cake Input-Milk Output (Yield) Curves of Cows of Various Yield Capacities

These curves are shown in tabular form in Tables 33 and 34. Table 33 represents a tabulated input (cost) surface, where the quality of the cows is one independent variable and input is the other. The inputs of cake equivalent above maintenance are given in 4-lb. intervals, in order to demonstrate the magnitude of the deficiency in output as compared with conventional expectations.

The similarity of the output estimates of the lower input classes, as between Shorthorns and Friesians of equal yield capacity, is perhaps due to the method of drawing the lower parts of the curves. In actual fact there may be differences between the breeds at these levels of input, but they cannot be large and would be very difficult to define with any precision.

In the higher input classes, Friesians of comparable yield capacities appear to produce considerably more milk from given inputs of cake equivalent than do Shorthorns. The differences seem to lessen with increasing yield capacity.

In Table 33, the centre column of each yield capacity class expresses the given output as a percentage of the conventional standard of 1 gallon for each 4 lb. of cake. If those percentage figures are examined it will be seen that the present yield estimates tend to exceed the conventional ones by 2—5 per cent at low inputs. Increased inputs will produce yields varying from 61 per cent of the conventional figure for "average" Shorthorns to 70 per cent for very high-capacity ones, and from 78 per cent to 82 per cent of that figure for average and top-grade Friesian cows. It would also appear that top-quality Shorthorns (of 1,300—1,500-gallon yield capacity) can produce over 500 gallons, and their Friesian counterparts (of 1,500—1,700-gallon capacity) over 1,200 gallons, on the average of conventional inputs; and that this category of cow can produce considerably—perhaps 250 gallons—more milk on average inputs which are only slightly above the conventional standards.

Table 34 takes the form of a tabulated production surface where yield capacities of the cows are one independent variable and outputs the other, inputs (of cake equivalent) being dependent on both. It presents estimates of the daily averages of lactation inputs which would be necessary for given yields from cows of various productive capacities, together with the corresponding inputs per gallon and the marginal inputs per half-gallon. Average inputs appear to rise from about 3.8 lb. cake equivalent per gallon at low average yields to over 6 lb. when low-capacity Shorthorns are forced to over 900 gallons per lactation. Medium-yielding Shorthorns and Friesians capable of over 900 gallons will need on the average over $5\frac{1}{4}$ lb. for each gallon if forced—the former to over 1,100 gallons and the latter above 1,200 gallons. Cows with yield capacities exceeding 1,100 gallons can be forced to produce considerably beyond that output on average inputs of below 5 lb. per gallon.

The Best Average Input

The marginal inputs given are not "point" ones but averages for half-gallon increments. In order to be directly useful, however, marginal curves in tabular form would have to be presented in very small intervals. It is therefore more convenient to define the best economic inputs graphically from marginal input curves. In Figure XI are shown the marginal

TABLE 33

Average and Lactation Output Schedules, Shorthorn and Friesian Cows of Different Yield Capacities

Estimated Yields which can be obtained at various levels of feeding and as a percentage of those which would be expected at the standard basis of 4 lb. cake per gallon

Daily Average Input above Maintenance equivalent to Cake	Yield Capacity of Cows, Gallons														
	700—900			900—1100			1100—1300			1300—1500			1500—1700		
	Output per Cow, Gallons														
	Average daily	% of Standard	Total lactation	Average daily	% of Standard	Total lactation	Average daily	% of Standard	Total lactation	Average daily	% of Standard	Total lactation	Average daily	% of Standard	Total lactation
lb.	Shorthorns														
8	2.03	101.5	619	2.05	102.0	622	2.18	104.0	634	2.10	105.0	641	—	—	—
12	2.57	85.7	784	2.84	94.7	866	2.98	99.3	909	3.02	100.7	921	—	—	—
16	2.87	71.8	875	3.30	82.5	1007	3.67	91.7	1119	3.88	97.0	1183	—	—	—
20	3.06	61.2	933	3.52	70.4	1074	5.00	80.0	1220	4.42	88.4	1348	—	—	—
24	—	—	—	3.68	61.3	1122	4.21	70.2	1874	4.70	78.3	1434	—	—	—
28	—	—	—	—	—	—	4.36	62.3	1330	4.89	69.9	1491	—	—	—
	Friesians														
8	—	—	—	2.04	102.0	693	2.18	104.0	634	2.10	105.0	641	2.10	105.0	641
12	—	—	—	2.88	96.0	878	2.98	99.3	909	3.05	101.7	930	3.12	104.0	951
16	—	—	—	3.52	88.0	1074	3.72	93.0	1135	3.88	97.0	1183	4.01	100.2	1223
20	—	—	—	3.90	78.0	1190	4.28	85.6	1305	4.56	91.2	1391	4.80	96.0	1464
24	—	—	—	—	—	—	—	—	—	4.93	82.0	1504	5.37	89.5	1638
28	—	—	—	—	—	—	—	—	—	—	—	—	5.73	81.9	1748

TABLE 34

Input-Output (Cake) Curves

Shorthorn and Friesian Cows of Different Yield Capacities

When Yield:		Yield Capacity of Cows, Gallons														
		700—900			900—1100			1100—1300			1300—1500			1500—1700		
		Input equivalent to lb. cake														
Total lactation, gallons	Average daily gallons	daily	marg.*	per gall.	daily	marg.*	per gall.	daily	marg.*	per gall.	daily	marg.*	per gall.	daily	marg.*	per gall.
		Shorthorns														
610	2.0	7.8		3.90	7.8		3.90	7.7		3.85	7.6		3.80	—	—	—
760	2.5	11.3	3.5	4.52	10.1	2.3	4.04	9.9	2.2	3.96	9.8	2.2	3.92	—	—	—
915	3.0	18.4	7.1	6.13	13.0	2.9	4.33	12.2	2.3	4.07	12.0	2.2	4.00	—	—	—
1070	3.5	—	—	—	18.3	5.3	5.23	14.8	2.6	4.23	14.2	2.2	4.06	—	—	—
1220	4.0	—	—	—	—	—	—	20.0	5.2	5.00	16.7	2.5	4.18	—	—	—
1370	4.5	—	—	—	—	—	—	—	—	—	20.9	4.2	4.54	—	—	—
1525	5.0	—	—	—	—	—	—	—	—	—	(31.0)	(10.1)	(6.20)	—	—	—
		Friesians														
760	2.5	—	—	—	10.1		4.04	9.7		3.88	9.6		3.84	9.4		3.76
915	3.0	—	—	—	12.8	2.7	4.27	12.1	2.4	4.03	11.8	2.2	3.90	11.5	2.1	3.83
1070	3.5	—	—	—	15.9	3.1	4.54	14.7	2.6	4.20	14.2	2.4	4.06	13.7	2.2	3.91
1220	4.0	—	—	—	21.3	5.4	5.33	17.7	3.0	4.45	16.7	2.5	4.18	16.0	2.3	4.00
1370	4.5	—	—	—	—	—	—	22.2	4.5	4.93	19.6	2.9	4.36	18.4	2.4	4.09
1525	5.0	—	—	—	—	—	—	—	—	—	23.7	4.1	4.74	21.2	2.8	4.24
1675	5.5	—	—	—	—	—	—	—	—	—	—	—	—	25.2	4.0	4.58

*Marginal input per half-gallon

input curves from Table 34 on a "per gallon" basis: that is to say, the inputs per half-gallon increment in that Table have been multiplied by two.

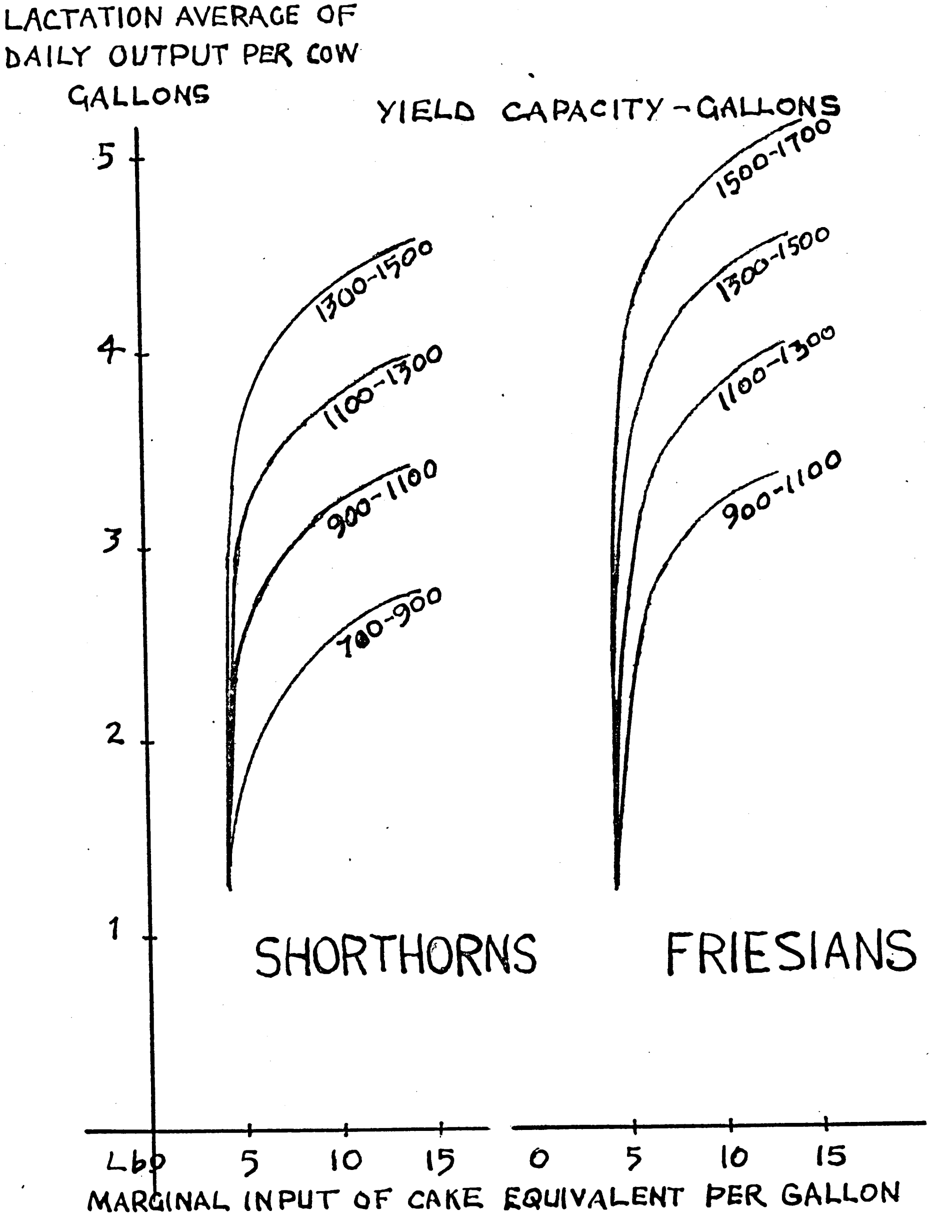

FIGURE XI

Marginal Input Curves of Cake Equivalent

Lactation Averages of Cows of Various Yield Capacities

The best average input is obtained when the average price of milk is equal to the cost of the marginal input of feed. In farming practice the calculation of the economic equilibrium in feeding will hardly ever be concerned with bulky foods or grazing, i.e. with "cake equivalent". The marginal input will nearly always be in the form of concentrates—"4 lb. per gallon cake" in this instance. The economic equilibrium can be found by first dividing the average value of one gallon of milk by the average cost per lb. of cake; this will give the maximum quantity of cake which it would pay to feed per additional gallon of milk produced per average day of a lactation. When in Figure XI this marginal input of cake is then read horizontally from its point on the curves on the vertical scale, the optimum average yield according to the given yield capacity of a cow is obtained.

For example, when milk is worth on the average 38d. per gallon net of charges, and cake is 3.6d. per gallon, the best marginal input will be $\frac{38}{3.6} = 10.55$ lb. The best average yield per Shorthorn cow would then range from about 2.6 gallons daily, i.e. 790 gallons per lactation for a 700—900 galloner, to about 4.4 gallons, i.e. 1,340 for a potential 1,300—1,500 galloner. The best yield for Friesian cows would range from an average of about 3.25 gallons daily, equivalent to 900 gallons per lactation, for a cow of 900—1,100 gallons' capacity, to 4.95 gallons daily, equivalent to about 1,500 per lactation for a potential 1,500—1,700 galloner. These best average outputs would be somewhat greater for autumn-calving cows and less for spring-calving ones, owing to the differences in the average prices of milk. They would have a tendency to be still larger in the case of autumn calvers, since the spring flush of grass is inclined to force up the yields of cows in the later stages of lactation beyond what would be economic levels of feeding concentrates, by providing them with nutrients above the requirements for their moderate yields.

There is difficulty in assessing a cow's yield capacity. Past performance on conventional rations obviously under-estimates it. In the majority of cases the most reliable pointer is the cow's yield at the peak of a lactation. In the last part of this study a method will be suggested by which yield capacity can be estimated from peak of lactation yields with the help of lactation curves.

Although there is as yet only a vague economic appreciation of a cow's yield capacity among both farmers and cattle-dealers, sooner or later more definite estimates of this capacity will have to be made and will find expression in the prices of cows. Until a cost or value can be put upon it, product contours in which it is a variable may be deemed to be of mostly theoretical interest. Such product contours can, however, help to assess in a rough manner the value of cows of yield capacities that are "known" or that it is possible to estimate. Although the product contours can be read from the tabulated production surfaces (horizontal lines of average daily inputs in Table 34), it is more convenient to use the graphic form presented in Figure XII.

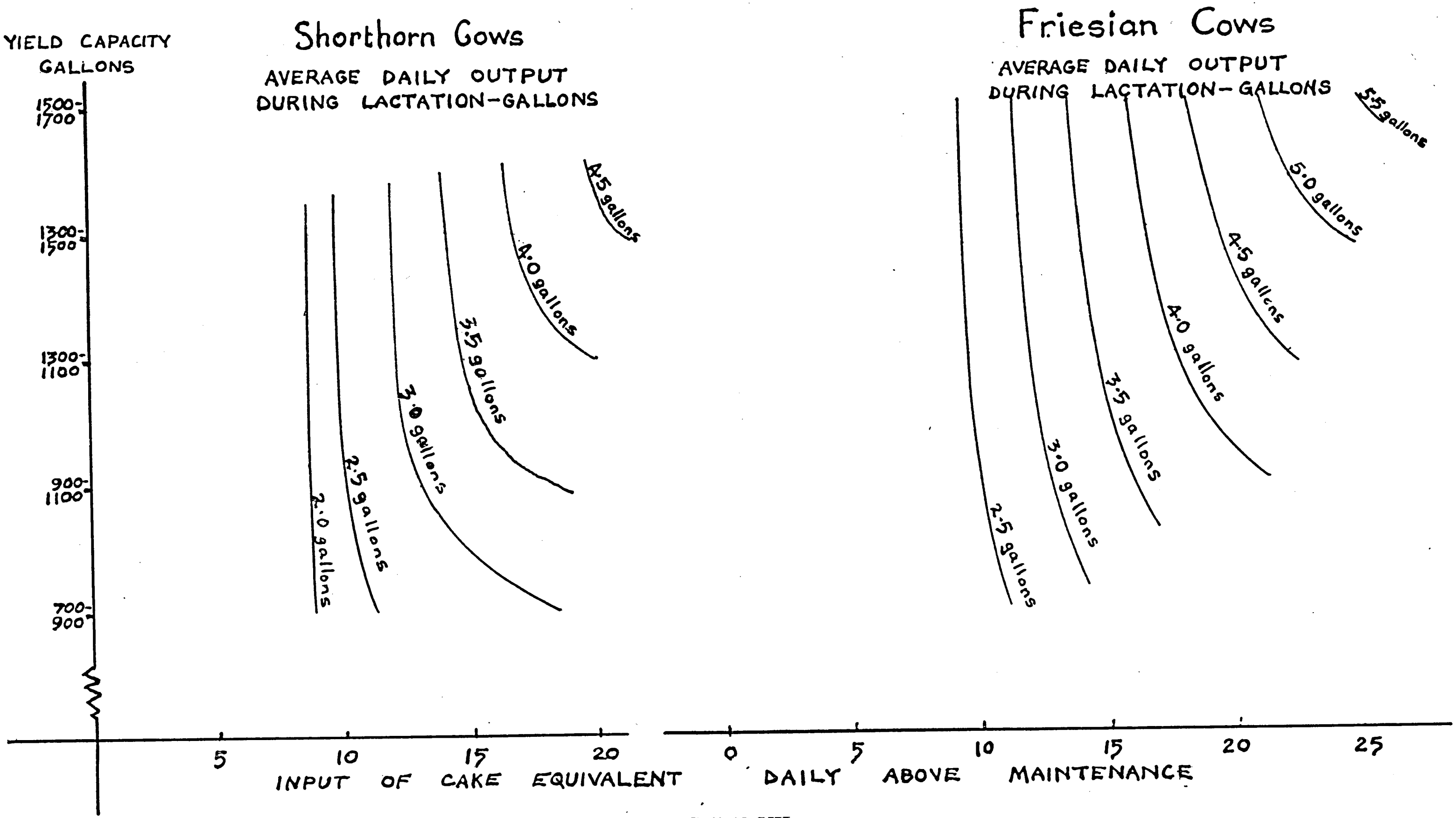

FIGURE XII

Product Contours of Cows' Yield Capacity and Input

An application of the product contours in comparing the values of two Friesian cows under given conditions is shown in a footnote.[1] More complicated applications are possible.

ESTIMATED LACTATION OUTPUTS OF COWS FED TO CONVENTIONAL STANDARDS

The inter-lactation input-output analysis would not be complete without an estimate of the outputs of cows of different yield capacities when they are fed to the conventional standard of 4 lb. of cake per gallon. This has been done separately for Friesians and Shorthorns by plotting the average inputs of cake equivalent per gallon on the horizontal axis, against the yield associated with the given average inputs on the vertical axis. Horizontal lines, through the points of intersection of the curves by the vertical line which goes through the 4 lb. cake per gallon scale, give the yields associated with those rates of input. The curves are not shown here, but the results are set out in the upper part of Table 35.

The estimates do not take into account the contribution which grazing usually makes to the achievement of more adequate inputs than those foreseen by conventional standards. This contribution varies a great deal, according to the quality of the grass. Even when cows calve in the spring, and their lactation curves follow the grass output of pastures, this output tends to be under-estimated in terms of nutrients: i.e. if pasture contained only the equivalent of 2.5 lb. SE with 0.55 lb. PE for each gallon "produced" from it, fewer gallons would be produced. It may be assumed that the output of spring-calving cows fed to conventional standards will

(1) *Question:* would it pay better to buy an average cow (estimated capacity over 1,000 gallons) for £80 or a very good cow (estimated capacity over 1,400 gallons) for £120 ?

Assumptions: cow will remain in herd for 5 years and then sell for £40. Yield aimed at: about 1,200 gallons. Daily average of 2.5 gallons obtained from grazing and homegrown food. Capital charged at 6 per cent interest.

	£80 Cow		£120 Cow	
Depreciation, yearly	$\frac{£80—£40}{5}$	= £8.0	$\frac{£120—£40}{5}$	= £16.0
6 per cent interest, yearly	on £80	= £4.8	on £120	= £7.2
Capital cost per cow, yearly ..		£12.8		£23.2
Requirements of cake equivalent, daily	21.3 lb. cake		16.6 lb. cake	
Less grazing and homegrown food equivalent to	10.1 " "		10.1 " "	
Average net input of cake, daily ..	11.2 " "		6.5 " "	
In 305 days of lactation	3416 " "	=£51.24	1983 " "	=£29.75
Plus capital cost per cow		£12.8		£23.2
Total cost per cow		£64.04		£52.95
Yearly balance in favour of better and dearer cow				£11.09
				£64.04

TABLE 35

Estimated Average Daily and Lactation Output of Shorthorn and Friesian Cows of Different Yield Capacities Fed According to Conventional Standards, and Corrections for Contribution above Maintenance from Grazing

(Input Equivalent to 4 lb. Cake per gallon over Maintenance)	Yield Capacity of Cows, Gallons									
	700—900		900—1100		1100—1300		1300—1500		1500—1700	
	Estimated Output, Gallons									
Output:	daily	lactation	daily	lactation	daily	lactation	daily	lactation	daily	lactation
	Shorthorns									
Theoretical*	2.10	641	2.37	723	2.70	824	3.05	930	—	—
Corrected for:										
Autumn Calvers (+ 5%) ..	2.21	673	2.49	759	2.84	865	3.20	976	—	—
Spring Calvers (+ 10%) ..	2.31	705	2.61	795	2.97	906	3.36	1023	—	—
	Friesians									
Theoretical*	—	—	2.38	726	2.90	885	3.40	1037	3.90	1190
Corrected for:										
Autumn Calvers (+ 5%) ..	—	—	2.52	763	3.05	929	3.57	1089	4.10	1250
Spring Calvers (+ 10%) ..	—	—	2.62	799	3.19	974	3.74	1141	4.29	1308

*These would be the outputs if grazing did not influence the total input during part of a cow's lactation.

be about 10 per cent higher than the theoretical ones shown in the upper part of Table 35, owing to the under-estimate of spring grass which is implicit in those standards. For the same reason, autumn-calving cows will increase their yields during the spring flush of pasture (although it is still widely believed that this is due to some unknown factor in the spring grass), since they can consume a great deal more nutrients in grass than it would be practicable to feed them in the form of concentrates at the given, latish stage of their lactation. But their yield capacity at these late stages of lactation is lower than it would be if they were newly calved, and for that reason outputs 5 per cent higher than the theoretical ones in Table 35 have been assumed for autumn calvers. The outputs, duly adjusted for the spring flush of grass, are given in the lower part of Table 35.

Although, in the nature of things, the above estimates are only crude approximations, they show up the order of magnitude of the differences involved in the system of feeding according to conventional standards as against that of making controlled use of cows' own yield capacities. They can also serve as a basis for the calculation of the economic advantage of the latter system. The two systems are compared in Tables 36 and 37.

An Economic Comparison of Feeding Cows to Conventional Standards or According to Yield Capacity and the Law of Diminishing Returns

In Table 36 estimates are made for Shorthorns. The assumptions are that milk is 36d. per gallon and ordinary cake 3.6d. per lb. The optimum marginal input of cake will therefore be $\frac{36d.}{3.6d.}$ = 10 lb. per gallon. From the marginal input curves in Figure XI can be read the associated average daily outputs estimated for the breed. Alternatively, the daily inputs associated with the above outputs can be read from Figure XI. The assumption is then made that the contribution to output from grazing and bulky foods amounts to 2 gallons daily, i.e. 610 gallons during a lactation. It follows that cake "equivalent" to the production requirements of this quantity of milk will not be needed. To the above estimates of input and output the given prices have been applied, in order that the surplus of revenue over the cost of cake may be calculated for cows with different yield capacities.

In the lower part of Table 36 is shown the estimated output which would be associated with feeding according to conventional standards. On the assumption that milk production in a herd is kept equal, irrespective of season, the relevant estimates in Table 35 have been corrected for the contribution from bulky food and grazing by adding to them 7.5 per cent —the mean of the estimated effect on spring and autumn calvers. After allowance has been made for the saving of cake equivalent due to bulky food and grazing, the requirements of cake have been calculated at 4 lb. per gallon.

Similar estimates for Friesian cows have been worked out and are shown in Table 37.

TABLE 36

Shorthorn Cows Fed According to New Theory or to Conventional Standards.
Comparison of Estimated Input, Output and Surplus over Cost of Cake

Estimates of	Yield Capacity of Cows, Gallons: 700—900	900—1100	1100—1300	1300—1500
	Shorthorn Cows Fed According to New Theory			
Best Economic Output, daily ..	2.57 galls.	3.22 galls.	3.73 galls.	4.37 galls.
Best Economic Output, 305 days ..	784 ,, = £118	982 ,, = £147	1138 ,, = £171	1333 ,, = £200
Input of Cake Equivalent, daily ..	12 lb.	15 lb.	16.7 lb.	19.5 lb.
Input of Cake Equivalent, 305 days	3660 lb.	4575 lb.	5094 lb.	5998 lb.
Input of Cake less bulk and grazing for 2 gallons = cake equivalent ..	—2379 lb.	—2379 lb.	—2379 lb.	—2379 lb.
Net Requirement of Cake	1281 lb. = £19	2196 lb. = £33	2715 lb. = £41	3619 lb. = £54
Surplus over Cost of Cake	£99	£114	£130	£146
	Shorthorn Cows Fed According to Conventional Standards			
Output in 305 days	689 galls. = £103	777 galls. = £117	886 galls. = £133	1000 galls. = £150
Requirement of Cake net of Equivalent 2 galls. from bulk and grazing	1378 lb. = £21	1554 lb. = £23	1772 lb. = £27	2000 lb. = £30
Surplus over Cost of Cake	£82	£94	£106	£120
	Difference Between Above Surpluses			
	£17	£20	£24	£26

Note on prices taken: Milk at 36d. per gallon, ordinary cake at 3.6d. per lb.

At these prices according to the new theory best marginal input of cake would be at the rate of 10 lb. for the last gallon. At Conventional Standards marginal input = average input = 4 lb. cake per gallon.

TABLE 37

Friesian Cows Fed According to New Theory or to Conventional Standards.
Comparison of Estimated Input, Output and Surplus over Cost of Cake.
(Prices and Best Marginal Input as in Table 36)

Estimates of	Yield Capacity of Cows, Gallons 900—1100	1100—1300	1300—1500	1500—1700
	Friesian Cows Fed According to New Theory			
Best Economic Output, daily	3.22 galls.	4.33 galls.	4.94 galls.	5.43 galls.
Best Economic Output, 305 days	982 ,, = £147	1321 ,, = £198	1507 ,, = £226	1656 ,, = £248
Input of Cake Equivalent, daily	13.9 lb.	20.3 lb.	23.1 lb.	24.5 lb.
Input of Cake Equivalent, 305 days	4240 ,,	6192 ,,	7046 ,,	7473 ,,
Input of Cake less bulk and grazing for 2 gallons = cake equivalent	—2379 ,,	—2379 ,,	—2379 ,,	—2379 ,,
Net Requirements of Cake	1861 lb. = £28	3813 lb. = £57	4667 lb. = £70	5094 lb. = £76
Surplus over Cost of Cake	£119	£141	£156	£172
	Friesian Cows Fed According to Conventional Standards			
Output in 305 days	781 galls. = £117	952 galls. = £143	1115 galls. = £167	1279 galls. = £192
Requirements of Cake net of Equivalent 2 galls. from bulk and grazing	1562 lb. = £23	1904 lb. = £29	2230 lb. = £38	2558 lb. = £38
Surplus over Cost of Cake	£94	£114	£134	£154
	Difference Between Above Surpluses			
	£25	£27	£22	£18

Note on prices taken: Milk at 36d. per gallon ordinary cake at 3.6d. per lb.

The differences in surplus revenue over purchased concentrates as between the two systems of feeding are given at the bottom of each Table. They appear to range from £17 to £26 per Shorthorn cow, and they increase with yield capacity. For Friesian cows they increase with yield capacity from £25 to £27 and then diminish to £18 per cow. The narrowing of the difference with increasing yield capacity is due to the flattening-out which the increase produces in the input-output curve. At high yield capacity the input requirements at the top of the curve are hardly larger than those at the beginning of it. Since the input-output curves of the Shorthorns appear to be flatter than those of the Friesians and are only shown up to 1,500 gallons' yield capacity as against the Friesians' 1,700 gallons, the differences between their additional revenues do not begin to diminish within the given range of yield capacity.

It must be stressed again that the estimates given in Tables 36 and 37 are only rough approximations. They are presented here in order to demonstrate the method of calculation for practical problems and the order of magnitude of the differences involved as between the two methods of rationing.

Genuinely valid circumstances may exist in which it would not be worth while to ration cows individually or even in groups. It is often assumed that, at least on larger farms, the advantage lies with the easier methods of management and the smaller labour requirements of a simplified conventional system of feeding dairy cows. Using the input-output estimates presented in the above Tables, it should be possible to arrive at rational appraisals of the economic gain or sacrifice connected with the adoption of a particular method of rationing.

The total lactation and average daily input-output data have been fully analysed within the inter-lactation function. Unless they can be broken down into daily estimates for the different stages of one lactation, their practical use for calculating rations will be very small. Even their value for budget calculations will be only academic: they will be essays in policy without practical executive instructions on how to carry it out.

The final chapter will be devoted to the task of "decoding" the inter-lactation functions and of building up, from their elements, intra-lactation functions which are necessary for day-to-day rationing of cows, individually or in groups.

A NEW SYSTEM OF RATIONING (*b*)

THE INTRA-LACTATION PRODUCTION FUNCTION

The Lactation Curves

The samples on which this study has been based provide no information which might indicate how to relate the inputs and outputs with the various stages of a lactation. Yet it is only when they are related to those stages that it becomes possible to determine the nutrient requirements of cows while they yield differently at various stages of one lactation. It is therefore necessary to know the shape and "elevation" (distance from the horizontal axis) of the lactation curves and the variations, if any, due to differences in breeds, yield capacity, time of calving, age, etc.

Little is so far known about lactation curves, but several sets of such curves exist which can provide the minimum necessary information. The most important of them is given in bulletin 815 of the U.S. Department of Agriculture.[1] Although in that study inputs and outputs were analysed only as averages of total lactations, nevertheless the relationship of output to the various stages of lactation, in weekly intervals, is given at three different levels of input: namely, for "heavy-, medium- and light-fed groups of cows". The lowest-fed groups reached the peak of production in the fourth week, the medium-fed ones in the fifth and the highest in the sixth week. The peak of production varied consistently with the level of feeding, i.e. the highest peak output was attained by the heaviest-fed cows and, conversely, the lowest peak was reached by the lowest-fed animals. The trend of the slopes down from the peak was quite similar and very consistent for all groups. This writer found that the inclination of the slopes resembled that in a lactation curve presented by the M.M.B.;[2] as well as those in a few other known curves of this kind.

The Jensen-Woodward lactation curves, originally presented in terms of lb. of 4 per cent fat corrected milk, have been re-calculated here in terms of gallons of 3.7 per cent fat milk. They are given as thick dotted lines in Figure XIII. Around them has been constructed a family of theoretical lactation curves (drawn in thin lines) for cows of different yield capacities.

When the thick-lined (American) curves are examined it will be noted that from peak to mid-lactation the middle curve runs practically parallel to the bottom one, while the top curve is somewhat flatter near the peak of lactation. This is due to the fact that these curves do not really represent the lactations of cows with different yield capacities, but rather those of animals which had *ex hypothesi* similar yield capacities. They can be regarded as the lactation curves of one cow given varying feed inputs, when

(1) Jensen, Woodward *et al.* op. cit. p. 53.
(2) Report of the Production Division No. 1, 1950, p. 21 ("cows, March calvers").

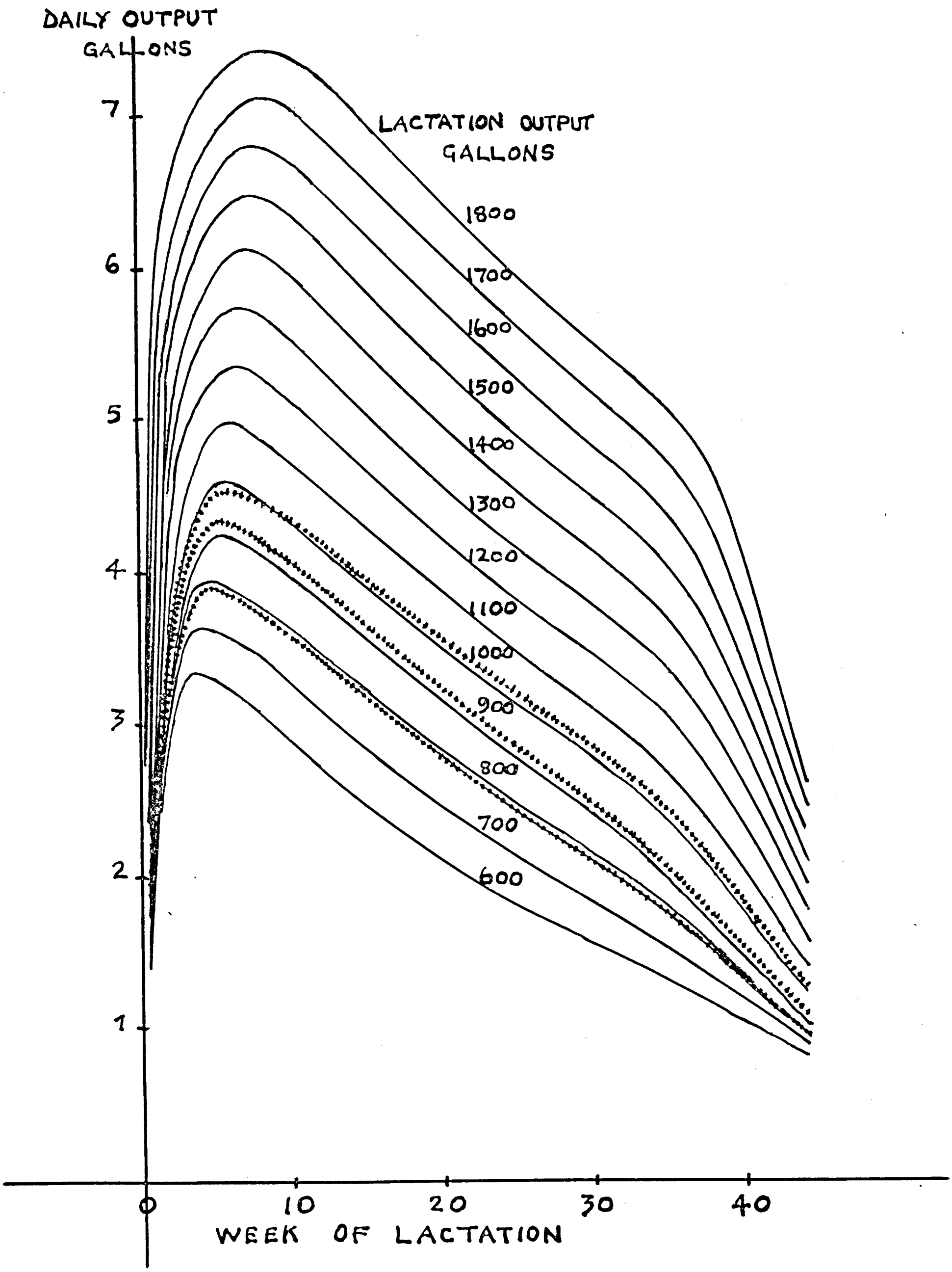

FIGURE XIII

Lactation Curves for Cows of Different Yield Capacities
Estimated from 3 Jensen-Woodward Curves (dotted lines)

diminishing returns lower the response from high inputs of feed. This would imply that a family of lactation curves like those constructed in Figure XIII can represent only "average" lactation curves of cows with different yield capacities. In actual fact, cows of any given yield capacity would have not one single lactation curve but bundles of them, each representing a lactation connected with a different level of feeding. It is obvious that any full presentation of such bundles of curves would be very difficult. The problem of the variations in input-output connected with different yield capacities has therefore been treated by a method similar to that applied in the previous section to the sample averages.

The importance of the American lactation curves lies in the fact that they reveal the "elevation", shapes and slopes of milk curves for three input-output levels in comparison with one another and the differences between them. There appear to be large differences as between the elevations of these curves. But for the first 30 weeks after peak lactation the two lower ones are practically parallel to each other; and the highest is very nearly parallel to them, in spite of diminishing returns. The family of lactation curves constructed around them could therefore be drawn parallel to the lower American curves over the largest part of each lactation, tapering off pro rata at both their ends; even so, they could be taken to represent the lactations of cows having varying yield capacities and being fed different inputs. (In fact, each of these curves can represent not only the lactations of cows of given yield capacities which have been fed certain inputs, but also those of cows of lower yield capacity which have been forced to the given level of output by higher inputs. In the latter case, the over-estimate of output would be about one-tenth of a gallon daily at peak of lactation and would taper off to nought towards mid-lactation. This difference is negligible and has been ignored.)

From the lactation curves in Figure XIII rough estimates could be made of the yielding capacities of cows in relation to their peak-lactation yields. These are given in Table 38.

TABLE 38

Yield Capacities of Dairy Cows Estimated from Peak Lactation Yields

Yield at Peak Lactation	Yield Capacity	Yield at Mid-Lactation
gallons	gallons	gallons
3.23	600	1.98
3.57	700	2.32
3.90	800	2.62
4.25	900	2.95
4.60	1000	3.29
4.95	1100	2.62
5.30	1200	3.95
5.65	1300	4.30
6.00	1400	4.61
6.35	1500	4.93
6.65	1600	5.26
7.00	1700	5.58
7.35	1800	5.91
7.70	1900	6.24
8.00	2000	6.56

The above estimates will play a key role in the final assessment of feed requirements which is to be made later in this study. In the first place, however, it was important to realise that rough estimates of yields could be made from them for cows of various yield capacities at any stage of lactation. It was these links between yields and stages of lactation that made it possible to construct daily (intra-lactation) input-output curves. The lactation curves represent the break-down of any given total lactation output into its daily constituents. Lactation inputs could be similarly broken down and the connection between time, i.e. the state of lactation, and input could then be extended to form a direct link between inputs and outputs. The time factor could be left out when the relationship had been established.

At first, much effort was spent in breaking down the average (lactation) inputs of SE and PE separately on the product contours along ridgelines of high and low input combinations. This involved dealing with four separate curves, representing the highest and the lowest inputs of each nutrient. Laborious adjustments had to be made in order to reconcile the high input points of one nutrient with the low input points of the other, in the process of constructing the product contours of the newly found *daily* input-output (i.e. intra-lactation) functions. The method rested on two assumptions: that at the lowest input levels the intra-lactation curves would be practically identical with the inter-lactation curves of each breed, and that both curves had to start from the origin. The total input of nutrients, calculated for 305 days of a lactation, had to tally with the sum of inputs allocated (by trial and error) to each section of the input-output curve under construction.

This method proved to be very laborious. Eventually it was much simplified by the breaking-down of cake equivalent—i.e. the two nutrients balanced in the same proportion as in dairy cake—instead of the separate nutrients themselves. Thus it was only necessary to construct one curve for each breed of cow, instead of four. On a product contour map such a curve would be represented by the diagonal line cutting through the nutrient combinations of 62.5 SE and 13.25 PE. From a daily cake equivalent input curve (intra-lactation curve) approximate product contours for the separate nutrients can be computed, since the rates of nutrient substitution have been determined beforehand.

Daily Cake Equivalent Input Curves

The assumption underlying the break-down of the lactation inputs expressed in terms of cake equivalent is that, while daily outputs are higher than the average for a lactation, daily inputs must also be higher than the average; and that similarly they must be lower when daily outputs are below the average ones. The shape of a daily input curve is indicated by three points: it must run through the origin; outputs of, and below, 2 gallons must be connected with inputs below 8 lb. of cake equivalent (i.e. below 4 lb. per gallon); and it must pass through or closely above the plot of the average lactation input on the mid-lactation week.

The method of breaking down the lactation inputs is demonstrated below, on the sample of Friesians. On a 305-day basis, their average yield in the sample amounted to 872 gallons per cow and ranged up to 1,200 gallons. According to Figure X and Table 33 a Friesian cow with a 900-gallon yield capacity, which could be forced to 1,200 gallons by appropriate feeding, would have lactation data as follows:

Yield in 305 days	..	900 gallons	1,200 gallons
Daily average yield	..	2.951 gallons	3.934 gallons
Input of cake equivalent above maintenance:			
Daily average ..	..	12.51 lb.	20.32 lb.
in 305 days, total	..	3,815.6 lb.	6,197.6 lb.
per gallon ..	..	4.24 lb.	5.17 lb.

On a graph which gave daily inputs per cow of cake equivalent (above maintenance) on the vertical axis and weeks of lactation on the horizontal one, first approximation curves of daily inputs were plotted. The curves were divided into sections of at least 7 days of lactation, for which daily inputs would be conveniently averaged. These average inputs, as read from the curves, were multiplied by the number of days in each section and the results were added up. These "re-constructed" total lactation inputs were then compared with the calculated total inputs, as in the above lactation data. The discrepancies between the estimated total inputs and the calculated ones indicated the necessary corrections which had to be made before these estimates, by trial and error, could be made to tally with those calculated from the average inputs. As a rule the construction of three to five successive approximation curves was necessary. The step-by-step break-down of the lactation input of cake equivalent into a daily input curve for the "average" Friesian cow is given in the upper part of Appendix Table VII. On the left-hand side are shown the numbers of days in each section into which a 305 days' lactation has been divided. (The different numbers of days are due to the varying slope of the constructed curves, coupled with the necessity to take convenient measurements of the input averages of each section). The daily average and total inputs corresponding to each section are given in two separate columns, as estimated requirements both for an average lactation of (in round figures) 900 gallons and for one forced to 1,200 gallons.

The link of input with output at the various stages of a lactation has been established by a break-down of the lactation curves of 900- and 1,200-gallon outputs (as in Figure XIII) into the same sections as those used for the break-down of inputs. The average daily outputs corresponding to each section, for the two yields involved, are shown on the right of the input columns.

The computed daily inputs were plotted against the linked daily outputs derived from the relevant lactation curves. An estimated daily input-output curve, representing cake equivalent requirements (above maintenance) for the average Friesian cow, was obtained by this method. It also appeared that the 1,200 gallon input-output curve was an extension of the 900-gallon one. Attempts to re-adjust the final input estimates given in Appendix Table VI invariably results in a difference between the computed total inputs and the averages as calculated from the equation. This implied that an average cow forced to high daily outputs would, later in her lactation when outputs were lower, require similar inputs to those she would have needed for the same daily outputs at an earlier stage of an unforced lactation.

The same method was also applied to the Shorthorns. It was estimated that the average cow of 700 gallons' capacity, which could be forced to 900 gallons by adequate feeding, would have the following calculated lactation data:

Yield in 305 days ..	700 gallons	900 gallons
Daily average yield ..	2.3 gallons	2.95 gallons
Input of cake equivalent above maintenance:		
Daily average	9.8 lb.	17.3 lb.
In 305 days	2,989 lb.	5,277 lb.
Per gallon	4.26 lb.	5.86 lb.

The step-by-step computation of the closest approximation input curves is shown in the lower part of Appendix Table VII.

When the computed total inputs were eventually made to tally with the calculated ones and the results were plotted, it appeared that the input curve bifurcated between the origin and a spot near the two-gallon output point. At 3.6 gallons—the peak output of a 700-gallon lactation—the input was nearly 16 per cent less than for the same yield on the input curve for the 900-gallon lactation. Any attempt to synchronize the two curves had the result that the computed total inputs differed from the calculated one. This implied that the Shorthorn sample included a proportion of cows which could reach a total lactation of 900 gallons only if they were fed on a higher plane than for a 700-gallon lactation, not only at and near the peak output, but also all along the lactation curve; if, later in the lactation, they were given inputs equal to those of cows with lower lactation and peak yields but with similar daily outputs at the time, then their daily output would fall somewhat as a result and the total lactation yield of 900 gallons would not be quite attained. It would seem that this is merely a different manifestation of the used joint equation's "strong reaction" to any element of irrational (i.e. over-) feeding. But it is extremely unlikely that the marginal output due to relatively higher inputs along all the input curve, for 700-gallon capacity Shorthorns forced to 900 gallons, would make a significant contribution towards their reaching the full 900-gallon lactation. (Otherwise it would be possible to force yields significantly at the later stages of a lactation by increased inputs of feed.) It is much more probable that the achievement by intensive feeding of a peak yield of 4.25 gallons, and subsequent rationing along a curve joining this peak input-output point (4.25 gallons) with the lower one (3.6 gallons) of the 700-gallon curve, would result in a total lactation yield little short of 900 gallons. It may well be, therefore, that the "correct" input curve for the average Shorthorn would be a little higher, between the 2-gallon and the 4.25-gallon (peak) points. Since the implicit error cannot be significant and any correction would have to be arbitrary, it has been decided to use the 700-gallon curve extended to the 4.25-gallon peak of the 900-gallon curve.

CONSTRUCTION OF DAILY CAKE EQUIVALENT INPUT CURVES FOR FRIESIAN AND SHORTHORN COWS OF DIFFERENT YIELD CAPACITIES

The "basic" cake equivalent input curves were plotted and are shown in Figure XIV. From them it was possible to estimate daily input curves for cows of different yield capacities. The method used was similar to that devised for the estimates of the various yield capacities from the inter-lactation functions (page 69). Moreover, the intra-lactation cake equivalent curves had to run through the origin, and below the 4 lb. cake input point at 2 gallons of yield. Again the underlying logic is that, at more intensive inputs, a cow with higher yield capacity will produce more than one of lower capacity fed at the same rate. The construction of the daily cake equivalent input curves from the broken-down mathematical, i.e. "basic", curves is given, in Appendix Table VIII, separately for the two breeds.

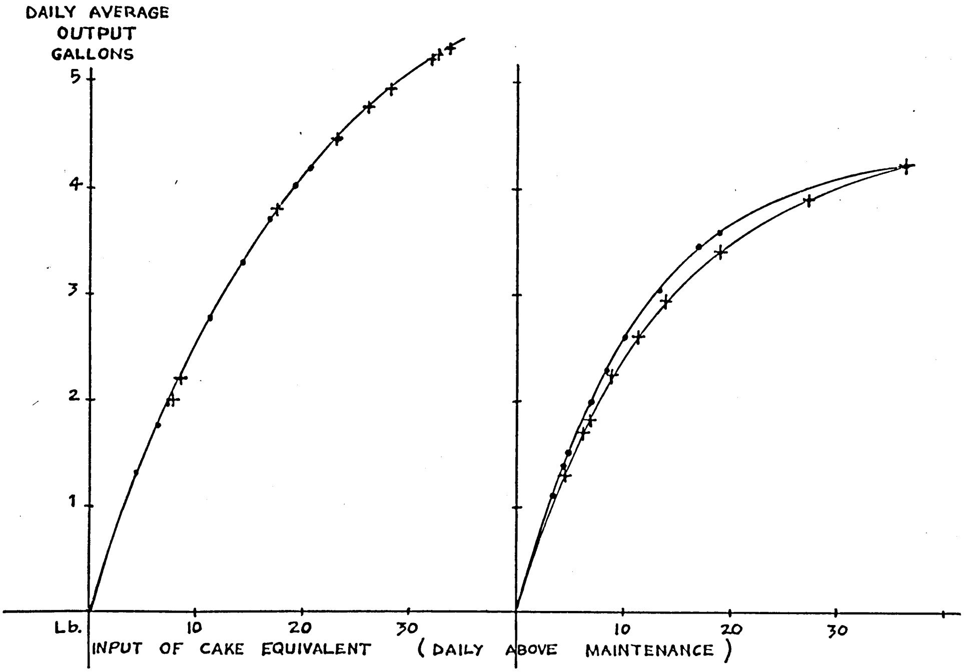

FIGURE XIV

Basic Daily Input Curves of Cake Equivalent for "Average" Friesian and Shorthorn Cows
(From Appendix Table IV)

The underlined outputs and inputs trace out the newly found "basic" curves. On top of the input columns the rates of input per gallon of output are given as derived from the basic inputs and outputs. Those rates per gallon have been applied to the various yields, at different yield capacities, in order that the daily input at the various stages of lactation may be estimated. For example, the "average" Friesian cow, with a yield capacity of 900 gallons, would produce in mid-lactation 2.95 gallons at an input of 12.3 lb. cake equivalent (above maintenance) = 4.17 lb. per gallon (see Figure X). It follows that a 1,600-gallon capacity cow, producing 5.26 gallons in mid-lactation at the same rate per gallon, will tend to require 5.26 x 4.17 = 21.93 lb. cake equivalent; and that a 800-galloner, yielding in mid-lactation only 2.62 gallons, would need 2.62 x 4.17 = 10.93 lb. cake equivalent. From the "basic" daily input curve it can be seen that the "average" cow at peak lactation (4.25 gallons) will tend to require 21.3 lb. cake equivalent, which is a rate of 5.01 lb. per gallon. Cows of different capacities at their various peak yields would tend to produce at a similar rate per gallon. For example, a 1,600-gallon capacity cow would yield about 6.65 gallons at peak lactation and at that stage would require 6.65 x 5.01 lb. = 33.32 lb. of cake equivalent. Finally, the average 900-gallon cow, if fed more intensively, could be pushed to 1,100 gallons with a peak yield of 4.95 gallons. It can be seen from the basic daily input curve (Figure XIV) that for this yield she would need 28.76 lb. cake equivalent, i.e. a rate of 5.81 lb. per gallon. Similarly, the 1,600-gallon cow pushed to 1,800 gallons with a peak yield of 7.35 gallons would, while yielding that amount, require 7.35 x 5.81 lb. = 42.7 lb. cake equivalent daily above maintenance.

It will be noted in Appendix Table VIII that higher capacity Shorthorns, pushed up 200 gallons, reveal input requirements which are distinctly beyond any cow's appetite. These have been calculated only in order to provide an additional point on the curve for the estimate of a more accurate slope in the region of highest outputs.

The construction shown in Appendix Table VII makes it possible to trace the daily cake equivalent input curves, for cows of several levels of yield capacity, through 4 points, namely: origin, mid-lactation, "ordinary" peak of lactation and "forced" peak of lactation. These curves, and several more in steps of 100-gallon capacity, are shown for Friesians in Figure XV and for Shorthorns in Figure XVII. Special marks indicate the three points on which the curves given in the Appendix have been constructed. The marginal inputs per half-gallon output are shown in Figures XVI to XVIII. They are "arc" marginal inputs, not "joint" ones.

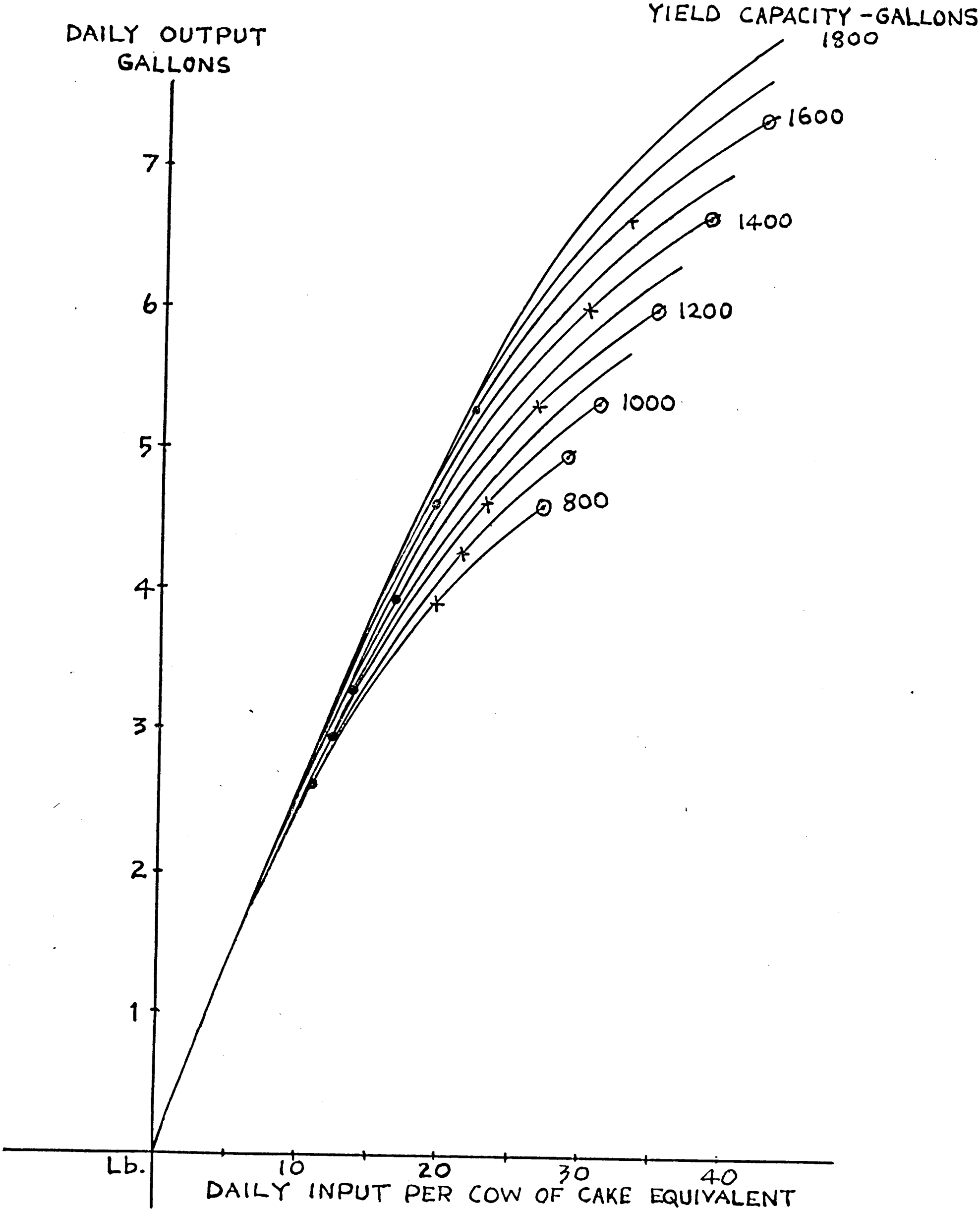

FIGURE XV

FRIESIAN COWS OF VARIOUS YIELD CAPACITIES

Daily Inputs of Cake Equivalent (above maintenance) at Different Levels of Output

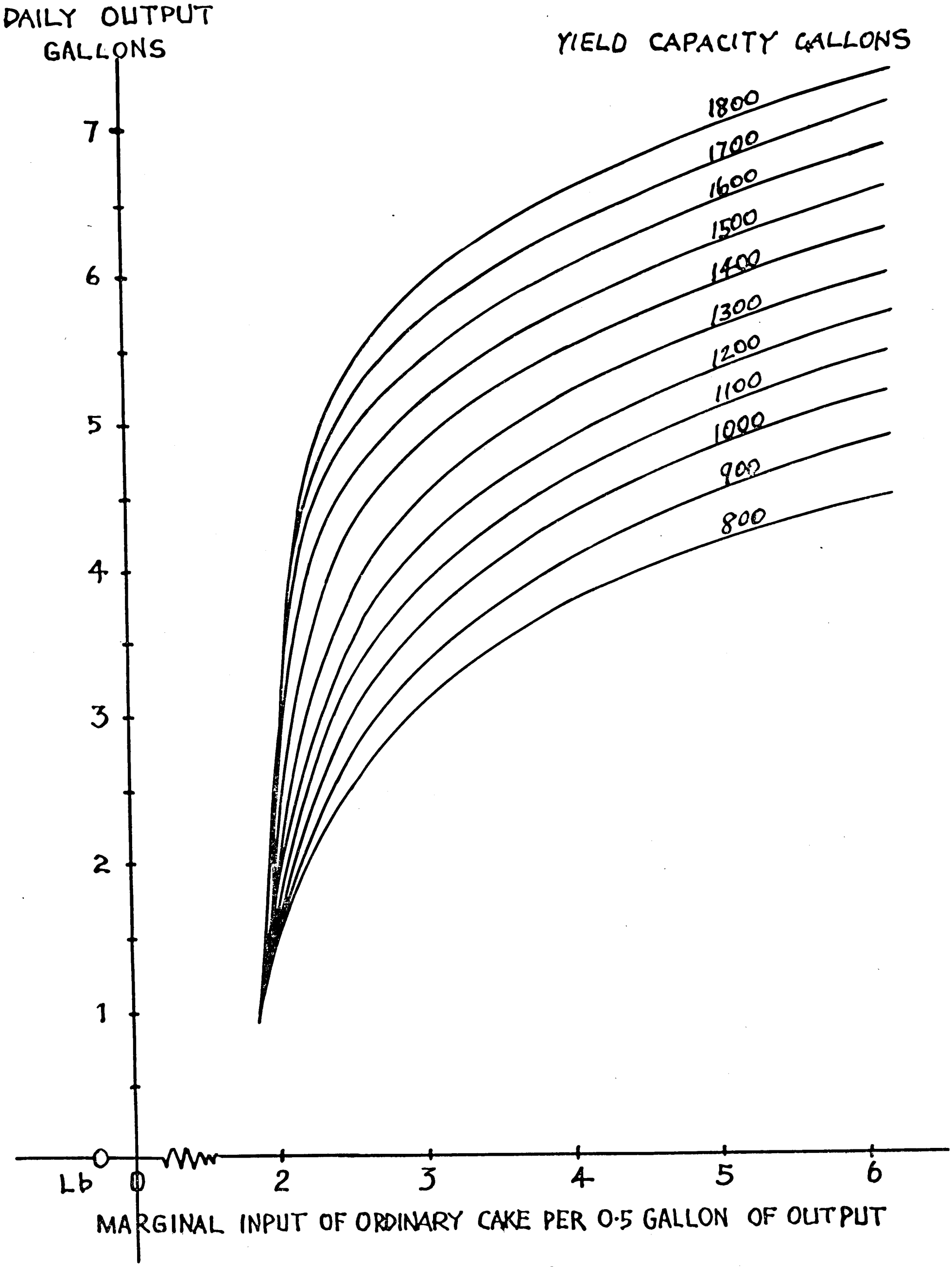

FIGURE XVI

FRIESIAN COWS OF VARIOUS YIELD CAPACITIES

Marginal Inputs per 0.5 Gallon at Different Levels of Output

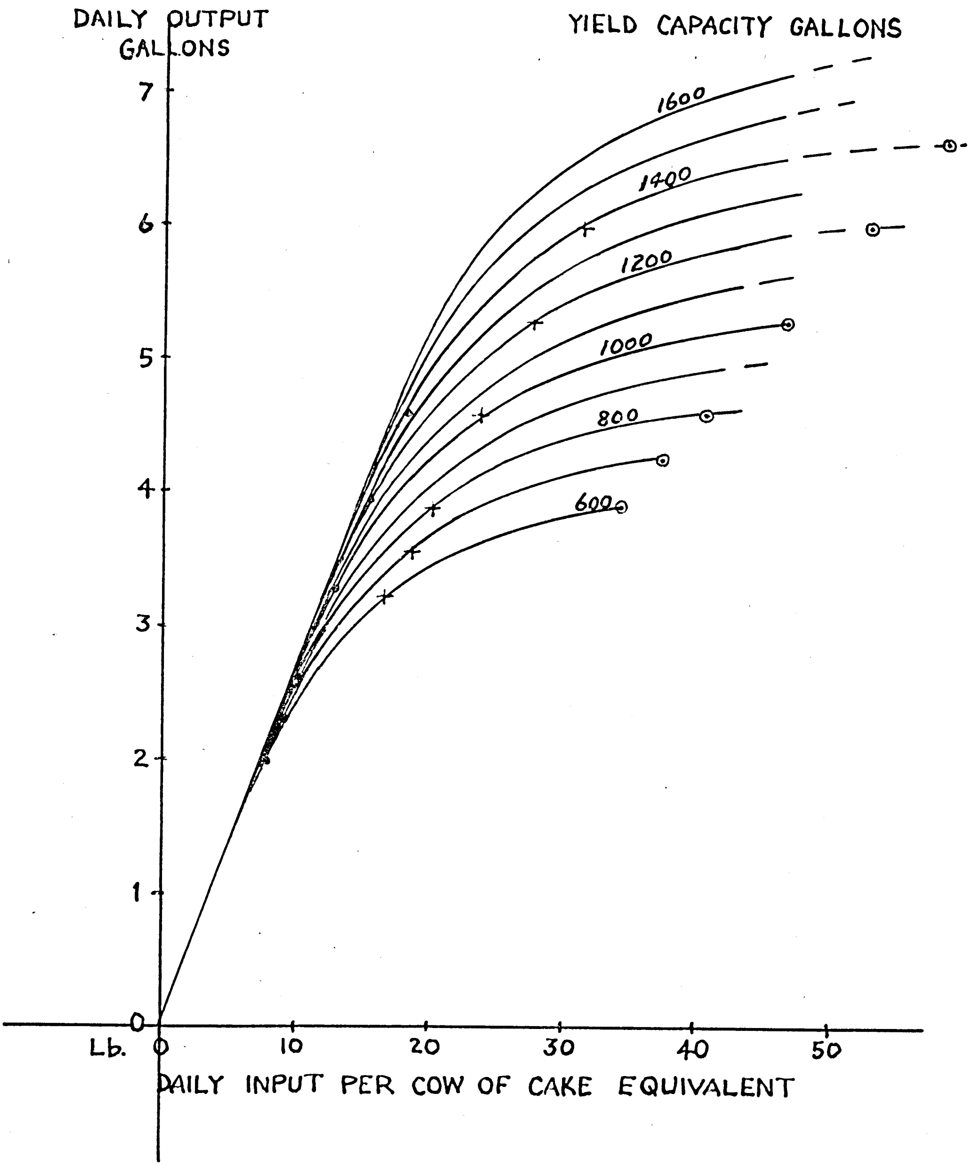

FIGURE XVII

SHORTHORN COWS OF VARIOUS YIELD CAPACITIES

Daily Input of Cake Equivalent (above maintenance) at Different Levels of Output

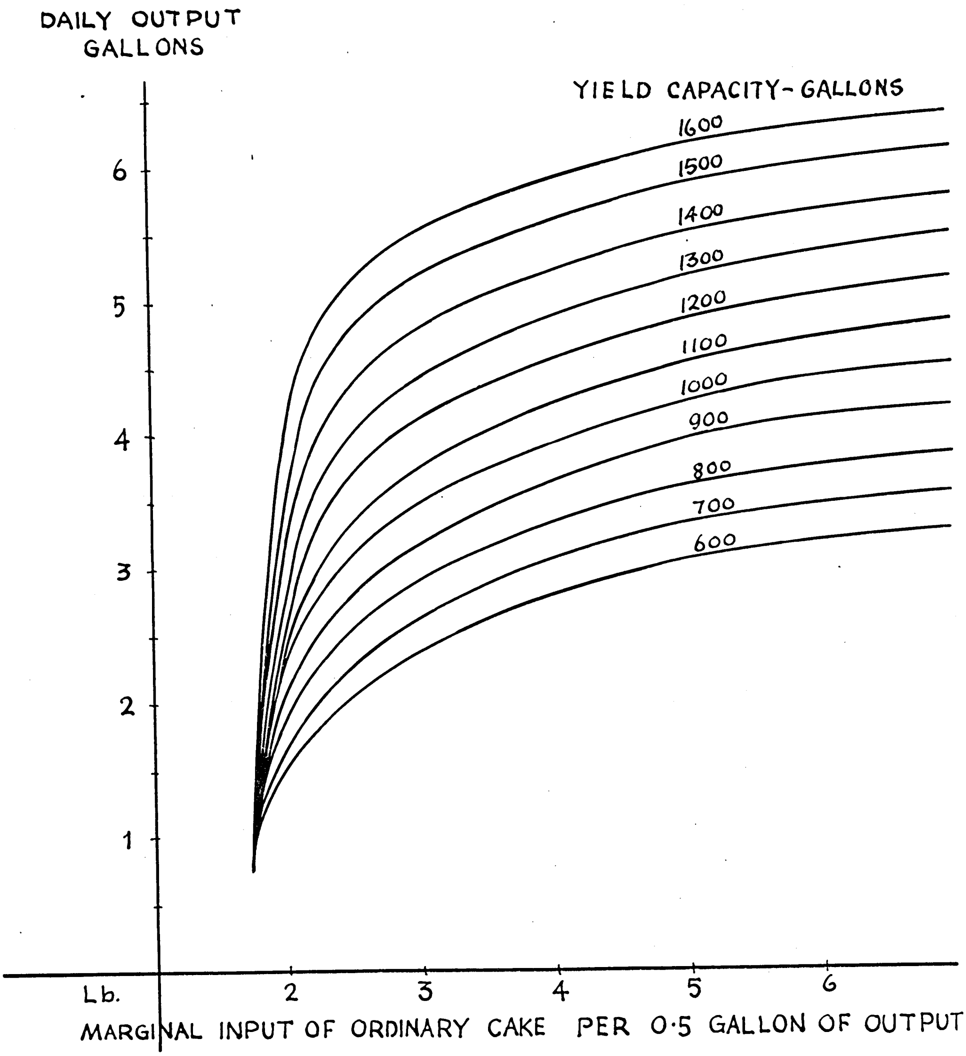

FIGURE XVIII

SHORTHORN COWS OF VARIOUS YIELD CAPACITIES

Marginal Inputs per 0.5 Gallon at Different Levels of Output

The corresponding input schedules are given in Table 39 for Friesian cows and in Table 40 for Shorthorns. The ranges of yield capacity shown are from 800 gallons to 1,600 gallons for the former and from 600 gallons to 1,400 gallons for the latter. For the sake of convenience, the estimated yield at peak of lactation is given for each yield capacity. The highest inputs tabulated would suffice to force yields to higher peak levels than those given. It should be borne in mind that the estimates of yield capacity from

TABLE 39

FRIESIAN COWS of Different Yield Capacities

Estimated Requirements of Ordinary Dairy Cake (or Equivalent) above Maintenance at Various Daily Output Levels

When Daily Yield per Cow	When Estimated Yield Capacity of Cows is Gallons																	
	800 (at peak 3.9 galls.)			900 (at peak 4.25 galls.)			1000 (at peak 4.6 galls.)			1200 (at peak 5.3 galls.)			1400 (at peak 6 galls.)			1600 (at peak 6.65 galls.)		
	INPUT above Maintenance of LB. ORDINARY DAIRY CAKE (or Equivalent)																	
Gallons	Daily per Cow	Mar-ginal*	Avge. per Gallon	Daily per Cow	Mar-ginal*	Avge. per Gallon	Daily per Cow	Mar-ginal*	Avge. per Gallon	Daily per Cow	Mar-ginal*	Avge. per Gallon	Daily per Cow	Mar-ginal*	Avge. per Gallon	Daily per Cow	Mar-ginal*	Avge. per Gallon
7.0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	37.3		5.3
6.5	—	—	—	—	—	—	—	—	—	—	—	—	36.0		5.5	31.4	6.0	4.8
6.0	—	—	—	—	—	—	—	—	—	34.9		5.8	30.0	6.0	5.0	26.9	4.5	4.5
5.5	—	—	—	—	—	—	33.5		6.1	28.6	6.3	5.2	25.3	4.7	4.6	23.4	3.5	4.25
5.0	—	—	—	29.2		5.8	26.8	6.7	5.4	23.8	4.8	4.75	21.7	3.6	4.3	20.6	2.8	4.1
4.5	25.5		5.7	23.5	5.7	5.2	22.0	4.8	4.9	20.0	3.8	4.4	18.7	3.0	4.2	18.2	2.4	4.0
4.0	20.1	5.4	5.0	19.1	4.4	4.75	18.2	3.8	4.6	16.9	3.1	4.25	16.2	2.5	4.1	16.0	2.2	4.0
3.5	16.1	4.0	4.6	15.6	3.5	4.5	15.0	3.2	4.4	14.2	2.7	4.1	13.9	2.3	4.0	13.8	2.2	3.9
3.0	12.9	3.2	4.3	12.7	2.9	4.25	12.3	2.7	4.2	11.8	2.4	3.9	11.7	2.2	3.9	11.7	2.1	3.9
2.5	10.2	2.7	4.1	10.1	2.6	4.0	9.9	2.4	4.0	9.6	2.2	3.8	9.6	2.1	3.8	9.6	2.1	3.8
2.0	7.8	2.4	3.9	7.8	2.3	3.9	7.7	2.2	3.9	7.6	2.0	3.8	7.6	2.0	3.8	7.6	2.0	3.8
1.5	5.7	2.1	3.8	5.7	2.1	3.8	5.7	2.0	3.8	5.7	1.9	3.8	5.7	1.9	3.8	5.7	1.9	3.8
1.0	3.8	1.9	3.8	3.8	1.9	3.8	3.8	1.9	3.8	3.8	1.9	3.8	3.8	1.9	3.8	3.8	1.9	3.8

*per half-gallon

TABLE 40

SHORTHORN COWS of Different Yield Capacities

Estimated Requirements of Ordinary Dairy Cake (or Equivalent) above Maintenance at Various Daily Output Levels

When Daily Yield per Cow	When Estimated Yield Capacity of Cows is Gallons																	
	600 (at peak 3.25 galls.)			700 (at peak 3.6 galls.)			800 (at peak 3.9 galls.)			1000 (at peak 4.6 galls.)			1200 (at peak 5.3 galls.)			1400 (at peak 6.0 galls.)		
	INPUT above Maintenance of LB. ORDINARY DAIRY CAKE (or Equivalent)																	
Gallons	Daily per Cow	Marginal*	Avge. per Gallon	Daily per Cow	Marginal*	Avge. per Gallon	Daily per Cow	Marginal*	Avge. per Gallon	Daily per Cow	Marginal*	Avge. per Gallon	Daily per Cow	Marginal*	Avge. per Gallon	Daily per Cow	Marginal*	Avge. per Gallon
6.0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31.1	6.5	5.2
5.5	—	—	—	—	—	—	—	—	—	—	—	—	32.0	8.0	5.8	24.6	4.0	4.5
5.0	—	—	—	—	—	—	—	—	—	33.0	10.2	6.6	24.0	4.7	4.8	20.6	2.9	4.1
4.5	—	—	—	—	—	—	34.2	12.3	7.6	22.8	5.0	5.1	19.3	3.3	4.3	17.7	2.3	3.9
4.0	—	—	—	26.7	8.7	6.7	21.9	5.8	5.5	17.8	3.5	4.5	16.0	2.5	4.0	15.4	2.1	3.9
3.5	20.9	6.6	6.0	18.0	4.6	5.1	16.1	3.7	4.6	14.3	2.7	4.1	13.5	2.2	3.9	13.3	2.0	3.8
3.0	14.3	3.8	4.75	13.4	3.2	4.5	12.4	2.7	4.1	11.6	2.2	3.9	11.3	2.0	3.8	11.3	2.0	3.8
2.5	10.5	2.7	4.2	10.2	2.5	4.1	9.7	2.2	3.9	9.4	2.0	3.8	9.4	1.9	3.8	9.4	1.9	3.8
2.0	7.8	2.2	3.9	7.7	2.1	3.9	7.5	1.9	3.8	7.5	1.8	3.8	7.5	1.8	3.8	7.5	1.8	3.8
1.5	5.6	1.8	3.8	5.6	1.8	3.8	5.6	1.8	3.8	5.6	1.8	3.8	5.6	1.8	3.8	5.6	1.8	3.8
1.0	3.8		3.8	3.8		3.8	3.8		3.8	3.8		3.8	3.8		3.8	3.8		3.8

*per half-gallon

peak lactation yields have been based on inputs increasing to approximately 5 lb. of cake equivalent per gallon. (It can be seen from the "basic" daily input curves in Appendix Table VII that the "average" Friesian cow received 5.01 lb. of cake equivalent per gallon at her peak output of 4.25 gallons. The "basic" Shorthorn at her peak of 3.57 gallons appears to have received 5.21 lb. per gallon.) But peak and subsequent lactation yields may be forced up by feeding above this accidental quantity.

For example, if a cow after calving is fed up to 5 lb. of cake equivalent per gallon, she may reach a peak yield of 4.6 gallons with a capacity for 1,000 gallons. If the same animal were fed up to 6.1 lb. per gallon, she could be forced to a peak of 5.3 gallons daily, when her yield capacity would exceed 1,200 gallons. But if fed only 4 lb. of cake equivalent per gallon this cow would yield barely 4.25 gallons at the peak of lactation, when her capacity would be about 900 gallons, and at this rate of feeding even such a yield as this would hardly be reached. To express it differently, we shall be able to speak of a 1,000-galloner fed to capacity, forced to 1,200 gallons or fed—below capacity—to only 900 gallons.

At the highest "forced" yield levels the requirements appear to be well over 5 lb. of cake equivalent per gallon. Towards the end of lactation they seem to fall to 3.8 lb. per gallon. At low yield levels, i.e. at the latter stages of lactation, the requirements per gallon appear to be similar as between cows of low and high yield capacity. But at higher yield levels these requirements tend to be considerably less for cows of high yield capacity than for those of low capacity.

In order to facilitate comparison between the input curves of Friesian and Shorthorn cows of similar yield capacities, four sets of such curves for yield capacities ranging from 800 gallons to 1,400 gallons are shown in Figure XIX. The curves for Friesian cows appear to be steeper in the area of higher input and output than do those for Shorthorns, which tend to flatten out quite considerably towards their peak ends. On the other hand, it would appear that in the medium and lower area of input and output Shorthorn cows require daily up to one lb. less cake than Friesians need for a similar output.

The above, relatively small, differences between the inputs connected with the given outputs are probably not significant, although it is not possible to test their significance in this study. The more steeply diminishing returns for the Shorthorns are, to a certain degree, confirmed by a similar trend displayed by the Hannah experimental group of Ayrshires. It may well be that a different sample of Friesians would agree more closely with this trend. The agreement between the curves in Tables 39 and 40 is, however, close enough to permit the assumption that, *ceteris paribus*, Friesian and Shorthorn cows of equal yield capacity tend to have similar input requirements for any given output within that capacity (although this may not be so for the ranges of "forced" yield capacity). It would appear that the significance of the difference between the input-output relationships of the two breeds, found earlier in this study, is based on the fact that the proportion of cows with high yield capacities is greater among Friesians than among Shorthorns.

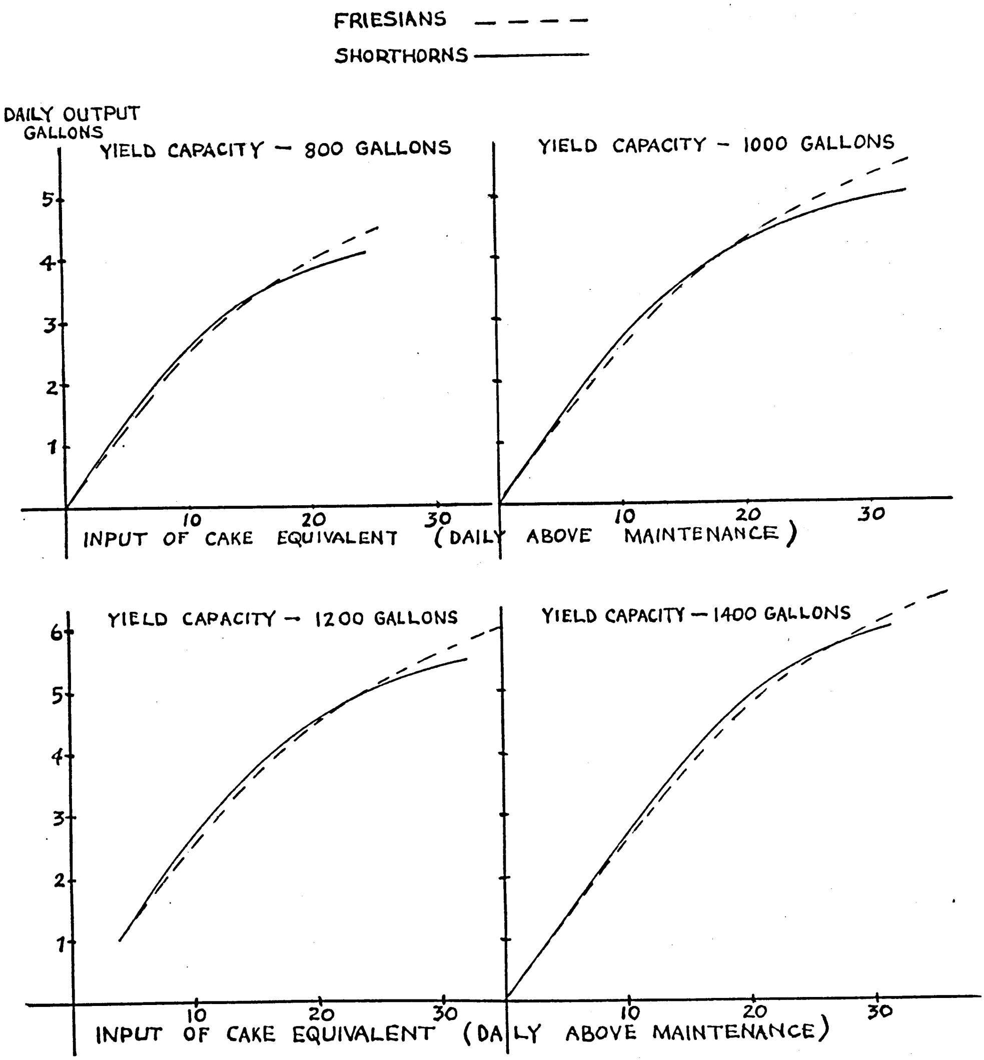

FIGURE XIX

Comparison of Input Curves for Friesian and Shorthorn Cows of Similar Yield Capacity at Four Different Levels

The close similarity of the separate estimates for Friesians and Shorthorns vindicates the methods used in breaking down the inter-lactation functions and the construction of separate input-output curves for cows of different yield capacities. It also increases the degree of reliability that can be attributed both to the sample on which this study has been based and to the method applied for the elimination of the data with (non-correlated) errors of observation.

The Peculiarity of Marginal Inputs and Outputs in Biological Production Processes

The normal definition of marginal input (or, *mutatis mutandis*, marginal output) is that it is the quantity that has to be added if another unit of output is to be obtained, or that can be saved if one unit of output is given up. When live animals are concerned in the process of production, the concept is often complicated by the varying and irreversible effect of time —which can be defined as ageing in an absolute or cyclical sense—on such biological processes. A newly calved cow will have a biological urge to produce milk even if fed less than the maintenance level. In such a case the input will be body tissue, and the cow's yield capacity will be vastly lower than those met with at commercial or experimental feeding levels. Nevertheless, an input-output curve could be determined, e.g. by measuring inputs in terms of losses of body weight, or of the energy requirements of the output. From such an input-output curve "marginal" inputs or outputs could be determined. But they would lack any practical value—in fact the very notion of them would be irrational—since it is in no one's power to "save" such "marginal input" by foregoing any of the output. One could only influence the process by increasing the inputs up to the level of full requirement. But it is extremely unlikely that a cow could be brought from such a predicament to her full potential yield capacity. In fact, after such treatment she would become a different animal—at least for the remainder of her current lactation, perhaps for the following lactation as well and sometimes for the remainder of her life.

Changes in feed input may have a significant influence on output in the first few months of a lactation. During this stage the meaning of marginal input is unambiguous: it can either add to, or lose, a given marginal output. But later in a lactation a considerable addition to input may increase output only by a negligible amount or not at all. On the other hand, the withdrawal of a marginal input will—after a time lag— diminish the output; and, although the re-inclusion of that input in the ration will (unless the interval has been too long) raise the output again, it will rarely reach its previous level. For example, if 2 lb. of cake equivalent were withdrawn from the ration of a cow in the eighth month of lactation, when she was yielding 2.5 gallons daily, her output might fall to 2 gallons. If the ration were restored to the previous level after a few days, yield would be likely to go up only to, perhaps, 2.2 gallons. The process would take a week or ten days, during which yield would have fallen by a fraction in any case; but the "natural" fall would tend to be smaller.

In fact, we have to deal here with different types of marginal input, which are derived from different functions. The inputs shown in Tables 39 and 40 are based on the intra-lactation function. Since one cannot turn back a lactation they may be termed "imperfect" ones. But on each day of a lactation these inputs can be increased or diminished; and this may influence daily yields, if only after a time lag. As has been pointed out before, early in a lactation the effect tends to be noticeable, and the functions measuring the results of input changes within one day and as between different days (i.e. the intra-day and intra-lactation functions) might tend to be similar. But in the later stages of lactation similar results as between the two different functions could only be expected if inputs were first made smaller and then, soon afterwards, raised to their previous level again. Increasing them directly from a given input would tend to produce very unspectacular responses or none at all. The higher input/900-gallon output curve of the "average Shorthorn", mentioned on page 89, appears to be an example of forcing output also later in a lactation—to a large extent wastefully.

An intra-lactation function might be compared with a taut violin string, which can be given various slopes by changing the position of the instrument. An intra-day function might be represented by the vibrations of such a string when plucked. No data available at present can give an insight into the nature of such a function, and it would not be easy to set up suitable experiments. Since it is known from experience that increased input has very little effect on output in the later stages of a lactation, the matter would not seem to be of much practical importance. But in fact the answer to the whole fundamental question of forcing the output of cows at any stage of lactation is closely linked with, and will have to wait for a solution of, this problem.

The Determination of the Most Economic Levels of Daily Inputs and Outputs

The most economic input-output level is obtained when the given price of milk is equal to the cost of the marginal input needed to produce a marginal quantity of output. In the previous chapter it has been argued that only the marginal inputs at the earlier stages of a lactation, i.e. only the higher marginal inputs given in Tables 39 and 40 for each yield capacity class, are "true" marginal inputs. The intra-day functions, from which would be derived such "true" inputs for more advanced lactation stages, are not known. Consequently the marginal inputs shown at the lower levels of these two Tables, and in the corresponding Figures XVI and XVIII, are merely approximations: they can be taken to represent points just short of the drastic steepening that could be expected of "true" intra-day marginal input curves.

It follows that precise estimates of the best input-output levels, according to the rule set out at the beginning of this chapter, are possible only for the relatively high levels of output, i.e. the early stages of lactation. At such levels marginal inputs are higher than at the lower ones, so that the *most* economic are also the *highest* economic levels of marginal input. It would not pay to exceed them irrespective of the effect on yield, with one exception which will be discussed later. But the highest economic marginal input levels will be too high for cows in more advanced stages of lactation. Under the price-cost conditions prevailing in this country since the last war, the marginal inputs given for the relatively lower yields in Tables 39 and 40 can be taken as close approximations to the best marginal inputs. In borderline cases the "best" marginal input is either the calculated higher economic marginal input or the marginal input as given in Tables 39 and 40—whichever is the less. The term "best" will be used in either case when brevity demands.

When the marginal quantity of output is one half-gallon the "best" (highest economic) marginal input can be determined by the formula: $\frac{\text{price of one half-gallon milk}}{\text{cost of one lb. concentrates}} = x$ lb. of "best" marginal input of concentrates. These "best" inputs have been calculated for various price combinations of milk and concentrates and are given in Table 41.

For example, when milk is worth 38d. per gallon, i.e. one half-gallon costs 19d., and one lb. of ordinary cake costs 3.6d., then the highest economic input per additional half-gallon will be 5.2 lb. With the same price of milk and "high production (3½ lb. per gallon) cake", costing 4d. per lb., it will not pay to feed more than 4.8 lb. for the last half-gallon produced by a cow in one day. On the other hand, a home-mixed concentrate of purchased corn and straight cakes, costing about 3d. per lb., will be most economic when 6.4 lb. of it are fed for the marginal half-gallon of milk.

After the best marginal input per half-gallon has been found, the best daily output for high yielding cows of different yield capacities can be determined graphically from marginal input curves. The bottom part of Figures XV and XVI show marginal input curves of ordinary cake for Friesian and Shorthorn cows respectively. In the relevant figure should be found, on the horizontal axis, the best marginal input, as in Table 41. This should then be traced vertically to the point of intersection with the marginal curve for the yield capacity in question and, after that, horizontally to the vertical axis. The point of intersection with this axis will give an estimate of the "best" daily yield per cow.

Estimates for "3½ lb. per gallon cake" are not given here. Since both the ordinary and the "high production" cake contain virtually identical quantities of nutrients at similar unit costs, so that input requirements per gallon of either cost about the same, the only practical difference between them is their volume. This is one-eighth less in the "3½ lb. per gallon cake" than in the ordinary one. Estimates for the former can therefore be made by adjusting estimates for the latter by 12.5 per cent.

The "best" level of output can also be estimated from Tables 39 and 40; but only rough approximations are possible, since the output interval in these Tables is one half-gallon and therefore too wide for precision.

For example, if we assumed that at the prevailing prices the "best" *marginal* input, for the last half-gallon which it would be economic to produce from a Friesian cow of 1,200-gallons' capacity and nearly at the peak of her lactation, would be 5.3 lb. of ordinary cake, then the "best" *daily* input per cow could be found by substituting 5.3 lb. for whatever marginal input, in the 1,200-gallon capacity column in Table 39, was next in size to 5.3 lb. The daily input per cow which lay a step below that marginal input, plus 5.3 lb., would be the required estimate. In the above example the marginal input next in size to 5.3 lb. is 4.8 lb., and the input per cow which lies a step below the marginal input of 4.8 lb. is 23.8 lb. The "best" daily input will therefore be 23.8 lb. + 5.3 lb. = 29.1 lb., which would correspond to an output slightly above 5.5 gallons. Although, on the assumptions made earlier, a cow of 1,200-gallons' capacity would

TABLE 41

Highest Economic Marginal Input per Half-Gallon at Different Prices of Milk and Concentrates
(Regardless of the level of yields higher marginal inputs than those given below would not pay)

When Price of Half-Gallon Milk is	When Price of Cake per Ton and per lb. is							
	£39.2 4.2d.	£37.3 4.0d.	£35.5 3.8d.	£33.6 3.6d.	£31.7 3.4d.	£29.9 3.2d.	£28.0 3.0d.	£26.1 2.8d.
	Highest Economic Marginal Inputs of Cake per Half-Gallon Output							
d.	lb.	lb.	lb.	lb.	lb.	lb.	lb.	lb.
21	4.8	5.2	5.6	5.8	6.2	6.6	7.0	7.4
20	4.6	5.0	5.2	5.6	5.8	6.2	6.6	7.0
19	4.4	4.8	5.0	5.2	5.6	6.0	6.4	6.8
18	4.2	4.6	4.8	5.0	5.2	5.6	6.0	6.4
17	4.0	4.2	4.4	4.6	5.0	5.4	5.6	6.0
16	3.8	4.0	4.2	4.4	4.8	5.0	5.4	5.8
15	3.6	3.8	4.0	4.2	4.4	4.6	5.0	5.4
14	3.4	3.6	3.8	4.0	4.2	4.4	4.6	5.0
13	3.0	3.2	3.4	3.6	3.9	4.0	4.4	4.8
12	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.4
11	2.6	2.8	2.8	3.0	3.2	3.4	3.6	4.0
10	2.4	2.6	2.6	2.8	3.0	3.2	3.4	3.8

tend to yield only 5.3 gallons at the peak of her lactation, under the given circumstances it would pay to force the peak output of such a cow to about 5.6 gallons. To do this it would be necessary to increase input per gallon from 5 lb. to 5.2 lb. of cake when the cow approached her peak output.

When milk prices are low and/or prices of concentrates are high, the "best" inputs will not be sufficient to attain or sustain a cow's potential yield capacity. This fact is illustrated by the estimates of input-output optima, for cows of different yield capacities at four price-cost combinations, which are presented in Table 42.

The upper part of the Table deals with ordinary cake, costing 3.6d. per lb., and the lower with a home-mixed concentrate having a similar nutrient content, but costing 3.1d. per lb. In both parts, the figures in the top division pertain to a milk price of 38d. per gallon and those in the lower division to one of 31d. per gallon. For each price-cost combination, estimates are given of the highest economic marginal inputs, the "best" daily inputs and outputs per cow and the total lactation yields which can be expected, with adequate feeding later, when the highest inputs do not exceed those given in the Table.

In the case of Friesians, and on the assumption that purchased cake costs 3.6d. per lb. and milk 38d. per gallon, the "best" daily input per cow will tend to be from 2.7 lb. to 3.9 lb. of cake higher—the difference increasing with yield capacity—than when the price of milk is 31d. per gallon. With home-mixed concentrates costing 3.1d. per lb. and milk worth 38d. per gallon, the "best" daily input of such a mixture will tend to be from 2.2 lb. to 4 lb. more—depending on yield capacity—than at a milk price of 31d.

Taking as "standard" the estimates of yield capacity from the peak of lactation yields in Table 38, the above "best" inputs for the higher price of milk should force the peak output of low-capacity cows about one-third of a gallon above "standard", and should just about suffice for the "standard" peak outputs of high capacity cows. The "best" inputs at the lower price of milk would do for the "standard" peak output of low-capacity cows, and for about one-third of a gallon less than the peak output of high-capacity cows.

The lactation yields of cows given the highest economic (i.e. the "best") inputs during their early lactations, and fed adequately at other times, have been estimated from the associated "highest economic yields" at the peak of lactation.

With cake costing 3.6d. per lb. and milk at 38d. per gallon, the "best" daily inputs will tend to permit cows of low yield capacity to attain total outputs which are about 10 per cent higher than the "standard" lactation yields; while cows of very high yield capacity may reach the "standard" lactation yield or a little less. If cake is fed when milk is 31d. per gallon the total lactation yield attainable, in view of the lower peak output, will tend to be equal to, or slightly below, the "standard" with cows of low yield capacity and to diminish to nearly 10 per cent below "standard" with cows of very high yield capacity.

TABLE 42

Estimates of Input-Output Optima for Friesian and Shorthorn Cows of Different Yield Capacities at Four Price-Cost Combinations

Highest Economic Outputs and Inputs of Cake Equivalent above Maintenance

	Estimated Yield Capacity, Gallons											
	FRIESIANS						SHORTHORNS					
	800	900	1000	1200	1400	1600	700	800	900	1000	1200	1400
When Ordinary Cake costs 3.6d. per lb.	When gallon milk = 38d. best Marginal Input per half-gallon = 5.2 lb. of cake $\left\{\frac{38d.}{3.6d. \times 2}\right\}$											
Highest Economic Yield, gallons	4.20	4.55	4.90	5.45	5.95	6.50	3.35	3.65	3.95	4.25	4.85	5.55
Highest Economic Input, lb.	22.2	24.0	25.8	28.1	29.6	31.9	16.5	17.7	18.8	19.8	22.2	25 0
Total Yield per Cow, gallons	880	970	1090	1250	1400	1550	640	700	810	900	1070	1275
	When gallon milk = 31d. best Marginal Input per half-gallon = 4.3 lb. of Cake $\left\{\frac{31d.}{3.6d. \times 2}\right\}$											
Highest Economic Yield, gallons ..	3.9	4.2	4.5	5.05	5.60	6.15	3.15	3.45	3.70	4.05	4.65	5.35
Highest Economic Input, lb.	19.5	20.8	22.1	24.3	26.0	28.0	14.5	15.5	17.0	18.1	20.5	23.3
Total Yield per Cow, gallons	800	880	970	1130	1290	1450	570	660	740	840	1020	1220
When Home Mixture* costs 3.1d. per lb.	When gallon milk = 38d. best Marginal Input per half-gallon = 6.12 lb. of Cake $\left\{\frac{38d.}{3.1d. \times 2}\right\}$											
Highest Economic Yield, gallons ..	4.45	4.80	5.15	5.70	6.30	6.80	3.5	3.75	4.10	4.40	5.00	5.7
Highest Economic Input, lb.	24.9	26.7	28.4	29.9	33.4	35.0	18.2	18.8	21.0	21.3	23.8	26.8
Total Yield per Cow, gallons	960	1060	1160	1320	1490	1650	680	750	860	940	1020	1320
	When gallon milk = 31d. best Marginal Input per half-gallon = 5 lb. of Cake $\left\{\frac{31d.}{3.1d. \times 2}\right\}$											
Highest Economic Yield, gallons ..	4.15	4.50	4.80	5.35	5.85	6.45	3.3	3.65	3.90	4.25	4.85	5 50
Total Economic Input, lb.	22.7	23.6	24.8	27.1	28.5	31.0	15.9	17.4	18.3	19.8	22.2	24.7
Total Yield per Cow, gallons	870	970	1060	1220	1420	1540	620	720	800	900	1080	1260

*Purchased corn and straight cakes with a nutrient content equivalent to that in ordinary dairy cake (= 62.5 SE and 13.25 PE).

Note: The estimates of total yield do not take account of the influence of grazing on yields of cows in more advanced stages of lactation. An Autumn calver might have total yields 50 gallons higher than in the above estimates, solely owing to the very high intake of nutrients in Spring pasture. Similarly, Spring calvers might increase their yields when grazing kale or fed *ad lib*. hay or silage.

When feeding the cheaper mixture, a milk price of 38d. per gallon will permit the forcing of cows to over half a gallon above "standard"; when the price of milk is 31d., the "best" input will enable low capacity cows to raise their peak output by one-third of a gallon above "standard", and high-capacity cows to attain their "standard" peak of lactation yield. In the first instance, low-capacity cows could attain a total lactation output of about 20 per cent above the "standard" and high capacity ones could achieve 15 per cent above "standard". In the second case, the former could still reach lactation outputs of about 9 per cent above "standard" and those of the latter could still be a few per cent above "standard".

A comparison of the highest economic inputs and the associated yields of the Shorthorns with those of the Friesians shows that, at similar yield capacities, the highest economic inputs of the Shorthorns tend to be from under 25 per cent to over 30 per cent lower for the higher price of milk and from under 15 per cent to over 25 per cent lower for the lower milk price. The larger difference pertains to cows of lower yield capacity and vice versa. The difference between the "best" quantities of the dearer cake and the cheaper mixture tend to be of a smaller order. The associated differences between the best peak yields per cow tend to be about half a gallon daily. But it would appear that the lactation yields of Shorthorn cows, fed in their early lactations to the highest economic level on purchased cake at 3.6d. per lb., tend to be over 100 gallons below their "standard" yield capacity, and still further below the yields of Friesian cows of equal capacity. Shorthorn cows, fed to the best level of the cheaper mixture at 3.1d. a lb., tend to have outputs which are nearer their "standard" capacity, but 150—200 gallons below those of Friesian cows with similar yield capacity.

Although these differences between Shorthorn and Friesian cows can only represent the rough trend, they explain the marked loss of popularity which Shorthorns have suffered in this country since the end of Hitler's war.

The Highest Economic Levels of Daily Inputs in Connection with the Best Total Output for a Lactation

Milk prices tend to vary as between winter and summer. If the highest profit per cow for one week, month or even quarter were the aim to be pursued, without any regard for the general effect over the whole lactation, then the static theory of profit maximisation could be applied to the problem without further discussion. But a lactation curve reflects the biological factor of ageing. In view of the time factor involved, and since the cycle of milk prices does not necessarily coincide with that of yield levels, a dynamic approach is needed when profit maximisation is calculated for a full lactation.

When a cow's peak of lactation coincides with high prices of milk, i.e. in the autumn and winter, the best daily input at, or near, the peak yield will also maximise the total lactation profit. As milk prices fall with the approach of spring, so the cow's yield will tend to fall with the progress of time since her calving, and daily marginal input requirements will drop below the levels of input-output equilibrium.

If, however, a cow's peak of lactation coincides with low milk prices in the spring, the best input in any one month will tend to depress peak (and near) yields. As a result, the cow may not reach a peak that might be economic at higher prices of milk. Yield will tend to fall until output is matched by the input. At this stage output will tend to be lower than it would have been if higher peak-of-lactation inputs had been applied. Although in the meanwhile prices of milk will be rising, it nevertheless will not be possible to force output to any extent by increasing inputs, since the lactation curve will already have been "fixed" at a lower level. Consequently the total lactation output will be relatively lower.

In spite of the fact that the highest economic inputs comparable with the low milk prices will have been applied, the loss of production resulting from the subsequent depression of the yield curve at a period of higher milk prices will tend to lower the profit for the whole lactation. It would be good economics somewhat to exceed the highest economic inputs of cows in their early lactations at periods of low milk prices. Although, at the time, such a policy would lower the daily profit per cow, it would keep up her yield curves for the remainder of her lactation when prices of milk were higher and—if done judiciously—increase the total profit for the lactation.

The best levels of inputs and peak yields which, when synchronised with the various points of a milk price cycle, would result in the maximisation of total lactation profits, can be roughly computed from estimates of the behaviour of the relevant lactation curves, following different inputs at and around the peak of lactation. This is a lengthy process and the author hopes to make it the subject of a separate paper. A computation has been made for Friesian cows, of 1,000 gallons' capacity, reaching peak lactation in May. It appears that at 1959 prices of cake and milk, in order to maximise the profit over the whole lactation of such a cow, 3 to 4 lb. daily of ordinary dairy cake should be given in May and June in excess of the "best" input. This quantity may be taken as a minimum for cows of higher yield capacity, or at a time when milk prices are higher, or when costs of cake are lower; and it has been recommended elsewhere.(1)

Nutrient Substitution in Intra-Lactation Functions

It does not seem that any useful purpose would be served at this stage by presenting iso-product curves of SE and PE for the daily input-output functions of cows with different yield capacities. The space required for it would be considerable. In practice, the balancing of various nutrient combinations is conditioned by the fact that purchased concentrates tend to have fixed proportions of nutrients, and that it would not usually be convenient to change the composition of concentrate mixtures too frequently. It seems to be much easier to balance bulky rations with high-protein concentrates for the lowest level of output, and thereafter to ration in terms of balanced cake. For example, if bulky foods contain sufficient

(1) M. B. Jawetz: "Maximising Profits from Feeding Concentrates to Dairy Cows". Shortened version of a paper given at the National Agricultural Advisory Service Milk Production Conference at Cheltenham on 1st December, 1959.

SE for maintenance and 2 gallons and sufficient PE for only 1.5 gallons daily per cow, the deficiency could be made up by 2 lb. of high-protein cake (72 SE with 30 PE) which would balance the ration for an output of 2.5 gallons; for yields exceeding 2.5 gallons a balanced cake of known nutrient content would then be used.

When some special circumstances necessitate the balancing of all nutrients at the same stage, e.g. when a ration is to consist entirely of good silage and low-protein dried grass, it will be necessary to use iso-product curves in order to find the "best" combinations. Product contours can then be roughly determined from the input-output curves of cake equivalent (Tables 39 and 40) and from adjusted slopes of the product contours found from the inter-lactation functions (Figures V and VI) on the principle that these contours are flattest when yields are at, or near, their peak irrespective of yield capacity, and that they steepen with declining yields. Although estimates thus obtained would be very rough approximations, they could nevertheless be quite useful. More precise estimates may be calculated from the points where the above product contours intersect the ridgelines which delimit the scatter of observations.

Part Four

CONCLUSION

This study is concerned with the relationship between the feeding of dairy cows and milk yield. It is suggested that the conventional feeding standards are unsatisfactory and a new system is proposed in their place.

The sample on which this study has been based was small and had defects. The estimates derived from it, therefore, can only be taken as approximations. Nevertheless, they can command a degree of confidence from the facts that the basic analysis of the data was conducted separately for two breeds of cow, that these were checked and compared throughout with each other and with a set of experimental data for a third breed from an entirely different source, and that in their final synthesis they gave closely similar and logical results. The theory of feeding dairy cows which emerges seems to fit any known situation, and to provide reasonable explanations for its divergencies from the existing theories and beliefs concerning this subject. Its disadvantage is its complexity in comparison with the simplicity of the conventional feeding standards.

One of the defects of the data is that the observations referred to three-monthly averages which—fortunately—could be taken to fall in mid-lactation and therefore roughly to represent averages for the whole lactation. Moreover, the number of observations was sufficient only for the statistical determination of the average lactation functions for Friesian and Shorthorn cows of "average" milking capacity.

Preliminary analysis revealed differences between Friesian and Shorthorn cows in the outputs of milk which they produced from inputs of both SE and PE. Statistical analysis showed these differences to be significant. Multi-variable regression analysis showed milk output to be a joint function of the input of both nutrients.

Production surfaces were constructed from the "basic" equations. They demonstrated that at low average yields the average inputs per gallon over the lactation tend to be somewhat less than the conventional standard of 2.5 lb. SE and 0.55 lb. PE; but that when yields are high average inputs are considerably above these standards—particularly with Shorthorns.

Product contour maps (iso-product curves) were also constructed, showing the extent to which SE can be substituted for PE. The amount of PE which can be replaced by 1 lb. of SE appears to vary, in the case of Shorthorns, from about 0.2 lb. at low inputs to less than 0.1 lb. at high inputs and, in the case of Friesians, from about 0.2 lb. to just under this figure.

Diminishing marginal rates of nutrient substitution mean that the best balance of SE and PE in cakes and mixtures—the "nutritive ratio" of conventional feeding standards—is not constant, but varies with the relative prices of SE and PE in the form of straight concentrates. At present (1960) price conditions in this country, it would appear that the most economic balance in ordinary cake of 62.5 SE would be achieved with about 15 PE instead of the prevailing 13.25. The saving in weight would amount to about 8 per cent, whereas the cost per lb. would increase by only a fraction.

In practice it will be convenient to supplement bulky feed with as little high-protein concentrate as will balance it for production, and thereafter to ration in terms of balanced cake or equivalent mixture. For this reason, and also for ease of presentation, further estimates of inputs have been given in terms of "cake equivalent", i.e. in combinations of SE and PE which are equal to their ratio in ordinary dairy cake (62.5 SE and 13.25 PE).

The amalgamation of the two variable inputs has been further necessitated by the need to consider the yield capacity of cows as yet another variable. This capacity seems to depend on several factors. The quality and amount of grazing and feeding in the preceding lactation, the method of steaming up and the level of feeding during the first weeks after calving all seem to influence it in degrees that so far are hardly known. Under these changeable influences, the yield capacity of a cow alters from lactation to lactation and sometimes even during a single lactation. It would therefore seem that a definition of yield capacity based on a cow's past performance would be grossly inaccurate. It would also appear, however, that yield capacity can be estimated from a cow's yield at the peak of her lactation; and a method has been evolved by which this can be done. Since, in the short term, yield capacity can be influenced by the amount fed between calving and the peak yield, a "standard" practice for this period has to be assumed. The one suggested here is that feed input, over maintenance requirements, should start at the equivalent of 4 lb. of ordinary dairy cake per gallon at calving and rise by 0.2 to 0.25 lb. per gallon each week, until the equivalent of 5 lb. per gallon is being fed at the time of peak yield. If this standard is adopted one might speak of a cow's producing "below capacity" when less than 5 lb. of cake equivalent per gallon is fed at the peak of lactation, and of "forced yield capacity" when more than 5 lb. is fed at this time.

Methods have been devised which have made it possible to estimate the following curves from the "basic" functions:—

(*a*) Lactation input-output curves of cows with different yield capacities. These curves show the differences between various cows in their total input and output during a lactation, and provide a basis for farm planning and budgeting and for the evaluation of grassland productivity.

(*b*) Daily input-output curves according to yield capacity. Curves of this kind indicate the differences in feeding required over the course of an individual cow's lactation, i.e. the daily input necessary while she is producing various outputs at different stages of her lactation. These curves can be used as feeding standards.

Marginal and average inputs are also shown for both types of curve. Provided that the difference between lactation and daily input-output curves—be they total, daily average, marginal or average per gallon—is kept in mind, only the daily ones need be discussed at this stage.

The higher the yield capacity of a cow the better is her productivity, i.e. the lower is the input necessary for a given output.

The difference between high- and low-capacity cows can be ignored at low levels of yield (towards the end of a lactation); but it increases progressively and is very marked early in a lactation, when yields are at their peak. Estimates of input requirements are given in this study for Friesian and Shorthorn cows of different yield capacities while they are yielding various quantities of milk, i.e. in various stages of their lactations. At low yields, e.g. towards the end of a lactation, inputs per gallon appear to be somewhat below the conventional 4 lb. of ordinary cake (or equivalent). With increasing yields inputs increase proportionately more than outputs do, particularly when cows are "forced" towards their highest limits of capacity. At yield levels approaching these limits, input requirements of cake per gallon appear to be over 50 per cent above the conventional standard for cows of low yield capacity, and over 25 per cent above that standard for cows of high yield capacity. Cows with high yield capacities tend to produce considerable yields—though not their best—at average input rates of not more than 4 lb. of cake equivalent per gallon. (Even at a peak yield of 8 gallons, a potential 2,000-galloner will often require only a fraction over 4 lb. of cake equivalent per gallon.) This explains the practical observations that very high outputs can sometimes be achieved by feeding to conventional standards.

It would appear that, as between cows of similar yield capacity, Friesians have a higher productivity (i.e. conversion rate of feed to milk) than Shorthorns at the highest levels of yield, whereas the latter may have a slight advantage over Friesians at lower yields. As a result, higher yields are obtainable more economically from Friesians than from Shorthorns of similar yield capacity. The advantage of Friesians over Shorthorns becomes greater as the price of milk rises and/or the (nutrient) cost of concentrates falls.

It is necessary to know not only what yield to expect from a cow for a given input, but also what yield it is economic to produce from her during the various stages of her lactation and in any part of the season. The most economic input depends not only on the biological productivity of a cow, but also on the seasonal price of milk and the cost of concentrates. It can be determined with the help of the marginal input curves presented. The study is concluded by a discussion, including examples of profit maximisation under various circumstances.

Finally, some of the typical situations, which are explained or confirmed by the theory outlined above, may be summed up as follows:

The ranges of nutrient substitution which have been found provide a rough theoretical explanation for the relative success of both the Rex Patterson and the Boutflour systems of milk production. The first, based as it is on an extensive use of grazing and bulky foods, substitutes large quantities of SE for relatively small ones of PE. On the iso-product curves given in Figures V and VI and in Table 27 the average nutrient requirements of such a system, for various levels of yield, are represented by the higher limits of SE with lower PE. The Boutflour system, which depends on an intensive use of concentrates, substitutes relatively small amounts of PE for large quantities of SE. Its nutrient requirements appear to be near the lower limit of SE with higher PE.

Very high-yielding cows tend to produce on conventional standards and sometimes on even less. This is due to the fact that high yield capacity is connected with above-average feed conversion.

Conventional inputs are sometimes claimed for high-yielding herds. These results occur in the following cases:

(*a*) Cows of very high capacity, which are under-fed and therefore yield perhaps 150—200 gallons less than they could, but are still high yielders. The conventional rationing in such cases is often bolstered by intake to appetite of grass or kale, or by bulky feed whose weight or nutritive value may be under-estimated.

(*b*) Cows of high yield capacity, which are fed sufficiently owing to considerable under-estimates of the utilised value of pasture or kale or bulky feed. A typical example would be "maintenance + 2 gallons" from very good grazing in May or June, plus 4 lb. ordinary cake per gallon, for cows yielding 6 gallons. In fact pasture may be sufficient for 4—5 gallons during these months, so that the cows may really be receiving not $6 \times 4 = 24$ lb. of cake equivalent according to conventional standards, but over 32 lb., i.e. over 8 lb. more than expected. Another example would be 20 lb. of hay and unlimited kale for "maintenance plus 1 or $1\frac{1}{2}$ gallons" plus cake. If full value is given in such cases to the homegrown foods, the inputs appear to be consistent with the estimates for high yield given in this study. The drawback of such rule-of-thumb feeding methods is that they tend to result in over-feeding at medium levels of output.

Adherence to conventional standards often results in unsatisfactory results which are then attributed to some defect in the utilised value of bulky food. Voices are frequently heard to the effect that it is impossible to obtain from high-yielding cows more than maintenance plus $1\frac{1}{2}$ or, at the most, 2 gallons from silage or kale with hay. But a cow yielding, say, 6 gallons may obtain silage sufficient for maintenance plus $3\frac{1}{2}$ gallons, i.e. the equivalent of about 14 lb. of cake over maintenance; when the conventional requirement of $2\frac{1}{2} \times 4 = 10$ lb. cake is added, the total production ration of 24 lb. of cake equivalent will be only 3 lb. short if the cow's yield capacity at peak has been 6.65 gallons, but 6 lb. short when it happens to be 6 gallons. It is obvious that in such cases the fault lies not in bulk feeding but in the conventional feeding standards.

APPENDIX TABLE I

Nutritive Value of Foodstuffs

Foodstuffs	DM	SE	PE
Ground Nut Cake (undecorticated)	89.7	57.0	27.2
Ground Nut Cake (decorticated)	89.7	73.0	42.0
Ground Nut Meal (undecorticated)	92.4	44.0	28.8
Ground Nut Meal (decorticated)	92.4	70.0	45.0
Coconut Cake	88.6	77.0	16.4
Cotton Cake (undecorticated)	88.0	40.0	15.2
Sunflower Cake (undecorticated)	92.9	50.0	16.5
Linseed Cake	89.0	74.0	25.5
Palm Kernel Cake	89.0	73.0	17.0
Fish Meal (white)	87.0	59.0	53.0
Blood Nitrogen	86.0	63.0	68.2
Malt Culms	86.0	43.4	16.0
Compound Dairy Cake (NCF No. 1 and No. 2)	86.0	59.5	13.2
Calf Nuts (NCF No. 3)	88.0	68.0	16.5
Calver Nuts (NCF No. 4)	86.0	60.0	12.0
Grain Balancer	88.0	60.0	19.0
High Protein Cake	89.0	63.0	30.0
Dried Grains	90.0	48.0	12.6
Dried Grains (beet pulp)	90.0	60.5	5.2
Molasses, etc.	75.0	51.6	1.2
Molassine Meal, etc.	90.0	58.3	4.6
Molasses Palm Kernel	85.0	65.0	12.0
Wet Grains	32.4	18.4	5.4
Wet Grains (Ale and Porter)	27.0	14.8	4.7
Oats	86.7	59.5	7.6
Dredge Corn	86.0	65.5	7.5
Dredge Corn (with about 20% Pulses)	86.0	65.0	10.0
Barley	85.0	71.4	7.3
Beans	86.0	66.0	19.7
Peas	86.0	69.0	18.2
Linseed	93.0	119.0	18.8
Linseed Meal	88.0	64.0	30.4
Maize, Flaked	89.0	86.0	9.2
Maize, Gluten	90.0	76.0	19.2
Weatings	86.0	56.5	10.8
Bran	87.0	43.0	10.0
Oatfeed (estimate)	90.0	50.0	3.0
Dried Grass 16% C.P.	90.0	51.0	10.7
Dried Grass 14% C.P.	90.0	49.5	8.8
Silage Arable	25.0	12.8	1.6
Silage Grass	24.0	12.6	1.7
Silage Grass (poorer)	25.0	10.5	1.5
Hay, medium/good	85.0	32.0	4.6
Hay, first quality	85.0	48.0	7.8
Hay Seeds (good)	85.0	40.0	7.1
Hay Seeds (poorer)	86.0	30.0	4.9
Straw, Oats	86.0	20.0	0.7

APPENDIX TABLE I (continued)

Nutritive Value of Foodstuffs

Foodstuffs	DM	SE	PE
Linseed Threshed Dust	88.0	50.0	2.6
Flax Chaff	87.0	60.0	3.5
Kale, marrow stem	15.0	9.0	1.3
Kale, thousand head	15.8	10.0	1.5
Mangolds	12.0	6.0	0.4
Turnips	8.5	4.4	0.4
Swedes	11.5	7.3	0.7
Cabbage	13.0	8.0	1.3
Rape	14.0	7.0	1.8
Parsley, Parsnips	15.0	11.0	0.8
Carrots	13.0	9.0	0.8
Potatoes	24.0	19.0	0.9
Sugar Beet Tops	16.0	8.6	1.2
Sugar Beet Pulp (Wet)	15.0	12.0	1.0

APPENDIX TABLE II

Percentage of Nutrients from Various Food-Categories in 5 Yield Groups of 10 Herds each

(3 monthly means) Daily Yield per Cow in Milk	Source of Nutrients																			
					Concentrates						Succulents									
	Total Concentrates		Total Roughage		Cakes and Meals		Grains		Hay and Straw		Silage and Kale		Roots		Total Succulents					
	SE	PE	SE	PE	SE	PE	SE	PE	SE	PE	SE	PE	SE	PE	SE	PE				
gallons	%	%	%	%	%	%	%	%	%	%	%	%	%	%	%	%				
3.2	50.3	63.0	49.7	37.0	32.3	49.4	18.0	13.7	29.5	24.1	10.8	8.9	9.4	3.9	20.2	12.8				
2.6	41.3	52.0	58.7	48.0	31.4	43.7	9.9	8.4	35.6	30.6	17.4	15.1	5.7	2.3	23.1	17.4				
2.4	35.3	47.6	64.7	52.4	21.3	35.3	14.0	12.4	43.0	36.5	15.8	13.3	5.9	2.5	21.7	15.8				
2.0	37.8	46.9	62.2	53.1	24.8	35.0	13.0	11.9	48.2	42.9	9.5	8.3	4.5	2.0	14.0	10.3				
1.7	29.3	38.8	70.7	61.2	20.8	30.5	8.5	8.3	44.7	41.3	16.7	15.1	9.3	4.8	26.0	19.9				

APPENDIX TABLE III

Test of Significance of the Difference between the Slopes of Regression of SE on Output of 34 Shorthorn and 21 Friesian Herds

Coefficients of regression: Shorthorn b_s, Friesian b_f

Sum of squares of error of regression: $SS_e = S_y{}^2 - \frac{(S_{xy})^2}{S_x{}^2}$

Variance of coefficient of regression: $V_b = \frac{SS_e}{S_x{}^2(n-2)}$

Variance of difference between coefficients of regression: $V(b_s - b_f) = V_{b_s} + V_{b_f}$

Standard error of difference: $s(b_s - b_f) = \sqrt{V_{b_s} - V_{b_f}}$

$$t = \frac{b_s - b_f}{s(b_s - b_f)}$$

34 Shorthorn herds	21 Friesian herds
$b_s = 5.8555$	$b_f = 3.1574$
$SS_e = 207.826$	$SS_e = 150.58$
$V_{b_s} = 1.09817$	$V_{b_f} = 0.960754$

Difference between b_s and b_f:

$V(b_s - b_f) = 2.05892$ $\quad$ $S(b_s - b_f) = 1.43492$

$$t = \frac{5.8555 - 3.1574}{1.43492} = 1.8003 \text{ with 53 d.f.} = \text{significant at .1 level.}$$

APPENDIX TABLE IV

Calculation of the Standard Errors of the Variance (V_e) and Variance Ratio (VR)

(Significance test of fit of regression equations)

(*a*) For joint regression equation with additional term (C):

$$V_e = \frac{S_{x_1} - (hS_{x_1x_2} + eS_{x_1x_3} + gS_{x_1x_4})}{n - 4}$$

Example: 21 Shorthorn herds.

$S_{x_1} = 11 = 3.4694$
$S_{x_1x_2} = 12 = 19.582$
$S_{x_1x_3} = 13 = 3.849$
$S_{x_1x_4} = 14 = 54.686$

$$V_e = \frac{3.4694 - (3.490119 + 5.341523 - 5.435132)}{n - 4} =$$

$$= \frac{3.4694 - 3.39651}{17} = \frac{0.07289}{17} = 0.00428765$$

Sum of Squares of		*n*	Variance
Total	3.4694	20	
Regression	3.39651	3	1.13217
Error	0.07284	17	0.00428765

$$VR = \frac{1.13217}{0.00428765} = 264.054^{***}$$

(*b*) For 3-variable quadratic equations:

$x_4 = x_2{}^2{}_1\ x_5 = x_3^2$
$S_{x_1x_4} = 258.16$
$S_{x_1x_5} = 11.64$
($S_{x_1} - x_3$ as in (*a*) above)

$$V_e = \frac{11 - (b_2 12 - b_3 13 - b_4 14 - b_5 15)}{n - 5}$$

Example: 21 Shorthorn herds.

$$V_e = \frac{3.469.4 - (-10.65465 - 15.6263 + 15.48573 + 12.28497}{n - 5} =$$

$$= \frac{3.4694 - 1.48975}{16} = \frac{1.97965}{16} = 0.123728$$

Sum of Squares of		*n*	Variance
Total	3.4694	20	
Regression	1.48975	4	0.37243
Error	1.97965	16	0.123428

$$VR = \frac{0.37243}{0.123728} = 3.0174^{*}$$

APPENDIX TABLE V

Linear Production Surfaces in Tabular Form

When PE = 1lb. / SE = 1lb.	1.2	1.5	1.8	2.1	2.4
	Yield per Cow, Gallons				
	21 Shorthorn Herds*				
11	2.68	2.78	2.88	2.98	3.09
9	2.48	2.59	2.69	2.80	2.89
7	2.29	2.39	2.49	2.60	2.70
5	2.10	2 20	2.30	2.40	—
3	1.90	2.00	2.11	—	—
	10 Friesian Herds*				
13	—	—	3.61	3.81	4.01
11	2.98	3.18	3.37	3.57	3.77
9	2.74	2.93	3.13	3.33	3.52
7	2.50	2.69	2.89	3.04	—
5	2.25	2.45	2.65	—	—

When PE = 1lb. / SE = 1lb.	1.8	2.3	2.8	3.3
	Yield per Cow, Gallons			
	Holmes' Ayrshires: 5 Groups†			
14.5	3.64	3.75	3.86	3.97
13.5	3.62	3.73	3.84	3.96
12.5	3.60	3.71	3.82	3.94
11.5	3.58	3.70	3.81	3.92
	Holmes' and Others' Ayrshires: 6 Groups†			
14.5	3.78	—	—	3.97
11.5	3.17	—	—	3.82

*3-monthly means in mid-lactation

†3-monthly means early in the lactation.

APPENDIX VI

(Joint) Production Surfaces of Shorthorn (S), Friesian (F) and Ayrshire (A) Cows.

3 monthly averages of daily inputs above maintenance and outputs (S and F in mid-lactation, A in early lactation) and average inputs per gallon.

Input PE lb. →	0.9			1.2			1.5			1.8			2.1			2.4			2.7			3.0			3.3		
↓ SE lb.	S	F	A	S	F	A	S	F	A	S	F	A	S	F	A	S	F	A	S	F	A	S	F	A	S	F	A
	Output per Cow—Gallons																										
4	1.83	—	—	2.04	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
5	1.95	—	—	2.14	2.19	—	2.32	2.46	—	2.51	(2.72)	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7	2.18	—	—	2.33	2.51	—	2.47	2.73	—	2.62	2.96	—	2.76	3.18	—	—	—	—	—	—	—	—	—	—	—	—	—
8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9	—	—	—	2.52	2.83	—	2.62	3.01	—	2.73	3.20	—	2.83	3.38	—	2.93	3.56	—	—	—	—	—	—	—	—	—	—
10	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
11	—	—	—	—	(3.15)	—	2.77	3.29	—	2.83	3.44	(3.55)	2.90	3.58	3.63	2.96	3.73	3.70	—	—	3.76	—	—	—	—	—	—
12	—	—	—	—	—	—	—	—	—	—	3.68	(3.63)	2.93	3.69	3.69	2.97	3.82	3.75	—	—	3.81	—	—	—	—	—	—
13	—	—	—	—	—	—	—	—	—	—	—	—	—	3.79	(3.75)	—	3.90	3.80	—	—	3.85	—	—	3.89	—	—	—
14	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	(3.85)	—	—	3.89	—	—	3.92	—	—	3.96
15	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3.93	—	—	3.95	—	—	3.97
	lb. SE Input per Gallon at above Outputs																										
5	2.56	—	—	2.34	2.28	—	2.16	2.03	—	1.99	(1.83)	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7	3.21	—	—	3.00	2.79	—	2.83	2.56	—	2.67	2.36	—	2.54	2.20	—	—	—	—	—	—	—	—	—	—	—	—	—
9	—	—	—	3.57	3.18	—	3.44	2.99	—	3.30	2.81	—	3.18	2.66	—	3.07	2.53	—	—	—	—	—	—	—	—	—	—
11	—	—	—	—	(3.49)	—	3.97	3.34	—	3.89	3.20	(3.10)	3.79	3.07	3.03	3.72	2.95	2.97	—	—	2.93	—	—	—	—	—	—
13	—	—	—	—	—	—	—	—	—	—	—	—	—	3.43	3.47	—	3.33	3.42	—	—	3.38	—	—	3.34	—	—	—
15	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3.82	—	—	3.80	—	—	3.78
	lb. PE Input per Gallon at above Outputs																										
5	0.46	—	—	0.56	0.55	—	0.65	0.61	—	0.72	(0.66)	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7	0.41	—	—	0.52	0.48	—	0.61	0.55	—	0.69	0.61	—	0.76	0.66	—	—	—	—	—	—	—	—	—	—	—	—	—
9	—	—	—	0.47	0.42	—	0.57	0.50	—	0.66	0.56	—	0.74	0.62	—	0.82	0.67	—	—	—	—	—	—	—	—	—	—
11	—	—	—	—	(0.34)	—	0.54	0.46	—	0.64	0.52	(0.51)	0.72	0.59	0.58	0.81	0.64	0.65	—	—	0.72	—	—	—	—	—	—
13	—	—	—	—	—	—	—	—	—	—	—	—	—	0.55	(0.56)	—	0.62	0.63	—	—	0.70	—	—	0.77	—	—	—
15	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0.69	—	—	0.76	—	—	0.83

The figures in brackets are at the fringe of the available observations.

APPENDIX TABLE VII

Computation of Daily Input Curves

Inputs of Cake Equivalent above maintenance, Estimated by Trial and Error. Outputs at given stages of lactation estimated from lactation curves in Figure XIII.

Stage of Lactation		Input of Cake Equiv. Average daily	Input of Cake Equiv. Total	Output average daily	Input of Cake Equiv. Average daily	Input of Cake Equiv. Total	Output average daily
		lb.	lb.	gallons	lb.	lb.	gallons
		Average FRIESIAN COWS					
days:		at 900 gallons yield			at 1200 gallons yield		
First	7	6.6	46.2	1.76	8.7	60.9	2.20
Following	21	17.0	357.0	3.70	26.4	554.4	4.75
,,	14	21.1	295.4	4.21	32.8	459.2	5.25
,,	14	21.0	294.0	4.19	33.9	474.6	5.31
,,	14	19.7	275.8	4.04	32.2	450.8	5.22
,,	35	16.9	591.5	3.68	28.4	997.5	4.92
,,	35	14.6	511.0	3.29	23.3	815.5	4.45
,,	70	11.4	798.0	2.75	17.9	1253.6	3.80
,,	70	7.6	532.0	1.94	13.8	910.0	2.98
,,	25	4.5	112.5	1.30	7.8	195.0	2.00
Total	305	12.51*	3813.4†	2.95*	20.32*	6170.9†	3.93*
		Average SHORTHORN COWS					
days:		at 700 gallons yield			at 900 gallons yield		
First	7	5.0	35	1.50	7.0	49	1.80
Following	21	17.1	359	3.45	27.7	582	3.85
,,	14	19.0	266	3.59	36.7	514	4.20
,,	14	17.1	239	3.45	36.7	514	4.20
,,	49	13.7	671	3.08	27.8	1362	3.86
,,	35	10.5	368	2.60	19.4	679	3.37
,,	35	8.9	312	2.28	14.0	522	2.92
,,	35	7.4	259	1.95	11.7	409	2.60
,,	35	6.2	217	1.67	9.0	315	2.20
,,	35	4.8	168	1.34	6.2	217	1.67
,,	25	3.8	95	1.07	4.4	110	1.28
Total	305	9.8*	2989†	2.30*	17.3*	5273†	2.95*

*Weighted averages.

†The calculated inputs of cake equivalent are:

for Friesians 3816 lb. and 6198 lb. respectively.

for Shorthorns 2989 lb. and 5277 lb. respectively.

APPENDIX TABLE VIII

Friesian and Shorthorn Cows of Different Yield Capacities

Construction of Daily Cake Equivalent Input Curves from Broken-Down Mathematical Curves (*Italics*)

Yield Capacity	I* Daily Yield in Mid Lactation	II† Cake Equivalent Input at I	III Daily Yield at Peak of Lactation	IV‡ Cake Equivalent Input at III	V When Total Yield pushed up 200 gal. to	VI Peak Yield of V pushed up to	VII§ Cake Equivalent Input at VI
gal.	gal.	lb.	gal.	lb.	gal.	gal.	lb.
				Friesian Cows			
		(at 4.17 lb. per gal.)		(at 5.01 lb. per gal.)			(at 5.81 lb. per gal.)
800	2.62	10.93	3.90	19.54	1000	4.60	26.73
900	*2.95*	*12.30*	*4.25*	*21.30*	*1100*	*4.95*	*28.76*
1000	3.29	13.72	4.60	23.05	1200	5.30	30.79
1200	3.95	16.47	5.30	26.55	1400	6.00	34.86
1400	4.61	19.22	6.00	30.06	1600	6.65	38.64
1600	5.26	21.93	6.65	33.32	1800	7.35	42.70
				Shorthorn Cows			
		(at 3.97 lb. per gal.)		(at 5.21 lb. per gal.)			(at 8.85 lb. per gal.)
600	1.98	7.86	3.23	16.83	800	3.90	34.52
700	*2.32*	*9.20*	*3.57*	*18.60*	*900*	*4.25*	*37.60*
800	2.62	10.40	3.90	20.32	1000	(4.60)	(40.71)
1000	3.29	13.06	4.60	23.97	1200	(5.30)	(46.91)
1200	3.95	15.68	5.30	27.61	1400	(6.00)	(53.10)
1400	4.61	18.30	6.00	31.26	1600	(6.65)	(58.85)

*From Table 38

† ‡ § Calculated at rate per gallon as found from broken-down mathematical curves:

	Friesian Cows	*Shorthorn Cows*
†	$\frac{12.3 \text{ lb.}}{2.92 \text{ gal.}} = 4.17$ lb. per gal.	$\frac{9.2 \text{ lb.}}{2.32 \text{ gal.}} = 3.97$ lb. per gal.
‡	$\frac{21.3 \text{ lb.}}{4.25 \text{ gal.}} = 5.01$ lb. per gal.	$\frac{18.60 \text{ lb.}}{3.57 \text{ gal.}} = 5.21$ lb. per gal.
§	$\frac{28.9 \text{ lb.}}{4.97 \text{ gal.}} = 5.81$ lb. per gal.	$\frac{37.6 \text{ lb.}}{4.25 \text{ gal.}} = 8.85$ lb. per gal.